Thirty-Ninth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

PME-NA 39 2017

SYNERGY AT THE CROSSROADS

Editors:
Enrique Galindo
and Jill Newton

HOOSIER AMTE

CONFERENCE PROCEEDINGS

Indianapolis, IN | October 5-8, 2017

Proceedings of the Thirty-Ninth Annual Meeting of the North American Chapter of the International Group for the

Psychology of Mathematics Education

Synergy at the Crossroads: Future Directions for Theory, Research, and Practice

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Revision

These proceedings have been revised as follows.

- Duplicate page numbers were removed from chapter 13.
- The correct paper for the working group "Developing a Research Agenda of Mathematics Teacher Leaders and their Preparation and Professional Development Experiences" has been included in chapter 14.
- The table of contents for chapter 14 has been updated.

PME-NA History and Goals

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of the International Group of Psychology of Mathematics Education. The first PME-NA conference was held in Evanston, Illinois in 1979.
The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

Welcome

On behalf of the 2017 PME-NA Steering Committee, the 2017 PME-NA Local Organizing Committee, and the Hoosier Association of Mathematics Teacher Educators (HAMTE), we welcome you to the 39th Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter held at the Crowne Plaza Indianapolis Downtown Union Station in Indianapolis, Indiana.

The theme of this year's conference is *Synergy at the Crossroads: Future Directions for Theory, Research, and Practice*. The metaphor of crossroads was inspired by the conference venue - the historic Indianapolis Union Station, as well as by the State motto, a reference to how Indiana is connected to the rest of the United States. PME-NA 39 includes research presentations, discussion, and reflection focusing on four driving questions connecting to the metaphor of crossroads: 1) What have we learned from the routes we have traversed, what are potential routes for mathematics education research in the future, and what considerations are relevant as we make choices about future directions in mathematics education? 2) How do we address issues of access and equity within mathematics education today? 3) How can we lay the groundwork for future crossroads or intersections between theory, research, and practice? and 4) What barriers within research traditions, educational policy, and teaching practice impede researchers', students' and teachers' success and how can we work to overcome these barriers?

Rochelle Gutiérrez will present the opening plenary talk on Thursday evening, *Living Mathematx: Towards a Vision for the Future*, into which she brings ideas from ethnomathematics, postcolonial theory, aesthetics, biology, and Indigenous knowledge in order to propose a new vision for practicing mathematics. Edd Taylor will serve as discussant for the talk. In the Friday afternoon plenary session, Les Steffe will present several crucial radical constructivist research programs to argue that rather than repeat attempts to make wholesale changes in mathematics education based on mathematical knowledge for adults, what is needed is to construct mathematics curricula *for* children that is based on the mathematics *of* children. Two of Dr. Steffe's former students, Erik Tillema and Amy Hackenberg, will serve as discussants, providing varied perspectives on the continuation of his work. Saturday's plenary session, *Elementary Mathematics Specialists: Ensuring the Intersection of Research and Practice* will include a historical overview by Maggie McGatha followed by a discussion panel composed of Dionne Cross and Jane Mahan, facilitated by Sheryl Stump. A panel discussion of technology in mathematics education with representatives from the three PME-NA member countries will complete the plenaries on Sunday: Ana Isabel Sacristán (*Digital Technologies in Mathematics Classrooms: Barriers, Lessons and Focus On Teachers*); Nathalie Sinclair (*Crossroad Blues*); and Karen Hollebrands (*A Framework to Guide the Development of a Teaching Mathematics with Technology Massive Open Online Course for Educators [MOOC-ED]*).

This year's conference will be attended by about 550 researchers, faculty and graduate students from around the world including the US, Mexico, Canada, Turkey, Australia, South Korea, Malawi, and Iran. We received 529 submissions. The acceptance rate was 39% for research reports as research reports, 57% for brief research reports as brief research reports, 78%

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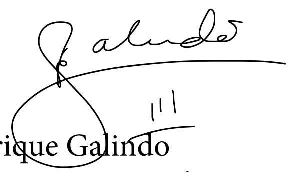
for posters as posters, and 100% for working groups. The accepted proposals included 75 research reports, 142 brief research reports, 167 posters, and 13 working groups. Continuing the efforts started at last year's conference there will be some presentations in Spanish, as well as simultaneous oral interpretation (from English to Spanish, and from Spanish to English) for selected sessions.

We would like to thank the many people who generously volunteered their time over the past year in preparation for this conference. This includes members of the PME-NA Local Organizing Committee, the PME-NA Steering Committee, Purdue Conferences, strand leaders, proposal authors and reviewers. We appreciate all of your hard work and dedication, and your commitment to ensure a high-quality conference program. We also wish to thank the generous financial support of the HAMTE member universities across Indiana.



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Contents

Chapter 1

Plenary Papers

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

LIVING MATHEMATX: TOWARDS A VISION FOR THE FUTURE[i]

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This paper offers specific implications for teaching and learning and brings into conversation ideas from ethnomathematics (including Western mathematics), postcolonial theory, aesthetics, biology, and Indigenous knowledge in order to propose a new vision for practicing mathematics, what I call mathematx. I build upon the work of sustainability in mathematics education and suggest we need to think not only about more ethical ways of applying mathematics in teaching and learning but question the very nature of mathematics, who does it, and how we are affected by that practice.

Keywords: Equity and Diversity, Instructional Activities and Practices

> We need to be constantly considering the forms of mathematics and what they seek to deal with. As society presents new demands, new technologies, new possibilities, we must ask ourselves whether our current version of mathematics is adequate for dealing with the ignorance that we have (Gutiérrez and Dixon-Roman, 2011, p. 32).
>
> The ecology of knowledges enables us to have a much broader vision of what we do not know, as well as what we do know, and also to be aware that what we do not know is our own ignorance, not a general ignorance (Santos, 2007, p. 43).
>
> We are all the product of our worldview—even scientists who claim pure objectivity…Science and traditional knowledge may ask different questions and speak different languages, but they may converge when both truly listen to the plants (Kimmerer, 2013, p.163, 165).

Everyday, we accumulate more evidence that humans are destroying the planet. We need only look at the increasing levels of air pollution, climate change, destruction of the ozone layer, and the elimination of various plant and animal species throughout the world to know that we cannot continue with the forms of living we have come to consider "normal." However, not until recently has the public become aware that the effects will deeply impact us in our lifetime (Kolbert, 2015). One might ask: what role(s) should mathematics play in stopping or slowing the rate of such destruction of the environment? The field of mathematics might serve mainly to: describe the nature of the global problem; offer excellent models for prediction; or provide efficient data analysis and statistics for calculating risk. Mathematics might also offer something else altogether. In what way(s) are current forms of mathematics teaching and learning consistent with the kinds of environmental crises we face? Do we need to think differently about our relationship between mathematics, humans, and the planet? And, if so, how?

In this article, I seek to bring into conversation ideas from ethnomathematics (including Western mathematics), postcolonial theory, aesthetics, biology, and Indigenous knowledge in order to propose a new vision for practicing mathematics, something I refer to as *mathematx*. I do so in order to promote interaction between different knowledges, different ways of knowing, and different knowers. I build upon the work of sustainability in mathematics education and suggest we need to think not only about more ethical ways of applying mathematics in teaching and learning but question the very nature of mathematics, who does it, and how we are affected by that practice. I introduce the concepts of In Lak'ech, reciprocity, and Nepantla to suggest we learn from other-than-human persons, which, in turn, may change our relationships with them. Along the way, I underscore with examples from biology the potential limitations of current forms of mathematics for understanding/interacting with our world and the potential benefits of considering other-than-human persons as having different knowledges to contribute. Finally, I suggest implications for teaching and learning.

Identifying the Problem

The relationship between mathematics, humans, and the planet has been one steeped too long in domination and destruction (O'Neil 2016; Martinez 2016). Due in large part to the way research is funded, the field of mathematics is often in the service of warfare and economics (BooB-Bavnbek and Hoyrup 2003; Gutiérrez 2013; Martinez 2016; O'Neil 2016; Porter 1995). With an emphasis on quantifying, categorizing, and reducing complex and multi-layered relationships between persons to mere abstractions, mathematics often supports a fallacy that modeling, big data, and software can solve anything. Some might suggest there is nothing inherent in the practice of mathematics that leads to domination; we simply need to follow more ethical practices in applying mathematics in the world around us.

Highlighting this role of domination and arguing for a new form of teaching mathematics, Coles and colleagues (2013) note,

> The history of humanity's relationship with the natural environment, at least in the West, can be summarized in one word: domination. The natural environment has been seen as a source of food and raw materials all to be placed in the service of human projects. Where the natural environment gets in the way of such projects, we simply blast our way through… (p. 4)

In an attempt to change this relationship, Coles and colleagues suggest we begin by altering the forms of teaching and the curriculum to which students are exposed. By situating mathematical problems in contexts that relate to such issues as climate change, students will have the opportunity to develop a new relationship to mathematics and new uses of mathematics in making life decisions. That is, students can be encouraged to analyze real-world statistics of temperatures in different regions to make conclusions about both the rates by which the climate is changing and the probabilities that the climate will continue to change. In this way, students would also be allowed to ponder such questions as what kind of mathematical information is necessary to address climate change? What mathematics should the average citizen know in order to make informed decisions about the consequences of their actions and the actions of others? Learning mathematics in real world social and political contexts can help students see relationships between the decisions humans make and the destruction of the planet, thereby urging them to take action to save the planet. In this way, mathematics education can more clearly highlight the roles of ethics (e.g., Atweh 2013; Boylan 2016) and practicality as they relate to the practice of mathematics. Thus, shifting the curriculum to more sociopolitical contexts (Gutiérrez 2010/2013[ii]), what some would refer to as teaching mathematics for social justice (Frankenstein 1990; 1995; Gutstein 2006), could broaden the service of mathematics beyond economics and warfare.

However, attending to *when* and *how* mathematics is in the service of sustainability or ethics may be a necessary but insufficient step towards new relationships between humans, mathematics, and the planet (Gutiérrez, 2002). This, for me, has been one limitation of social justice mathematics (Gutstein 2003; 2006; 2007), as it tends to assume we will keep intact as "classical" what I refer to as "dominant" mathematics rather than challenging whether that version or any single version should remain central. In the social justice mathematics tradition, students are taught to use classical mathematics as a tool to read and write the world, in order to develop their sociopolitical consciousness and mathematical proficiencies. But, in general, the tool itself is not questioned. Recognizing the limitations of using the master's tools to dismantle the master's house (Lorde 1984) leads me to argue that we must also be willing to question and reconceptualize what counts as mathematics in the first place, thereby taking up issues of epistemology and ontology.

I am not alone in suggesting we need to reconsider our definitions of mathematics in light of our current state of global crises. For example, Appelbaum (2016) suggests a different approach through curriculum, where a key component is questioning what counts as mathematics.

> …one key curriculum question that can no longer be pushed to the side is how very narrow, Western, "rational" conceptions of what mathematics "is" have continued to be wielded implicitly as tools of epistemicide, obliterating alternative epistemologies of number, size, quantity, possibility, shape, algorithmic problem solving, analogic representation, and other extended components of mathematical thinking and living. (p. 5)

Similarly, Boylan (2016) considers the role of mathematics in relation to the planet and argues,

> An ecological ethics calls not only for an environmentally informed critical mathematics education but also for a critique of the social construction of mathematics itself as separate and disconnected from the earth (p. 9).

The Program Ethnomathematics offers a useful starting point for broadening the definition of mathematics, something I will discuss later in this article.

Not only must we: a) be conscious of the ways mathematics can dominate and b) constantly question what counts as mathematics and who decides, we must also c) think about how we, as living beings, practice mathematics as we interact with others and ourselves. As we begin to reimagine mathematics, we have the opportunity to reimagine the mathematician—who is considered a mathematician as well as how are mathematicians influenced by the mathematics they do? Many of the current efforts to reconsider mathematics and its role in our global society tend to rely upon a utilitarian version of mathematics that allows us to better survive on this planet. I am suggesting that a form that describes moving through the world and relates to all living beings is more likely to change our relationships with each other in this universe or in others. We need a definition that acknowledges mathematics as a verb and how that practice relates to our bodies, minds, and intentions. For that, we might consider our philosophical stance.

Much of the philosophical research produced in mathematics education centers on European thinkers. For example, we are abundant with theories of postmodernism, poststructuralism, and psychoanalysis that regularly draw upon such writers as Deleuze and Guattari, Ranciere, Foucault, Lacan, Badiou, Derrida, and Freud. As a Chicanx scholar, a cis gender female with Rarámuri[iii] roots, I seek to decenter the field's overreliance on Whitestream views. I use the term Chicanx (as opposed to Chicano, Chicana/o, or Chican@) as a sign of solidarity with people who identify as lesbian, gay, bisexual, transgender, queer, questioning, intersexual, asexual, and two-spirit[iv] (LGBTQIA2S). Chicanx represents both a decentering of the patriarchal nature of the Spanish language whereby groups of men and women are normally referred to with the "o" (male) ending as well as a rejection of the gender binary and an acceptance of gender fluidity. The "x" signifies a variable to represent any gender form. My choice to use this term reflects my respect for how people choose to name themselves.

In this article, I introduce three Indigenous concepts that have guided my work over the years—In Lak'ech, Nepantla, and reciprocity—and suggest they can serve as guiding principles of a new practice of mathematics.

Indigenous Epistemologies

Why privilege Indigenous concepts when considering the relationship between mathematics, humans, and the planet? The answer to that question lies partly in the way (Western) mathematics is viewed as universal (being able to explain everything in reality) and highly valued in society. When challenges of discontinuity or undecidability arise, mathematicians often protect the universal view by suggesting that mathematics still holds true if we simply begin with different axioms (Barrow 1992). Yet, no knowledge could fully describe or attend to our universe and our relations with/in it. If we look to the role that Aboriginal[v] knowledges have played in the reading of signs of distress from the land (i.e., predicting the global crises we face), the preservation of biodiversity, and the role of survival in general, we see the limits of Western mathematics/science practices as a means for intervention (Berkes et al. 2000; Brayboy and Maughan 2009; Cajete 1999; Deloria 1979; González 2001; Heinrich, et al. 1998; LaDuke 1994; Little Bear 2000; 2009; Tallbear 2013; Watson-Verran and Turnbull 1995). I claim neither that all Western thought is colonizing/hegemonic nor that all Indigenous thought does not have the ability to dominate. However, modern Western thinking has been hegemonic in ways that erase Indigenous thought. In this way, I use the term "Western" to refer to the modern version that has tended to colonize and "Indigenous" or "Aboriginal" to refer to the version that has tended to be erased throughout history.

Acknowledging the limits of Western mathematics is not to discount the value of mathematical knowledge in other realms. However, such limitations suggest that, in contrast to the global push to get more students to enter Science, Technology, Engineering, and Mathematics (STEM) fields in order to deal with the complexity and challenges in our world, we cannot fully address our problems through a reliance on Western mathematics/science.

Santos (2007) suggests that the problem of domination may lie not in *which* knowledge is authoritative, but rather in our overreliance on any single knowledge as authority. As such, he suggests an epistemology of knowledges, underscoring the view that all knowledge is legitimate, partial, and interdependent. In fact, with respect to ignorance, learners do not just lack knowledge, they have "misknowledges" (i.e., stereotypes, incorrect knowledge) about others (Kumashiro 2001). And, those misknowledges may not easily be replaced by the introduction of new knowledge because desconocimiento (ignorance) can be a "refusal to know" when what is new disrupts what was previously believed to be true (Anzaldúa 2000).

Yet, from a postcolonial perspective, it is important to unlearn what one thinks one knows, both to recognize a form of epistemological arrogance (thinking that one's ways of knowing are superior to others') and to learn to see oneself in relation to others (Andreotti, Ahenakew, and Cooper 2011). Such a perspective acknowledges that our ignorance is our own, not a general form that cannot be known or is not yet known (Santos 2007). That is, just as there is no unity of knowledge, there is no unity of ignorance. Each of us has knowledge and ignorance that is, to a certain extent, unique. Consistent with this epistemological pluralism, some scientists have argued against trying to develop a theory of everything (Gleiser 2015).

Ecology of knowledges does not follow a single abstract universal hierarchy among knowledges. Rather, it sees knowledge practices as context dependent. In that sense, it recognizes that different knowledges can address our understanding and ability to relate to one another depending upon our different purposes (e.g., the ways we aim to connect, the problems we seek to solve, the ways we invite joy into our lives) (Little Bear 2009). For example, by seeking to be predictive, generalizable, reductionist, and quantifiable in nature, Western perspectives tend to privilege knowledge as a form of (re)presentation and explanation of reality (Aikenhead and Michell 2011). Yet, given the global crises we face, we might be better served by knowledge as action—a form of intervention (Santos 2007; Andreotti 2011).

Given these different purposes, it is important to create inter-knowledges, whereby learning another's knowledge does not negate knowing one's own knowledge (Santos 2007). In this way, learning how other living beings perform mathematics does not eliminate what is known in terms of academic mathematics. But, it does help us know what we do not know. Recognizing these inter-knowledges can go a long way towards embodying humility and establishing the need for responsibility, and therefore reciprocity, *toward* another, as opposed to *for* another (Spivak, 1987).

While Santos is referring to an epistemology of knowledges that would include scientific/mathematical versus social scientific, I am arguing that within mathematics, we might acknowledge and value an epistemology of knowledges. That is, mathematically, we might come to see that different ways of knowing, different knowers, and different forms of knowledge are all legitimate, partial, and interdependent. Epistemological pluralism recognizes that there will be tensions, contradictions, and politics in translating Indigenous knowledges into Western categories/languages (Andreotti, et al. 2011). As such, an epistemology of knowledges is destabilizing because it interrogates the politics of knowledge and, unlike Western knowledge, does not presume causal outcomes—that is, that we can know the potential from any given actual. Therefore, the production of knowledge is an ongoing process that is not cumulative but relational.

Centering Indigenous Knowledges

To be clear, there is no universal "Indigenous worldview." Within the US, alone, there are 567 peoples federally recognized as American Indian and many more that are not recognized. Within México, there are 62 peoples recognized as Indigenous, comprising 13 percent of the nation's population. Within Canada, there are 634 peoples recognized as First Nations, plus peoples who are Métis and Inuit, all accounting for 5.6 percent of the nation's population. And, these populations cover only North America, not the globe. The use of particular languages and ties to particular lands create unique views held by Aboriginal peoples throughout the world and by individuals within those groups. And, many Aboriginal writers refuse to refer to themselves as Indigenous, Indian, or First Nations, as those categories are reflections of a colonizing history that blurs specificity. Even so, at times, "strategic essentialism" (Spivak 1987) is important for joining peoples and advancing common resistance tactics. As such, I speak of commonalities across the range of Indigenous knowledges. The perspectives I share are my view and do not necessarily reflect the views of others.

Indigenous knowledges recognize that we are part of a system of intelligent and sentient beings, also referred to as persons, with interconnected spirits, including rocks and bodies of water. Plants, for example, have lived on this planet for millions of years before humans. In that sense, plants are our older brothers/sisters and have developed ways of efficiently using space, relating with other living beings, and sustaining life not just for themselves but for others, often with few resources at any given moment. They have been able to withstand long droughts, communicate about impending dangers, and collaborate in order to protect others in the community in ways that appear to be selfless acts. They have much to teach us; and we may have something to teach them. Breaking with a human/non-human binary is consistent with queer theory, which recognizes the violence that is justified when some are viewed to be more human than others (Chen 2012).

Our choice to destroy the planet to serve our immediate/capitalistic/technology needs is a form of settler colonialism that perpetuates violence. That is, because a Western worldview does not consider plants, animals, and rocks as living beings of equal value with the same rights to this universe as humans, the result is that plants, animals and rocks suffer the same treatment as Indigenous peoples have endured throughout time. For example, like American Indians who were stripped of their lands and communities and forced to live in boarding schools, plants are yanked from their families and forced to assimilate into Western ways of doing things (e.g., to become suburban gardens). By respecting animals, plants, and even rocks as living beings, we can avoid some of the human/material binary that has plagued the sciences in the past.

By referring to humans as a young species, I do not mean to imply a sense of posthumanism or transhumanism. That is, I am not looking to make humans better or into a fuller version of themselves by combining with technology, fiction, or art (Haraway 1990; but also Chela Sandoval's extension). An Indigenous perspective, for me, seeks not to transform humans into another form of being; rather it serves to help us recognize our place in this world as the younger brothers/sisters of animals, plants, and rocks who have much to teach us about making sense of and remaining connected to this planet and possibly other planets. In this sense, by changing our world view—how we move through this world and possibly into others—we will necessarily change ourselves, but not in a way that is separate from other living beings, not in a way that is necessarily tied to technology. There may be things we cannot yet access or understand because we are a young species. Other persons may have ways of accessing information that can be helpful for us.

While our Elders have long spoken of the sentient capabilities of plants and rocks and of the collective spirit they/we share, only recently have modern scientists begun to acknowledge that claim with experiments that prove this to be the case, suggesting trees are sentient and intelligent (Haskill 2017; Jahren 2017; Wohlleben 2016). For example, tracing isotopes of carbon dioxide gas offered to sample trees shows they turn that carbon dioxide into sugars that travel down through the trunk and use a complex system of roots, fungi, and mycelium to share that resource with other trees nearby, even trees of a different species (Simard et al. 2012). Similarly, when a tree is injured or attacked by pests, it is able to communicate by way of pheromones to nearby trees to tell them to start changing the chemistry of their leaves to be unfavorable to the intruder (Wohlleben 2016). And, mother trees are able to both reduce their root system to make room for their offspring as well as send defense signals through their mycorrhizal network to increase the resistance of their offspring to future stress (Teste, et al. 2009).

Beyond embracing the intelligence and sentience of other living beings, Indigenous epistemologies connect place, body, spirit, and consciousness. They reflect understandings of land, history, culture, identity relationships, and therefore, politics (Deloria 1979). Many Indigenous knowledges have been developed with roots in survivance; that is, not surviving in the colonialist depiction of escaping catastrophe or being positioned as victims, but resisting dominance in a way that renews Indigenous knowledges that are particular and have always been present (Vizenor 2008). While there are many Indigenous concepts that could be fruitful to revisioning mathematics, I present three that have been important in my upbringing. I do so in order to set the stage for an epistemology of knowledges that can guide our practice of mathematics.

In Lak'ech

The Mayan definition of human being (huinik'lil) translates to "vibrant being" in recognition of the idea that all human beings are part of a universal vibration (Arguelles 1987; Paredez 1964). Acknowledging that all beings are connected, Mayan philosophy includes the important concept of *In Lak'ech* woven into everyday thought and action. When a person meets another, they begin with the saying In Lak'ech (You are the other me), to which the receiver responds with Ala K'in (I am the other you). This greeting highlights for all persons (human and other-than-human) their connection with each other and the need to protect each other. Consistent with Indigenous knowledge, I use the terms "living beings" and "persons" interchangeably, as each term refers to all things living.

Seeing a version of oneself in other living beings or persons is a powerful reminder to move through the world with compassion, gratitude, and interdependence. For me, In Lak'ech suggests that if we look closely, we can see ourselves in others and others in us, but not in a way that implies an erasure of our uniqueness, even while recognizing that uniqueness does not imply a sense of self without others. To be clear, In Lak'ech does not translate to "I am you; You are me." Seeing a version of oneself in others and others in us is a kind of mirror, an affirmation; while the concept also recognizes we are not exactly the same. In the same way that a mirror refracts light, produces words that are backwards, and has imperfections from the glass, In Lak'ech reminds us that each person is unique. In this sense, other persons also serve as a kind of window, a way of viewing another world, another self, another (possibly better) you.

Over time, Chicanx scholars have brought the concept of In Lak'ech into poems and theater as reminders of how we should move through the world.

> Tú eres mi otro yo.
> You are my other me.
> Si te hago daño a ti,
> If I do harm to you,
> Me hago daño a mi mismo.
> I do harm to myself.
> Si te amo y respeto,
> If I love and respect you,
> Me amo y respeto yo.
> I love and respect myself. (Valdez and Paredez n.d.)

Through this poem and other writings (e.g., Valdez 1971), Valdez highlights the ways in which Chicanx might relate to others in order to move with the cosmos. The meaning of In Lak'ech is similar to the Lakota saying Mitakuye Oyasin "we are all related" (Cajete 1999 cited in Hatcher et al. 2009). The idea that we are all related can, in some ways, bring us joy, a simultaneous affirmation of self and others. Building upon the idea that we are all interconnected, an Indigenous production of knowledge to benefit others is in opposition to knowledge production as performance that benefits mainly oneself and that is seen in most White institutions or places that value Western thought. Brayboy and Maughan (2009) remind us,

> Indigenous communities have long been aware of the ways that they know, come to know, and produce knowledges, because in many instances knowledge is essential for cultural survival and well-being. Indigenous Knowledges are processes and encapsulate a set of relationships rather than a bounded concept, so entire lives represent and embody versions of IK (p. 3).

Reflecting these relationships, In Lak'ech focuses not on description of reality but on movement through the world and metaphysics. By metaphysics, I simply mean a set of first principles by which we make sense of the world around us (Deloria 1979).

Reciprocity

Extending the idea of In Lak'ech, the second concept upon which I draw is reciprocity. The concept of reciprocity highlights the idea that different persons have different strengths and needs, and thus must rely on others for what they lack. More than simply recognizing that reciprocity enables persons to do things they could not otherwise do alone, it underscores a kind of ethic that is valued in maintaining harmony of the cosmos. In this sense, reciprocity is not only the productive thing to do, it is the right thing to do. Whereas In Lak'ech acknowledges the nature of the relationship between self and others, reciprocity highlights the actions that should result.

As a botanist and a member of the Citizen Potawatomi Nation, Kimmerer (2013) weaves the view of a scientist with an Indigenous view on the role of reciprocity and suggests that when we honor other living beings (e.g., plants), it changes our relationships with them. She says,

> When I speak of the gift of berries, I do not mean that Fragaria virginiana has been up all night making a present just for me, strategizing to find exactly what I'd like on a summer morning. So far as we know, that does not happen, but as a scientist I am well aware of how little we do know. The plant has in fact been up all night assembling little packets of sugar and seeds and fragrance and color, because when it does so its evolutionary fitness is increased. When it is successful in enticing an animal such as me to disperse its fruit, its genes for making yumminess are passed on to ensuing generations with a higher frequency than those of the plant whose berries were

> inferior…what I mean is that our human relationship with strawberries is transformed by our choice of perspective…when we view the world this way, strawberries and humans alike are transformed. The relationship of gratitude and reciprocity thus developed can increase the evolutionary fitness of both plant and animal. (p. 29-30)

Can we come to understand mathematics as a living practice that needs actors and can respond to their needs? Are there already ways in which these concepts play into mathematics?

Kimmerer highlights how in the Thanksgiving Address, humans are reminded of the importance of balance and harmony, "We have been given the duty to live in balance and harmony with each other and all living things" (p. 107) and she asks the non-Native reader, "What would it be like to be raised on gratitude, to speak to the natural world as a member of the democracy of species, to raise a pledge of *inter*dependence?" (her emphasis, p. 112)

This is very similar to Cajete's notion of laws of interdependence. What might it look like to view mathematics (what it is, how we practice it, who is considered a mathematician, what knowledge we produce) as having a basis in interdependence? Kimmerer expands,

> Cultures of gratitude must also be cultures of reciprocity. Each person, human or no, is bound to every other in a reciprocal relationship. Just as all beings have a duty to me, I have a duty to them. If an animal gives its life to feed me, I am in turn bound to support its life. If I receive a stream's gift of pure water, then I am responsible for returning a gift in kind. An integral part of a human's education is to know those duties and how to perform them. (Kimmerer, p. 114)

If we keep in mind our duties to others, might we think about the forms of mathematics we are producing and practicing as well as how those forms impact other persons, not just ourselves or other humans?

In describing the relationship between beans, corn, and squash, referred to collectively as Las Tres Hermanas (the Three Sisters), Kimmerer highlights, for me, the particular way in which these sisters perform mathematics.

> The corn stands eight feet tall; rippling green ribbons of leaf curl away from the stem in every direction to catch the sun. No leaf sits directly over the next, so that each can gather light without shading the others. The bean twines around the corn stalk, weaving itself between the leaves of corn, never interfering with their work. In the spaces where corn leaves are not, buds appear on the vining bean and expand into outstretched leaves and clusters of fragrant flowers. The bean leaves droop and are held close to the stem of the corn. Spread around the feet of the corn and beans is a carpet of big broad squash leaves that intercept the light that falls among the pillars of corn. Their layered spacing uses the light, a gift from the sun, efficiently, with no waste. The organic symmetry of forms belongs together; the placement of every leaf, the harmony of shapes speak their message. Respect one another, support one another, bring your gift to the world and receive the gift of others, and there will be enough for all. (p. 131-132)

Phyllotaxis, the study of the ordered position of leaves on a stem, highlights the fact that many plants grow in ways that mirror "Fibonacci[vi]" numbers and the ratios of two consecutive numbers tend towards the golden ratio (Douady and Couder 1992). Interestingly, scientists who have studied Las Tres Hermanas have documented that when grown together, they out-produce what the plants would if cultivated individually (Mt. Pleasant 2006). That is, the corn makes light available; the squash reduces weeds; and the beans turn atmospheric nitrogen into mineral nitrogen fertilizer. Reciprocity is modeled in their relationship. This form of reciprocity is also present in research methods used by indigenous scholars and scholars of color (e.g., Dance, Gutiérrez, and Hermes 2010; Kovach 2009; Rigney 1999; Smith 1999).

Drawing upon ten years of teaching integrative science that acknowledges both Western science and Indigenous sciences, Hatcher et al., (2009) argue that knowledge is only passed on from one living being to another when a relationship between the two is formed and when the receiver is ready. In this sense, knowledge is a verb; teacher and learner both play constructive parts in it, highlighting the role of reciprocity. In fact, the Mi'kmaq word netukulimk means to "develop the skills and sense of responsibility required to become a protector of other species." While a Whitestream view might privilege the problem solving/utilitarian aspect of reciprocity, I see reciprocity (along with In Lak'ech) as related to experiencing connections and joy—knowing that one's actions are positively affecting oneself and others.

The overall point I am making is for us to live in harmony, without domination, as a form of metaphysics, and to continue to note the similarities and differences between our modes of being and those of other-than-human living beings. Recognizing other persons as having something to "teach" us is not to begin with a stance that other living beings are a means to our end, in order to better ourselves and our time on this planet or in our multiverse, though that can be a byproduct. Rather, this stance is simply reflective of a deep belief that we must show respect for others, a form of ethics, because in doing so, we are showing respect for ourselves, a frame of mind consistent with In Lak'ech.

Nepantla

Nepantla is the third concept upon which I draw. Nepantla is the Nahuatl (Aztec) term for the interstitial space between worlds. Gloria Anzaldúa explains,

> Nepantla can be seen in the dream state, as well as in transitions across borders of class, race, or sexual identity. Nepantla experiences involve not only learning how to access different kinds of knowledges—feelings, events in one's life, images in-between or alongside consensual reality. They also involve creating your own meaning or conocimientos[vii]. (Anzaldúa 2000; p.267)

In many ways, Nepantla serves as a space of tensions, of multiple realities. Anzaldúa highlights those tensions, explaining how as a lesbian Chicana poet, she is neither fully accepted by her White feminist colleagues who do not acknowledge her Indigeneity nor by the Chicano community who does not recognize her as a lesbian. She is neither and both at the same time; she is in Nepantla. The same could be said for people who identify as two-spirit, a translation of niizh manidoowag, the Anishinaabe (Ojibwe) term for spiritual people who walk in two worlds, one foot in female and one foot in male. In fact, Nepantla has been compared to the action of walking, whereby one is constantly in motion and where each step shifts the center of gravity so there is no solid grounding. Anzaldúa highlights this movement and potentiality,

> Nepantla, where the out boundaries of the mind's inner life meet the outer world of reality, is a zone of possibility. You experience reality as fluid, expanding and contracting. In Nepantla, you are exposed, open to other perspectives, more readily able to access knowledge derived from inner feelings, imaginal states, and outer events, and to "see through" them with a mindful, holistic awareness. (Anzaldúa and Keating 2002, p. 544).

For Anzaldúa, being able to see through human acts of identity, knowledge, and construction allows us to question when/if the actions of some violate the actions of others, thereby attending to issues of dehumanization.

It is not simply the "space" of Nepantla that is powerful, but the power of being a Nepantlerx[viii]-one who chooses to live in a place of tensions—as a border crosser, so as to birth new knowledge.

> For Nepantleras[ix], “to bridge is an act of will, an act of love, an attempt toward compassion and reconciliation, and a promise to be present with the pain of others without losing themselves to it.” (Anzaldúa and Keating 2002; p. 4)

Bridging between two different views requires deep intellectual and emotional work. It means being willing to hold two or more contradictory views in one’s mind at the same time with the goal of not quickly coming to a conclusion that subsumes both ideas under an umbrella but maintains some of those views and reaches a third space that is neither and both of those views. The idea of Nepantla is consistent with Aboriginal knowledge of the metaphoric mind where we have the ability to hold two completely different thoughts simultaneously (Cajete 2000).

Nahua metaphysics recognizes the shared collective consciousness of the cosmos. As such, a person is both *in* Nepantla and *is* Nepantla. That is, I am situated within a space of tensions and multiple realities that is called Nepantla. And, by virtue of being in that space, I am also the thing called Nepantla; I contribute to its essence. Therefore, Nepantla dictates how we move through the world. We are conscious of the multiple realities and energy in which we participate and to which we contribute as well.

Elsewhere, I have argued that Nepantla can help mathematics education researchers think differently about knowledge (Gutiérrez 2012) and provide a guiding principle for teacher education (Gutiérrez 2015). Here, I am suggesting that Nepantla can help us interrogate the idea that mathematics is both a universal endeavor and not a universal endeavor. That is, the practice of mathematics is not universal in the sense that it is always localized and particular to the needs of those who practice it (e.g., D’Ambrosio 2006; Ascher 2002; Gerdes 1997; Powell and Frankenstein 1997; Knijnik 2007; Restivo 2007). Yet, many of the forms that are practiced throughout the world have been identified as falling within six general forms: counting, locating, measuring, designing, playing, and explaining (Bishop 1988).

For Hatcher et al. (2009), this is two-eyed seeing, learning to see with one eye through Indigenous ways of knowing and the other eye on Western ways of knowing.

> The principles of Two-Eyed Seeing are used for the purposes of collateral learning or colearning where Western Scientific concepts are constructed side by side with minimal interference and interaction with Indigenous Scientific concepts (p. 149).

Unlike Hatcher’s goals, I choose to privilege the view of a Nepantlerx—seeing the interconnectedness between Indignenous and Whitestream knowledge of mathematics. I choose the term Whitestream instead of European American to highlight the role of global White supremacy in the enterprise of mathematics education. Like Hatcher et al., Ogawa (1995) advocates for a kind of multi-science teaching, seeing from multiple views. Aikenhead (2017) echoes this focus on seeing more than one reality, saying,

> Indigenous cultures, for instance, generally share presuppositions characterized as value-laden, contextualized, cultural, ideological, mostly subjective, and embracing multiple truths. (p. 29)

In embracing these multiple truths, he suggests that students need to learn to be “cultural border crossers” (Aikenhead 1997), reminiscent of Anzaldúa’s Nepantleras.

I choose to talk about knowledge from the point of view of a Nepantlerx because it highlights metaphysics and the choice for persons to stay in tensions rather than choosing one view over the other. A critical theorist might suggest an omnipotent perspective from above, a single version of mathematics that would be necessarily less oppressive and best at addressing ethics. In contrast, a post-structural view might suggest a relativist position where there is no one truth and all possibilities are viable for addressing ethics. For me, neither of these options is productive, as each requires a form of collapsing under one umbrella. From the view of a Nepantlerx, one is always trying to find ways of staying in the tensions long enough to birth new knowledge. The value of Nepantla is reminding us to seek multiple realities and to hold those in view because they help us generate new knowledge.

Embracing Nepantla would mean allowing these differing views to remain separate but in relation. Anzaldúa refers to this state of interdependence and solidarity as nos/otras,[x] meaning us/them intertwined. [See Gutiérrez (2012) for an explanation of nos/otr@s as it relates to mathematics education.] Like Nepantla, mathematics is always in motion and embodying principles that could be considered contradictory. Mathematically, the relationship between abstraction and contextualization is an example, as the definition of each relies upon the other.

Mathematx

Combining the views of In Lak'ech, reciprocity, and Nepantla allows us to raise new questions about a vision of practicing mathematics that might move past previous notions of Western versus other mathematics, past an idea of mathematics as either oppressing or liberating, beyond a mathematics that is either discovered or invented, towards an idea that allows us to deal with today's complexity and uncertainties. Towards that end, I am calling for a radical reimagination of mathematics, a version that embraces the body, emotions, and harmony.

Seeking/Performing Patterns for Problem Solving and Joy

Mathematx is a way of seeking, acknowledging, and creating patterns for the purpose of solving problems (e.g., survival) and experiencing joy. Beginning with the principles of recognizing self and/in others, responsibility towards others, and valuing tensions, several things stand out as different from the typical way Western mathematics is conducted or experienced by students in school. First, although some mathematicians experience pleasure as a result of solving previously unsolved problems, that aspect of joy is often a very small percentage of the time and almost always absent from the "mathematical product" (e.g., new theorem, new proof) that is valued by the community. Yet, mathematics education researchers who study aesthetics highlight this domain as essential to human meaning making and to the insights that mathematicians develop (Sinclair 2009).

Aesthetics join emotion, pleasure, and understanding for humans as they relate to their world (Dewey 1934). For mathematicians, aesthetics may serve as a precursor for intuition, whereby they do not rely upon a sense of logic and deduction but upon some general sense of how things connect together (Burton 1999), often illuminating a unity of meanings and values. In this sense, intuition and wonder may lead to joy and discovery (Sinclair and Watson 2001). That is, we seek what is surprising and wonderful, yet events must fit into a broader scheme; the parts must fit with the whole (Gadanidis and Borba 2008). In fact, because humans have had to discern patterns in their world in order to survive, we may be predisposed to attend to just "enough complexity to engage the mind but…not overwhelm it with incomprehensible irregularity or diversity" (Sinclair 2009, p. 52). Although much of this intuitive/aesthetic work remains at the subconscious level for many mathematicians, mathematx is intricately tied to what is pleasing and rewarding in a connected way, not just a utilitarian or "problem solving" manner. This perspective is consistent with Boylan's (2016) call for putting passion and pleasure at the heart of mathematics education. For me, "pleasing" includes not just the playful way in which many "pure[xi]" mathematicians invent new workspaces by beginning with different axioms, (e.g., 8-dimensional space) but also how other persons perform mathematx for/with us. This version of play deviates from Bishop's definition surrounding games because play does not necessarily involve an organized game, but includes a kind of frivolous activity with value perhaps only for the one performing it.

Like plants, humans also have a way of expressing ourselves (our tastes, our values) and our sense of beauty through patterns (e.g., braiding hair, creating symmetry in our surroundings, walking, dancing, speaking, dressing, creating balance in a home). These patterns are both playful (useless) and purposeful (useful) at the same time because they have the potential to connect us with others. Reviewing the work of Dissanayake, Sinclair (2009) highlights that this form of expressing ourselves through aesthetics helps indicate that we are special. In terms of patterns, it might not be just regularity that matters for persons. Biologists have noted that the ability to embody opposites (Nepantla) is consistent with living systems that show simultaneous stability and plasticity, incomplete separation between internal and external topology, prolonged stages of criticality, and the co-existence of future and past (Soto et al. 2016; Longo and Montévil 2011; Montévil et al. 2016). Again, broadening our definitions of living beings may yield insights for mathematicians who seek to discern, appreciate, and reciprocate patterns.

Current versions of what count as "beautiful" in mathematics tend not to reflect the diversity in our world. Instead, they tend to relate to truth (Stewart 2007), implying universals rather than uniqueness/expression that would align with performance or a plurality of epistemologies. If we can recognize that cultural theses of modes of living are aesthetic choices (Popkewitz 2002; 2008) and some aesthetics are not superior to others, then the means for controlling or dominating is lessened. The opportunity to appreciate another's values is the embodiment of In Lak'ech. In other words, approaching life in this way of appreciating and looking for similarity is what helps us grow and also recognize difference. Ethics and aesthetics join in mathematics when we have guiding principles like In Lak'ech, reciprocity, and Nepantla.

Intervention in Reality

Second, whereas mathematics tends to be thought of as a noun (e.g., a body of knowledge, a science of patterns, a universal language), mathematx is performance and, therefore, a verb. Just as identity is not something that you are, but rather something you do (Butler 1999), mathematx emphasizes the guiding principles and the process as opposed to the product. Drawing upon the concept of reciprocity, mathematx is an intervention-in-reality (action) as opposed to a representation-in-reality (explanation) (Santos 2007). The starting point for Western mathematicians would be to begin with embracing the joy/emotions and seeking In Lak'ech, reciprocity, and looking for opportunities to be a Nepantlerx while doing mathematics. Let us consider an example. A common theme in combinatorics is to start with an object P, and define some sort of counting function to P, which makes sense for taking in positive values because it results in a polynomial. Then, negative values are substituted into the counting function and it is recognized as a new counting function for a different/new mathematical object. For mathematicians, this work is known as combinatorial reciprocity (Meléndez 2017). In fact, Beck and Sanyal (2017) ascribe animacy to the process by referring to it as moving from "your world" to "my world." The new counting function has offered something that the original counting function could not. Is the mathematician grateful for the offering of this new counting function? Is there some joy in noting that functions can give back to each other? How might that starting point extend to other forms of reciprocity in doing mathematics with other persons?

The idea of mathematx as verb is consistent with many Aboriginal languages that are largely verb-based and may relate to how persons practice mathematics (Lunney Borden 2011). Mathematx is an activity that cannot be extracted from the living being(s) in the process of solving problems and/or experiencing joy--the mathematxn. Although ethnomathematics tends to take into consideration the idea that different cultures do different mathematics, the unit of analysis normally remains at the level of the group and what they have produced, possibly promoting the unintended message that all members of that culture do the same things for the same purposes. Mathematx acknowledges this group relation, but recognizes the meaning that each person ascribes to what is being experienced.

The x at the end of the word signifies movement, an openness, the x being a variable that could be represented by anything. In this sense, mathematx is constantly evolving, depending upon what is represented with that x. This framing is consistent with the choice to use "x" as an ending (e.g., Latinx) to represent any gender performance instead of privileging a patriarchal view or ascribing to a binary of male/female.

I choose mathematx instead of mathematix in order to distinguish between the two when spoken aloud. In Nahuatl, the "x" is pronounced "sh." So, the word is pronounced mathematesh. The x is also political in the sense of Malcolm X, the human rights activist who took on the x to represent all of the unnamed ancestors and their cultures that had been lost through slavery. For me, mathematx is a political statement about reclaiming the persons who have been lost when humans remain at the center. As such, mathematx seeks to intervene in the status quo of mathematics.

Living Mathematics

The title of this article suggests a vision of living mathematx. What might it mean to live mathematx? Living mathematx means both that we live a version of mathematx as well as we are a living version of mathematx. This framing is consistent with Nahua metaphysics that suggests one is both <u>in</u> Nepantla and one <u>is</u> Nepantla. Living mathematx means moving through the world with other living beings, acknowledging, appreciating, and reciprocating the patterns produced. If we look to animals and plants for some insight, we see that Brassica oleracea (Romanesco cauliflower) performs itself in both utilitarian (compact) and non-utilitarian (pleasing) ways that may get us to pay attention to its form and to continue to cultivate it. On the one hand, Romanesco cauliflower performs a version of the "Fibonacci" sequence that maps onto Western mathematics, and the elegance of the pattern brings joy while at the same time solves problems of space. Yet, like all persons, every brassica oleracea, performs itself in a way, and over its lifetime, that shows variance and suggests a departure from a pre-determined set of possible outcomes programmed by genomes (Montévil et al. 2016). We might ask ourselves, why is a grove of trees, each with similar but not perfect versions of fractals more pleasing than a computer-generated version of a grove of trees that precisely follows expanding symmetry? Is there something more in our relation that triggers a sense of pleasure, appreciating the aesthetics that plants perform? Are we able to discern and appreciate asymmetry along with symmetry? And, in what way(s) might this relate to aesthetics, intuition, or insight? Are there patterns in the ways in which our pleasure is communicated back to plants, for example, through pheromones or other means we are not yet able to understand or describe?

Do other persons remind us of the importance of beauty in imperfection, of not relying upon a defined algorithm? That is, although they offer good approximations of such things as shorelines of oceans, fractals in Western mathematics do not map perfectly onto the universe around us. Moreover, not all symmetry is inherently beautiful or "natural." Marcelo Gleiser refers to this phenomenon as the aesthetics of the imperfect. He notes that while synthesizing amino acids in a laboratory setting, biologists achieve approximately 50 percent right-handed chiral[xii] formations and 50 percent left-handed formations. Yet, in living creatures, virtually all amino acids are left-handed. This asymmetry is critical for protein folding and reproduction. The same is true for the asymmetry of occurrence in matter and anti-matter in physics[xiii]. So, asymmetry, not just symmetry, may be a form of performance by living beings to which we need to pay greater attention. Perhaps this asymmetry has aspects of a pattern that are complex enough without being overwhelming to initiate surprise or wonder.

Can our older brothers and sisters in this universe (and others) teach us something based on how they have developed and organized themselves to relate with each other to please and solve problems? From a practical point of view, are there ways in which we can organize our living spaces to draw upon visions such as the Three Sisters and other geometric formations that our older brothers and sisters use? In some respects this idea of learning from our older brothers and sisters is not new. Researchers have begun to rely upon biomimicry, copying the forms observed in "nature," in order to solve complex problems of space, design, and efficiency. For example, termites have taught architects in Harare, Zimbabwe, how to erect buildings with patterns that create effective internal climate control systems; Kingfisher birds have taught engineers how to construct high speed trains that will move through the air with less noise; plants and insects are teaching aerospace engineers about miori folds in order to tightly package and then deploy enormous complex origami versions of sun shades to block the light and allow telescopes to take more accurate pictures; similar folds in the universe are helping physicists understand neighboring galaxies. However, all of this biomimicry is taking place in research labs, not in schools with students. We are missing an opportunity to expose students to plants, animals, and other persons as our teachers, and perhaps also our opportunity to reciprocate actions.

In terms of recognizing and performing patterns—living mathematx—marine creatures such as salmon, sea turtles, trout, and eels have the ability to read magnetic fields in the earth and use them in migration (Pennisi 2017). Animals such as bears, dear, elk, great apes, macaws, lizards, and fruit flies are able to read (communicate with) plants in order to self-medicate when they have diseases (Shurkin 2014) or develop high levels of toxins in their skin and use other chemical signals to communicate and ward off predators (Hagelin and Jones 2007). Several tree species such as oak, spruce, and beech are known to communicate among themselves and with each other in order to ward off disease, share resources, and protect each other (Wohlleben, 2016). Like Las Tres Hermanas (corn, beans, squash) mentioned earlier, many of our cousins seem to recognize/acknowledge patterns and create new ones while collaborating and valuing reciprocity. To date, many researchers rely upon Bishop's (1988) classification of six forms of mathematics: counting, locating, measuring, designing, playing, and explaining. I urge us to consider what forms of classification might we develop in looking to other-than-human persons and the ways in which they live mathematx in their local contexts? Which new forms of mathematics might arise?

From a philosophical perspective, perhaps it is neither that we have come to appreciate the "natural" patterns present in plants, animals, and rocks, as Platonists would have us believe (i.e., that they have taught us patterns that were programmed within them or that they developed), nor that we simply project our own aesthetics onto our living cousins (i.e., that we see the mathematics we want to see in our environment) as Realists would have us believe. More likely, our relations and the tensions between us provide the multiple lenses on reality and instability. We are constantly in motion like a Nepantlerx. This is consistent, though different, from Barad's (2001) notion of "intra-action." If, instead of perpetuating a human/non-human binary, we consider the shared consciousness between all living beings, the greater unity to which we belong, we are more likely to value mathematx for what it offers us. We can acknowledge both the potential for domination between living beings while also opening up the possibility of harmony and reciprocity in the practice of mathematics.

As we look for new structures and forms of mathematics to help solve the global crises we encounter as well as to experience joy, we might consider how other living beings might offer lessons and insights. We have developed new structures and physics concepts by studying intently such things as symmetry and conservation laws in the physical world. Even using a narrow definition of living beings, biologists have noted that all organisms (uni-cellular or multicellular) do not simply follow prescribed rules or programming. They develop their own norms/rules in a way that balance between plasticity and robustness; that is, they show spontaneous organization and variance that does not appear in physics (Soto et al. 2016). If we broaden our understanding of living beings beyond the organism, we might find even further insights.

Reflecting an Ecology of Knowledges

Building upon Andreotti, Ahenakew, and Cooper's epistemic plurality (2011) and Santos' (2007) call for a new ecology of knowledges, I suggest that mathematx guide our work in mathematics. Because mathematx acknowledges that all persons will seek, acknowledge, and create patterns differently in order to solve problems and experience joy, multiple knowledges are valued and sought. These multiple knowledges are important, given that all knowledge is partial and each offers us a different angle and understanding on the world. The goal is not to work towards a summative understanding, as if by simply adding the different knowledges we will have a complete or perfect view. Rather, our work is to locate ourselves in others and others in us, as we attempt to understand our world through patterns. Doing mathematics in this way offers us the opportunity to unlearn our epistemological arrogance. The concept of reciprocity draws upon complementarity in recognizing that different knowledges contribute something others do not. Mathematx nurtures a view of mathematics that always considers strengths and limitations for particular purposes. For example, we might ask ourselves: which forms of mathematics can our brothers/sisters perform for which we do not have a way to express? In looking to other persons, might we be more open to multiple versions of knowing that are constantly open to new axioms and even non-axiomatic mathematics?

While others have noted that Western mathematics—sometimes referred to as Platonist mathematics or European mathematics or European American mathematics—is in opposition to Indigenous mathematics, I am not seeing that mathematx would be in opposition; rather it would include Aboriginal mathematizing. In the same way that Latinx rejects the gender binary, mathematx rejects the epistemological binary. Mathematx allows for a variety of expressions without suggesting one is "normal," superior, or the reference point for erasing other epistemologies. However, mathematx is not everything and anything. It privileges a particular way of moving through the world that acknowledges and produces patterns that align with the collective consciousness and energy of the cosmos and respects other persons. Mathematx is less a way to describe how we currently do mathematics and more a goal for how to approach our relations with each other in the practice of mathematics. In this sense, mathematx is a quest for intersubjectivity and systems thinking, not unity.

Moreover, mathematx acknowledges Nepantla by underscoring the fact that there is no absolute universalism or absolute relativism. That is, there is no umbrella term under which all forms of mathematics can collapse and explain everything in reality. When we move from a global universal mathematics to a form of mathematx, whereby we acknowledge epistemological pluralism and are guided by first principles of In Lak'ech, reciprocity, and Nepantla, we are likely to see changes in not only mathematical activity (and products) but also in mathematxns.

Philosophers, sociologists, and anthropologists who study mathematics have long argued that "school mathematics" is but one small version of the many forms of mathematics practiced in the world and that such mathematics does not operate outside of individuals, morals, or politics (Brown 1994; Clarke 2001; Ernest 1994, 2000; Fitzsimons 2002; Restivo 1994; 2007; Turnbull 2000; Verran 2001). Often, in making these claims, researchers point to the field of ethnomathematics to highlight the fact that all cultures do mathematics in localized ways. In some respects, I am arguing for an extension of ethnomathematics to include animals, plants, rocks, bodies of water, and other persons. Mathematx is consistent with a focus on peace, education as relation, a recognition of the imprint of Western thought in dominant mathematics, and a language through which people could be more creative (D'Ambrosio 2007; Francois and Van Kerkhove 2010; Gerdes 1988; Powell and Frankenstein 1997). Even so, I choose mathematx as opposed to "ethnomathematics with the inclusion of other-than-human persons" because I aim to avoid some of the pitfalls of previous understandings and implementations of ethnomathematics (Cimen 2014; doCarmite and Pais 2009; Vithal and Skovsmose 1997). For example, I am not looking to use Western mathematics or a Platonist view as the standard by which we judge other persons to live mathematx or to suggest a kind of essentialization of humans (Gutiérrez 2000; Francois and Van Kerkhove 2010). Moreover, I do not wish for the knowledge of our older brothers and sisters to simply be acknowledged/sanctioned and shared (Mesquita and Restivo 2013); I want such knowledge to be valued and applied. Although D'Ambrosio broadened his definition of "ethno" to include "all culturally identifiable groups with their jargons, codes, symbols, myths, and even specific ways of reasoning and inferring" (p. 17, cited in Francois and Van Kerkhove), people have continued to think about ethnomathematics as practiced by ancient ethnic (non-Western or non-White) cultures[xiv] or collapsed it into a form of cultural appropriation. By introducing mathematx, I also seek to decenter the notion of "tics" (technologies), which, for me, do not capture the body/spirit (feminine) and the ways we move through the world in the same metaphysical manner (Haraway 1988; Harding 2008). Mathematx is more than explaining and understanding in order to survive (D'Ambrosio 1990); it attends to aesthetics and the body.

Implications for Teaching and Learning

Elsewhere, I have argued that the practice of school mathematics in the US regulates the child by privileging: algebra/calculus over geometry/topology/spatial reasoning; rule following over rule breaking; Western mathematics (culture free) over ethnomathematics (recognizing that even academic mathematicians are a culture); the "standard algorithm" over invented or international algorithms; abstraction over context ("just pretend this is real world"); mind over body; logic over intuition; and encouraging students to "critique the reasoning of others" over appreciating their reasoning (Gutiérrez, in preparation). Not only can these repeated practices over a lifetime serve to dehumanize students and teachers in classrooms, the narrative about mathematics being a pure discipline, reflective of the natural world around us, universal, with an almost unilaterally positive relationship to society's advancement, leaves many humans unable to challenge this narrative to consider other ways of doing mathematics. In this way, school mathematics comes to normalize and valorize particular practices and to make others seem deviant and in need of fixing (Skovsmose 1994; Walkerdine 1994). By continuing to privilege data analysis and probability over other kinds of spatial patterning, even if that data analysis concerns itself with issues such as climate change, we run the risk of limiting new ways of doing mathematics and our relationships to the practice.

In contrast, what might teaching and learning look life if mathematx were embraced? First, students need time to relate with other-than-human persons in order to develop a familiarity with the kinds of patterns that exist outside of themselves—things that are both another version of us and yet not exactly us—so they can provide mirrors onto ourselves and windows onto another's world. Rather than education happening within school walls, students might be asked to head outdoors. In lieu of a purely dominant mathematics curriculum (Gutiérrez 2002), students might be asked to investigate: How do we acknowledge, understand, and relate to the patterns in bird song? What are the patterns/signs/codes that allow some animals to relate to their plant relatives for the purpose of self-medication? What are some of the patterns that occur as insects package their wings and bodies? And, in what way(s) might those forms solve problems and bring joy? How do those packages of wings and bodies relate to other packages in humans, in other species, in the imagination? Where does the search for patterns fail to capture other meanings in these practices? These are all questions for which most teachers will not have answers. Therefore, different from the portrayal of the math teacher as the credentialed professional who has acquired the "knowledge base" and who is inserted into the child's life in a coercive relationship whose success is conditional upon pre-set performance measures and criteria, living mathematx would involve the passing of knowledge only when the knowledge receiver is ready and a relationship is formed between giver and receiver, as suggested earlier by Hatcher et al. (2009).

In some respects, seeking to understand how we and our older brothers and sisters live mathematx can serve as both a problem solving exercise (in mental manipulation, spatial reasoning, and other things that might map easily onto current forms of humans doing mathematics), but it is also likely to deviate from the language we have to understand or describe. In this way, students will be learning how to be open to other forms of being and for recognizing the tools necessary for reading and responding (reciprocity) to those forms and also being fully present in the beauty of such performances. Such an education would shift the dynamics from an objectifying description and problem-solving manner towards one that includes joy, respect for the person, and the desire to act (reciprocate) in a way that is responsive to the particular situation at hand, thereby changing the individual learner in the process. In the same way that we might see traditional mathematics classrooms move away from students being taught to "critique" the reasoning of other students, as is called for in the Common Core State Standards in Mathematics (National Governors' Association 2010) towards what I refer to as "appreciating" the reasoning of other students (i.e., being able to stand in their shoes), we might see that process occur across all persons.

Some researchers have started to bridge the gap between aesthetics and mathematics through the online game *Fold It* where players find pleasure in folding proteins in compact ways and earn game points (Cooper et al. 2010). The players' unique folds are analyzed by researchers who then apply puzzle solutions to real world problems in the medical industry. In fact, this form of crowd sourcing has developed insights and answers to problems concerning the AIDS epidemic that researchers and computer-generated approaches alone had failed to solve. Researchers involved in the project are studying the intuition of players and how they approach the folding process in order to improve algorithms generated by computers. This form of pleasure and "learning" occurs outside of the school walls. However, combining versions of exploring the world to relate with other persons and then playing such games may help us identify certain trends that would have been difficult using our eyes alone. That is, there may be ways in which relating with plants, animals, rocks, or other persons inspire us to develop intuition in approaching the visual display of computer-generated objects that can be both pleasing for us as well as build upon the mathematx that other persons live in order to generate biomedical solutions to health problems.

Learning through mathematx accedes that all knowledge is based on particular worldviews and ways of knowing that close down other possible choices; that is, knowledge is a political process, not a neutral product. Rather than mathematics being seen as the pursuit of truth in the sense of a unifying theory of reality (e.g., the unique solution to string theory) and, therefore, the means to control, learners embracing mathematx might come to see that the mathematics performed by humans is but one form that describes part of our world, but not all. Through living mathematx, teachers and students would practice walking alongside of other living beings, revising their understandings based upon their relations with them. In this sense, students would have opportunities to unlearn their epistemological arrogance. Teachers would focus upon helping create opportunities for learners to engage in an aesthetic experience—seeking surprise both in how similar something is, but also how it differs—to wonder about how other living beings seek, acknowledge, and perform patterns for their own survival and joy. Teachers might also encourage students to search for patterns that are felt/experienced (at the macro level), not just conceptually identified (at the micro level). What are the aesthetic preferences that help us define and understand the concept of pattern? Through mathematx, learners are likely to become more reflective about their learning and their relations in the world—what they know, what they do not know, as opposed to what can be known.

Because mathematx involves the Nepantla state of both/neither when discussing problem solving and joy, learners will need to become comfortable with such uncertainty. In other words, they will come to know and practice mathematics as neither purely problem solving, nor as purely joy, but also not both in a cumulative sense. Learning mathematics in this way means being able to, at times, acknowledge one side over the other, but always seeing the two in relation.

Teachers' roles would necessarily shift from telling/showing and towards living alongside of students and other persons. Teachers should be asking themselves, "Am I conducting mathematical activity with an eye towards reciprocity, Nepantla, In Lak'ech? Am I doing mathematics to see myself in others and others in myself, to give and to receive from my universe, to acknowledge multiple ways of knowing and multiple kinds of knowers?" Students would be learning to move through the world, appreciating, noting the forms, packages, and connections that plants, animals, rocks and other persons develop. In a sense, we are apprenticing learners to become "mathematxns" by providing guiding principles—In Lak'ech, Nepantla, and reciprocity. We are preparing them to look for what we already acknowledge/sanction as some humans doing mathematics with how other persons (human and other-than-human) live mathematx. In doing so, we must recognize that ignorance might not just be a lack of knowledge but an active refusal to know because it disrupts one's previous beliefs. If we start early with young learners, it may be easier to disrupt what humans have come to consider normal in the practice of mathematics. That is, like learning a new language, young students often are able to absorb new ideas and new ways of gaining knowledge.

Mathematx is not a rival body of formal knowledge to mathematics. Rather, mathematx is a worldview that surrounds and guides whatever it is that we are trying to accomplish mathematically. However, because of the performativity of mathematx, this new approach is likely to produce new structures and forms that academic mathematicians might acknowledge as new mathematics. Indigenous epistemologies value context and relationships, recognizing that our strength comes from understanding ourselves not with universal principles but in relation to particular lands and particular living beings. One could argue that the individual cannot be extracted from its environment and understood in any meaningful way. Biologists would agree, suggesting that because biological systems operate under different theoretical principles, a focus on living beings is likely to require different forms of mathematical modeling (Montévil, 2017). For example, breaking something down into its parts in order for study does not necessarily lead to anything meaningful about the results of a model when inserted back into its context. We saw this was the case with synthesized amino acids versus ones occurring in nature. So, our definition of a "useful mathematical model" may need to be reexamined when we include all living beings as performers of mathematx, including ones that would not be classified as organisms.

I am not suggesting that humans have gotten it all wrong and that by turning to other-than-human persons, we will get it right. My goal is not to get closer to some absolute truth about our world. Rather, learning with other persons opens the door for us to have different lenses for viewing and relating with our universe and others. And, in doing so, we have the opportunity to learn how different approaches (mathematics or mathematx) make im/possible certain forms of knowing the world, recognizing that all of these forms are provisional, local, and legitimate. Even so, given the history of particular knowledges, knowers, and ways of knowing that have dominated in our history with respect to mathematics, it is important to give greater focus to the ways other-than-human persons live mathematx.

I recognize the potential limitations of attempting to use a term like mathematx that is difficult to both say and spell, even if one understands conceptually what it can offer. The term ethnomathematics, even when being explicit that all cultural forms of mathematics are "ethno" has not prevented many researchers and teachers from continuing to use Western mathematics *in opposition to*, instead of *as a version of* ethnomathematics. That is, neither do we tend to refer to Western mathematics as such nor do we refer to other mathematics as Eastern, Mexican, Northern, or American. Ethnomathematics seems to encourage researchers and teachers to create a binary between Western and Indigenous, rather than recognizing a variety of forms, some with overlapping goals and principles. Moreover, ethnomathematics also has not been well incorporated into the school mathematics curriculum. So, some might wonder, what is to prevent the same phenomena with mathematx?

To avoid these potential pitfalls, I have suggested we expand our view to all living beings, thereby providing us with the ability to consider how some humans live mathematx differently from each other as well as from other persons, creating new lines of solidarity (In Lak'ech) or difference (and the need for reciprocity), or contradiction/tension (Nepantla). By expanding to other living beings, mathematx can avoid the trap of Western versus "other" mathematics and open the door for new categories to be drawn. For example, in what ways do humans live mathematx that are consistent or compatible with how trees live mathematx? And, how are individual humans affected by considering trees to be simultaneously another version of us (In Lak'ech) and not a version of us (Nepantla), but in need of our reciprocity? In what ways are we incompatible? What are the new knowledges and sensibilities we need to fully develop to live in harmony? Moreover, because mathematx is not a description of the world, but rather a set of first principles in doing mathematics, it differs from ethnomathematics in that it sets out a form of intervention.

Although the vision of living mathematx that I have outlined may sound outlandish, we need only remember Clarke's (1973) third law: "Any sufficiently advanced technology is indistinguishable from magic." In fact, I argue that mathematics as a field and as a human endeavor need only look to other sciences to see it is late to evolve. The field of physics used to promote the idea that there was a single time-space continuum. Then, Brian Greene (2011) introduced the concept of infinite parallel universes and physicists are now imagining how humans could participate in more than one space at one time. Moreover, the cosmologist Alexander Vilenkin has proposed a theory of our universe sitting within a bubble of other universes (Vilenkin and Tegmark 2016), the implication being that other universes may have different laws of physics. In a similar vein, I am suggesting that we may have different forms of mathematx in which we participate, but to which we are largely blind and numb. When we move [1]through the world seeking connections and reciprocity, our views of ourselves and of others change. I ask us to open our minds to envision how such a view could change the relationship between humans, mathematics, and this universe with/in which we currently live.

Endnotes

[i] I am grateful to Federico Ardila-Mantilla, Kimberly Seashore, Andrés Vindas Meléndez, and Diana Zambrano at San Francisco State University; and Brandon Singleton at the University of Georgia for providing helpful comments on an earlier version of this article.

[ii] I cite this article as 2010/2013 because it was published online through JRME in 2010 and some researchers began citing it as such then. It was not released in print until 2013, and some researchers have cited it as such since. Because the focus of the article is on a particular point in history, the work should reflect the earlier date.

[iii] My maternal grandmother was a woman of Rarámuri (Tarahumara) descent. My ancestors are located in the Copper Canyon region of Northwestern México.

[iv] Two-spirit is an Aboriginal term.

[v] I use Indigenous and Aboriginal interchangeably. US authors tend to use the term Indigenous, whereas authors from Canada, Australia, and New Zealand tend to use the term Aboriginal. In Canada, Aboriginal includes First Nations, Métis, and Inuit peoples.

[vi] I place Fibonacci in quotes to highlight the presence of settler colonialism. That is, although the Italian Leonardo Pisano (Fibonacci) receives credit for the pattern, many cultures and persons throughout the world, including Pingala in 200BC in India, had already known/performed the same pattern many years earlier. In fact, if humans are no longer the center, we might credit nautilus

pompilius (Nautilus shell), pinus coulteri (pinecone), or helianthus annus (sunflower) with the "discovery."

[vii] Conocimientos translates to "knowledges" in English.

[viii] Similar to the use of Chicanx, Nepantlerx indicates solidarity with people who identify as LGBTQIA2S. In the Spanish language, the –ero/-era ending of a word typically signifies "one who…" As such, a Nepantlerx is one who chooses to reside in Nepantla.

[ix] Anzaldúa's terms do not reflect the "x" because she was writing before such language was common. She used a version that privileges a feminist perspective and therefore ends in "a" instead of "o."

[x] Anzaldúa's terms do not reflect the "x" because she was writing before such language was common. She used a version that privileges a feminist perspective and therefore ends in "a."

[xi] I place pure in quotations to suggest that there is no such purity to mathematics. When we use terms like pure mathematics or fundamental mathematics, we are "othering" different forms of mathematics in ways that make them sound primitive or deviant. An Aboriginal stance would call into question whether any form of mathematics could be seen as pure, as it will always have a purpose and a grounding—cultural context—to start.

[xii] Chirality refers to the geometric structure of a molecule, in particular how four different entities connect to a carbon center. Like hands, chiral molecules cannot be superposed onto their mirror image.

[xiii] See, for example, Paul Dirac's prediction of anti-matter that contradicted classical quantum physics where systems were thought to only have positive energy.

[xiv] Noted exceptions include the work of Gelsa Knijnik (2011), who has chronicled the Peoples Land Movement in Brazil.

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PSYCHOLOGY IN MATHEMATICS EDUCATION: PAST, PRESENT, AND FUTURE

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Starting with Woodworth and Thorndike's classical experiment published in 1901, major periods in mathematics education throughout 20th century and on into the current century are reviewed in terms of competing epistemological and psychological paradigms that were operating within as well as across the major periods. The periods were marked by attempts to make changes in school mathematics by adherents of the dominant paradigm. Regardless of what paradigm was dominant, the attempts essentially led to major disappointments or failures. What has been common across these attempts is the practice of basing mathematics curricula for children on the first-order mathematical knowledge of adults. I argue that rather than repeat such attempts to make wholesale changes, what is needed is to construct mathematics curricula for children that is based on the mathematics of children. Toward that end, I present several crucial radical constructivist research programs.

Keywords: Learning Trajectories, Research Methods, Cognition, Curriculum

> The accent must be on auto-regulation, on active assimilation – the accent must be on the activity of the subject. Failing this there is no possible didactic or pedagogy which significantly transforms the subject (Piaget, J., 1964).

Mathematics Education—1900-1950

Behaviorism and Faculty Psychology

The classical experiment. The classical experiment by Woodworth & Thorndike (1901) at the beginning of the 20th Century introduced the "scientific movement" in education and it was considered as the death knell of faculty psychology, the doctrine of "mental discipline" (e.g., Whipple, 1930; Thorndike, 1922). In faculty psychology, the mind was viewed as a collection of separate modules or faculties assigned to various mental tasks, such as reason, will, concentration, memory, or language and it was thought that training in one faculty would transfer to another. As a result of their experiment, Woodworth & Thorndike (1901) concluded that, "The improvement in any single mental function rarely brings about equal improvement in any other function, no matter how similar, for the working of every mental function-group is conditioned by the nature of the data in each particular case" (p. 250). The lack of transfer led Thorndike (1903) to develop his theory of identical elements: "The answer which I shall try to defend is that a change in one function alters any other only in so far as the two functions have as factors identical elements" (pp. 80-81). Once this idea was accepted, "arithmetic was on its way to being analyzed into elements so that the stimulus-response theories of Thorndike could be more readily applied" (Van Engen & Gibb, 1956, p. 1).

Cartesian epistemology. There was also a separation or duality between the mind and the body in faculty psychology in that it was thought that mental discipline of the intellect would lead to control of the will and emotions, a duality that has become known as "Descartes error" (Damasio, 1994, pp. 248)—"I think, therefore I am." It is interesting to me that this philosophical rationalism of faculty psychology was regarded as falsified by means of a "crucial experiment" that was conducted in the context of a competing paradigm, empiricism.[i] Although I don't wish to defend faculty psychology, in retrospect I believe that a basic reason why faculty psychology was abandoned transcended Woodworth and Thorndike's classic experiment. In empiricism, the doctrine that the world imprints itself on the mind, there is a duality that is similar to the mind-body duality between an endogenic (mind centered) view versus an exogenic (world centered) view (Konold & Johnson, 1991). This mind-reality duality in the main explains why faculty psychology was rejected and why empiricism was so widely embraced. In behaviorism, no explanation of mind was needed nor was it sought so there was already a major conflict in the two views of mind in faculty psychology and in the behaviorism of Woodworth and Thorndike[ii]. That is, there was already a paradigmatic rejection of faculty psychology by the empiricists and the classical experiment corroborated the philosophical rejection. Furthermore, in empiricism, something is true, "only if it corresponds to an independent, 'objective' reality" (von Glasersfeld, 1984. p. 20). So, the idea that the functioning of one faculty would be transferred to the functioning in another faculty would have to be validated by such functioning in objective reality, which is the crux of the classical experiment.

Behaviorism and Progressive Education

Progressive education. Although faculty psychology was abandoned as a psychological rationale in education, there was a competing paradigm to the scientific movement during the period of time that was known as Progressive Education. Under the leadership of John Dewey, the Progressive Educational Association was formed in 1919 and it served as a counterpoint to the scientific movement. Progressive Education promoted the idea of a child-centered education as well as other aspects of education.[iii] As early as 1902 John Dewey wrote;

> Abandon the notion of subject matter as something fixed and ready-made in itself, outside of the child's experience; cease thinking of the child's experience as also something as hard and fast; see it as something fluent, embryonic, vital; and we realize that the child and the curriculum are simply two limits which define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction. (Dewey, 1902, p. 11)

This quotation might be interpreted as Dewey introducing a duality between the child and the subject matter. Dewey's (1902) distinction here is the subject matter as known by the scientist and the subject matter as known by the teacher.

> Every subject thus has two aspects: one for the scientist as a scientist; the other for the teacher as a teacher. These two aspects are in no sense opposed or conflicting. But neither are they immediately identical. (p. 22)

For Dewey (1902), subject matter for scientists represented a given body of truths, whereas for the teacher,

> He is concerned not with the subject matter as such, but with the subject matter as a related factor in a total and growing experience. Thus to see it is to psychologize it. (p. 23)

Two concepts of number. Dewey's emphasis on psychologizing subject matter was quite different than that of the behaviorists. The difference is well illustrated in how Dewey and Thorndike regarded number. For McLellan & Dewey (1895),

> Number is not a property of the objects which can be realized through the mere use of the senses, or impressed upon the mind by so-called external energies or attributes…In the simple recognition, for example, of three things as three the following intellectual operations are involved: *The recognition of the three objects as forming one connected whole or group*—that is, there must be a recognition of the *three* things as individuals, and of the *one*, the unity, the whole, made up of the three things. (p. 24)

So, Dewey was not an empiricist. Recognition is an indication of assimilation, which, for Piaget (1964), is the essential relation involved in learning. Recognizing the three things as individuals is the result of using an operation of the mind, the unitizing operation (von Glasersfeld, 1981), and recognizing the three things as the one, the unity, is the result of using the operation of uniting the three things into a composite unity. Unitizing sensory material from two or more sensory channels into experiential wholes stands in contrast to the assumption that the world imprints itself on the mind, an assumption on which Thorndike's psychology of number was based.

Thorndike (1922) identified three meanings of numbers—the series, collection, and ratio meanings—and he credited McLellan and Dewey for the ratio meaning. However, he made no attempt to engage in an analysis of the operations of the mind that produce these meanings. Of the collection meaning, he wrote:

> Or we may mean by knowledge of the meaning of numbers, knowledge that two fits a collection of two units, that three fits a collection of three units, and so on, each number being a name for a certain sized collection of discrete things, such as apples, pennies, boys, balls, fingers, and the other customary objects of enumeration in the primary school. (pp. 2-3)

As an empiricist, number was taken as a given in reality and imprinted itself on the mind through the senses. Rather than being concerned with the mathematical experience of the child, for Thorndike (1922), "The psychology of the elementary school subjects is concerned with the connections whereby a child is able to respond to the sight of printed words by thoughts of their meanings…" (p. xi).

Thorndike's influence. The influence that Thorndike had in mathematics education is illustrated in the twenty-ninth yearbook of the National Society for the Study of Education.

> Mainly, the main psychological basis is a behavioristic one, viewing skills and habits as fabrics of connections. This is in contrast, on the one hand, to the older structural psychology [faculty psychology][iv] which has still to make direct contributions to classroom procedure, and on the other hand, to the more recent Gestalt psychology, which, though promising, is not yet ready to function as a basis of elementary education. (Knight, 1930, p. 5)

Knight's attempt to separate the behaviorist approach to elementary education and that of the faculty psychologists was spurious because it is difficult to distinguish faculty psychology's educational model (mental discipline) and Knight's development of a behavioristic educational model. In faculty psychology, it was thought that the best way to strengthen the minds of younger students was through drill and repetition of what we might now call the basic skills in order to cultivate the memory[v], which is quite similar to Knight's interpretation of Thorndike's (1922) Psychology of Arithmetic. Thorndike thought that arithmetical knowledge should be treated as an organized interrelated system, whereas his students, of which Knight was one, focused on the mechanics of arithmetic (Van Engen & Gibb, p. 10). Knight also wrote of avoiding progressive education in the same introduction to the yearbook.

> Some readers may feel that the spirit of this Yearbook is too conservative, that it lacks a bold and daring spirit of progressiveness. There has been a conscious attempt to avoid the urging of any point of view not supported by considerable scientific fact. (Knight, 1930, p. 2)

A contentious relationship. The contentious relationship between progressive educators and educators who held the opinion that the function of the school was to train the working class, be they empiricist or faculty psychologists, appeared prior to the publication of the twenty-ninth yearbook. In 1918 Harold Rugg and John Clarke critically analyzed attempts to reconstruct ninth-grade mathematics and presented their own program in the last chapter of their study. "[T]he construction of a continuous mathematical course, worked out around two basic principles, one mathematical and the other psychological" (p. 176) was a major component of their program. They did cite a classic textbook series (Wentworth, Smith, & Brown, 1918) as an attempt to reconstruct ninth grade mathematics, but such texts were regarded as coming up short. In a perusal of the cited text I found that basic algebra was as formal, rule bound, and manipulative as one would expect in a text designed to train students in algebra.

The contentious relationship continued on after Rugg and Clarke's 1918 study, this time directed toward Harold Rugg's social study textbooks. Rugg eventually became one of a small group of progressive educators at Teachers College, Columbia University where he published a social study textbook in 1929 from a social-justice perspective titled, "Man and his changing society," that became widely used. Being a social studies textbook, it was appropriate that there was a focus on social problems in the Unites States and the author encouraged students to explore potential solutions. Rugg was eventually accused of socialism and conservative patriotic business groups who did not want school children raising questions about the capitalistic economic system censured his books.

> By the end of the decade Rugg's books and several others were condemned by the American Legion, the Advertising Federation of America, and the New York State Economic Council. In 1940, in a speech to the leaders of the oil industry, H.W. Prentis, the President of the National Association of Manufacturers (NAM), complained that public schools had been invaded by "creeping collectivism" through social science textbooks that undermined youths' beliefs in private enterprise.[vi]

Progressive education was repudiated and, during the decade of the 1950's, it disintegrated as an identifiable movement in education.[vii] Although the movement may have disintegrated, that doesn't mean that the involved principles died with it.

Mathematics Education 1950-1970: The Era of Modern Mathematics

After World War II, wide spread concern for the state of the education of scientists and engineers emerged when compared with that of the Russians. As a result, the mathematics community became integrally involved in the reeducation of college teachers of mathematics (Price, 1988). The concern soon shifted to the education of precollege mathematics (and science) teachers, especially after the Soviet Union launched Sputnik I in October of 1957. Buttressed by the National Science Foundation, a concerted effort was made by several mathematicians to upgrade the precollege mathematics curriculum in order to educate college capable students (CEEB, 1959; Price, 1988). Classical idealism (the doctrine that reality, or reality as we can know it, is fundamentally mental) replaced empiricism as the dominant philosophical position among the reformers and mathematics textbooks were written from the point of view of a mathematician's mathematics (e.g., Allendoerfer & Oakley, 1959; School Mathematics Study Group, 1965).

However, among the curriculum reformers the belief was, and it still is by most contemporary mathematicians, that mathematics is discovered rather than invented by human beings (Stolzenberg, 1984). So, despite a major shift from empiricism to idealism, Cartesian epistemology was still the prevailing epistemology of the curriculum developers and others primarily involved in the modern mathematics movement, including researchers in mathematics education. Behaviorism was rejected and problem solving along with learning by discovery became the major psychological emphases (Pólya, 1945, 1981) for which Wertheimer's[viii] (1945) work on productive thinking served as a basic psychological rationale. Wertheimer considered productive thinking, or the solving of problems, as based on insight and criticized reproductive thinking such as repetition, conditioning, and habits, all of which are emphasized in behaviorism.

Teaching Modern Mathematics

Interestingly enough, during the modern mathematics movement of the 1960's, mathematics teachers in the main did not change their traditional, behavioristic ways of teaching mathematics. There were at least three reasons for this state of affairs. First, mathematics teachers were not knowledgeable about what was purported to be the psychological emphases of the modern mathematics programs. Institutes for mathematics teachers were held, but the institutes did not offer courses on problem solving or learning by discovery. The primary emphasis in the institutes was on upgrading the mathematical preparation of mathematics teachers.[ix,x] Second, the modern curricula emphasized mathematical structure and the logical, deductive presentation of ideas rather than problem solving and learning by discovery.[xi] There were minimal attempts to psychologize the subject matter in these ways, which was a major oversight because of the influence textbooks have on the classroom teaching of mathematics. Finally, behaviorism is a common sense psychology. Although I would say that few mathematics teachers, including myself, had a working knowledge of Thorndike's psychology of arithmetic or algebra, or of behaviorism more generally, being held accountable for four or five classes of 25-35 students per class can easily lead a teacher to using common sense psychology in teaching without being reflectively aware of doing so. What I mean by a common sense psychology is amply demonstrated in the following citation from an introduction to Thorndike's psychology of algebra.

> Suffice it to say here that it emphasizes the dynamic aspect of the mind as a system of connections between situations and responses; treats learning as the formation of such connections or bonds or elementary habits; and finds that thought and reasoning—the so-called higher powers—are not forces opposing those habits but are those habits organized to work together and selectively. (Thorndike, Cobb, Orleans, Symonds, Wald, & Woodyard, 1926, p. v)

Piaget's Genetic Structures as a Psychological Rationale

It is very interesting that Piaget's genetic structures and stage theory of cognitive development served as a psychological rationale for the modern mathematics programs at the elementary school level (Bruner, 1960). This was primarily due to the logical-mathematical structural emphasis in the modern mathematics programs that left the programs without a psychological rationale. Piaget's *constructivism* did not serve as an epistemological basis for the modern mathematics programs nor was it even emphasized in a conference devoted to Piaget's work and the modern programs that was held at Cornell University in 1964 (cf. Ripple & Rockcastle, 1964). Instead, the interest was in Piaget's stage theory and his formalizations of the thinking of children within the stages as can be seen by Bruner's (1960) citation of Bärbel Inhelder, Piaget's close collaborator, in *The Process of Education*:

> Basic notions in these fields are perfectly accessible to children of seven to ten years of age, provided that they are divorced from their mathematical expressions and studied through material the child can handle himself. (p. 43)

Inhelder's idea was that children in the *concrete operational stage*[xii] were ready to learn, and indeed could learn, "basic notions in these fields". This idea served as the basis of Bruner's (1960) famous concept of the readiness to learn the basic structures of mathematics:

> Any subject can be taught effectively in some intellectually honest form to any child at any stage of development. (p. 33)

Bruner (1960), however, conflated basic structures of mathematics and Piaget's genetic structures when he referred to "less able students":

> Good teaching that emphasizes the structure of the subject is probably even more valuable for the less able students than for the gifted ones. (p. 9)

By "less able students," I take Bruner as referencing children in Piaget's preoperational stage, children who's thinking was not explained by Piaget's Grouping structures. In this quotation, he seemed caught in Cartesian anxiety.

> [Cartesian anxiety] is an anxiety that permeates all metaphysical and epistemological questions concerning the existence of a stable and reliable rock upon which we secure our thoughts and actions. As Bernstein explains: "Either there is some support for our being, a fixed foundation for our knowledge, or we cannot escape the forces of darkness that envelope us with madness, with intellectual and moral chaos (p. 18)." (Konold & Johnson, 1991, p. 2)

In spite of using Piaget's psychology as a rationale for the emphasis on mathematical structure, Piaget was considered to be an observer rather than a teacher, and the elasticity of the limits of children's minds was not considered as having been established:

> These reformers (and I speak now not only of SMSG) have been so successful in teaching relatively complex ideas to young children, and thus doing considerable violence to some old notions of readiness, that they have become highly optimistic about what mathematics can and should be taught in the early grades. (Kilpatrick, 1964, p. 129)

I had no problem with Kilpatrick's assertion for children who were in Piaget and Inhelder's more advanced concrete operational stage.[xiii] But I did not accept Bruner's famous hypothesis about the readiness to learn for the "less able" children nor did I accept Kilpatrick's assertion for children in Piaget's preoperational stage. Consequently, the way in which Piaget's grouping structures might be relevant in the mathematics education of children became a major problem for me soon after I earned my Ph.D. from the University of Wisconsin in 1966. At that point, research in mathematics education was still based in empiricism and to work scientifically meant to use experimental and statistical methods (Stanley & Campbell, 1963) in the test of hypotheses in a way that was quite similar to Thorndike and Woodworth's classical experiment.

Applying Piaget's Psychology

After joining the Department of Mathematics Education in 1967, I turned to working for a period of approximately eight years in an attempt to reject Bruner's famous hypothesis concerning the readiness to learn mathematics for children who were in Piaget's pre-operational stage. In this effort, I functioned as an experimental researcher with little awareness that Piaget (1980) rejected empiricism.

> Fifty years of experience has taught us that knowledge does not result from a mere recording of observations without a structuring activity on the part of the subject. (Piaget, 1980, p. 23)

My efforts were directed toward *applying* Piaget's psychology in the mathematics education of preoperational children in a "scientific" manner. Although I experimentally rejected Bruner's readiness hypothesis for these children (e.g., Steffe, 1966, 73), the children rather forcefully taught me that I had no insights into the psychology of their mathematical thinking (Steffe, 2012). I considered myself as doing pseudo-science and making only accretional progress if I was making any progress at all. The relationships with the mathematics students that I taught as a mathematics teacher was missing. That is, my contributions to the mathematical thinking and reasoning of the children who were my "subjects" in the experiments was not being realized.

So, rather than rely on Piaget's Grouping structures as a psychology of the child, I returned to my identity as a mathematics teacher and taught two classes of first-grade children over the course of a school year so the children could teach me how they think when engaging in mathematical activity (Steffe, Hirstein, & Spikes, 1976).[xiv] The involved children taught me that counting was their primary and spontaneous way of operating in discrete quantitative situations and that counting could have quite different meanings for different children. Piaget had not explained children's counting, so this finding corroborated abandoning attempts to *apply* Piaget's *psychology* in children's mathematical education. It also led to throwing off the straight jacket that controlled experimentation and statistical methodology had on my conception of doing science in mathematics education. In fact, it led to developing the teaching experiment as a method of doing research and using teaching as a method of scientific investigation (Cobb & Steffe, 1983; Steffe, 1983; Steffe & Thompson, 2000b; Steffe & Ulrich, 2013).

The shift to using teaching as a method of scientific investigation was a major shift in doing research and, to my knowledge, at the time it was unprecedented in the United States. I learned later that researchers in the Academy of Pedagogical Sciences in the USSR had already used versions of teaching experiment in their work (Kilpatrick & Wirszup, 1975–1978). Not only did their work provide academic respectability for what then was a major departure in the practice of research in mathematics education in the United States, it was also a departure in the goals of the research. In El'konin's (1967) assessment of Vygotsky's (1978) research, the essential function of a teaching experiment is the production of models of student thinking and changes in it.

> Unfortunately, it is still rare to meet with the interpretation of Vygotsky's research as modeling, rather than empirically studying, developmental processes. (El'konin 1967, p. 36)

So, the new problem that faced me was to construct explanations of the mental processes that are involved in children's counting and, further, to construct explanations of how children might construct those mental processes. I had constructed a typology of the units children create in counting that they taught me. However, I could not explain the processes that are involved in children's construction of these unit types other than Piaget's account of children's construction of what he called arithmetical units (Piaget & Szeminska, 1952). That is*, I realized that it was I who had to construct a psychology of the mathematical children that I taught rather than attempt to apply a psychology that had been constructed for a different purp*ose. That was a major breakthrough in my conception of what it meant to do research in mathematics education.

Mathematics Education 1970-2000: The New Progressive Educators

Interdisciplinary Research on Number

The modern mathematics era ended circa 1970 and behaviorism came roaring back into mathematics education. When von Glasersfeld and I started to work on the project, Interdisciplinary Research on Number (IRON), he had just published his manifesto on radical constructivism (von Glasersfeld, 1974) and it was his intention to start an epistemological revolution that would eliminate the duality between mind and reality in Cartesian epistemology. It was also his intention [and mine] to countermand the stranglehold that behaviorism once again had on mathematics education throughout 1970's and 1980's. Radical constructivism emerged as an epistemology in mathematics as well as in science education (e.g., Driver, 1995) throughout the 1980's and played a role similar to that of progressive education during the first one half of the century. But the role was essentially based on von Glasersfeld's (1989) first principle that, "knowledge is not passively received but actively built up by the cognizing subject" (p. 182) rather than on the research that we were doing in IRON. In fact, I frequently was told that joining radical constructivism was like joining a political party. Few progressive educators appreciated the implications von Glasersfeld's (1989) second principle that, "the function of cognition is adaptive, and serves in the organization of the experiential world, not the discovery of ontological reality" (p. 182), which was the "radical" part of radical constructivism that eliminated the Cartesian dualism between mind and reality (von Glasersfeld, 1974, 1984).

The Standards Movement and the "Math Wars"

Mathematics education was a conceptual wasteland during the 1970's, so it was no surprise that another crisis in education emerged that was marked by the publication of *A Nation at Risk* (National Commission on Excellence in Education, 1983). Influenced by this newly perceived crisis, the constructivist revolution, and the recommendation that problem solving be the focus of school mathematics in the 1980's (National Council of Teachers of Mathematics, 1980), the standards movement in mathematics education officially began in 1989 with the publication of the *Curriculum and Evaluation Standards for School Mathematics* (CESSM; National Council of Teachers of Mathematics, 1989). The influence of Cartesian epistemology was still strong among the progressive educators, so CESSM was a strange mixture of realism and constructivism in spite of the commission claiming a constructivist view of learning, where learning was thought to, "occur through active as well as passive involvement with mathematics" (CESSM, p. 9).

The National Science Foundation funded ten curriculum projects based on the CESSM that were published circa 2000, curricula that unfortunately became known as "constructivist curricula." The publication of these curricula extended the famous "math wars" between conservative mathematicians and progressive mathematics educators that erupted in California (cf. Klien, http://www.csun.edu/~vcmth00m/). The "math wars" had their origin in the 1985 *California Mathematics Framework* (California State Department of Education, 1985). This framework,

> [W]as considered a progressive document—an antecedent of the 1989 NCTM *Standards.* California's professional teacher organization, the California Mathematics Council, was one of the most progressive teacher organizations in the country, and one of the most enthusiastic adopters of the spirit of the 1989 *Standards.* When the next adoption cycle came, the 1992 *California Mathematics Framework* (California State Department of Education, 1992) "pushed the envelope" a good deal further: it emphasized reform, focusing on "mathematical power" and collaborative and independent student work while de-emphasizing traditional skills and algorithms. (Schoenfeld, 2007)

The attempts of the constructivist curricula writers to focus on student work were realized in part through their social agenda, "Mathematics for All," and concomitantly, how they regarded mathematics learning and teaching. In this agenda, it was assumed that all students could learn the mathematics specified in the content standards of CESSM.

> If all students do not have an opportunity to learn this mathematics, we face the danger of creating an intellectual elite and a polarized society. The image of a society in which a few have the mathematical knowledge needed for the control of economic and scientific developments is not consistent either with the values of a just democratic system or with its economic needs. (CESSM, 1989, p. 9)

The social agenda of the writers of the so-called constructivist curricula was based on social constructivism (Bauersfeld, 1995, 1996; Cobb & Yackel, 1996, Voight, 1989). The orientation that shaped the social agenda and the recommendations for teaching is cogently caught in a comment made by Bauersfeld (1995) that, "We can understand the development of mathematizing in the classroom 'as the interactive constitution of a social practice'" (p. 150). This sociological emphasis is compatible with von Glasersfeld's (1989) first principle of radical constructivism if "interactively" is included in "actively." It doesn't, however, take into account von Glasersfeld's (1989) second principle. The reason is that, although interaction is a fundamental assumption in radical constructivism, there are two types of interaction: within subject and between subject interaction (Steffe & Thompson, 2000a). The social constructivists emphasize between subject interaction and make few attempts to model what might go on inside of the heads of children, which is where learning and development take place.

The social agenda served to exacerbate the dissatisfaction the mathematical critics had with the "constructivist" curricula.

> [T]here is a unifying ideology behind "whole math." It is advertised as math for all students, as opposed to only white males. But the word all is a code for minority students and women (though presumably not Asians). In 1996, while he was president of NCTM, Jack Price articulated this view in direct terms on a radio show in San Diego: "What we have now is nostalgia math. It is the mathematics that we have always had, that is good for the most part for the relatively-high socioeconomic anglo male, and that we have a great deal of research that has been done showing that women, for example, and minority groups do not learn the same way. They have the capability, certainly, of learning, but they don't. The teaching strategies that you use with them are different from those that we have been able to use in the past when ... we weren't expected to graduate a lot of people, and most of those who did graduate and go on to college were the anglo males." (Klein, 2000)

Klein went on to say that; "I reject the notion that skin color or gender determines whether students learn inductively as opposed to deductively and whether they should be taught the standard operations of arithmetic and essential components of algebra" (Klein, 2000). So, not only did Klein critique the standards in CESSM and the mathematics that was involved in the "constructivist" curricula, he was also a critic of how teaching was conceptualized and practiced. Essentially, the "math wars" were reminiscent of the contentious relationship between conservative patriotic business groups and progressive educators concerning Rugg's social science textbooks.

Mathematics Education 2000 and Forward: Outcome-Based Education

Klein's rejection of the standards and the social agenda of the constructivist curricula writers foreshadowed the mission of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association for Best Practices and Council of Chief State School Officers, 2010). The release of the CCSSM helped thaw the "math wars" (Lobato, 2014; Norton, 2014) primarily, in my view, because of the presence of more rigorous curriculum standards. We find the following statement in the introduction to CCSSM.

> The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy. The Common Core State Standards provide a consistent, clear understanding of what students are expected to learn, so teachers and parents know what they need to do to help them (CCSSM, 2010, Introduction).

The CCSSM, similar to the CEEB in 1969, was designed primarily for college bound students.[xv] It has carried the emphasis on outcome-based education forward to the present time, whose beginning was marked in mathematics education by the publication of the CESSM in 1989. It might seem surprising that I would say that CESSM ushered in outcome-based education given that it also was an impetus of the constructivist curricula that was so severely criticized in the "math wars". However, one of the main criticisms of the "constructivist" curricula and CESSM by the mathematicians was that the involved standards were weak, not that there were not any standards.

Outcome-based education is based on Cartesian epistemology with its requirement that something is true only if it corresponds to an independent, objective reality, where the standards constitute that objective reality. The neo-behaviorism of outcome-based education along with the national emphasis on standards-based education by the No Child Left Behind Act of 2001 has had the effect of standardizing precollege mathematics education. For example, students are required to take standardized test throughout their years in school[xvi] and these tests are used in evaluating teachers, a practice that has become known as Value Added Measures [VAM's] of teacher performance. This surge of neo-behaviorism in mathematics education during the first years of the 21st century is exemplified in the report of the National Mathematics Advisory Panel (2008) with its emphasis on *rigorous scientific research*. The research conducted in IRON concerning children's number sequences and fraction schemes and how they are used in the construction of adding, subtracting, multiplying, and dividing schemes that has been published in books and various articles (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988; Steffe, 1992; Steffe & Olive, 2010) was not even mentioned in that report. So, obviously, the authors of the report did not consider that research as scientific research if they considered it at all.

Given the ubiquity of the influence of outcome-based education, one might think that there should be another major effort by progressive educators to countermand that influence similar to the era of the modern mathematics programs or to the era of the constructivist curricula. While that may be of critical importance given the current state of mathematics education in precollege education, essentially the attempted wholesale changes in mathematics education that were made following national reports were abandoned after the changes led to major disappointments and failures. If this history can be used to predict what might happen if another round of national reform in mathematics education is attempted, a strong argument can be made that what is needed is to construct mathematics curricula *for* children that is based on the mathematics *of* children rather than continue on with the historical practice of basing mathematics curricula for children on the first-order mathematical knowledge of adults. Simply put, if lasting progress in mathematics education is to be made, researchers must establish the construction of mathematics curricula for children as an academic field. I think of constructing mathematics curricula for children that is based on the mathematics of children as a result of intensive and longish periods of teacher/researcher interactions with children. Toward that end, I present several radical constructivist research programs that are tailored toward constructing mathematics curricula for children that emerge from the work in IRON. Before presenting the programs, I present several basic concepts that I feel will help understand the research programs.

Radical Constructivist Research Programs

Basis Concepts

First- and second-order models. I understand *children's mathematics* as a result of maturation coupled with what children have constructed as a result of interacting in their social-cultural milieu in all of its aspects. The assumption that children construct mathematical knowledge is an assumption of an observer.[xvii] Children's mathematics is thought of as first-order knowledge, which are, "the hypothetical models that the observed subject constructs to order, comprehend, and control his or her own experience (Steffe, et al., 1983, p. xvi). An observer psychologizes children's mathematics by constructing second-order models, which are, "the hypothetical models observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et. al. 1983, p. xvi). The second-order models are referred to as the *mathematics of children* and the children's first-order models are referred to as *children's mathematics.*[xviii] The concept of children's mathematics is based on the belief that mathematics is a product of the functioning of human intelligence (Piaget, 1980). The mathematics of children, which is an explanation of children's mathematics, is a legitimate mathematics to the extent that teachers/researchers can find rational grounds to explain what children say and do.

Epistemological analysis and conceptual analysis. Conceptual analysis is the method by which the second-order models that constitute the mathematics of children are produced. Conceptual analysis is an analysis of mental operations. In explaining conceptual analysis, von Glasersfeld (1995) drew from his experience with Silvio Ceccato's Italian Operational School, whose goal was to, "reduce all linguistic meaning, not to other words, but to 'mental operations'" (p. 6). The main goal of conceptual analysis is defined by a question from Ceccato's group: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (p. 78). Thompson & Saldanha (2000) reformulated the goal in a way that is more relevant to constructing second-order models of children's language and actions. Their goal is to describe, "conceptual operations that, were people to have them, might result in them thinking the way they evidently do" (p. 315). Although I have extensively engaged in conceptual analysis in the construction of the mathematics of children, I know of no papers that have been written that address the problem of how one might creatively use the analytical tools that are available in radical constructivism in conceptual analysis of children's mathematical concepts and operations.

When conceptual analysis is used in the construction of second-order models, I refer to it as a second-order conceptual analysis. Thompson & Saldanha (2000) included what I refer to as first-order conceptual analysis in their discussion of epistemological analysis, that is, an analysis of one's own mathematical concepts and operations (cf. Thompson, 2008). According to Thompson & Saldanha (2000), epistemological analysis, "is used to model what might be called systems of ideas, like systems of ideas composing concepts of numeration systems, functions and rate of change, or even larger systems like those expressed in quantitative reasoning" (p. 316). First-order conceptual analysis is inextricably involved in second-order conceptual analysis of children's mathematical language and actions. Thompson & Saldanha (2000) also included a teacher/researcher analyzing their own concepts and operations relative to children's concepts and operations in interactive mathematical communication. This kind of analysis involves the teacher/researcher operating as Maturana's (1978) second-order observer; that is, an "observer's ability through second-order consensuality to operate as external to the situation in which he or she is, and thus be observer of his or hers circumstance as an observer (p. 61).

In the following quotation, if "intentionally isomorphic" is interpreted as imputing operations to a mathematically operating child, what I said about making explanations is similar to Maturana's second part of the scientific method.

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed to be intentionally isomorphic to (models of) the systems that generate the observed phenomena. In fact, an explanation is always an intended reproduction or reformulation of a system or phenomenon. (Maturana, 1978. p. 30).

Maturana's second part of the scientific method emphasized second-order conceptual analysis and his first part emphasized first-order conceptual analysis, which was, "observation of a phenomenon that, henceforth, is taken as a problem to be explained" (Maturana, 1978, p. 29). Of the observer, he commented,

> Yet we are seldom aware that an observation is the realization of a series of operations that entail an observer as a system with properties that allow him or her to perform these operations, and, hence, that the properties of the observer, by specifying the operations that he or she can perform determine the observer's domain of possible observations. (Maturana, 1978, p. 30)

Like Maturana, I take the subject dependent nature of science in mathematics education as a starting point. But I expand on it in two ways. First, the primary reason for engaging children as a teacher/researcher is to allow children to teach one how and in what ways they operate mathematically and, as commented by Thompson & Saldanha, to *create* operations that if a child had those operations, the child would operate as observed. Second, as a teacher/researcher kind of scientist, my *contributions* to children's ways and means of operating mathematically by teaching them is a constitutive part of a conceptual analysis of children's mathematical language and actions. In the words of Steier (1995);

> Approaches to inquiry … have centered on the idea of worlds being constructed … by inquirers who are simultaneously participants in those same worlds. (p. 70)

This understanding of the subject dependent nature of science in mathematics education provides researchers with the power to create images of unrealized possibilities in the mathematics education of children. But these possibilities are subject to the constraints of children as self-organizing systems—the mind organizes the world by organizing itself (Piaget, 1935/71).

Learning and development. A central goal that runs throughout each research program is to learn how to operationalize children's mathematics learning and development as spontaneous processes in mathematics teaching. A virtue of teaching that is focused on constructive itineraries of children's mathematics in which the teacher/researcher is a participant is that it allows the teacher/researcher to become aware of children's constructive processes, which are understood as the construction of schemes and the accommodations that children make in them (cf. von Glasersfeld, 1980). Because of continual interaction with children, a teacher/researcher is likely to observe at least the results of those critical moments when restructuring is indicated by changes in children's operations and anticipation (Tzur, 2014). Major restructuring of mathematical schemes is compatible with a vital part of Vygotsky's (1978) emphasis on studying the influence of learning on development.

Unlike Vygotsky, however, I regard both learning and development in the context of accommodations that children make in their schemes (Steffe, 1991b). But there is a difference in the two kinds of accommodations. Learning is captured by the functional accommodations that occur in a scheme in the context of the scheme being used, whereas development is captured by metamorphic accommodations that occur independently in no particular application of a scheme. A metamorphosis of a scheme is thought to be the result of autoregulation of the process of interiorizing the scheme (cf. Simon, Saldanha, McClintock, Akar, Watanabe, & Zembat, 2010, for a related view).

Learning and development are not spontaneous in the sense that the provocations that occasion them might be intentional on the part of the teacher/researcher. In children's frames of reference, though, the processes involved are essentially outside of their awareness. This is indicated by the observation that what children learn or develop often is not what was intended by the teacher/researcher. It also is indicated when a child learns or develops when a teacher/researcher has no such intention. Even in those cases where children learn what a teacher/researcher might intend, the event that constitutes learning arises not because of the teacher's actions. Rather, teaching actions only occasion children's learning (Kieren, 1994). Learning as well as development arises as an independent contribution of the interacting children. So, although I do not use "spontaneous" in the context of learning and development to indicate the absence of elements with which children interact, I do use the term to refer to the non-causality of teaching actions, to the self-regulation of the children when interacting, to a lack of awareness of the learning process, and to its unpredictability. Because of these factors, I regard learning and development as spontaneous processes in children's frame of reference.

Trajectories of the constructive activity of children. The construction of trajectories of children's learning and development is one of the most daunting but urgent problems facing mathematics education today. It is also one of the most exciting problems because it is here that we can construct an understanding of how teacher/researchers can profitably affect children's mathematics (Steffe, 2004). By building an understanding of children's mathematical concepts and operations and how a teacher/researcher can engage children to bring forth changes in those concepts and operations, a vision of children's mathematics education can emerge in which children engage in productive mathematical learning and development and teacher/researchers engage in productive mathematical teaching. The principle of self-reflexivity[xix] compels teacher/researchers to consider their own knowledge of children's mathematics, including accommodations in it, as constantly being constructed as they interact with children as the children construct mathematical knowledge. Through the construction of trajectories of children's learning and development that are coproduced by children and teacher/researchers, it is possible to construct trajectories that include an account of teacher/researchers' ways and means of acting and operating relative to children's ways and means of acting and operating (Ellis, 2014). Such an account entails the teacher/researcher operating as a second-order observer.

A trajectory of children's learning and development includes a model of the children's initial concepts and operations, an account of children's constraints and necessary errors, an account of the observable changes in children's concepts and operations as a result of their interactive mathematical activity in situations that are used by a teacher/researcher when interacting with children, an account of the situations relative to a teacher/researcher's models of the involved children's mathematics and the teacher/researcher's goals and intentions, and an account of the involved mathematical interactions. A similar historical account of what transpires in between observed changes is critical not only to understand the changes, but also to provide estimates of the length and the nature of the plateaus in children's mathematical learning and/or development.

Trajectories of the constructive activity of children are third-order models that include the second-order models that constitute the mathematics of children, the first-order models of the teacher/researcher, and relationships between them. In the following research programs that I present, I assume that the models that constitute the mathematics of children produced by IRON will be used at least as starting places in the construction of the trajectories. Because of the nature of the trajectories, I will refer to them as mathematics curricula for children throughout the rest of the paper (Steffe, 2007). Concentrating on constructing mathematics curricula for children does not exclude research programs that center on teacher/researchers working with classroom teachers of children.[xx,xxi] In fact, each stated research program can be reformulated so that it is a research program that involves teacher/researchers working with classroom teachers of children.

The First Research Program

The first research program is to construct mathematics curricula for children who enter their first grade as counters of perceptual unit items over the course of their first eight years in school.[xxii]

The second-order models that were constructed in IRON concerning children's number sequences and how the number sequences are used in the construction of adding, subtracting, multiplying, and dividing schemes have been published in books and various articles (e.g., Steffe, et. al., 1983; Steffe & Cobb, 1988; Steffe, 1992; Steffe & Olive, 2010). Ulrich (2015-16) has published two very readable papers that provide an introduction to the units, schemes, and operations that were constructed in IRON as well as to some of the work that has extended the basic work (e.g., Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009; Hunt, Tzur, &

Westenskow, 2016; Norton & Wilkins, 2013; Tillema, 2013; Ulrich, 2012). To start, I provide a brief summary of the first two stages in the construction of children's number sequences.

The first stage is a sensory-motor or pre-numerical stage that comprises pre-counters, counters of perceptual unit items (CPUI), and counters of figurative unit items (CFUI). Counters of perceptual unit items are restricted to counting items that are in their perceptual field, such as the toys in their toy box, their steps, their heartbeats, or the chimes of a Grandfather clock. For example, an interviewer covered six of nine marbles with his hand and asked Brenda, a six-year-old child, to count all the marbles. Brenda first counted the interviewer's five fingers and then counted the three visible marbles. The interviewer pointed out that he had six marbles beneath his hand and Brenda replied, "I don't see no six!" (Steffe, & Cobb, 1988, p. 23)

Counters of figurative unit items might attempt to count the items in a closed container when told that there are, say, seven items in the container, by touching the container where they believe items might be hidden in synchrony with uttering number words. Because they concentrate on generating images of the items they are counting, they can easily become lost in counting and stop fortuitously. Counting figurative unit items is a step in interiorizing the countable items, which produces abstract unit items (Steffe et.al, 1983). If the child also interiorizes the acts of counting, I mark this monumental event by referring to it as the stage of the initial number sequence (INS). *Spontaneously* counting-on is the indication of the INS (Steffe, & Cobb 1988).

To illustrate some of the constraints that I experienced when teaching CPUI, I recount my experience teaching three such children at the start of their first grade in school. I taught them approximately 60 times in teaching episodes over their first two school years to explore their progress in the construction of counting-on (Steffe, & Cobb, 1988). Although these children also participated in their regular mathematics classrooms, they did not spontaneously count-on in spite of my best efforts to provoke it and, presumably, the best efforts of their teachers. It wasn't until their 3rd Grade that at least one of them had constructed counting-on. Based on my experience in working in teaching experiments and teacher education at UGA and data that were supplied to me by Professor Bob Wright of Southern Cross University, Australia, who started the Mathematical Recovery Program (Wright, Martland, & Stafford, 2000; Wright, Stewart, Stafford, & Cain, 1998), I estimate that 40% of entering first graders in the United States are CPUI. Of this estimate, Professor Wright commented that, "I think that is a good estimate for the number in the perceptual stage or lower, that is the children who can't yet count perceptual items. I think the percentage would be lower in Australia and New Zealand, say about 30%" (Personal Communication).

Of the 40% who enter the 1st Grade as CPUI, I expect that a majority of them to construct counting-on during their 3rd Grade [Wright estimated that from 5 to 8% might not be counting-on by the 3rd Grade]. From that point on the relative percentages are not certain, but because of the length of time and the great difficulties we had in teaching experiments in engendering progress beyond counting on (Biddlecomb, 2002, Hackenberg, 2005; Tillema, 2007), my best estimate is that approximately 30% of the children entering the 6th Grade will be only able to count-on. And those who are at that stage will remain there until their 8th Grade. Wright's estimate was, "that about 30% of kids entering the 6th Grade in the US will only be able to count-on" (Personal Communication).

I consider this program as the most important research program in mathematics education today. My appeal to those who choose to work in such an intractable but crucial research program is to learn how to teach these children in such a way that they do not lose confidence. My practitioner's maxim is that children are never wrong; even children who are CPUI. An adult can easily induce "mistakes" in these children, but my basic and pervasive assumption is that children are rational beings and our responsibility is to find ways of acting and interacting that are not only harmonious with their ways and means of operating, but will also affect them in productive ways. It is crucial to re-establish the NCTM's vision of mathematics for all.

The Second Research Program

The second research program is to construct quantitative mathematics curricula for children who enter their first grade as CFUI or children who can only count-on (1) in the construction of operative measuring schemes, and (2) in the construction of adding and subtracting schemes as reorganizations of their operative measuring schemes during their first two grades in school.

Children who enter their first grade as CFUI have a quite different constructive trajectory than those who enter as CPUI. It is possible for CFUI to construct the INS by means of a metamorphic accommodation by the end of their first grade in school (Steffe & Cobb, 1988, pp. 308ff). By the end of the second grade, it is possible for their INS to undergo another metamorphic accommodation in the construction of the explicitly nested number sequence (ENS), which is indicated when children spontaneously count-up-to (Steffe, 1992, 94: Steffe & Cobb, 1988).

There are three principal *operations* of the ENS that were not available to children who have constructed only the INS. The first is that units of one have been constructed as *iterable* units; for example, at noon a grandfather clock strikes one twelve times in contrast to simply making 12 chimes. The second is that any initial segment of a (finite) number sequence can be disembedded—"lifted"— from the complete sequence without destroying the sequence.[xxiii] The remainder of the initial segment in the sequence can be also disembedded from the sequence and the numerosity of the remainder can be found by counting its elements starting with "one." This way of counting is referred to as *the recursive property* of the ENS in that children can take the number sequence as its own input (Steffe & Cobb, 1988). That is, children who have constructed the ENS can willfully create their own countable items using elements of their number sequence and count these elements using the same number sequence that was used to create the countable items. It is as if the child has two number sequences "side by side," one to use to create countable items and the other to count the countable items. ENS children have more "mathematical power" than do INS children, to borrow a phrase from the California Mathematics Framework. So, there are three distinct stages in children's construction of their number sequences entering their first grade in school; CPUI, CFUI and the INS, and the ENS. There is a more advanced number sequence that only rarely can be observed that is referred to as the generalized number sequence (GNS; Ulrich, 2014, 2016)

My best estimate is that children who enter their first grade as CFUI or who can only count-on comprise 45% of the first-grade population. Table 1 contains my best estimates of the percent of children who enter their first grade in each of the three number sequence types. The question of whether stage shifts can be engendered by means of specialized interactions has been

Table 1: Number Sequence Type Across Grades for Children Who Enter their First Grade Counting-on (INS) or as CFUI.

Grade/N Seq.	CFUI or INS	ENS	GNS
First	≈ 45 Percent	≈ 10 to 15 Percent	≈ 0 to 5 Percent
Second	≈ 30 Percent	≈ 25 to 30 Percent	≈ 0 to 5 Percent
Third	≈ 5 Percent	≈ 45 to 50 Percent	≈ 0 to 10 Percent

worked on by Norton and Boyce with an *eleven-year old* child (2015). These authors did demonstrate that by working intensively with the child individually in 14 teaching sessions, he did make progress in reasoning from one level of units (INS) to two levels of units (ENS). The authors note, however, that,

> Cody did not seem able to coordinate units in continuous contexts in the same way he could in discrete contexts…We conjecture that that limitation is due to the lack of physical referents for the embedded units within composite units that are continuous. For example, a tablespoon contains three teaspoons, but these three units are not as evident within the tablespoon as they would be with three chips within a cup. Rather, students have to produce the units within a continuous composite unit through some kind of segmenting or partitioning activity (Steffe, 1991a), which involves breaking down the composite unit. (Norton & Boyce, 2015, p. 229)

Children who have constructed the ENS and, hence, two levels of units, do use their number concepts spontaneously in partitioning continuous units. So, there is always an issue of the generality of the learning process when the situations used in the teaching experiment are with only one type of quantity. According to some authors, a fundamental question that pervades mathematics education today is whether mathematical thinking begins with counting or with comparisons of quantity (Sophian, 2007). Based on the work of Davydov (1975) and influenced by Doughtery (2004), Sophian (2007) commented that, "The most fundamental idea I have derived from those papers is the idea that mathematical thinking begins, not with counting, but with comparisons between quantities, in particular the identification of equality and inequality relationships" (p. xiv). This notion of quantity is based on Davydov's (1975) formal definition that a quantity is any set for the elements of which criteria of comparison have been established. However, establishing the quantitative property of a composite unit called its numerosity and the quantitative property of a continuous item called its length precedes a need for comparing the numerosity of two collections or the length of two continuous items (Steffe, 1991a). So, it's not a matter that mathematics begins with comparisons between quantities be they discrete or continuous. Rather, one might say that mathematics begins with establishing the quantitative properties of objects (Steffe, 1991a). This fits with Thompson's (1994) notion of a quantity as, "composed of an object, a quality of that object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). This idea of quantity, both discrete and continuous, leads to the following reorganization hypothesis.

> *Reorganization Hypothesis*: Operative measuring schemes and their use in constructing adding and subtracting schemes can emerge as reorganizations of children's INS.[xxiv]

In this hypothesis, the main goal is for children to use their INS in measuring activity in order to transform the measuring activity, such as described in CCSSM standard 1.MD.2 stated below, into operative measuring schemes and to what Thompson, Carlson, Byerley, & Hatfield, (2014) referred to as additive measurement.

> Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.

It is important to note that this CCSSM standard is written in such a way that emphasizes the activity of measuring. After actually measuring linear objects to establish how to measure and the units used in measuring, INS children can engage in *operational* measuring activity such as finding the length of a 64-inch string after it is increased by seven inches. If operational measuring is generalized across other quantities such as time, money, temperature, weight, etc., children can construct operational measuring schemes that they could use as if they were using the INS in discrete quantitative situations. They could also be asked to find, say, how many tablespoons of powder could be made from nine teaspoons of powder to engender the construction of composite units—or units of units—which, at this point, I consider as essential in engendering a metamorphosis of the "INS measuring schemes." Furthermore, in the case of discrete quantity, *children construct adding and subtracting schemes as reorganizations of their number sequences* (Steffe, 2003). So, by the children using their INS in the construction of operative measuring schemes, they can in turn use their measuring schemes in the construction of operative adding and subtracting schemes across different quantitative contexts. My hypothesis is that if a stage shift is observed from an INS to an ENS measuring scheme in the case of one type of quantity, a corresponding stage shift will be observed in all of the measuring schemes that the INS was used in establishing. Such a constructive generalization would lead to considerable mathematical power of the children, to borrow a phrase from the California standards.

For children who are CFUI, engaging in measuring activity that includes counting activity extends the goals, situations, activities, and results of their figurative counting schemes. Similar to the INS children who use their counting schemes in measuring activity, the effects of the CFUI using their figurative counting schemes in measuring activity is yet to be determined. Still, it is possible that their measuring activity could serve in engendering metamorphic accommodations like that which produces the INS (cf. Steffe, & Cobb, 1988, pp. 306 ff) if for no other reason than a teacher/researcher could capitalize on children's need to measure things in such a way that provokes monitoring re-presentations of measuring activity.

The Third Research Program

> *The third research program is to construct quantitative mathematics curricula for ENS[xxv] children in the construction of extensive quantitative measuring schemes and their use in constructing adding, subtracting, multiplying, dividing, and numeration schemes in which strategic reasoning and relationships between quantities are of primary importance.*

I agree with Smith & Thompson (2007) that an emphasis on quantitative reasoning needs to begin early on in children's mathematics education and that building quantitative reasoning skills for the majority of students is not a one or two-year program. Their paper concerned how a shift in current school curricula could emphasize quantitative reasoning, whereas my emphasis is on constructing a quantitative mathematics for children based on abstractions from actually teaching children to establish learning trajectories in the sense that Ellis (2014) explained. In this context, it is critical to understand what schemes can be considered as extensive quantitative schemes, which I refer to as genuine measuring schemes. Rather than think of extensive quantities as substances as would be the case when considering 5/4 as referring to a point on the number line, von Glasersfeld & Richards (1983) pointed out that Gauss focused on extensive quantities as relations.

> To forestall the idea that the extensive quantities he is referring to are a matter of inches or degrees, Gauss hastens to add that mathematics does not deal with quantities as such, but rather with relations between quantities. These relations he calls "arithmetical" and in arithmetic, he explains, quantities are always defined by how many times a known quantity (the unit), or an aliquot part of it, must be repeated in order to obtain a quantity equal to the one that is to be defined, and that is to say, one expresses it by means of a number" (pp. 58-59).

The ENS is the first numerical counting scheme that qualifies as an extensive quantitative scheme in that any number such as 50 can be conceived of as one fifty times as well as 50 ones. The other operations of the ENS are also critical in constituting this scheme as an extensive discrete quantitative measuring scheme. So, by viewing the construction of measuring schemes more generally as reorganizations of the operations that produce the ENS, the hypothesis is that the measuring schemes will emerge as extensive quantitative schemes.

The Fourth Research Program

> *The fourth research program is to construct quantitative mathematics curricula for children in (1) the construction of quantitative measuring schemes as reorganizations of their fraction schemes,*[xxvi] *and (2) the construction of multiplicative and additive measuring schemes as reorganizations of their fraction schemes.*

A reorganization hypothesis that was fundamental in the work of IRON that centered on children's construction of fraction schemes was that *children's fraction schemes can emerge as accommodations in their numerical counting schemes.* The fraction schemes that emerged were of a different genre than the number sequences that were used in their construction primarily because children used their number sequences (or concepts) in partitioning in their construction of fraction schemes.[xxvii] Two basic fraction schemes that emerged were the partitive and the iterative fraction schemes.

The partitive fraction scheme. When ENS children use their number concepts in partitioning, they establish an equi-partitioning scheme (Steffe & Olive, 2010, p. 75ff). For example, when the number concept five is used in partitioning a candy bar, say, an estimate can be made of where to mark off one of five equal parts. Once a mark is made, the child can disembed the marked part (mentally or physically), use it in iterating to make five equal parts, and mentally compare the five parts to the original bar to test if the five parts together are equivalent to the original bar. If a child considers that the disembedded part is one out of five equal parts, or a fifth of the candy bar, this produces the first genuine fraction scheme that is referred to as the partitive fraction scheme (PFS; Tzur, 1999).

The iterative fraction scheme and fractional numbers. For children who have constructed the ENS and the PFS, it would seem that the CCSSM Standard 4.a under Number and Operations—Fractions would be appropriate for these children.

> Understand a fraction a/b as a multiple of $1/b$. For example, use a visual fraction model to represent 5/4 as the product 5×(1/4), recording the conclusion by the equation 5/4 = 5×(1/4).

This standard was meant to illustrate how *multiplying a fraction by a whole number* might be modeled by a mathematics teacher in a straightforward way. But it doesn't explain the operations that are involved in children constructing fractions as *fractional numbers*. There is a scheme in the fractional knowledge of children, the *iterative fraction scheme* (IFS), where the fraction 5/4 is constituted as a fractional number; as five times one fourth of the candy bar (Steffe, & Olive, 2010, p. 333ff). The structure of the "candy bar" produced consists of a unit of units of units. That is, as a composite unit containing a composite unit comprised by 4/4 of the candy bar and one more partitive unit fraction. Once constructed, children can use the scheme to produce fractional connected number sequences {1/4, 2/4, 3/4, 4/4, 5/4, 6/4, …} that are constructive generalizations of their explicitly nested number sequence (Steffe, & Olive, 2010, p. 333ff) . This is the first fraction scheme that can be judged as an *extensive quantitative scheme*. The PFS constructed using the ENS is still constrained to the fractional whole. The construction of fractional numbers is not in the zone of potential construction of the children who have constructed the PFS in any short-term sense because it involves a stage shift from two to three levels of units coordination.

The splitting scheme. The splitting scheme, which is a reorganization of the equi-partitioning scheme, is used in the construction of fractional numbers. The splitting scheme is indicated when children can mentally produce a hypothetical stick that can be iterated seven times when given a stick and told that the given stick is seven times longer than their stick and are asked to make their stick. After the splitting scheme is constructed, if a child mentally splits a stick into, say, 48 parts, the child knows that one of the parts would be one forty-eighth of the whole stick because the whole stick is 48 times as long as the part. The result of the scheme is an inverse multiplicative relation between the part and the partitioned whole in the sense that Gauss specified extensive quantitative relations (cf. also Thompson, & Saldana, 2003).

Assessments of fifth through eighth grade children. With this brief introduction to the PFS and the IFS, I now turn to assessments of fifth, sixth, seventh, and eighth grade children concerning these schemes. Norton & Wilkins (2009) found that only 34% of the fifth graders and 35% of the sixth graders in their sample could engage in splitting, which is an indication of the presence of the operations that produce three levels of units[xxviii]. Of those same children, only 14% and 20%, respectively, provided some indication of having constructed the iterative fraction scheme.[xxix] In other assessments, Norton & Wilkins (2010) found that only 13% of their seventh grade sample and 19% of their eighth grade sample could produce the fractional whole when given, say, a stick partitioned into three parts and told that it was three sevenths of a candy bar and asked to draw the whole candy bar, which I consider as an assessment of fractional numbers.[xxx] In their earlier study Norton & Wilkins (2009) reported similar percentages for their fifth and sixth grade samples (14% and 18%). These data are consistent with an analysis of the percentages of children at one, two, and three levels of units that I present in Table 2 in which Norton's and Wilkins' data are included.

Table 2: Estimated Percent of Children at Each Level of Units by Grade

Grade/Level	One Level	Two Levels	Three Levels	IFS
Third	45	45	10	
Fifth	35	40	25 (34%)[NW]	(14%)[NW]
Sixth	30	30	40 (35%)[NW]	(18%)[NW]
Seventh				(13%)[NW]
Eighth				(18%)[NW]

It is especially disconcerting that only approximately 15.5% of Norton & Wilkins' seventh and eighth grade sample indicated that they had constructed a fraction as a multiplicative concept. It's disconcerting because, based on my own estimates, at least 40% of this sample should be able to construct a fraction as a multiplicative concept; that is, they should have been able to construct the IFS. But this expectation is tempered by the realization that the children in the fractions project constructed the iterative fraction scheme by working with us in teaching experiments. The fraction standards of the CCSSM are stated by grade level and as such underestimate what children who have constructed three levels of units can accomplish. On the other hand, children who have constructed two levels of units are constrained to constructing the PFS, a scheme that children use to construct proper fractions. What this means is that approximately 45% of the third-grade population, 40% of the fourth grade population, and 30% of the sixth grade population are able to construct partitive fractions, but not fractional numbers. When combined with the children who have constructed only one level of units throughout these three grade levels, we see that approximately only 15% of the third graders, 25% of the fourth graders, and 40 percent of the sixth graders will be able to construct the IFS and engage in producing fractional numbers.

Recommendations of the NMAP. Children's construction of fractions as well as the teaching of fractions must be changed. In the report of the National Mathematics Advisory Panel (2008), the following comment was made.

> Difficulty with learning fractions is pervasive, and is an obstacle to further progress in mathematics and other domains dependent on mathematics including algebra. … Conceptual and procedural knowledge about fractions with magnitudes less than 1 do not necessarily transfer to fractions with magnitudes greater than 1. Therefore, understanding of fractions with magnitudes in each range needs to be taught directly and the relation between them discussed. (p. 28)

Apparently, the authors of this report believed that fractions (proper and improper) can be taught directly to children regardless of the levels of units the children have constructed. The report of the panel, as I interpret it, exemplifies an empiricist as well as a neo-behavioristic agenda in the teaching of mathematics in precollege education that harks back to Thorndike's influence on the teaching of mathematics during the first one-half of the last century. Still, I do agree with the writers of the report concerning the pervasive difficulty that the learning of fractions presents to schoolboys and schoolgirls and also to the pervasive difficulty that the teaching of fractions presents to their mathematics teachers. Resorting to direct teaching in an attempt, for example, to raise children who have constructed only the PFS to the IFS could be interpreted as a more or less empirical enterprise and as generating a whole industry of empirical research on mathematical learning, to paraphrase Michael Cole's (2004) comments concerning the training studies of the 1960's that were conducted to prove Piaget wrong. In contrast, for the children who have constructed at least the partitive fraction scheme, my hypothesis is that quantitative measuring schemes can emerge as reorganizations of children's fraction schemes.

This hypothesis is similar to the hypotheses in the second and third research programs that additive measuring schemes can be constructed as reorganizations of children's number sequences. It is quite different, however, in that partitioning is a fundamental operation in the construction of the measuring schemes, which opens the way for children to construct measuring schemes involving two levels of units; for example, meters and centimeters, minutes and seconds, pounds and ounces, weeks and days, etc. Measuring systems in multiple levels of units might still be problematic. It is especially crucial to investigate possible changes that indicate *fundamental* transitions between reasoning with two levels of units and three levels of units induced in the construction of quantitative measuring schemes and their use in the construction of multiplicative and additive measuring schemes.

The Fifth Research Program

> *The fifth research program is to construct quantitative mathematics curricula for children in their construction of the rational numbers of arithmetic and the rational numbers, and the schemes and operations entailed in and by these constructions.*

Fractional numbers are a major achievement of children who can use three levels of units as assimilating operations, but fractional numbers are not equivalent to the Rational Numbers of Arithmetic nor to the Rational Numbers. Constructing the rational numbers of arithmetic involves the operations that generate the generalized number sequence (cf. Ulrich, 2014, p. 256). To exemplify those operations, an eight-year old child, Nathan, was presented with copies of a string of three toys and a string of four toys and asked to make 24 toys. Nathan reasoned out loud as follows,

> Three and four is seven; three sevens is 21, so three more to make 24. That's four threes and three fours! (Steffe & Olive, 2010, p. 278)

In solving the task, Nathan integrated a unit of three and a unit of four into a unit of seven, iterated the unit of seven three times to produce 21, increased 21 by three to produce 24, disunited 21 into three threes and three fours, integrated the additional three with the three threes, and produced four threes and three fours. These operations are operations of a GNS. In a GNS, any composite unit can be taken as the basic unit of the sequence in such a way that the composite unit implies the sequence just as the unit of one implies the ENS. Similar to the ENS, in the GNS a child can establish two number sequences "side by side", a sequence of units of three and a sequence of units of four and combine the basic units of each sequence together to produce another sequence of units of seven. What this amounts to is the coordination of two three-levels of unit structures.

The rational numbers of arithmetic can be regarded as those operations that can be used to transform a given fraction into another given fraction; that is, the operations that are involved in quotitive fraction division. Quotitive fraction division involves the coordination of two three-levels of units structures; units within units within units. For example, consider a case where a child is given a segment that is said to be 1/5 of a unit segment and another segment that is said to be 1/3 of the same unit segment, and asked to use the 1/3-segment to produce the 1/5-segment. If the child partitions the 1/3-segment into five parts, takes one of these parts as a 1/15-segment and iterates this segment three times to produce the 1/5-segment, and if the child abstracts the operations as 3/5 of 1/3, then 3/5 is referred to as a rational number of arithmetic. After operating, I would also want to know if the child knows that 3/5 of the 1/3-segment is the 1/5 segment without actually taking 3/5 of the 1/3-segment. I would also want to know if the child can engage in reciprocal reasoning and understand that 5/3 of the 1/5-segment is the 1/3-segment (Hackenberg, 2010, 2014; Thompson & Saldanha, 2003; Thomson, et. al., 2014).[xxxi] The child is aware of the operations needed, not only to reconstruct the unit whole from any one of its parts, as in the case of fractional numbers, but also to produce any fraction of the unit whole starting with any other fraction, which are the operations involved in quotitive fraction division. (cf. Olive, 1999, for an interpretation of the schemes and operations involved in the production of the rational numbers of arithmetic). My hypothesis is that construction of the rational numbers of arithmetic entails a metamorphic accommodation relative to fractional numbers, and learning how to engender this accommodation and the constructive possibilities it entails is included in the first part of the fifth research program.

One might think that the distinction between the rational numbers of arithmetic and the rational numbers is "simply" that the latter involve negative as well as positive rational numbers of arithmetic. But that is not the case at all. My hypothesis is that a scheme of recursive distributive partitioning operations is involved in constructing rational numbers. In general, distributive partitioning operations are those operations that allow a student to share n units among m people and interpret one share as *n*/*m* of one unit and as 1/*m* of all *n* units (Liss, 2015; Steffe, Liss, & Lee, 2014; Lamon, 1996). Distributive partition operations are involved in what Thompson et al. (2014) referred to as "Wildi Magnitudes". The power of Wildi's definition of magnitude is that it makes explicit the fact that, "the magnitude of a quantity is invariant with respect to a change of unit" (Thompson, et. al., 2014, p. 4). So, if a quantity measures 22 inches, and if there are 12 inches/foot, then the quantity also measures 22inches/(12 inches/foot), whose transformation into 22*(1/12 foot) or 22/12 feet involves rational number of arithmetic operations. It also involves use of a scheme of recursive distributive partitioning operations because, according to Thompson (2014), "When a person *anticipates* that any measurement of Q[xxxii] with respect to an appropriate unit can be expressed in *any other* (emphases added) appropriate unit by some conversion without changing Q's magnitude, she possesses Wilde's meaning of magnitude" (p. 4)

When the scheme of recursive distributive partitioning operations can be used to produce what I would consider an equivalence class of fractional numbers, I would judge that the child has constructed a rational number.[xxxiii] I hypothesize that the construction of the rational numbers constitutes a stage shift relative to the rational numbers of arithmetic, and learning how to engender this stage shift and the constructive possibilities it entails is included in the second part of the fifth research program. The scheme of recursive distributive partitioning operations that is involved in the construction of rational numbers is also involved in the construction of intensive quantity (Liss, 2015; Steffe, et al., 2014). The main difference is that intensive quantity involves relative magnitude, which means that a quantity is measured using a quantity of a different nature (Thompson, et al., 2014). In the case of rational numbers, a quantity is measured using a unit quantity of the same nature as the quantity to be measured.

The Sixth Research Program

> *The sixth research program is to construct quantitative mathematics curricula for children in their construction of integers and rational numbers as measures of change in an unsigned quantity, where "unsigned" refers to the magnitude of the quantity, and operations with them.*

Based on work by Thompson & Dreyfus (1988), Ulrich (2014) defined an integer as a measure of change in an unsigned quantity, where "unsigned" refers to the magnitude of the quantity. Concerning integer addition, Ulrich (2014) commented that,

> Unlike in unsigned addition, in which the second addend can have a different quality than the first addend, the addends in this case need to be of the same type in the mind of the student. Depending on the relative magnitudes, the sum could be a subsequence of either addend. … I hypothesize that a student will need to have constructed the GNS in order to conceptualize addition in this way, precisely because both addends need to be reified composite units (which seems to correspond to iterability and the ability to disembed while maintaining a nested relationship) so that the sum can be disembedded from either addend (p. 256).

Ulrich's hypothesis concerning the operations that are needed to construct integer addition leaves open the question of the operations that are needed to construct the concept of an integer other than her comment concerning "reified composite units." I interpret the meaning of a "reified composite unit" in terms of Thompson's (1994) hypothesis that, "an integer is a reflectively abstracted constant numerical difference" (p. 192). So, Ulrich's hypothesis concerning the operations needed to construct integer addition also pertains to the construction of the concept of integers. Although it might seem unusual that the operations needed to construct integers are two steps beyond the operations that are needed to construct the natural numbers of the ENS as extensive quantities, all of the operations of the ENS have to be reorganized and extended to produce an integer as a difference of two such natural numbers. That is, as a reflectively abstracted concept, an integer is the difference of any two signed quantities *a and b,* denoted by $a - b$, such that $a - b$ is a constant number of units between *a* and *b* in the direction from *b* to *a.* This concept of an integer is crucial in algebraic reasoning and should not be finessed by using the sum of a and the additive inverse of b as the definition of a difference $a - b$ like it is done in CCSSM.

I extend this way of regarding integers to the construction of signed rational numbers, where rational numbers are regarded as magnitudes in the way that I regard them in the above text. Based on my experience teaching middle school children in teaching experiments as well as teaching prospective middle school mathematics teachers, finding sums and differences of signed quantities whose magnitudes are rational numbers will require at least a constructive generalization of integer operations. Furthermore, although the product and quotients of signed quantities are rarely considered in studies of children's mathematics, they are fundamental as preparation for more general algebraic reasoning and involve constructive generalizations of rational number of arithmetic operations. Constructive trajectories also need to be established in which students establish the laws of signs for products as a logical necessity as well as patterns of reasoning that might be recognized as distributive, associative, and commutative reasoning.

Finally, because of the preponderance of children who are yet to construct the rational numbers of arithmetic or even fractional numbers in the middle school and beyond, it is essential to explore what a quantitative mathematics curricula involving signed quantities might look like for children who have constructed only three levels of units. This problem is especially acute for children who have constructed only two levels of units.

The Seventh Research Program

The seventh research program is to construct quantitative algebraic curricula for children in the construction of basic algebraic knowing.

The first aspect of the program is to learn the operations that are involved in children's construction of combinatorial reasoning. My hypothesis is that the concept of natural number variable is essential. Even children who can reason with three levels of units make extensive lists when finding the possible outcomes of two or more events that occur together rather than reason with compositions of natural number variables (Panapoi, 2013). Further, my hypothesis is that the multiplicative principal of combinatoiral reasoning and the dimensionality involved in spatial coordinate systems (Lee, 2017) both involve recursively coordinating two three levels of units structures. Lockwood (2015), in her work with college students, and Panapoi (2013) and Tillema (2007, 2013, 2014), in their work with middle grade students of differing levels of units, have made substantial progress in this program. But extensions of their work are needed to establish mathematics curricula for children involving combinatorial reasoning across differing levels of units.

The second part of this research program is to extend the fifth and sixth research programs to working with operations on quantities of unknown measurements, which could be considered as "generalized arithmetic." An extensive quantitative unknown refers to the potential result of measuring a *fixed* but unknown extensive quantity before actually measuring it (Liss, 2015, p.30). An intensive quantitative unknown refers to the potential result of enacting the operations that produce a fixed but unknown equivalent ratio. The production of such a ratio implies the availability of the operations needed to produce an equivalent ratio and, thus, a proportional relationship (Liss, 2015, pp. 31-32). Hackenberg (2005, 2010, 2013, 2014), Hackenberg & Tillema (2009), Hackenberg & Lee (2015), and Liss (2015) have made substantial progress in this program by working with students of differening levels of units. An extension of this work is needed so that quantitative algebraic curricula for children are established across differing levels of units.

The third part of this research program is highly related to the second part. It is to construct quantitative algebraic curricula for children concerning the construction of the basic rate scheme and its use in the construction of linear functions. Given two co-varying quantities, I consider a rate as the result of enacting the operations that produce a ratio equivalent to a unit ratio at any but no particular time (Steffe, et al., 2014, p. 52). The basic rate scheme can be considered as a metamorphosis of intensive quantitative unknowns and proportional reasoning. One might consider the result of enacting a rate formally as an equivalence class of ratios, but that doesn't say anything about the involved metamorphic accommodation that produces rate. Toward that end, Thompson's (1994) commented that, "A rate is a reflectively abstracted constant ratio, in the same sense that an integer is a reflectively abstracted constant numerical difference" (p. 192). Although I agree with this way of thinking about a rate, it too doesn't specify the operations that children use to produce the reflective abstraction. There are various studies that contribute to understanding such mental operations (Ellis, Özgüra, Kulowa, Williams, & Amidonba, 2015; Hackenberg, 2010; Hackenberg & Lee, 2014; Johnson, 2012, 2014; Liss, 2015; Moore, 2014, Thompson, 1994; Tillema, 2013). But how teacher/researchers might provoke such a reflective abstraction is a fundamental problem in establishing quantitative algebraic curricula for children across differing levels of units.

Endnotes

[i] I surmise that, in part, it was because of what was considered as sufficient to falsify a theory during that period of time. According to Lakatos (1970) all justificationists, "whether the intellectualists and empiricists, agreed that a 'hard fact' may disprove a universal theory" (p. 94).

[ii] Thorndike considered himself a connectionist, which I regard as a form of behaviorism, but not radical behaviorism.

[iii] There was also an emphasis on social interaction, active citizen participation in all spheres of life, and democratization of public education.

[iv] Comment in brackets is added to the quotation.

[v] (http://www.eds-resources.com/facultytheory.htm)

[vi] (http://schugurensky.faculty.asu.edu/moments/1938rugg.html)

[vii] (http://www.uvm.edu/~dewey/articles/proged.htm)

[viii] Wertheimer was one of the three founders of Gestalt psychology along with Kurt Koffka and Wolfgang Köhler.

[ix] I attended a sequential summer institute for secondary school mathematics teachers during the summers of 1961, 62, and 63 at Kansas State Teachers College, Emporia, Kansas. There were no courses on teaching via problem solving that emphasized discovery learning by students although we did solve a lot of mathematical problems!

[x] James W. Wilson offered a course on problem solving for MEd and Ph.D. students at the University of Georgia for many years.

[xi] There were modern programs that did emphasize experiential learning of mathematics (Davis, 1990).

[xii] Piaget's grouping structures served as an abstracted model of the reasoning of children in what Piaget called the concrete operational stage.

[xiii] Piaget thought that the construction of the length unit was more advanced than the construction of the arithmetical unit.

[xiv] I am indebted to Dr. Larry Hatfield for his colleagueship and insight that led us to teach 1st and 2nd grade children in order to learn children's thinking.

[xv] A mathematician writer of the content standards told me that the standards are designed so that students can take college mathematics courses.

[xvi] In some cases, students can opt out of taking these tests.

[xvii] In constructivist research, Maturana's concept of the observer is essential. According to Maturana (1978), "Everything said is said by an observer to another observer who can be himself or herself" (p. 31).

[xviii] "Students" can be substituted for "children". I use "children" throughout the paper to be consistent.

[xix] Self-reflexivity involves applying one's epistemological tenets first and foremost to oneself.

[xx] Cf. Student-Adaptive Pedagogy for Elementary Teachers: Promoting Multiplicative and Fractional Reasoning to Improve Students' Preparedness for Middle School Mathematics, Dr. Ron Tzur, Principal Investigator.

[xxi] Cf. AIMS Center for Math and Science Education

[xxii] Cf. the work of Dr. Robert Wright's US Math Recovery Council.

[xxiii] Twenty-nine, say, can be disembedded from fifty while leaving it "in" fifty.

[xxiv] In stating this hypothesis, I assume that in the case of continuous quantity, children will primarily use units like inches, pounds, etc., in segmenting.

[xxv] This research program is not restricted to six-year-old children.

[xxvi] Cf. Hackenberg, Norton, & Wright (2016) for an excellent start on this problem.

[xxvii] A number concept such as five is a composite unit containing five arithmetical unit items containing records of counting "1, 2, 3, 4, 5."

[xxviii] Hackenberg (2007) found that some children who constructed only two levels of units could engage in splitting.

[xxix] These authors referred to this scheme as the generalized measurement scheme for fractions (GMSF).

[xxx] These authors referred to this scheme as the measurement scheme for proper fractions (MSPF).

[xxxi] Reciprocal reasoning of the kind Thompson, et al. (2014) identified involves coordinating two three-levels of units structures.

[xxxii] Q is taken as 22 inches in length.

[xxxiii] I did not observe a child construct what might be called an equivalence class of fractions even in the case of the GNS children (Steffe, & Olive, p. 337ff)

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THREE FACETS OF EQUITY IN STEFFE'S RESEARCH PROGRAMS

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The NCTM research committee made a recent, urgent call for mathematics education researchers to "examine and deeply reflect on our research practices through an equity lens." With this in mind, we use this paper to reflect on the ways in which Steffe's work has contributed to three facets of equity. We also suggest opportunities for researchers working within this framework to deepen their commitments to issues of equity.

Keywords: Equity and Diversity, Cognition, Learning Theory

The percentages that Steffe (2017) gives in his plenary paper are alarming because they indicate that current standards, and curricular materials based on these standards, are insufficient for a large percentage of students in grades K-8. For example, a majority of students entering 6th grade are not structuring number and quantity in ways that are required for the significant multiplicative reasoning that is the target of most middle school mathematics standards and curricular materials (e.g., developing proportional reasoning, an understanding of rates, etc.). For us, this phenomenon is fundamentally an issue of equity: As it stands, current standards and curricular materials are inequitable if they do not meet the learning needs of a significant number of elementary and middle school students.[i]

So, we take this opportunity to discuss Steffe's research in relation to the NCTM Research Committee's recent, urgent call for mathematics education researchers to "examine and deeply reflect on our research practices with an equity lens" (Aguirre et al., 2017, p. 125). We start with the important caveat that Steffe has not explicitly analyzed the ways in which race, culture, ethnicity, gender, and socio-economic status impact learning opportunities for students in school mathematics either at a broad level or in his specific interactions with students. This caveat may lead some mathematics education researchers to simply dismiss Steffe's work; after all, isn't this omission simply another way of saying that Steffe has studied mathematics teaching and learning using a colorblind framing that does not account for contextual or cultural factors in the teaching and learning process? We think that this conclusion is far too dismissive given Steffe's: a) profound commitment to unpacking what he terms students' mathematics; b) his drive to work with cognitively diverse students over long periods of time to learn this mathematics; and c) his repeated pushes to interrupt dominant discourses about what mathematicians, mathematics education researchers, and curriculum writers think should constitute school mathematics (e.g., Steffe, 1992, 1994; Steffe & Olive, 2010). We see this paper, then, as an opportunity to reflect on the ways that Steffe's research addresses facets of equity, as well as a space to call for researchers using similar frameworks to deepen their commitment to issues of equity in the context of critiques of de-contextualized and/or colorblind framings of mathematics teaching and learning (e.g., Martin, Gholsson, & Leonard, 2010; Martin, 2009).

In our view, Steffe's research addresses at least three critical facets of equity: positionality and power relations, what counts as mathematics, and access and achievement. We start with an overview of how Steffe's research addresses these facets of equity. Then we provide data excerpts to illustrate each facet of equity. The excerpts are of middle school students who have interiorized one level of unit because their mathematical ways of operating are rarely reflected in current curricula and standards.

Three Facets of Equity

Positionality and Power Relations

In sociology, *positionality* refers to the "occupation or adoption of a particular position in relation to others, usually with reference to issues of culture, [race], ethnicity, or gender" (Oxford Dictionary on-line). Those who articulate their positionality are articulating their stance or viewpoint on themselves, others, and interaction between people, often with respect to societal identifiers. In his research, Steffe articulates his stance on himself as a teacher/researcher and on those with whom he interacts (students). This stance starts with self-reflexivity, which is a version of Gutiérrez's mirror test (2016): "The principle of self-reflexivity compels teacher/researchers to consider their own knowledge of children's mathematics, including accommodations in it, as constantly being constructed as they interact with children as the children construct mathematical knowledge," where "Self-reflexivity involves applying one's epistemological tenets first and foremost to oneself" (2017).

Although this positionality does not address culture, race, ethnicity, or gender, we argue that it does address relations of power between a teacher and students. As Cobb (2007) points out, Steffe's research paradigm is an actor-oriented perspective, concerned with "small scale" human interactions that are useful (although not sufficient) for instructional design at the classroom level, not an observer-oriented "large scale" view of societal structures that focuses on how people participate (or are barred from participating) in cultural practices. So, the power that Steffe addresses has to do with power relations in student-teacher relationships. Although some may see that as a limited view of power from the perspective of social science more broadly (e.g., Foucault), it nevertheless bears directly on the idea that power is intertwined with knowledge and that those in power (including mathematics education researchers) are those who determine what we count as knowledge.

Steffe's orientation is that his own mathematics (his own first-order knowledge) is insufficient to understand children's mathematics (their first-order knowledge). For example, Steffe states: "I usually find it inappropriate to attribute even my most fundamental mathematical concepts and operations to children" (2010b, p. 17). Instead of doing that, he positions students as rational mathematical thinkers who have mathematical knowledge to which he does not have direct access. Steffe positions himself, as a teacher/researcher, as someone who must learn from children "how and in what ways they operate mathematically" (2017) and who must *"create* operations that if a child had those operations, the child would operate as observed" (2017). This statement is a statement about making a second-order model of a student's thinking (the mathematics of students), which he views as mathematical knowledge—as legitimate mathematics. Indeed, for Steffe, second-order knowledge is social knowledge co-constructed by him and children (2010b). Thus, the students with whom he interacts have power to determine what we count as knowledge—in fact, students are primary in his student-teacher relationships because he could not learn students' mathematics (i.e., create the mathematics of students) without interacting with them. This stance positions students as the generators of knowledge.

What Counts as Mathematics

There are numerous examples of the creation of second-order knowledge in Steffe's research, starting with the five counting sequences that model how children undergo significant reorganizations in creating and structuring units and quantity in their construction of what we call whole numbers (e.g., Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). In this paper we give an example of fractional knowledge that Steffe learned from students: the partitive fraction scheme (Steffe, 2002, 2010a). Often standards documents and curricular materials define fractions as parts out of wholes (e.g., 4/5 is four parts out of five parts) or as multiples of unit fractions (e.g., 4/5 is 4 × 1/5) (CCSSM, 2010). These definitions do not reflect students' ways and means of operating as they construct fractional knowledge because they omit a lot and, as Steffe (2017) points out, they may ask students to conceive of fractions in ways that are not within their current possibilities in the near term (cf. Norton & Boyce, 2013).

Students who have interiorized only two levels of units have the potential to construct partitive fraction schemes (Steffe, 2010a). Students who construct this scheme create fractions from iterating (repeating) a unit fraction some number of times. So, for example, if asked to draw 4/5 of a granola bar they partition the bar into five equal parts and then take one of those parts four times. This activity looks like these students see 4/5 as 4 × 1/5—they are repeating 1/5 four times, after all. However, the 4-part bar that is the result of their activity is, *for them*, four parts out of five parts—it has a part-whole meaning. So, when these students are asked to draw 7/5 of a granola bar, they will often object that doing so does not make sense because you can't take seven parts out of five (Olive & Steffe, 2001). Constructing partitive fractions is an advance over fractions conceived of only as parts in relation to wholes. However, students who have constructed only partitive fraction schemes do not yet see fractions as consisting of sequences of fractional numbers (e.g., 1/5, 2/5, 3/5, 4/5, 5/5, 6/5, 7/5, etc.), as Steffe (2017) points out. In addition, students who have constructed partitive fraction schemes have just begun to think of fractions as measurable extents—they have not completed this process (Steffe & Olive, 2010).

Yet rather than position students who have constructed partitive fraction schemes as deficient or behind, Steffe argues that these students' mathematics is a legitimate mathematics that should be the basis for developing curricula and instruction in schools. That is, he states: "rather than assume a God-like stance regarding 'school mathematics,' I assume that I must intensively interact with my students to learn what their mathematics might be before I can begin to think about what 'school mathematics' might be" (1992, p. 261). He critiques school mathematics texts—even reform texts—as being based on the writers' first-order knowledge of school mathematics. This phenomenon

> "places the mathematics of schooling outside of the minds of the students who are to learn it and is manifest in the univocal expression of concepts like multiplication and division. One searches the school mathematics books in vain for a mathematics of children, and school mathematics is taken to be the way it is rather than the way students make it to be" (1992, p. 260).

So, Steffe views school mathematics—mathematical knowledge—as something that should be squarely based on students' mathematical ideas.

Access and Achievement

In our view, developing curricular tasks and instructional materials for students who have constructed partitive fraction schemes is about the issue of access. Gutiérrez (2009) characterizes *access* to be about

> the resources that students have available to them to participate in mathematics, including such things as: quality mathematics teachers, adequate technology and supplies in the classroom, a rigorous curriculum, a classroom environment that invites participation, and infrastructure for learning outside of class hours (p. 5).

She characterizes *achievement* as about student outcomes, including "participation in a given class, course taking patterns, standardized test scores, and participation in the math pipeline (e.g., majoring in mathematics in college, having a math-based career)" (p. 5). She positions access and achievement at the ends of the "dominant" axis, and power and identity at the ends of the "critical" axis in her framework on equity.

Steffe's research does not directly address some items in Gutiérrez's (2009) list of resources for access, such as adequate technology and supplies in the classroom, or infrastructure for learning outside of class hours. However, Steffe's research is about increasing access to mathematical ideas and participation because it redefines what is being accessed. Rather than position mathematics as something outside of the minds of students to be accessed, he positions mathematics as being created by students, and so it is something that they have access to already, in a sense. Thus, his job as a teacher/researcher to facilitate this access is to create second-order models of students' ways of operating that allow him (and others who work in a similar vein) to interact with students so that their mathematics can surface and so that they can build on their ways of thinking from wherever they are. And, further, his job is to create learning trajectories, which he refers to as third-order models (2017), as curricula that would constitute school mathematics. Steffe's call to base school mathematics on the mathematics of students means students' achievement is defined as making progress from where they are—as learning.

Three Examples

In this section we illustrate each aspect of equity with data of student-teacher interactions. We aim to paint a picture of what student-teacher interactions based on models of students looks like, because we argue that, done well, these interactions open significant opportunities for participation and learning. Although this statement is true for all students, it is striking for those who have interiorized only one level of unit in middle school, because these students' ways of thinking are typically not reflected in or addressed by school mathematics (e.g., Hackenberg, 2013). So, all three examples in this section are of students who have interiorized one level of unit.

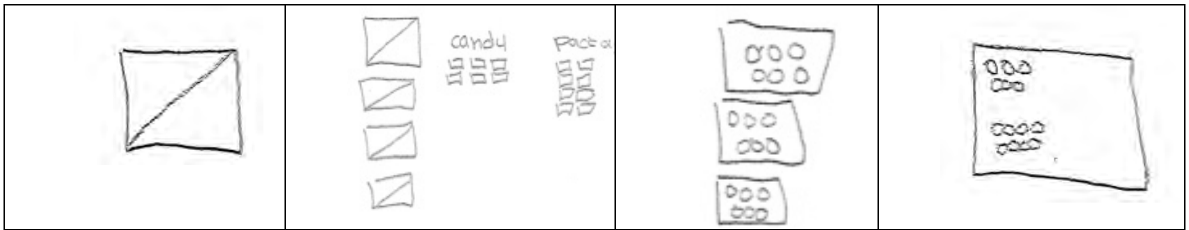
Here we present a few aspects of our second-order models of these students to help readers interpret the data: Students who have interiorized one level of unit view numbers as composite units (units of units)—e.g., 5 is five 1s and also one 5. However, for these students there is not a multiplicative relationship between the units of 1 and composite units. In addition, these students have yet to construct disembedding operations, whereby they can lift part of a number out of the number and not destroy the number, e.g., take 10 out of 14 while keeping 14 intact. Since these students cannot yet disembed, they don't reason strategically when combining numbers additively. For example, to determine 14 + 18, they typically count on by 1s from one of the numbers. In contrast, students who have constructed disembedding operations can separate 14 into 10 and 2 and 2, combine one 2 with the 18 to make 20, and then add on the 10 and 2 to get 32. This strategic additive reasoning is not in the province of students who have interiorized one level of unit. To make assessments of students' levels of units, we often use problems that involve embedded units, as we demonstrate next.

Power and Positionality: Hal Coordinating one Level of Unit

We examine Hal's response to the *Candy Factory Problem* to show what it looks like when a 7th grade student has interiorized only one level of unit. We then analyze how the teacher positioned himself in relation to Hal and the impact of this positionality on power dynamics in a student-teacher relationship.

> *Candy Factory Problem:* A candy factory puts 6 candies in each package, puts 8 packages in each box, and puts 4 boxes in each crate. Make a picture to show the number of candies in one crate.

Data Excerpt 1: Hal solves the *Candy Factory Problem*.[ii]
[The teacher reads the problem to Hal. Hal draws Figure 1a.]



(a)	(b)	(c)	(d)

Figure 1. Hal's response to the *Candy Factory Problem*.

T: Okay. What do you got there?
H: A crate.
T: And can you, like, the candies…so it says 6 candies in each package, puts 8 packages in each box, and puts 4 boxes in each crate. So can you draw what would be inside the crate? [Hal draws Figure 1b.] So these are your six candies, eight packages, and four crates [points to each part of Figure 1b]? [Hal nods.]
….
[The teacher asks Hal to re-read the question. The teacher then asks Hal to draw a single package containing six candies. The teacher asks Hal to draw a second package, and then a third package (Figure 1c).]
T: How many packages does it say would be in one box?
H: Eight.
T: Yeah. So could you draw everything that would be in one box?
H: Six candies and eight packages [draws Figure 1d.]
….
[The teacher returns to asking Hal to draw a fourth package with six candies in it, adding on to Figure 1c. He then asks Hal how many total candies there would be if he had a fifth package.]
H: Thirty.
T: Thirty? Okay. How did you know it was 30?
H: Because 5 times 6 is 30.

This excerpt illustrates that Hal initially did not consider candies, packages, or boxes to be contained in a crate—he drew a single crate (Figure 1a). He subsequently drew the candies and packages outside of the original crate, re-interpreted the one crate as a box, and drew three more boxes (Figure 1b). These drawings provide indication that he assimilated the situation using a single level of unit (e.g., a crate or a box or a package or a candy). With support from the teacher/researcher, Hal established a drawing where candies were contained within packages (Figure 1c). However, this structure seemed ephemeral for him because when the teacher asked him to use it to show "everything that would be in one box," he drew six candies and eight packages, separately, inside of the box (Figure 1d). Doing so indicates that he did not use a two-levels-of-units structure, a package containing six candies, when he created his box.

Interestingly, the last part of the excerpt demonstrates that Hal could use multiplication facts to solve problems—in fact, later in the interview it was evident that he knew and could use many multiplication facts, which is not atypical for middle grades students, even those that are coordinating solely one level of unit (Norton & Boyce, 2015). However, this phenomenon does not mean that "knowing multiplication facts" for students like Hal results from the structure and imagery that is typically assumed when students use such facts.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

We contend that middle grades students like Hal are often silenced or invisible in the classroom; they are often positioned as deficient and behind. In fact, even for the teacher/researcher (who was an experienced middle school teacher), Hal's response to this problem was surprising, and it took significant adjustment on his part in order to be responsive to Hal in the moment. Ultimately the teacher/researcher abandoned the original problem as it was stated in favor of presenting problems that were related but did not involve all of the levels of units as the original problem. The teacher/researcher did so because he interpreted his primary goal of interacting with Hal to be to learn Hal's mathematics. This goal, when taken seriously, can be quite humbling, because even an experienced teacher can quickly realize the insufficiency of his or her own mathematical thinking in bringing forth productive mathematical reasoning on the part of the student. So, positioning oneself as Steffe (2017) does is not at all a simple challenge for mathematics education researchers.

Indeed, we think such an orientation needs to be learned anew in each student-teacher interaction in order for mathematics education researchers and teachers to avoid positioning themselves in a "God-like" role. When a teacher ceases to position themselves as learner (for example, by assuming they know what a student should learn prior to interacting with a student), they reify their prior knowledge as the knowledge to be learned rather than entering interactions openly. Notably, however, this does not mean that teachers or researchers should enter interactions with students unprepared, but rather with a genuine openness to students' contributions to these interactions. In the interaction with Hal, it would have been possible to simply "coach" him through creating a representation for solving the problem where the teacher would have learned very little about the structure Hal attributed to the situation. We have witnessed many middle school students who have interiorized one level of unit experience this kind of coaching in schools.

Expanding What Counts as Mathematics: Kianna solves the Coordinate Points Problem

We turn now to a second 7^{th} grade student who was part of the same study, and who had also interiorized one level of unit, to examine how she worked through and solved the *Coordinate Points Problem*.

Coordinate Points Problem. You have number cards that have the numbers 1 through 8 on them. You draw a card, replace it, and draw a second card to create a coordinate point (e.g., 1, 2).

a. How many coordinate points could you make? Represent these points as an array.
b. Suppose you added one additional number card that has the number 9 on it. How many *new* coordinate points could you make?

In typical curricula this problem could be considered to be about "finding the difference of two squares"—the difference between 8^2 and 9^2. Successive iterations of this problem (e.g., starting at 9 and adding the 10 card) could open the possibility for students to consider that the difference between two squares is non-constant, and that the difference of these differences is constant (it is 2). We suggest that seeing the task as univocally about "finding the difference of squares" elides students' mathematics. We use data from Kianna to illustrate what a different characterization affords in terms of seeing what the challenges and successes were for a student who has interiorized one level of unit, and how in turn this characterization serves to expand what counts as mathematics.

We anticipated that, for Kianna, creating pairs and taking them as countable items would be a challenge. Therefore, we asked her to list aloud the new pairs as she was creating them and to keep track of how many pairs she had created. Kianna started the problem by listing aloud the new coordinate points, putting up a finger of her left hand each time she said a new coordinate point and reusing the fingers of her left hand once she had used all five of them. Then the teacher/researcher asked her how many she had created. Kianna seemed uncertain. She listed several calculations that she appeared to think were relevant (e.g., 9 + 9 and 9 × 9). Instead of asking her to compute, the teacher/researcher responded as follows.

Data Excerpt 2: Kianna solves the *Coordinate Points Problem*.

T: You want to just say them out loud again? You had a pretty cool method before. You want to just keep using that?

K [smiles]: Yeah. Okay. So, one nine, two nine, three nine, four nine, five nine [puts up a finger of her left hand each time she says a coordinate point until all five fingers on her left hand are raised], six nine [she begins to reuse the fingers of her left hand each time she says a coordinate point], seven nine, eight nine, nine...[is about to say nine-nine, but stops herself]. I mean, one nine [re-states one-nine instead of saying nine-one], two nine [finishes using the fingers on her left hand a second time], three nine, four nine, five nine, six nine, seven nine [finishes using the fingers on her left had a third time], eight nine, nine nine [puts up the thumb and index finger of her left hand]. That's twelve new ones. Yeah, that's twelve.

T: Twelve? Let's try one more time.

K: Ah. Okay. [Smiles broadly].

T: Do you want to try one more time?

K [emphatically]: Yeah.

T: Okay. You're so close.

K: Okay. One nine, two nine, three nine, four nine, five nine, six nine, seven nine, eight nine, one nine, two nine, three nine, four nine, five nine, six nine [is keeping track on her left hand in a similar manner as the previous attempt]... I said that wrong [realizing she has said one nine, two nine, instead of nine one, nine two, etc.]. Okay. One nine, two nine, three nine, four nine, five nine, six nine, seven nine, eight nine, nine one, nine two, nine three, nine four, nine five, nine six, nine seven, nine eight, nine nine [keeps track in a similar manner on her left hand as previously]. It would be eighteen?

T: Eighteen. You're so close.

W: How'd you get eighteen?

K: I was trying to count them on my fingers. I have a problem when I go past fifteen.

T: You're doing great.

W: Can I ask a quick question?

K: Yeah.

W: How many past fifteen did you get?

K: I think I got two more.

W: Okay. So, what's two more than fifteen?

K: Seventeen.

W: Yeah.

For Kianna, solving the problem appeared to be both challenging and satisfying. Kianna often said she did not like mathematics because she could not "see" herself in her mathematics classes. Her attitude about the interview, however, differed significantly from that: She smiled throughout, acknowledged and accepted the challenges presented to her, and successfully solved problems that were hard for her. We argue that this was possible because the teacher/researcher planned activity for her based on the mathematics of students who have interiorized one level of unit and can make two levels of units in activity, and he used this model as a basis for being responsive to her in the moment. For example, he asked Kianna to verbally list the pairs, which meant he supported her to produce pairs in her activity. He planned this activity for her because creating a pair (coordinate point) in activity is similar to creating a two-levels-of-units structure in activity: Both involve counting two units as a single unit (Tillema, 2013).

After watching Kianna make two attempts at enumerating the pairs, and arriving at 12 and then 18 pairs, the witness-researcher suspected that she was having difficulty coordinating the number of times she had counted by five on her left hand (three) with the number of left over fingers she had (two). Her response of 12 likely stemmed from a lack of differentiation of the number of times she had used all the fingers on her left hand and the two remaining fingers she used—when she reviewed her activity, she equated the two remaining fingers with the number of times she had used all of the fingers on her hand (two): Two hands and two leftover fingers would give 12. Her response of 18 likely stemmed from a similar lack of differentiation, except that in this response she seemed to substitute the three times she used her hand for the number of leftover fingers; three hands and three leftover fingers would give 18. The witness-researcher interacted responsively with her, assuming that this might be the conflation that she was making and so supported her to review the number of leftover fingers she had beyond 15 to determine she had counted 17 coordinate points.

Kianna's response of 17 coordinate points is numerically equivalent to the difference between 9^2 and 8^2. However, we think that claiming that she found the difference of two squares does not match well with Kianna's mathematics. In fact, we think it conflates the mathematics of the observer with the mathematics of a student by failing to differentiate between the two.

Instead, we think that Kianna's problem entailed creating and counting pairs that contained nine in either the first or second position, where her counting activity involved coordinating the number of times she used the five fingers on her left hand (three) with the number of leftover fingers she had on her left hand at the end of her count (two). There was not evidence that she established in a single structure the total number of pairs that could be created with nine number cards (9^2), the number of pairs that could be created with eight number cards (8^2), and the 17 newly created pairs that had the number nine in either the first or second position. Coordinating these three quantities in a single structure could be initial evidence for considering a students' mathematics to be compatible with something that might be called "finding the difference of two squares." Even though Kianna did not do this, her way of operating was fundamentally interesting to us and to her, involved challenging mathematics for her, and imbued her with a sense of mathematical power—she was a participant in producing the solution to what she considered a challenging problem.

Access and Achievement: Alyssa's Work on Symbolizing her Reasoning

We now turn to an example from a different study within a 5-year project to investigate how to differentiate instruction for middle school mathematics students, as well as relationships between students' rational number knowledge and algebraic reasoning. The current phase of the project involves the second author in co-teaching 25-30 day classroom units with a classroom teacher in which the teacher and project team design to differentiate instruction. In the first of these classroom design experiments, the 20-student 8th grade pre-algebra class consisted of five students who had interiorized one level of unit, 13 students who had interiorized two levels of units, and two students who had interiorized three levels of units. The focus of the instruction was equivalence in algebraic contexts, following the *Say It With Symbols* unit from the 3rd edition of the Connected Mathematics Project (Lappan et al., 2014).

One of the students who had interiorized only one level of unit, Alyssa, struggled with most of the ideas in the unit. For example, in class on Day 7 students were learning to factor expressions based on "reversing" the Distributive Property. To factor $6 + 2x$ Alyssa wrote $3(2 + x)$. Even after two conversations between the second author and Alyssa's group about what products they were aiming for, Alyssa still wrote $3(2 + x)$ while her groupmates had expressions like $2(3 + x)$. Later, on Day 14 students were solving an equation to find the break-even point in a situation that involved a school group selling boxes of greeting cards. Because Alyssa and another group mate were struggling to solve the equation, the second author worked with them on understanding the situation—finding the profit when different amounts of boxes were sold: 1, 5, 10, and 20. After seeing that all of these amounts resulted in losing money, the group mate was ready to increase the number of boxes to find the break-even point, but Alyssa suggested that they try 3 boxes or 15 boxes. In general, keeping track of the multiple quantities involved in determining profit (number of boxes, revenue, expenses) was challenging for Alyssa.

During a mid-unit interview, the second author posed to Alyssa a question similar to one worked on in class about developing an expression for the amount of money a swimmer raised in a swim-a-thon, where each sponsor gave the swimmer \$10 to start and \$2 per lap. There were 15 sponsors. Alyssa wrote "$10 + 2x + 15$," where x was the number of laps. Her rationale was that the swimmer was getting more, so "then you're adding, is what I thought." When asked how much money one sponsor gave the swimmer, Alyssa suggested "$10 \times 15 + 2x$", She explained as follows: "the 15 is how many sponsors and then they start with \$10 so I did 10 times 15 to give the amount of money that she's getting." She added the $2x$ because "for every lap they're giving her more money." But then she was concerned about the \$150 because it seemed like too much money from one sponsor. So, although she had just identified the 150 as coming from 15 sponsors, she then thought it was from just one.

With questioning support similar to what we have shown in the prior two data excerpts, Alyssa developed correct numerical responses for the swimmer swimming 4 laps with 1 sponsor and then 2 sponsors. However, she appeared to be "in" the activity of reasoning through these specific outcomes and did not stand above them in order to abstract a structure that she could represent algebraically. The second author expected this phenomenon, to some degree, based on her second-order model of students like Alyssa working on algebraic problems (Hackenberg, 2013). So, the second author drew from evolving second-order knowledge of Alyssa's ways of thinking that opened possibilities for Alyssa to be mathematically active—i.e., to access her mathematical ways of thinking in the context of the problem, and thereby to participate mathematically. In contrast, in math class Alyssa often followed along with the responses of group mates and did not seem mathematically active. In other words, she often did not seem to access her mathematical ways of thinking.

Interestingly, like Kianna, Alyssa appeared to find the interview pleasing in that in the school days that followed she asked the second author why her math class couldn't be like the interview because what they were doing in math class did not make sense to her, implying that the interview was sensible and even enjoyable. So, in the interaction during the interview, Alyssa appeared to experience herself as capable of doing mathematics in a way that she did not regularly experience in mathematics classrooms. Her comments and demeanor further support the claim that in the interview she had access to mathematical activity in a way that was pleasing and unusual for her.

If the second author had had more time with Alyssa on the swim-a-thon problem (e.g., if the problem were a classroom task), she would have continued to work with numerical examples for the amount of money earned in swimming 4 laps with different numbers of sponsors to learn whether Alyssa could abstract a pattern from her activity that she could represent algebraically. Exploring these possibilities with Alyssa would have promoted Alyssa's achievement in the sense of learning. If Alyssa's classroom tasks were designed similarly to this task, how would her access to mathematical activity and her mathematical achievement, or learning, change? We can't say for sure, of course. However, students who tend to feel like mathematics makes sense and who feel that their ideas are valued are certainly more likely to participate regularly and actively, in comparison with students who generally feel that they don't understand in mathematics classrooms, which was the case for Alyssa. Being active mathematically is certainly necessary for learning and achievement more generally, although it does not guarantee any particular learning or achievement.

Concluding Remarks

We have aimed to show how Steffe's research programs address three aspects of equity: positionality and power relations in student-teacher relationships, what counts as mathematics, and access and achievement. In doing so, we have seen how intertwined these three aspects are—it is hard to draw a boundary between them, because each mutually influences the other. For example, by positioning students as the generators of mathematical knowledge, Steffe expands what counts as mathematical knowledge: Students' mathematics counts as mathematics, or rather, the mathematics of students, since that is what he creates based on his interactions with students. Steffe advocates that this mathematics become the basis for curricular design, which has implications for access in the sense of how mathematically active a student might be in their classroom interactions with a teacher, and achievement in the sense that this helps to re-define what success might look like in mathematics classrooms.

We do note that, throughout his work, Steffe focuses on cognitive diversity. Over the course of his career he has worked with students from diverse backgrounds including different racial, cultural, ethnic, gender, and socio-economic backgrounds. Thus the participants in his studies have been diverse in these ways, but this has not been the focus of his analyses. We see this observation as an opportunity for researchers working within this tradition to continue to expand their analytic lens. We see at least three promising possibilities for such an expansion: a) explicit analyses of student-teacher interactions that account for how race, culture, ethnicity, gender, or socio-economic status impact the mathematics that a teacher-researcher is able to bring forth in interactions with students; b) design or teaching experiments that embed the goal of making second order models of students mathematics in situations that address a substantial social issue; and c) explicit attempts, based on second order models of students' mathematics, to influence policy discussions.

To examine what letter *b* might look like, we highlight one of our current graduate students who has used Steffe's framework as a basis for selecting students into a design experiment in which he created mathematical problems that opened the way for students to consider racial bias in jury selection (Gatza, in press). Gatza is working to unpack how, for example, different students' understanding of randomness or a limiting process (including differences based on level of units coordination) impact how they reason about issues of racial bias, and in turn how their understanding of issues of racial bias impact their understanding of randomness or a limiting process (in ways that may not strictly be accounted for based on differences in units coordination). We see unpacking these complex relationships as one avenue for researchers working in this tradition to deepen their commitments to issues of equity.

Endnotes

[i] Steffe (2017) might say that this statement is true for all students because standards and curricular materials under-challenge students who have interiorized three levels of units.

[ii] In the data excerpts, T stands for teacher/researcher, H for Hal, K for Kianna, and W for witness-researcher. Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher/researcher's perspective. Ellipses (…) indicate a sentence or idea that seems to trail off. Four periods (….) denote omitted dialogue.

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ELEMENTARY MATHEMATICS SPECIALISTS: ENSURING THE INTERSECTION OF RESEARCH AND PRACTICE[i]

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This paper provides a historical overview of the role and impact of elementary mathematics specialists as well as current implications and opportunities for the field. Furthermore, suggestions are offered for the mathematics education field for ensuring the intersection of practice and research.

Keywords: Teacher Education-Inservice/Professional Development, Teacher Knowledge, Elementary School Education

Historical Background

Over the years, many groups and leaders have seen the need for supporting teachers of elementary mathematics. In 1981, the National Council of Teachers of Mathematics (NCTM) Board of Directors recommended that state certification agencies offer teaching credentials for elementary school teachers that included mathematics specialist endorsements. The intent of this recommendation was to prepare elementary teachers to assume the primary responsibility of teaching mathematics, typically in the intermediate grades. At that time, certification boards across the country did not positively respond to this suggestion by creating mathematics specialist endorsements (Dossey, 1984). Since that time, a number of recommendations for the use of elementary mathematics specialists (EMSs) have emerged (see Figure 1).

Year	Recommendation
1981	The National Council of Teachers of Mathematics (NCTM) Board of Directors recommends that state certification agencies offer teaching credentials for elementary school teachers that include mathematics specialist endorsements.
1983	The National Science Board Commission on Precollege Education in Mathematics, Science and Technology recommends mathematics specialists in grades 4-6 in *Educating Americans for the 21st Century.*
1984	An article in *The Arithmetic Teacher* by John Dossey, entitled *Elementary School Mathematics Specialists: Where Are They?* discusses the importance of mathematics specialists in the elementary school.
1989	The National Research Council in *Everybody Counts* recommends that states alter certification requirements to encourage the use of mathematics specialists in elementary schools.
2000	*The Principles and Standards for School Mathematics* (NCTM) discusses the importance for mathematics teacher-leaders and specialists especially in grades 3-5.
2001	The National Research Council in *Adding It Up* recommends that mathematics specialists should be available in every elementary school.
2001	*The Mathematical Education of Teachers* (CBMS) calls for mathematics specialists starting at the fifth grade.

2003	Johnny Lott's Presidential Message entitled *The Time Has Come for Pre-K-5 Mathematics Specialists* advocates for mathematics specialists at the elementary level.
2003	An article in *Teaching Children Mathematics* by Reys and Fennel, entitled *Who Should Lead Mathematics Instruction at the Elementary Level? A Case for Mathematics Specialists* makes the case for mathematics specialists using both models.
2003	NCTM and the National Council for Accreditation of Teacher Education (NCATE) release standards for Elementary Mathematics Specialists programs.
2006	Francis (Skip) Fennell's Presidential Message entitled *We Need Elementary Mathematics Specialists NOW* outlines the need for mathematics specialists/leaders.
2008	The National Mathematics Advisory Panel (NMAP) releases their report in which they call for research to be conducted on the use of mathematics specialists in elementary schools.
2009	NCTM Research Brief describes 9 research studies focused on mathematics specialists and coaches and calls for additional research.
2010	Association of Mathematics Teachers Educators (AMTE) releases *Standards for Elementary Mathematics Specialists* which outlines program standards for teacher credentialing and degree programs. Revised in 2013.
2010	AMTE, Association of State Supervisors of Mathematics (ASSM), National Council of Supervisors of Mathematics (NCSM), NCTM joint position statement recommends that every elementary school should have access to an EMS.
2012	Conference Board of Mathematical Sciences (CBMS) *The Mathematical Education of Teachers II* outlines the increased use of EMSs.
2012	NCTM/CAEP *Standards for Elementary Mathematics Specialists* (Advanced Preparation) are released.
2013	Linda Gojak's Presidential Message entitled, *It's Elementary: Rethinking the Role of the Elementary Classroom Teacher,* advocates for mathematics coaches and specialists at the elementary level.
2015	Updated NCTM Research Brief describes 24 research studies focused on mathematics coaches and calls for additional research.
2017	AMTE releases the 2nd book in their Professional Development Series, *Elementary Mathematics Specialists: Developing, Refining, and Examining Programs that Support Mathematics Teaching and Learning*

Figure 1. Recommendations for Mathematics Specialists and Coaches. Adapted from Fennell, F. S. (2017). We need elementary mathematics specialists now: A historical perspective and next steps. In M. B. McGatha & N. R. Rigelman, (Eds.). *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning.* Charlotte, NC: Information Age Publishing. Reprinted with permission.

Although these recommendations use the term *mathematics specialist*, they describe models that include working with students, teachers, or both. Some of the recommendations distinguish between the models by using different titles and others do not. In fact, the title of these teacher leaders varies from state to state and even from district to district. In an effort to provide some clarity on these titles, my colleague and I (McGatha & Rigelman, 2017) offered a general overview of the work in which these teacher leaders engage and suggested some common language that could be used in referring to these positions (see Figure 2).

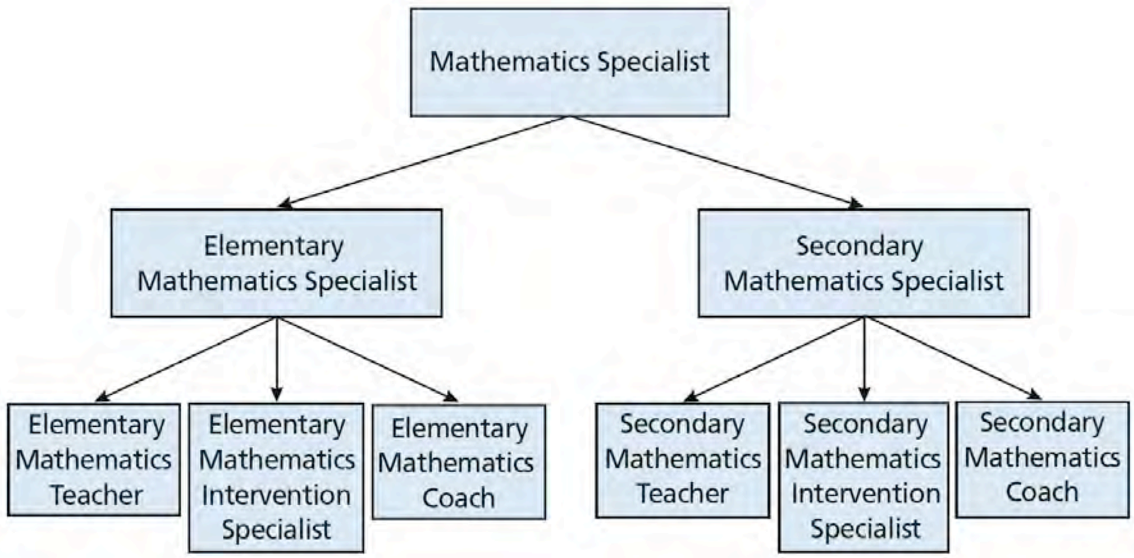

Figure 2. Mathematics Specialists' Titles. Source: McGatha, M. B. & Rigelman, N. R. (2017). Introduction. In M. B. McGatha & N. R. Rigelman (Eds.). Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning. Charlotte, NC: Information Age Publishing. Reprinted with permission. Copyright IAP. All rights reserved.

The titles under EMS and Secondary Mathematics Specialist (SMS) describe the major roles in which these teacher leaders engage: (a) mathematics teacher, a professional who teaches mathematics to students; (b) mathematics intervention specialist, a professional who works with students in "pull out" or "push in" intervention programs; and (c) mathematics coach, a professional who works primarily with teachers (McGatha & Rigelman, 2017).

Regardless of the title used to describe these teacher leaders as indicated in Figures 1 and 2, the mathematics education community has recognized a need for mathematics specialists at the elementary level for over 35 years. These recommendations stimulated several initiatives in schools and districts across the country.

Practice: What is Happening in the Field?

In 1988, ExxonMobil launched the *K-5 Mathematics Specialist Program* in which grants were given to 120 districts across the country to train and place mathematics specialists in elementary schools. However, the model in this program was actually the mathematics coach model since teachers were trained to be "proactive resources for other teachers, administrators, and parents" (ExxonMobile, n.d.). This corporate-based program was one of the first large-scale mathematics coaching initiatives in the United States. The state of Virginia took advantage of the ExxonMobile grants and became an early leader in supporting the work of EMSs. Various stakeholders and organizations in that state began work as early as 1992 and that work still continues today (http://www.vacms.org). More recently, the Elementary Mathematics Specialists & Teacher Leaders project (ems&tl), supported by the Brookhill Institute of Mathematics, was created in 2009 to support a core group of EMSs in Maryland. The project studies the impact of mathematics specialists and also hosts a nationally recognized clearinghouse (www.mathspecialists.org). Other large scale projects (e.g., Mathematics Coaching Project, Examining Mathematics Coaching Project) have, and continue to, support EMSs. This is in addition to the many district-based programs that exist across the US.

Another important aspect of work in the field, focuses on the ongoing support of the three national mathematics education professional organizations (AMTE, NCSM, NCTM). Arbaugh, Mills, and Briars (2017) outlined this important work and presented a representative list of activities from each organization (see Figure 3).

With the increased attention on EMSs and projects to support their work, AMTE felt it was important to address credentialing and degree programs for these mathematics professionals. In 2010, AMTE released *Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentialing and Degree Programs.* When the standards were published, there were only nine states that had a credential for EMSs while nearly every state has a credential for reading specialists. Currently, 20 states have some sort of credential for EMSs. While this growth is impressive in just seven years, we need every state to support the credentialing of EMSs.

Unfortunately, the number of schools or districts that have implemented mathematics coaching or specialist programs is unknown because a comprehensive national survey of such programs does not exist (National Mathematics Advisory Panel, 2008). However, the number of large-scale projects and the work of professional organizations as described above clearly indicate a growing focus on EMSs. Since 2000 the number of sessions on mathematics coaching and specialists at the annual conferences for AMTE, NCSM, and NCTM has steadily increased. In addition, other anecdotal evidence provides insights into the growing popularity of mathematics coaches and specialists. For example, a search on the Internet for “mathematics coach” produced 21,900 hits in 2008 and 26,600,000 in 2017 and “mathematics specialist” produced 17,000 hits in 2008 and 615,000 in 2017. While the exact number of schools and districts using mathematics specialists or coaches is unknown, it is clear that these programs have become a preferred professional development strategy to improve the teaching and learning of mathematics. It is critically important that we research what is happening in the field to verify the impact of EMSs.

Research: What is Happening in the Field?

When the first NCTM research brief on mathematics specialists was published in 2009, there were only nine studies included in the report. Research in this area quickly gained prominence and there were 24 research studies included in the 2015 research brief. And, the research continues. The research included in this brief overview (2002-2017) has either been published in an educational journal, edited book, or presented at a research conference so it has undergone some sort of peer-review process. Additional research has been conducted and can be found in evaluation reports, program review documents, and dissertations.

	AMTE **AMTE.net**	**NCSM** **MathEdLeadership.org**	**NCTM** **NCTM.org**
Peer Reviewed Journals	• *Mathematics Teacher Educator* (*MTE*) (with NCTM)	• *NCSM Journal of Mathematical Leadership*	• *Mathematics Teacher Educator (MTE) (with AMTE)* • *Teaching Children Mathematics (TCM)* • *Coaches Corner* • *Reflect and Discuss* • *Journal for Research*

			in Mathematics Education (JRME)
Facilitated Learning Opportunities	• *Annual Meeting* • *EMS State Certification Conferences* • *EMS Research Conference* • *Webinars*	• *Annual Meeting* • *Summer Academies* • *Fall Leadership Seminars* • *Webinars*	• *Annual Meeting and Exposition* • *Regional Conferences and Expositions* • *Research Conference* • *PreConference workshops* • *Institutes provide a deep-dive into grade and/or topic-specific content* • *Innov8 Conferences focus on a particular problem of practice* • *Webinars and webcasts*
***Sample* Print and Electronic Resources**	• *Standards for Elementary Mathematics Specialists: A reference for Teacher Credentialing and Degree Programs* • *AMTE Professional Book Series* • *Jump Start -Formative Assessment (w/NCSM)* • *Connections newsletter* • *Contemporary Issues in Technology and Teacher Education (CITE) journal*	• *The PRIME Leadership Framework: Principles and Indicators for Mathematics Education Leaders* • *It's TIME: Themes and Imperatives for Mathematics Education* • *Professional Learning Module Resources* • *Illustrating the Standards for Mathematical Practice* • *Jump Start -Formative Assessment (with AMTE)* • *NCSM PLC: The Digital Mathematics Education PLC* • *Coaches Corner* • *Curriculum Materials Evaluation Toolkit (with NCTM)*	• *Principles to Actions: Ensuring Mathematical Success for All* • *Principles to Actions Professional Development Toolkit* • *The Elementary Mathematics Specialist's Handbook* • *A Guide to Mathematics Coaching: Processes for Increasing Student Achievement* • *Professional Development Guides and More4U that provide suggestions for using NCTM publications in professional learning.* • *5 Practices for Orchestrating Productive Mathematics Discussions*

		• *Position/White Papers on teaching/learning topics* • *NCSM Newsletter and eNews*	• *Online classroom resources* • *Research Briefs summarize research on mathematics teaching and learning* • *Position Statements address policy issues relevant to mathematics education* • *Curriculum Materials Evaluation Toolkit (Joint with NCSM)*
Scholarships and Grants	• *EMS Scholarship Program*	• *Iris Carl Travel Grants*	• *Mathematics Education Trust (MET) grants*

Figure 3. Representative AMTE, NCSM, & NCTM Support for Elementary Mathematics Specialists. Source: Arbaugh, F., Mills, V. L., Briars, D. J. (2017). The role of professional organizations: Advocacy, development and research. In M. McGatha & N. Rigelman (Eds.). *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning.* Charlotte, NC: Information Age Publishing. Reprinted with permission. Copyright IAP. All rights reserved.

Specialists as Mathematics Teachers

There are currently very few studies on EMSs working as MTs. McGrath and Rust (2002) studied the effectiveness of departmentalized mathematics at the elementary level. The study compared gain scores in achievement test data from students in self-contained classrooms and departmentalized classrooms in grades 5 and 6. For the mathematics subtest of the achievement data, there were no significant differences in student achievement data gain scores between departmentalized and self-contained classes. However, Gerretson, Bosnick, and Schofield (2008) found that using MTs at the elementary school level allowed teachers more time to effectively plan lessons and focus their professional development (PD). In addition, teachers in this study reported gains in student achievement as a result of using MTs. Nickerson (2010) also noted that achievement gains were greater in treatment schools with MTs as compared to control schools without MTs. The MTs in this study noted significant changes to students' persistence in solving mathematics tasks and increased interest in mathematics. Nickerson noted changes in MTs' instructional practice towards an inquiry-based approach, but pointed out that this took time.

More recently, Markworth (2017), examined the various content specialization models of MTs involved in team teaching within seven school districts. Similar to the Gerretson, Bosnick, and Schofield study (2008), the MTs acknowledged affordances to the content specialization models such as having more time to focus on fewer content areas, which allowed for more in-depth study and focused PD. The MTs believed this supported them in providing higher quality instruction. MTs also pointed out that sharing the responsibility for teaching was beneficial to students. Constraints to the model are also described including (a) scheduling issues not present when teaching in a self-contained class and (b) isolation can occur if there is only one content area teacher per grade level.

Specialists as Mathematics Coaches

The majority of the research on EMSs focuses on MCs. These studies answer three main questions: (*a*) How do coaches interact with teachers? (*b*) What knowledge do coaches need? and (*c*) What is the impact of mathematics coaching?

How Do Coaches Interact with Teachers? The answer to this question varies greatly because districts and schools are still trying to figure this out. Several studies have focused on this question in order to support schools in understanding the most beneficial coaching practices. The research focuses on coaching practice in one-on-one settings (one coach and one teacher) and group settings (one coach and multiple teachers).

Studies that reported on coaching in one-on-one settings, in general, have identified similar ways of interacting with teachers that fell along a continuum from *more-directive* to *less-directive.* While each study used different language to describe the ways of interacting, they all focused on similar ideas. On the *more-directive* end of the continuum, the coach shared knowledge by (*a*) modeling lessons, (*b*) telling teachers what to do, or (*c*) providing resources for teachers (Becker, 2001; Chavl et al. 2010; Polly, 2012). Toward the middle of the continuum, coaching interactions focused on collaborative activities such as co-teaching, co-planning, and providing support during teaching (Becker, 2001; Chavl et al. 2010; Gibbons & Cobb, 2017; McGatha, 2008; Polly, 2012; Race, Ho, & Bower, 2002). At the *less-directive* end of the continuum, the coach supported teachers in becoming reflective practitioners. Activities on this end of the continuum included collecting data from observed lessons, providing feedback, and engaging teachers in thoughtful reflections (Becker, 2001; Bruce & Ross, 2008; Chavl et al., 2010; Gibbons & Cobb, 2017; Harrison, Higgins, Zollinger, Brosnan, & Erchick, 2011; McGatha, 2008; Olson & Barrett, 2004; Olson, 2005; Polly, 2012; Race, Ho, & Bower, 2002). While all of these coaching interactions serve useful purposes, activities on the *less-directive* end of the continuum seem to be more powerful in supporting teachers in changing their instructional practice.

A second aspect of coaching practice is coaching in group settings, such as a coach working with grade-level teams or professional learning communities. Gibbons and Cobb (2017) identified potential group coaching practices from the research on professional development and teacher learning that included (*a*) doing mathematics, (*b*) analyzing student work, (*c*) analyzing classroom video, and (*d*) rehearsing high-leverage practices. They point out that these practices can serve as a beginning framework, but additional research is needed to understand the usefulness of these practices in group settings. Baker, Bailey, Larsen and Galanti (2017) used the potential coaching activities identified by Gibbons and Cobb (2017) as a framework to identify high-leverage coaching practices across other coaching studies. Baker et al. (2017) suggested that even though the practices were not identified in many of the coaching studies, it did not invalidate the list. They agreed with Gibbons and Cobb (2017) that more research is needed in this area.

A few studies have focused on group coaching situations. In these settings, it is important to have regularly scheduled meetings in order to build continuity and maintain momentum (Gibbons, Garrison, & Cobb, 2011). In addition, it is critical to focus group meetings on issues of practice such as student learning and best teaching practices. (Alloway & Jilk, 2010; Obara & Sloan, 2009; Gibbons, Garrison, & Cobb, 2011). Beyond regularly scheduled meetings, Gibbons (2017) reported on the use of math labs (similar to lesson study) as a coaching structure to support the collective learning of a group of teachers.

What Knowledge Do Coaches Need? The *Standards for Elementary Mathematics Specialists* (AMTE, 2010, 2013) offer detailed descriptions of three broad areas of knowledge necessary for mathematics coaches and specialists: (*a*) content knowledge for teaching mathematics, (*b*) pedagogical content knowledge for teaching mathematics, and (*c*) leadership knowledge and skills

(p. 4). Researchers generally agree that these three areas of knowledge are important. However, the research focuses more explicitly on the third category of leadership knowledge and skills.

Sutton, Burroughs, and Yopp (2011) outlined eight domains of mathematics coaching knowledge: "Assessment, Communication, Leadership, Relationships, Student Learning, Teacher Development, Teacher Learning, and Teacher Practice" (p. 16). At first glance, many of these domains seem aligned with the AMTE categories; however, the detailed descriptions reveal more focused attention on supporting *teacher* learning, which falls into the AMTE category of leadership knowledge and skills. Several research studies help to further define specific ways coaches can support teachers. For example, it is important for coaches to understand trajectories of teachers' development so they can offer differentiated experiences for teachers (Baldinger, 2014; Gibbons & Cobb, 2016; Sutton, Burroughs & Yopp, 2011) and create long-term goals for teachers' development (Gibbons & Cobb, 2016). Coaches should have a deep knowledge of instructional practice and theory so they can support teachers in (*a*) assessing their own practice (Gibbons & Cobb, 2016) and (*b*) making connections between theory and practice (Alloway & Jilk, 2010; Sutton, Burroughs, & Yopp, 2011). Campbell and Malkus (2013) reiterated the importance of adequate preparation for coaches to make sure they possess the knowledge necessary to be effective coaches.

What Is the Impact of Mathematics Coaching? Two major areas are discussed in the research concerning the impact of mathematics coaching: improving teacher instructional practice and improving student achievement. Teacher instructional practice is defined broadly to focus on best practices in teaching as described in NCTM documents (1991, 2007). Of course, each study reports on particular aspects of teacher instructional practice.

Across all the instructional practice studies, researchers saw improvements (in varying degrees) in teacher instructional practice including increases in teacher questioning (Polly, 2012; Race, Ho, & Bower, 2002); student engagement (Balfanz, MacIver, & Byrnes, 2006; Race, Ho, & Bower, 2002); and teaching for understanding (Becker & Pence, 2003; Bruce & Ross, 2008; Burroughs, E., Yopp, D., Sutton, J., & Greenwood, M, 2017; Neuberger, 2012). Increases were also noted in particular instructional formats such as cooperative learning (Balfanz, MacIver, & Byrnes, 2006; Becker & Pence, 2003); classroom discourse (Balfanz, MacIver, & Byrnes, 2006; Neuberger, 2012; Race, Ho, & Bower, 2002); and technology (Becker & Pence, 2003). Two studies in this category differed from the others in that their findings did not fall into the categories described above but were more focused on specific instructional practices. Rudd, Lambert, Satterwhite, and Smith (2009) focused on one particular instructional practice, teacher's use of math-mediated language in their lessons. After the professional development and coaching sessions, researchers saw an increase in teacher's use of math-mediated language. Krupa and Confrey (2010) noted increases in (*a*) effective use of class time, (*b*) accurate delivery of content, and (*c*) frequent use of formative assessment as a result of teachers working with coaches.

Seven studies looked at the impact of mathematics coaching on student achievement. In varying degrees and with a variety of methods, all the studies reported increases in student achievement. At the elementary and middle school levels, studies show that coaching positively impacted student achievement on state-level assessments during the first and second years of a coaching program (Conaim, 2010; Zolligner, Brosnan, Erchick, & Bao, 2010). Additional studies at the elementary and middle school levels focused on student achievement impact after four years of a coaching program and showed even stronger results (Balfanz, MacIver, & Byrnes, 2006; Brosnan & Erchick, 2010; Campbell, Griffin, & Malkus, 2017; Campbell & Malkus, 2011). Findings from these longer studies indicate that, in order to significantly impact student achievement, coaches needed both experience and sufficient time to interact with teachers. There is only one study conducted at the high school level (Alloway & Jilk, 2010) and it was not specifically designed to study student achievement; however, its authors noted that pass rates in algebra and geometry classes increased from 40% to

70% after the implementation of coaching. As we move forward in the field, it is imperative to ensure the intersection of practice and research.

Ensuring the Intersection of Practice and Research

Probably the most important way to ensure the intersection of EMS practice and research is to collaborate, collaborate, collaborate! We must emphasize the importance of ongoing research to identify best practices in the field that are making a difference in teacher practice and student achievement. We really can't describe research-based practices in the field quite yet. We need more research!

I propose four suggestions to support the field in ensuring the intersection of EMS practice and research:

1. *Identify districts using EMSs.* As noted above, the number of districts using EMSs is unknown because a comprehensive national survey of such programs does not exist. Such a survey needs to happen! Once we know where programs exist, we can encourage districts to share their successes and challenges to support other EMSs through conference presentations and articles in practitioner journals. In addition, we can support districts in conducting research on their EMS programs to inform the field.
2. *Provide adequate preparation and ongoing support for EMSs*. As noted throughout this paper, there are many initiatives focused on supporting EMSs in the field. These efforts need to continue and new initiatives need to emerge. There is an abundance of anecdotal evidence of districts utilizing EMSs without providing them any professional development or ongoing support. Research has shown that adequate preparation and ongoing professional development can positively impact student achievement (Campbell & Malkus, 2013).
3. *Increase the number of states with EMS certifications/endorsements.* As noted previously, there are currently only 20 states that offer an EMS certification/endorsement. As the number of states offering an EMS credential increases, we will see more EMSs in the field supporting the teaching and learning of mathematics. Receiving a credential should require some level of preparation which aligns with suggestion #2. And, of course, more well-prepared EMSs in the field will increase the research possibilities.
4. *Establish working groups focused on EMS research.* There are relatively few researchers focused on EMSs. They need opportunities to collaborate with other like-minded researchers to reflect on their practice and explore future research opportunities. A few such groups have emerged but we need more attention on focusing the EMS research agenda. Relatedly, two EMS research conferences have occurred recently (AMTE in 2015 and the Virginia Mathematics Specialist Initiative in 2016). Such conferences are another opportunity for researchers to share their work and form collaborations. Because the audience is relatively small, these conferences are not that expensive and funding is available to support these efforts. The research that emerges from these collaborations will provide insights for EMSs in the field.

It is exciting to be involved in an area of practice and research that is still emerging and growing! We have opportunities to influence the field in multiple ways. We also still have many challenges facing us. As we continue to find ways to ensure the intersection of practice and research, we will move the field forward in positive ways.

Endnotes

[i] Parts of this manuscript are adapted from *The Impact of Mathematics Coaching on Students and Teachers* published by NCTM (2015), http://www.nctm.org/Research-and-Advocacy/Research-Brief-and-Clips/Impact-of-Mathematics-Coaching-on-Teachers-and-Students/.

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A FRAMEWORK TO GUIDE THE DEVELOPMENT OF A TEACHING MATHEMATICS WITH TECHNOLOGY MASSIVE OPEN ONLINE COURSE FOR EDUCATORS (MOOC-ED)

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Mathematics teacher educators face a challenge of preparing teachers to use technology that is rapidly changing and easily available. Teachers have access to thousands of digital tools to use with students and need guidance about how to critically choose and use tools to support students' mathematics learning. Research provides guidance to teachers about what features to look for in a technology tool and suggestions are offered about how mathematics teacher educators and researchers can support teachers in using technology to teach mathematics.

Keywords: Technology, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development

Introduction

Technology is an essential component of today's workplace and a ubiquitous part of our society. 82% of high school students and 68% of middle school students have access to smart phones. 75% of these students would like to use their devices to support learning (Speak Up, 2014). Students report that using technology better engages them in learning. Parents state that the use of technology will better prepare their children for the workforce of tomorrow. While researchers have evidence to show the positive impacts the use of technology can have in classrooms, and while there are increasing numbers of freely accessible digital tools available to use, teachers' incorporation of technology has been slow.

We know technology, like any tool, must be selected and used carefully. Mathematics teachers have access to more open-access digital resources than ever before. While ten or twenty years ago, teachers were creators of activities, today, teachers search, find, select, and often modify activities they find using Google, Pinterest, or the Blogosphere. In addition, many teachers have access to digital resources that accompany their curricula. With so many activities easily available, many teachers have become curators rather than creators of digital resources. As teachers gather materials together to present to students, they need guidance to assure that what they have selected will meet the needs of their students and will achieve their learning goals. Teacher educators face a tremendous challenge in preparing teachers to use digital technology that is rapidly changing to support students' mathematics learning. For example, the resources available today may be different tomorrow. Research can provide guidance and advice to assist teachers in using technology in the mathematics classroom.

In this paper, a framework developed to guide the design of a Teaching Mathematics with Technology massive open online course for educators (MOOC-Ed) will be shared along with questions that teachers can consider when making decisions about using technology to teach mathematics.

Guiding Framework

The Didactic Triangle.

A framework was created to guide the development of a MOOC to support teachers in using technology to teach mathematics. The foundation of this framework is the didactic triangle. The didactic triangle is a representation that has been used by several researchers (e.g., Brousseau, 1997;

Freudenthal, 1991; Steinbring, 2005) to describe interactions that occur among a teacher, his or her students, and the content that is being taught. These interactions can be described in terms of pedagogical activities the teacher uses to engage students in learning content – in this case, mathematics. It is important to note that mathematics refers to mathematical topics like algebra, geometry, measurement, statistics, probability, and number, and also the mathematical processes students use when engaging with mathematics. These mathematical processes and practices include using representations, making connections, communicating reasoning, creating and critiquing arguments, attending to precision, solving problems, and mathematical modeling (NCTM, 2000, 2014; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010)

Within the didactic triangle many interactions take place. For the purpose of designing our MOOC, we focus on those interactions that are planned and used by the teacher. These include (but are not limited to) pedagogical activities related to 1) the selection and implementation of mathematical tasks, 2) questions teachers pose to push student thinking or probe their understanding, 3) the facilitation of mathematical discussions, and 4) assessment of student learning. We depict these four pedagogical activities at the center of our didactic triangle (See Figure 1).

Figure 1. A sample of activities that take place among students, teachers, and mathematics.

Although not explicitly mentioned, we acknowledge that there are many factors that influence classroom interactions such as classroom culture, norms, attitudes, and beliefs. These all influence the enactment of the pedagogical activities we have placed at the center of the triangle. We depict the addition of technology to the classroom by adding a vertex to expand the didactic triangle and create the didactic tetrahedron.

The Didactic Tetrahedron

To make explicit how one considers the role of technology among interactions with students, a teacher, and mathematics, the didactic triangle was extended by Tall (1986), and more recently by Olive et al. (2010) and Ruthven (2012). We can depict this influence by expanding our didactic triangle to create a didactic tetrahedron with technology as the fourth vertex (See Figure 2). Olive et al. state "the introduction of technology into the didactic situation could have a transforming effect on the didactical situation that is better represented by a didactic tetrahedron, the four vertices indicating interactions among Teacher, Student and Mathematical Knowledge, mediated by Technology" (p. 168).

It is important to define what we mean by technology. For some, technology is any object or tool that allows a user to accomplish a task. Others restrict the use of the term technology to refer to electronic or digital technology. Some researchers make distinctions between artifacts, tools, and instruments. According to Monaghan "an artefact is a material object, usually something that is made by humans for a specific purpose, e.g. a pencil. An artefact becomes a tool when it is used by an agent, usually a person, to do something" (Monaghan, Trouche, & Borwein, 2016, p.6). On the other hand, Trouche discusses the process that is involved in an artefact becoming an instrument. He states, "when an artefact has been appropriated by a user, I will name instrument the mixed entity composed of the artefact and the associated knowledge… a tool is a thing somewhere on the way from artefact to instrument" (ibid, p.8). In this paper technology will be used synonymously with tools as defined by Monaghan.

Figure 2. The didactic tetrahedron which includes technology.

When adding technology to mathematical pedagogical activities it is important for teachers to think about how the use of technology influences representations of mathematics and how the use of technology influences pedagogy.

The Influence of Technology on Representations of Mathematics

Mathematics is abstract and it is only through its representations to which we have access to it. Technology offers new and different representations for students and teachers to interact with and use. For teachers, we emphasize that when evaluating technology there are three important factors to consider (See Figure 3). In particular, it is important to determine if the representations technology offers, determine whether it has mathematical fidelity, and consider if technology will be used with students as an amplifier or as a reorganizer (Pea, 1985, 1987).

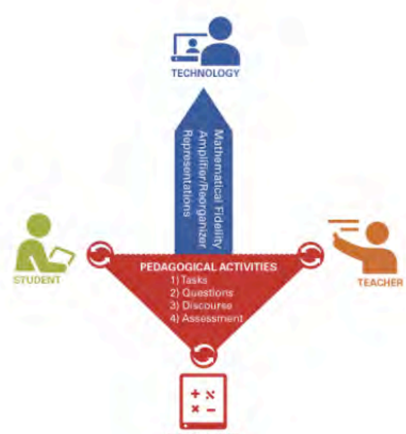

Figure 3. Three factors for teachers to consider when evaluating technology.

Mathematical Fidelity. When choosing technology to use in the classroom, teachers need to make sure that what is represented in the tools are accurate representations of the mathematics. Dick refers to this as, "mathematical fidelity" (Zbiek, Heid, Blume, & Dick, 2007). To illustrate, consider the sketch shown below that was created to show how to compute the slope of a line.

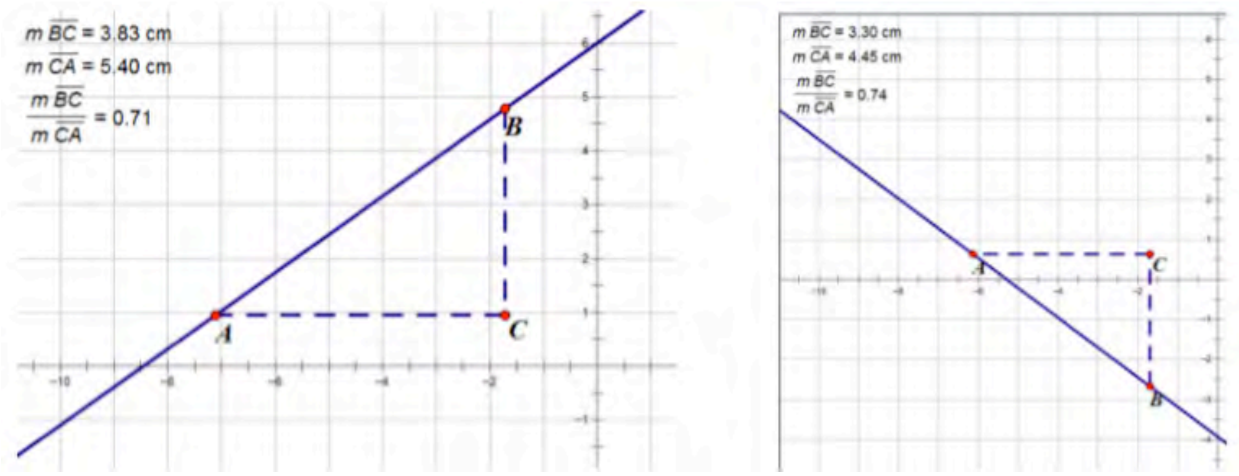

Figure 4. An example of a technology-based activity without fidelity.

Notice that the length of segment *BC* and segment *AC* are used to calculate the ratio of the "rise" to the "run." Segment lengths are always positive and this becomes problematic when the line has negative slope. In this case, the technology calculated the ratio, but cannot account for the negative slope – thus, the calculated ratio is not mathematically correct. The sketch is not faithful to the mathematics and thus lacks mathematical fidelity. It is important for teachers to select tools to use with students that have mathematical fidelity. In addition, teachers should also consider how students will interact with the technology.

Amplifier/Reorganizer. Pea (1985, 1987) used the metaphors of amplifier and reorganizer to describe how technology might be used. As an amplifier, technology performs many of the same actions that could be completed by hand, just more precisely, quickly, and efficiently. The question that is answered using by-hand methods or using technology is relatively unchanged. For example, students might be asked to create a table of values for the function with rule $f(x) = 3x+5$. This could be produced by hand or by using a spreadsheet. The results would be generated more quickly and accurately with the spreadsheet. As a reorganizer, technology changes the way students think about a question or mathematical idea. For example, a student may be provided with the graph of the function with rule $f(x) = 3x+5$ that is linked with a table of values and sliders that dynamically change the values of the slope and y-intercept parameters. By allowing the technology to quickly produce the graph and table, questions can be posed to shift a student's focus from producing the representations to conjecturing and reasoning about how changes in the parameters are related to changes in the graph and table.

When technology is used as a reorganizer, questions can be posed that align with and take advantage of the representations and actions afforded by the tools. Many technology tools that support mathematics learning provide multiple representations of mathematical objects (e.g., numeric, graphic, symbolic, pictorial) and allow the user to interact with the technology to dynamically adjust one representation and see the changes in other representations. This dynamic linking can influence how students reason with and make connections among different representations of mathematics (Kaput, 1987).

Representations. The use of multiple representations to support students' mathematical thinking has long been recognized as an important pedagogical activity (e.g., Kaput, 1992; NCTM 2000, 2014). Research suggests that the use of multiple representations can assist students in developing deeper understandings of mathematics and become more flexible problem solvers (Ainsworth, 1999). NCTM (2014) claims, "Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (p. 24). When one thinks about representations of mathematics, symbols, graphs, and tables often come to mind. Mathematical representations can also include pictures, diagrams, contexts, verbal descriptions, and physical objects. There are many ways to represent mathematical ideas.

Technology tools provide students with easy access to representations and these technology-based representations are often linked. That is, a change in one representation results in a change in other representations. For example, changing the function rule $f(x)=2x+1$ to $f(x)=3x+1$ can result in a corresponding change in its graph. From these interactions, students can better understand slope. However, it is important that the production of multiple representations is not the sole focus of an activity. Rather, multiple representations can be the centerpiece of productive mathematics discussions and making connections within and among different representations an important cognitive activity (NCTM, 2014).

With these important features of technology described, we created questions that teachers might consider when selecting tools to use in the mathematics classroom that are shown in Figure 5.

Technology-Mathematics	*Mathematical Fidelity*. Is the technology tool a faithful and true representation of the mathematics students are to learn? (Dick, 2008) *Amplifier/Reorganizer*. Does the technology allow the teacher and/or student to do the same work more effectively, efficiently, and quickly (amplifier)? Does the technology change the way the student and/or teacher thinks about mathematical ideas (reorganizer)? (Pea, 1985) *Representations.* How does the technology represent the mathematics? Does it provide linked representations for students to use? (Goldin & Kaput, 1996)

Figure 5. Questions teachers may consider when evaluating interactions between mathematics and technology.

The Influence of Technology on Pedagogy

Just as technology can influence the mathematical representations students interact with, it can also impact pedagogy. It is important for the teacher to be aware of the opportunities technology allows and consider how it influences the four pedagogical activities involved in 1) designing tasks, 2) posing questions, 3) facilitating discourse, and 4) assessing student learning.

Researchers have described tasks in terms of their cognitive demand (Henningsen & Stein, 1997) and mathematical richness. Technology can have an influence on both the richness of a task and its cognitive demand. A mathematically-rich task in a paper-and-pencil environment may be a completely different activity when students have access to technology tools. Consider the task of constructing an equilateral triangle. Doing so on paper with compass and straight-edge requires different thinking than doing so with a dynamic geometry program. Similarly solving a question such as the following requires different thinking if solved using paper and pencil or solved using a dynamic geometry program.

1. In the coordinate plane, line p has slope 8 and y-intercept $(0,5)$. Line r is the result of dilating line p by a factor of 3 with center $(0,3)$. What is the slope and y-intercept of line r?

A. Line r has slope 5 and y-intercept $(0,2)$.

B. Line r has slope 8 and y-intercept $(0,5)$.

C. Line r has slope 8 and y-intercept $(0,9)$.

D. Line r has slope 11 and y-intercept $(0,8)$.

Figure 6. A technology inactive question from https://parcc.pearson.com/resources/Practice-Tests/TBAD/Geo/PC1105806_GeoTB_PT.pdf

If the question is answered using paper and pencil, a student may plot $f(x)=8x+5$. They may then plot the point (0,3). They may determine the distance between (0,5) and (0,3) to be two and then multiply this distance by three to determine that (0,9) is the image of (0,5) under the dilation. This may be repeated for another point to find line *r* or a student may recall that the slope is invariant under dilation. Using a dynamic geometry program, students need to enact technological procedures, plotting a line, plotting a point, and performing dilation. The thinking needed is different given the different tools students have available for them to use. Thus, when teachers have access to technology, they need to think carefully about the tasks and questions they will pose and the thinking required of students when technology is used. Dick and Hollebrands (2011) stress the importance of questions in the context of technology by stating: "The value of technology to the teacher lies not so much in the answers technology provides but rather in the questions it affords. Indeed, "what questions could I ask that I could not ask before?" is the ruler by which we should judge what technology buys us as teachers of mathematics" (p. xvi).

Technology also allows teachers new tools to use when leading mathematical discussions. Collaborative tools such as Google Docs, Sheets, or Slides allow multiple students to share and discuss their work with the whole class. Mathematics specific technology tools like the TI-Navigator and Desmos allow the teacher to monitor and share student work. Orchestrating discussions using these types of tools requires teachers to focus on their mathematical goals and consider ways that they can reach them by selecting and sequencing students' work to assist them in making connections (Smith & Stein, 2011).

Finally, assessment in mathematics classrooms can look very different when teachers are using technology. Game-like assessment tools like Kahoot! Quizlet and Quizzies can be motivating for students and can provide feedback to teachers about what students know. Diagnostic assessments tied to learning trajectories can provide teachers with information about what students know or do not know and make informed decisions about what to do next to advance students' learning.

We present these as questions a teacher may consider when using technology in the mathematics classroom (See Figure 7).

In addition to considering how technology can influence mathematics and pedagogy there are ways in which technology interacts with the teachers and students that we highlight in our MOOC. These are described in the following sections.

Technology -Teacher Edge. Technology is used by teachers and students in the classroom in a variety of ways. Walk into any classroom and you may find teachers using a document camera, interactive white board, and a laptop equipped with a wide array of software applications. Dick and Hollebrands (2011) use the constructs of conveyance and mathematical action technology to assist teachers in making distinctions among the different technology they have available to use in the classroom.

Type of Technology. Conveyance technologies are used to "transmit and/or receive information" and are not math specific. These include presentation technology (PowerPoint, document cameras, interactive boards, projectors), communication technology (social media), collaboration technology (Google Docs), and assessment technology (clickers, educational games). Even though these technologies are not mathematics specific they can still have a significant impact on a mathematics classroom by providing opportunities for students to consider and critique each other's solutions and justifications.

Pedagogical Activities	**Questions to Consider**
Designing Tasks	What is the cognitive demand of this technology-based task? (Stein & Smith, 1998) How will the student interact with the task and technology? How does technology enhance student learning? What learning goals would be best served by this task? How might I prepare students to engage productively in this task?
Questions	What new questions does this technology allow me to ask? In what ways can I ask questions that will advance student thinking and probe what students are learning? What opportunities does the technology allow for students to pose their own mathematical questions? How might a structure my classes to help students feel comfortable generating and posing their own questions and responding to questions that other students generate?
Discourse	Does the technology allow for different solutions and/or different solution strategies? What would make a discussion of technology-based tasks productive? How can I use technology to facilitate a productive mathematics discussion?
Assessment	What type of feedback does the technology provide to the student? How can I build self-assessment into the tasks? How can I leverage the technology to determine what students are learning? How can I use the technology to assess what the students have learned?

Figure 7. Questions a teacher can consider when examining the effects of technology on pedagogy.

Mathematical action technologies are tools, software, and applets that can "perform mathematical tasks and/or respond to the user's actions in mathematically defined ways" (Dick & Hollebrands, 2011, p. xii). These technologies include: graphing calculators, computer algebra systems, and dynamic mathematical environments (GeoGebra, the Geometer's Sketchpad, Fathom, TinkerPlots). Often these technologies are used to perform computations, graph functions, plot data, and construct geometric figures. However, mathematical action technologies can also be used to allow students access to approaches and tasks that would not be possible without technology. Here technology can be used to develop students' mathematical understanding and support students as they explore patterns. Mathematical action technologies can also offer opportunities for teachers to pose questions and tasks that could not be asked in non-technological environment (Zbiek, Heid, Blume, & Dick, 2007). For example, in a dynamic geometry environment, a teacher can ask students to explore how a particular quadrilateral behaves when one of its vertices is dragged. This question is one that cannot be posed in a paper-pencil environment. Guiding questions a teacher can consider when selecting and evaluating technology tools are included in Figure 8.

Technology Consideration	**Issues to Consider and Questions to Pose**
Technology-Teacher	Conveyance/Mathematical Action Technology. Will the technology be used for the teacher to convey information to students (e.g., power point, internet)? Will the technology be used to allow students to perform mathematical actions? (Dick & Hollebrands, 2011) Is the technology readily available for the teacher? Is the learning curve minimal for the teacher?

Figure 8. Questions a teacher can consider when selecting technology tools.

Technology-Student Edge. When making a decision about whether to use a particular technology tool, thinking about how students interact with the technology is especially important (See Figure 9).

Figure 9. The edges of the tetrahedron.

Mathematical action technology often include mathematical representations students can directly manipulate. Direct manipulation allows users to use a mouse or their finger to interact directly with the representation of the mathematical object. The way that the object moves is continuous. There is no lag between the user's own movement and that of the object in the environment. The way that the objects respond is determined by mathematical rules. Thus, through direct interactions the student can observe and infer mathematical properties and theorems. This is one important feature that makes technology tools different from non-technology tools such as base-10 blocks or a ruler. By interacting with technology, students can learn mathematical rules and properties. Conveyance technology tools, on the other hand, typically do not offer mathematics representations and can sometimes be challenging for students to enter mathematics notation. However, these conveyance technologies can be designed by the teacher to provide students with feedback that they can use to gauge their learning and mathematical understanding. Some questions teachers may want to consider when thinking about students' interactions with technology are included in Figure 10.

Technology-Student	Interaction. How does the student interact with the technology? Is the technology available? Is the technology learning curve minimal or steep? Feedback. What types of feedback does the technology provide to students when they are interacting with it?

Figure 10. Questions teachers can consider related to the ways students interact with technology and the feedback it provides.

Conclusion

While technology is rapidly changing and evolving, there are general questions teachers can contemplate when making decisions about whether to select and use a particular technology tool. Teachers should consider whether the technology they are selecting to use is a conveyance or mathematical action technology. They should evaluate the types of representations the technology offers and determine whether those representations are faithful to the mathematics students are learning. They should also assess whether the technology is used to amplify or reorganize students' thinking. How technology effects the design of tasks, questions that can be posed, facilitation of discourse, and assessment of student learning should also be considered. Finally the ways students can interact with the technology and the feedback it provides to support students' mathematics learning should be taken into consideration.

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DIGITAL TECHNOLOGIES IN MATHEMATICS CLASSROOMS: BARRIERS, LESSONS AND FOCUS ON TEACHERS

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In this paper, drawing from data from several experiences and studies in which I have been involved in Mexico, I reflect on the constraints and inertia of classroom cultures, and the barriers to successful, meaningful and transformative technology integration in mathematics classroom. I focus on teachers as key players for this integration, calling for more teacher involvement in both professional development, and as co-constructors and collaborators in the design of technological implementations and resources.

Keywords: Technology, Teacher Education-In Service/Professional Development

Classroom Cultures, Teachers and Technologies

Throughout his career, Seymour Papert, a pioneer advocate of digital technologies for changing learning, criticized the way in which school systems constrain knowledge and learning. At the ICMI Study 17 study conference in 2006, in the final talk of his life (Papert, 2006), Papert denounced that educational systems ration every aspect by dividing learning into school grades, "cutting up the knowledge into the subjects" and ordering it; with schools being dictated by graphocentrism—i.e. by paper-and-pencil technology—and new technologies being used only to implement what was there before the newer technologies. He ridiculed that situation by saying: "We'd never have had airplanes...if we had constrained the new transportation to follow the schedules of the sailboats and the horse-drawn carriages; but that's what we are doing in our schools" (Papert, 2006).

Before delving into the issues of the constraints—or what I call the inertia—of educational systems, let us look at some evidence on how digital technologies and tools (DT) have been used and are being used in schools, using data from several studies carried out in Mexico over a decade, and from the research literature.

Uses of Digital Technologies in Classrooms in Mexico and Elsewhere

In a survey carried out in Latin-America in 2006 and reported in Julie et al. (2010), it was found that the most predominantly-used software in mathematics classrooms were software for word processing (Word, LaTex, PDF) and presentation (PowerPoint) – not mathematical tools, but communication ones. Other studies in Mexico at middle-school (Rodríguez-Vidal & Sacristán, 2011) and high-school levels (Miranda & Sacristán, 2012, 2016) showed similar results, with few teachers using technology in classrooms, and of those who did, for simply presenting information, projecting videos, plotting graphs or checking results produced in paper-and-pencil, with very rare use of technology by students. In those studies, the access to technology in classrooms was scarce.

Despite technology becoming more accessible in some schools, this year (2017), Luc Trouche and I visited a high-school in Mexico where we again observed a teacher using technology in a similar way: to simply project static function graphs using GeoGebra, completely omitting any of the dynamic and experimental possibilities (and main purposes) of such a "Dynamic" Geometry environments (DGEs). (In Sacristán, 2011, we also reported on a case where the teacher failed to transmit the dynamic function of a DGE, and students simply used the software for drawing static figures.) Furthermore, the 2017 teacher did not encourage—in fact, *dis*couraged—students from using technological tools in the classroom (interestingly, however, a couple of students ignored the teacher's recommendations, and did use a tablet to reproduce some of the functions demonstrated by the teacher).

Although in more developed countries, technological tools in the classrooms are becoming increasingly ubiquitous, the use of DT is not so different. For example, in the UK, a report edited by Clark-Wilson, Oldknow and Sutherland (2011), cites other reports that conclude that, despite considerable investment in DT in schools, these are underused within in secondary mathematics classrooms and, if used, their potential is generally underexploited. The report also points to classroom evidence suggesting that the use of DT has had emphasis

> on teacher-led use, using mainly presentational software such as PowerPoint and interactive whiteboard software. Revision software and online content services are also used, with the focus being on the computer teaching mathematics alongside practice exercises. Where digital mathematical tools such as graphing calculators, dynamic geometry, and spreadsheets are used, these are conceived primarily as presentational, visual and computational aids rather than as instruments to facilitate mathematical thinking and reasoning. (Clark-Wilson, Oldknow and Sutherland, 2011, p. 19)

On his part, Trouche (2016), while pointing to a lack of research at a large-scale for analyzing the real integration of technology in mathematics classrooms, reports that, in general, integration remains local with a huge difference between schools and teachers. It is also mostly teacher-centered (at least in the cases of England and France), with sometimes teachers showing students the use of the technology, or being unable to analyze the effects of the technology being used. Monaghan hypothesizes that, in the case of England, the increased teacher-centered use of DT in class in classrooms could be due to the increase of interactive whiteboards (IWBs) with "a very large proportion of the use of IWBs is teacher use of IWBs with PowerPoint (rather than interactive mathematics software) and the result is 'teacher demonstration,'" perhaps pointing to students "not being granted wide access to tools to explore mathematical relationships" (Monaghan, Trouche & Borwein, 2016, p. 388). I argue that teacher demonstration is due also to the inertia of old school practices and cultures that are teacher-centered, particularly in countries where this model is still prevalent, such as Mexico.

I summarized these observed predominant uses in classrooms of technology in Sacristán (2011; in press), as being for:

- Presentation or demonstration (e.g. PowerPoint, projecting graphs, videos, etc.)
- Easier visualization
- Easier, faster computation and accuracy
- Saving time (time optimization)
- Checking paper-and-pencil task results
- Information
- Student's revision, exercises or "drill and practice" through interactive, or online resources (such as pointed by Clark-Wilson, Oldknow & Sutherland, 2011, above)
- Communication (e.g. using email or Internet for sending homework)

With many times:

- Little innovation (doing, as Papert criticized, the same or similar tasks as with paper-and-pencil)
- A lack a sense or understanding of didactical and mathematical purposes for the use of digital tools in their classrooms; leading to technocentrism (showing or teaching about the tool itself—see Brennan, 2015—rather than using the tool for mathematical purposes)

- Ignoring the purpose or potential of the tools (e.g. no dragging or dynamism using DGE, as narrated above), indicating, again, the influence of graphocentrism
- Unlinked to other resources (Monaghan & Trouche, 2017)

Thus, when we look at the evidence of how digital technologies are used in schools, we see that in fact they are used in the direction that Papert (2006) claimed: to teach and serve the old (e.g. serve existing curricula), with much of their potential ignored.

On the other hand, the research literature is full of successful innovative practices with technology at experimental scale. But many authors, and at different education levels (see also, Clark-Wilson, Oldknow & Sutherland, 2011; Artigue, 2012), point to how

> despite over 20 years of research and curriculum development concerning the use of technology in mathematics classrooms, there has been relatively little impact on students' experiences of learning mathematics in the transformative way that was initially anticipated. (Clark-Wilson, Robutti & Sinclair, 2014, p.1)

In Sacristán (in press), I reflected on the gap between research results and what happens in classroom practices. Clark-Wilson, Robutti & Sinclair (2014) indicate that a response to this has been increasing research on the role of the teacher; and that will be the theme of the last section of this paper. But I will focus now on the reasons, such as different types of obstacles and barriers, impeding more meaningful technology integration in schools and practice.

Difficulties and Challenges for the Integration of DT for Math Learning in Classrooms

Between 1997 and 2006, a government-sponsored national program in Mexico called EMAT (Teaching Mathematics with Technology) was put into practice for gradual implementation of expressive computational tools, together with a pedagogical model, in the middle-school mathematics classrooms (see Sacristán & Rojano, 2009). We learned a lot from that program in terms of issues that emerge when attempting large-scale massive implementation of technologies in schools (even when carefully designed and planned through the expertise of an international team of mathematics education researchers, as was EMAT). Difficulties and obstacles were encountered at different levels: (a) the teacher, student and classroom level; (b) the school level; (c) the local authorities level; and (d) the national government level; and of different kinds related to:

- *changes in classrooms practices and cultures*: both teachers and students were unaccustomed to working in a more exploratory, student-centered, setting, and teachers had difficulties in adapting to the *proposed* pedagogical model
- *integration of technological tasks with the established curriculum*
- *time issues*: time (or lack of) for preparing the technology-based tasks, and for their implementation
- (teachers') *content knowledge of mathematics*: the use of technology made teachers aware of their deficiencies of their mathematical content knowledge, leading to two types of consequences: (i) some teachers did not want to continue working with technology; or (ii) in other cases, it *motivated* and helped teachers improve their content knowledge.
- *professional development* and support in terms of the tools – which was usually insufficient and without *continuity*
- *teachers and students' attitudes, beliefs and confidence* with regards to the use of the technological tools and programs: these have been shown to have an impact on students' learning with the tools (Sacristán, 2005). As we put it in Sacristán and Rojano (2009, p.213): "Putting it bluntly, 'good teachers' achieve good results: they are able to take

advantage of the *technological* tools and their students benefit from those experiences; but less experienced, poorly trained teachers, or simply teachers who dislike the technological tools, do not do so well."
- *technical difficulties*
- *administrative and bureaucratic issues, policies and political issues*, including lack of communication between the different levels of authorities. It is also worth noting that the program was discontinued in 2006 due simply to a change in government (change in policy). It did survive at some local levels, mainly in places where there would be some form of local support, such as a self-appointed regional coordinator.

In other studies (e.g. Sacristán, Sandoval & Gil, 2011; Miranda & Sacristán, 2016), there were similar findings pointing to reasons that impede the integration by teachers of DT. Among these: difficulties in accepting changes (even when they recognize possible benefits of DT) with many of them continuing doing the same as before; fears (e.g. of losing control of the class, of showing mathematical and technical deficiencies); difficulties in understanding how to integrate technologies in terms of the mathematical aims; lack of adequate infrastructure; and lack of time.

These difficulties are similar to those mentioned in the BECTA (2004) review on barriers to the uptake of information and communication technologies (ICT) by teachers. That report categorizes barriers (e.g. lack of access to resources—including lack of hardware, inappropriate organization, poor quality software—lack of time, lack of effective training, technical problems, lack of confidence, resistance to change and negative attitudes, no perception of benefits) into school-level barriers and teacher-level barriers, which can be external and internal barriers. Likewise, Clark-Wilson, Oldknow and Sutherland (2011, p. 20), cite a report from the UK's National Centre for Excellence in Teaching Mathematics where mathematics teachers' concerns about the use of DT are listed as related to:

- a lack of confidence with digital technologies;
- fears about resolving problems with the technology;
- fears about knowing less than their learners;
- access to digital technologies;
- inappropriate training;
- lack of time for preparation;
- a lack of awareness of how technology might support learning;
- not having technology use clearly embedded into schemes of work,

and include among the barriers to the more student-centered use of DT:

- an inadequate guidance concerning the use of technological tools in curriculum documentation;
- assessment practices;
- and "a perception that digital technologies are an add-on to doing and learning mathematics".

To these we can add the current overload of information and availability of resources of varying quality that are available to teachers through the Internet.

From Challenges to Trends and Lessons

The *NMC Horizon Reports* (www.nmc.org) takes a look every year (since 2012) at technology adoption, at both K-12 and higher education, enlisting (six for each) (i) key trends accelerating technology adoption, (ii) significant challenges impeding technology adoption, and (iii) developments in technology, poised to impact teaching, learning, and creative inquiry.

Among the trends that are identified in several of the reports (Johnson et al., 2015; Adams Becker et al., 2017; NMC/CoSN, 2017) that will be having an impact—in the short, mid and long terms—on technology adoption, are: an increasing use of collaborative learning approaches and of blended learning; a shift from students as consumers to creators and the recent push for coding literacy; the rise of STEAM learning, which seeks to engage students in interdisciplinary learning breaking down traditional barriers between different classes and subjects—one of the criticisms raised by Papert (2006), cited at the beginning of this paper; a rethinking of how schools work, shifting to deeper learning approaches (e.g. project-based learning, etc.) and a redesigning of learning spaces.

At the same time, one of the challenges that the *NMC Horizon Reports* consider difficult (even "wicked" at higher education level), is the changing role of teachers and educators, whose

> primary responsibilities are shifting from providing expert-level knowledge to constructing learning environments that help students gain 21st century skills including creative inquiry … acting as guides and mentors, … providing opportunities for students to direct their own learning trajectories. (NMC/CoSN, 2017, p. 7)

The nine years of EMAT led us to identify some of the key factors for success and for transforming school practices and teacher's roles, such as: adequate planning, gradual implementation, continuous professional development and support, and enough time (years) for assimilation and integration (Sacristán & Rojano, 2009). In relation to the latter, we found that even the most enthusiastic, committed towards the program, and supported teachers, needed at least three years in order to appropriate themselves of the tools and pedagogical ideas. But those who did became very successful in the future, continuing using the resources from the program for many years on their own, even until even this day. In fact, during the writing of this paper, I received an out-of-the-blue call from one of the teachers with whom I worked during the EMAT program. She told me of the limited resources in the school where she now works, in a very low-income area, but how, by implementing the EMAT materials in the last few years (more than a decade after they were developed), student achievement and assessments had improved dramatically, and she had even won two prizes for her work (one of them for her students' explorations with Logo of the four-color theorem). This case shows an appropriation by the teacher of the resources, tools and pedagogical ideas.

It is thus clear that the key player for successful implementations of technology-centered educational innovations is the teacher.

> the role of the teacher is very important, and his/her beliefs, insecurities and lack of mathematical and technical preparation affect the possible impact that the use in the classroom of these technologies can have on students' learning and even attitudes. The need for careful, considered and continuous work with teachers is thus extremely important. A priority in this kind of work should be the integration of digital technologies with the work that teachers are required to do, to take them into account at all steps of the implementation process, and to assist them in developing pedagogical strategies. (Julie et al., 2010, p.380)

With respect to the latter, I would like to reflect briefly on policy aspects and the changing role of teachers in the design of technological implementations.

A Reflection on Policies for Technology Integration, Societal Changes and Teachers

The EMAT program was a top-down design: a policy-driven decision that attempted to achieve technology integration in mathematics teaching and learning, and teachers were not involved in its design. It may have succeeded in small scales, but for an innovative educational program to catch on, it cannot be only about policy-driven implementations, no matter how carefully and well designed it is. For appropriation by teachers of new resources and pedagogical ideas, it may be a prerequisite to involve teachers in the design from the beginning, not just as participants, but as co-creators. In fact, rather than a technologically-driven model of technology integration, Hennessy, Ruthven, and Brindley (2005) point to the importance of teacher involvement, although also influenced by the teachers' working contexts, for effecting classroom change. Furthermore,

> approaches with the most potential to bring about genuine improvement in learning mathematics are those that resonate with teachers—with their interests, beliefs, emotions, knowledge, and practice. (Kieran, Krainer, & Shaughnessy, 2013, p. 364)

But involving and engaging teachers in the design of technological implementations is only one part of what is needed. There are dialectical forces at play here. On the one-hand, top-down policies do generate part of the change: they can initiate it and sow seeds of transformation (as in the case of teachers from EMAT, who 15 years later, continue working and transforming classrooms with what they learned); even if, as an imposition, it is unlikely it will resonate with the majority of teachers. On the other hand, changes that take place in society due, for instance, to technological advances—such as the trends mentioned in the NMC reports above—also influence policies. In any case, professional development and support is needed. In Trouche, Drijvers, Gueudet, and Sacristán (2013), we discussed the above and said:

> Merely providing access to technology is not enough for promoting educational change; support for teachers' professional development is a necessary precondition for a thoughtful and fruitful integration of technology. […] Policy shifts do not fall out of the blue, but reflect or intend to support underlying views on learning, and are mediated by new paradigms of teaching and learning. (Trouche et al., 2013, p. 756)

The issue of professional development was also touched upon in the NMC reports, in relation to the changing role of teachers:

> The evolving expectations also change the ways teachers engage in their own continuing professional development, much of which involves social media, collaboration with other educators both inside and outside their schools, and online tools and resources. Pre-service teacher training programs are also challenged to equip educators with digital competencies amid other professional requirements (NMC/CoSN, 2017, p. 7)

I will now focus on teachers as the core of the efforts for improving meaningful technology-integration and promoting changes in classroom cultures.

Improving Technology Integration in Math Classrooms: Focus on Teachers

Based on what was said in the previous sections, this focus on teachers has two aspects: (i) professional development, and (ii) enhancing teachers' involvement in generating changes, resources and decision-making.

I will begin by drawing from two experiences of in-service professional development programs of which I was part of, both of which emphasized self-reflection by teachers.

Two Experiences of Reflective Professional Development Programs for In-Service Teachers

As the EMAT experience taught us, there is a need to strengthen the mathematical content knowledge of teachers in our country, so the training and self-reflection processes in both programs addressed three aspects: the technological, mathematical and pedagogical.

Experience 1. From 2005-2010, I participated in a long-term professional development program for a small group of six in-service teachers in Mexico. As described in Sacristán, Sandoval and Gil (2011), our approach was for teachers to reflect on the changes in their practice through both training and classroom implementation of DT, document their findings and present them in seminar sessions to their fellow participants and the tutors. For this, we developed a professional development model (Fig. 1), based on works (e.g. Artzt and Armour-Thomas, 1999) that provide models in which teachers reflect on their instructional practice. In the pedagogical design of our model, we considered teaching as a constructive process that requires reorganizing and reinterpreting the subject matter and the practice as a result of experience (Thompson, 1992); and that the knowledge that is derived from social interactions in a (real-life) context is more valuable and significant for the teacher (Liu & Huang, 2005).

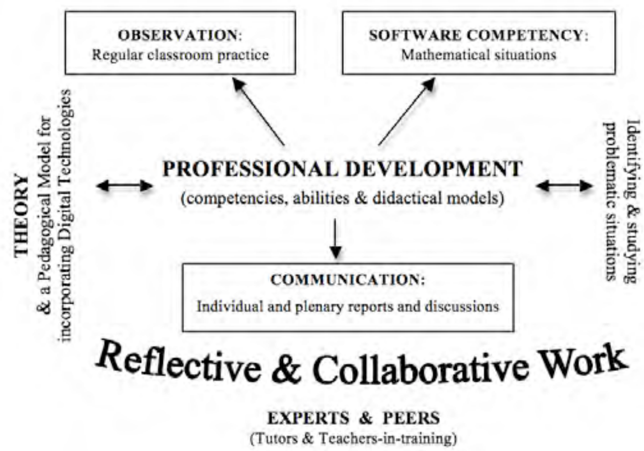

Figure 1. The professional development model described by Sacristán, Sandoval & Gil (2011).

In our program, teachers were involved in: (a) training and development of abilities, for the use of DT in the classroom (mainly Spreadsheets, Dynamic Geometry, CAS and Logo, as well as some applets); (b) the design and planning of teaching strategies and activities that integrate DT; and (c) engaging in observation and reflection-on-action (Bjuland, 2004) of the changes in their own teaching practice with the new tools. The participants also studied and discussed theoretical frameworks and pedagogical models for a meaningful incorporation of DT into the (mathematics) classroom. In parallel, the participants attempted to incorporate DT, as well as the pedagogical models studied, into their real-life classroom activities, analyzing and reflecting upon the potentials, limitations and changes brought forth by this incorporation of DT into their own practice, and that of their colleagues, from various perspectives. These activities and model are schematized in Figure 1. This experience, even though it was a top-down initiative, gave the participants the opportunity to reflect upon and share their personal experiences with the other participants. We consider that the diverse elements of the development model—training, continuous support, processes of reflection, self-observation, and promoting equally the technological, mathematical and pedagogical aspects—were significant for helping generate changes in the participants' professional practices, and enabled them to construct didactic strategies for the use of DT, more in accordance with the specific needs of their students and/or of other teachers. Half of them even appropriated themselves of our model's ideas, for peer training, designing and implementing a training program for other teachers and colleagues.

All six participants considered the educational system as very rooted in traditional ways and difficult to change; but they perceived a change from a technical and presentation use of DT, to more mathematically-centered uses, both in themselves, and in some of those they trained.

Experience 2. In 2009-2011, as described in Parada, Sacristán & Pluvinage (2013), we used a theoretical and methodological model called the Reflection-and-Action (R-&-A) Model, for promoting reflective processes—as a complement to professional development, and for strengthening the teachers' mathematical content knowledge—in two communities of practice (CoP) of mathematics educators (in-service teachers and researchers) in Mexico: one with 46 members; another with 125 members, which met through internet forums and periodically in person. The R-&-A model centered on a mathematical activity that was reflected upon before, during and after a teaching experience, i.e., through three reflective processes: (a) reflection-for-action; (b) reflection-in-action; and (c) reflection-on-the-action. The reflections that emerged enhanced teachers' mathematical content knowledge related to the specific activities, and also helped them recognize the need to adapt methodological and didactic resources, such as DT, to the purposes and characteristics of each student group.

Teachers as Active Collaborators in Meaningful Technology Integration

In the above section, I presented professional development experiences that promote teacher reflections and collaborations. The first one, though successful, was not a teacher initiative. The second one, included a researcher as mediator who proposed and coordinated the mathematical reflective processes. I believe that in order to generate change, teacher involvement and collaboration with researchers needs to take place in a way that makes teachers feel they are decision-makers. I consider this crucial in terms of motivation, beliefs, and overcoming affective apprehensions – which we have seen are areas that can be important barriers to integration and sustainability. Sustainable CoPs or networks involving both researchers and teachers are important and ICT makes it possible to share, discuss and remix resources online. A useful example from which we can draw lessons, is that of Sésamath (http://www.sesamath.net/) in France (see Trouche et al., 2013), which emerged as a bottom-up approach where mathematics teachers started to share and design resources and software. A bit over ten years ago Sésamath started to collaborate with researchers (Trouche et al., 2013). The quality of the resources at the beginning may not have been so good but through the sharing between teachers, and collaboration with researchers, these were greatly improved (Monaghan & Trouche, 2017).

> Explaining the reasons for the success of *Sesamath* requires specific research. The existence in France of the IREMs (Institutes for Research on Mathematics Education), a national network that involves many mathematics teachers, has played an important role. A similar project could perhaps not succeed in countries were such a network, linked with mathematics education, did not exist. (Trouche et al., 2013, p.772)

Concluding Remarks

In this paper I began by quoting Papert (2006) and his criticism of educational systems as a way of introducing the issue of the difficulties of creating meaningful change and technology-integration in classrooms, and the inertia of the classroom and the paper-and-pencil cultures that limit change. This was then expanded in listing some of the barriers identified from decades of research, to that change and integration. After a brief excursion into some of the lessons learned from technological and educational trends, I focused on the teacher as the key player for successful and transformative technology-integration and argued in favor of promoting models of collaboration (such as CoPs and networks) between teachers, researchers and policy-makers that both enhance teachers' professional development, empower them and provide a means for sharing, discussing and improving resources and their implementations, as well as overcoming some of the detected barriers. But one of those barriers is time: educational systems need also change (perhaps pushed by the trends of society) in a way that makes them more flexible for allowing teachers more time to engage in collaboration and innovation.

There are some other aspects that I did not cover in this paper, and that are worth reflecting upon. For instance, what role do MOOCs have, or will have, both in terms of changing the role of teachers and of technology for teaching and learning mathematics; as well as for teachers' professional development? Or are they a way of up-scaling current educational practices without truly innovating them?

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CROSSROAD BLUES

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In this paper I take up the questions posed by the conference organisers with respect to what we have learned and where we are going in technology-based research in mathematics education research. I begin by troubling the metaphors of crossroads and intersections and argue—through a wide range of considerations in relation to past research, to theory development, to teaching practices, to assessment and curriculum design and to concerns around access and equity—that there may be more fruitful metaphors for understanding our past and imagining our future.

Keywords: Technology, Learning Theory, Equity and Diversity

The metaphors of 'crossroads' and 'intersections' that were chosen for this conference are worth dwelling upon, in relation to research on the use of digital technologies. Crossroads are often used symbolically in literature, drawing on Sophocles' work *Oedipus*, to indicate a crucial moment of choice. In Oedipus there were three possible roads to follow, perhaps evoking past, present and future. But Sophocles' idea of choice may be less about making an independent decision, than about following the roads that have been carved by destiny: Oedipus was always going to end up killing his father. The invitation of the conference organisers to think in terms of crossroads provoked for me some questions about whether we really are at a decision point with numerous options available and whether one choice might be inevitable. The metaphor of intersections, which is similar in some ways, also has more mathematical connotations, ones that evoke significant ideas in geometry, especially around whether lines will intersect, how many times they will intersect and what it means to not intersect at all—all these questions being beautifully perturbed by moving to dimensions beyond the plane. But what could it be, in the context of research on technology in mathematics education, that could be seen as a crossroad or an intersection?

In considering this question, I was reminded of the work of the anthropologist Tim Ingold (2007), whose book *Lines: A brief history* traces the way in which the very idea of *line* functions metaphorically in Western society. He argues that it is so deep and entrenched that we can often find ourselves using it to describe a wide range of phenomena—often using words such as trajectories, paths, roads, trails, courses, routes—that might not actually be so linear or straight or one-dimensional. In his book, Ingold distinguishes two ways of thinking lines: as transporting and as wayfaring. In the former, we might think of getting from point A to point B and the line is the journey that gets us there. In the latter, the line is what one makes as one moves; there is no path independent of the travelling. Transposed to a theory of learning, the former would tend to conceive of learning as a sequence of journeys one might make from one concept to the next; the latter would focus on the act of tracing, on the direction that is taken and the new territory being explored. The former involves reaching successive destinations while the latter involves creating paths. Whereas crossroads and intersections, at least in my own imagery of them, have the past, the present and the future already laid out—you can go this way or that—I wonder whether it is possible to dwell in the present and so withhold the temptation to pre-determine a destination, let alone the journey that will take us there. Getting off the plane, we might even be able to think of creating paths that loop around like a Mobius strip or fan out into a surface or sprout into 17 dimensions, only four of which we might actually be able to see. As with most of our thinking around education, such an approach, which embraces multiplicity, indeterminacy and nonsense, may be the best way to handle the complexity of digital technology use in mathematics education.

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In what follows, I have attempted to address the questions and prompts offered by the conference organisers, not in a way that is exhaustive, but that is opportunistic—drawing on my own research and research interests in technology. I will try to keep the provocation of anti-crosswords alive throughout, inviting readers to think less in terms of the image in Figure 1a and more in terms of the image in Figure 1b, which is a replica of the cover of Ingold's book.

Figure 1. From transporting (intersections and crossroads) to wayfaring.

What we Have Learned from the Routes we Have Traversed

One way of seeing research in mathematics education is as an activity that enables us to answer questions, question such as: Should digital technologies be used in mathematics classroom? When is one technology better than another? What does a given technology change the way students learn? Another way of seeing research in mathematics education is as a practice of posing new questions, perhaps transforming the questions we started with so that they better respond to the complexities of the mathematics classroom. In this second kind of practice, the questions shift: new paths are created. Researchers have realised that the first question listed above, for example, depends less on empirical evidence than on assumptions about the goals of mathematics education. The second question may shift if one realizes that each technology might produce a different mathematical conception, in which case deciding on which is the best depends on many factors, ranging from aesthetic choices in mathematics to considerations of what might be evaluated on standardised tests. The third question listed above will also morph as researchers begin to appreciate that the student-technology dyad is a reductive focus, and that the role of the teacher, of the curriculum and of the classroom environment are also significant factors in what is learned.

To answer the question of what we have learned, it thus seems reasonable to consider how our questions have changed over the past few decades of research on the use of technology in mathematics education. I turn to the recently published *Second Handbook of Research on the Psychology of Mathematics Education* (Gutiérrez et al., 2016), which contains a chapter on technology (Sinclair & Yerushalmy, 2016) that considers the research published in the PME proceedings from 2006-2016. This is just one source—other Handbooks could also have been considered—but I have chosen it because it is international and because it explicitly compared research over the past decade with research conducted over the previous decade, which was reported in the first *Handbook of Research on the Psychology of Mathematics Education*, which was published in 2006.

The authors of the technology chapter report that while the 2006 Handbook had been structured into different topic areas (geometry, arithmetic and algebra), the research over the past decade was less amenable to such a categorization, in part because the research was *less explicitly concerned* with particular mathematical concepts. Instead, the primary concerns were theory, the role of the teacher, new technologies and the design of tasks and assessment. The authors found that while the majority of papers in PME proceedings were related to the use of well-known digital technologies such as dynamic geometry environments, computer algebra systems, graphing calculators and spreadsheets, these papers were less focused on the question of 'do they work?' than on questions such as: how do teachers integrate them? How might suitable tasks be designed for the use of a given technology? How might new theories help us understand the role that technologies play in teachers' and students' mathematical activity? Indeed, with respect to the first question, the authors remark on the attention not only to the teacher's role in using a given technology in the classroom, but to the challenge of orchestrating several types of resources: "Technology has opened up new challenges for teaching, not only in terms of their knowledge and beliefs, but also in terms of the complexities of integrating different kinds of resources" (p. 236). The shifting emphasis from the learner to the teacher can also be seen in the recently published edited collection entitled *The mathematics teacher in the digital age* (Clark-Wilson et al., 2014). This book was heavily oriented towards theory and professional development, but marked by a near absence of focus on mathematics. These new strands of research become entangled with prior foci of interest.

An entire section of the chapter is devoted to theorising. The authors cite Drijvers, Kieran and Mariotti's (2009) "plea for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives" (p. 89), especially ones that can extend and refine the two dominant theories found in European research: instrumentation theory and the theory of semiotic mediation. Sinclair and Yerushalmy report that while these two theories, which attend explicitly to the use of digital tools, were predominant, several other theoretical perspectives were used in the PME proceedings over the last decade, many of them *not* specifically attending to digital tools. The authors write that,

> With respect to the papers that do draw on theories, there has been significant development over the past decade, which suggests that the field of mathematics education related to digital technology has certainly matured; it has evolved from being an "experimentation niche" and has become an established domain of research that now carries a more solid message for the future (p. 251).

The authors go on to identify two issues related to theory use: first, they argue that theories related to the use of digital technologies need to be better coordinated with more general and established theories; second, while there has been a burgeoning of theory use and development, the concomitant development of associated methodologies has not kept pace. The idea of better coordination might imply some kind of intersections with other theories, but the simple crossing of one theory with another rarely does justice to the epistemological, ontological and axiological commitments of each.

One thing that we can say about "what we have learned from the routes we have traversed" is that the use of new theories has enabled us to ask different, more refined questions about the use of technology mathematical teaching and learning. For example, instead of asking "did the students learn fractions better?" an instrumental genesis approach might focus more on the new schemes that the students developed in using a given technology to work with fractions; a semiotic mediation approach might focus on the particular gestures that students made while using a technology and how they were transformed into mathematical signs by the classroom teacher. In both cases, there is not a revisiting of the initial question, but a re-layering of it. These questions focus less on the determining whether digital technologies should be used or whether they work better than other resources; they instead take technology use as a given and investigate the complex and often unexpected effects on how learners move their bodies, how mathematical concepts seem to arise and crystallise in new ways and how aspects of classroom activity, such as language use, student agency and material arrangements (of furniture, devices, bodies) change as well.

One final note on what we have learned relates to the evolution of research identified by Sinclair and Yerushalmy from the study of the use of "second wave technologies" [i], which are open in the sense that they do not contain embedded tasks, to an interest in *task embedded* digital technologies, "which direct the actions and uses to more specific purposes" and *evaluative* digital technologies, which "provide feedback on students' responses and actions" (p. 252). The inclusion of tasks and evaluative features may improve accessibility for teachers in that it takes care of some of the decisions that teachers would have to make with more open technologies such as identify and choosing problems and assessing student learning. Of course, the streamlining of open digital technologies may also have an adverse effect on classroom use, inasmuch as openness has often been taken as crucial for encouraging curiosity, expressiveness and agency. Nonetheless, we see in this evolution a complexifying of technology in which it is not simply the hardware/software device with strict boundaries, but instead a more amorphous entity that includes its associated tasks and modes of use. The question is less about technology A then it is about technology A using task B in setting C.

Addressing Issues of Access and Equity within Mathematics Education Today

For the most part, at least according to reports in the literature, the long-standing challenge of access—that is, whether students and teachers have access to computers and to software—is no longer the main hurdle in digital technology integration. Not only have computers become more common in classrooms, but many schools have embraced tablets; furthermore, the trend towards free software (including free versions of software programs that were originally licensed) has removed some hurdles for teachers, especially teachers in developing countries[ii].

As intimated above, the greater hurdle for technology integration relates to teaching practices, to curriculum and to assessment—and, in a sense to access to professional development (see Clark-Wilson et al., 2014). In terms of equity, there have been two main, different approaches to supporting diverse learners' needs through the use of technology. These seem to entail quite different understandings of what certain learners need in order to have more mathematical success. The development of new digital technologies addressing equity has focused mainly on students diagnosed with learning disabilities (MLDs), as well as deaf and blind students[iii]. In the area of the MLDs, for example, there have been several software programs created to help struggling children improve their number sense. These tend to be focussed on particular aspects of number and designed as *instructive*[iv] environment, which provide instant evaluative feedback and tend to target procedural skills. Such programmes aim primarily to address the deficits of the children; equity thus identifies the problem as belonging to the learner (rather than to the mathematics, the environment, etc.). Unfortunately, despite some promising results (Butterworth & Laurillard, 2010), researchers such as Goodwin and Highfield (2013) have shown that children working with the instructive digital technology were more focussed on receiving positive feedback than on discussing or reflecting on the embedded mathematical concept.

A different approach has been taken up by researchers in Brazil (see Fernandes et al., 2011; 2013; Santos, et al., 2013), who have studied the use of digital technologies in inclusive classrooms (that may include deaf, blind, seeing and hearing students), and have developed more *manipulative* technologies. Their design and research process seeks to identify different ways of interacting with mathematics that may help all learners, and not just those diagnosed with disabilities. Such work requires re-thinking mathematics (as something that can be heard, for example, instead of seen through symbols or graphs) instead of merely simplifying traditional mathematics or breaking it down into steps. Their approach to equity identifies the problem as belonging less to the learner than to the mathematics (or the ways it is taught). A similar approach was taken in the study reported by Cohen et al. (2017), which involved the use of the *manipulative*, mutitouch iPad app *TouchCounts* with grade 1 children identified as low-achievers in the mathematics classroom. The app, which enables tangible, visual, aural and symbolic modes of interaction, was used both in the whole classroom situation, but also in a smaller group setting with the identified children. The use of fingers, which enabled these children to improve their subitising and awareness of place value, was also helpful for the other children in the classroom.

Returning to the metaphors of crossroads, it seems that one image that drives the choice of technology used with MLDs is that the children cannot take one road, so they must take the other, thereby setting off a chain of entailments about two kinds of mathematics, two kinds of learners, two kinds of technology. Such an approach fails to consider the extent to which the traditional technology of mathematics (paper and pencil) is implicated in the very nature of school mathematics and the possibility that new technologies may change what school mathematics looks (and sounds and feels) like, and what mathematical actions might be valued in the classroom.

Barriers within Research Traditions, Educational Policy, and Teaching Practice that Impede Researchers', Students' and Teachers' Success

In the first section, I identified the recent burgeoning of theory that was evident in the last 10 years of research published in the PME proceedings, as well as the current tendency for digital technology-specific theories to be isolated from other theories in mathematics education. A similar phenomenon—the segregating of technology and non-technology research—can also be seen in peer-reviewed journal publication. This is evident when comparing articles published in JRME, FLM and ESM (three of the top-ranked, long-standing international journals in mathematics education). As Table 1 shows, there are relatively few articles that focus explicitly on the teaching and learning of mathematics using digital technology. The frequency of publication seems to be quite stable when comparing articles published in 1996, 2006 and 2016.

One reason for this low frequency is the fact that there are several journals in which authors can choose to publish their work, journals where technology is an explicit focus (for example, IJCML (now TKL), DEME, IJMTL, CJMSTE). Publications in these 'technology journals' may not be in conversation with publications in journals such as JRME, FLM and ESM, thus leading to a group of theories that specialise in the use of technology and another group of theories that more or less ignore issues relating to technology[v]. This has been partially true for the influential learning trajectory research, which though tending to a more Vygotskian perspective, which recognises the central importance of language and tools in learning, continues to identify and disseminate trajectories that do not specify the use of digital technologies. If technologies were used in any of the tasks studied by researchers, it is assumed that the stepping stones from one concept to another could be made *no matter what* technology is used—but the default technology is almost always paper and pencil. This point of view contradicts the Vygotskian premise, but also reifies a certain vision of mathematics teaching and learning that makes it more difficult for digital technologies to be taken up more widely—and thus contributing to the continued debate around "the basics" (see Roth, 2008).

Table 1: Comparison of Articles Focused on the Use of Digital Technology Across Journals

	JRME		FLM		ESM	
	Technology	Total	Technology	Total	Technology	Total
2016	1	12	1	29	3	61
2006	0	13	2	31	0	50
1996	0	23	1	23	2	32
Total	1	48	4	83	5	143
	2%		5%		3%	

The ignoring of technology has also tended to occur in the areas of curriculum design and assessment (some of which is based on learning trajectory research). While standards in most countries may have language that includes reference to the importance of technology, the actual concepts that are listed, and the order in which they are listed, are determined in a way that is absolutely independent of any particular digital technology. For example, in the area of geometry, which is my research focus, a curriculum or textbook that asks students to engage in geometric construction by drawing shapes that have numerically determined side lengths and angle measures is anti-dynamic[vi]. This after two decades of research showing the pedagogical benefits of using dynamic geometry environments in the teacher and learning of geometry.

Lines can be dangerous. Lines can begin as imaginary paths to be followed, but once carved, they can become troughs that are hard to escape. Research in the use of digital technologies can sometimes reinforce troughs, when it focuses more on how technologies make concepts more efficient or quick to learn, rather than underscoring the sometimes unexpected conceptual shifts that innovative digital technologies can occasion.

The issue of assessment may be particularly important in high school and undergraduate contexts, where the use of digital technology on tests is often disallowed, meaning that students may be learning with a given technology but are being assessed as if that technology was a disposable scaffold to learning. Sangwin et al. (2010) argue, "if a teacher encourages students to make extensive use of tools in a course but does not allow their use on the end-of-course test, are students being given the opportunity to show what they learned with the use of such tools?" (p. 229). The issue is complex, however: in a study of secondary school teachers in Canada, Venturini (2015) found that teachers were reluctant to use digital technology assessment tasks because they were concerned that the students would learn as they used the digital technology, which was seen to contradict the purpose of assessment.

In terms of teaching practices and teachers' success, there has certainly been a dearth of research in this area. As Sinclair and Yerushalmy write, "Compared with research on student learning with technology, research on the teacher has not been as well developed" (p. 260). Nascent theory development began with the framework of TPACK, which describes the different types of knowledge that teachers may use in their teaching practices, adding technology to the well-known pedagogical and content knowledge aspects. As a theory, it is rather limited. More recently, theories that provide a more analytic lens on the role of the teacher in teaching with digital technology have been developed, based on theories of instrumental genesis (such as instrumental orchestration). Ruthven (2014) has also proposed a framework for analysing the teaching expertise that underpins successful use of digital technology in the mathematics classroom. His framework highlights the tensions that arise for teachers when trying to integrate technology, that relate to the lack of articulation between digital technologies and other resources such as textbooks, curricula and assessment. Worth studying would be situations in which this articulation has been attempted (perhaps with a high-quality e-textbook (see Pepin et al, 2015) or with trajectories that have been elaborated using digital technologies).

Laying the Groundwork for Future Crossroads or Intersections Among Theory, Research, and Practice

When thinking about future crossroads or intersections, two recent, related developments in educational research come to mind, both of which are highly relevant to technology. One is the association of mathematics with computational thinking (CT) and the other is the emergence of the idea of STEM. Both developments have received substantial funding over the past decade (and have given rise to specialized conferences, journals and special issues) and will likely shape future discussions around the role of technology in mathematics education. In both cases, the role of technology shifts quite significantly from the way it has been conceived in research over the past two decades. Before commenting on whether or not we are at a crossroads, I would like to look more closely at each new development in turn.

In the case of CT, the research initiatives most closely associated with mathematics education have involved studying the use of computer programming as a means to support mathematical learning, much in the tradition of Papert (1980). For example, Benton et al. (2017) as well as Gadanidis et al. (2016) explore the use of Scratch programming in relation to concepts that are recognizably mathematical (e.g., angle, binomial theorem). In these two cases, the digital technology in question is one that was not designed specifically for the teaching and learning of mathematics, and that entails practices and values that are specific to the domain of computer science.

In the case of STEM, the nature of the "T" seems to be less precise than in CT, involving not so much the use of programming (or coding), but instead the use of digital tools. For example, in the STEM videos published by the Teaching Channel[vii], students use scientific tools such as digital thermometers or calculators as well as simulations (a programme for building and testing rollercoaster). In these cases, the technology is not vectored towards the learning of mathematics, but rather to the completion of what is essentially a science or engineering project. Whereas the CT connection privileges computer programming as the primary mode of engagement with digital technology, the STEM agglomeration features the use of digital technologies that are oriented towards their pragmatic value rather than their epistemic value (see Artigue, 2002 for a discussion of the distinction between these two values).

I bring up these two examples because of the stress they will likely place on the way digital technologies are used and researched in mathematics education. They displace technology from being constitutive of mathematics (à la Rotman, 2008), which may result either in the displacement of technology to something you do in your CT lesson, not in mathematics, or in the isolating of technology as one element in a STEM fruit salad of disciplines that shares little disciplinary value with mathematics. Again, a crossroads view of things encourages us to think about choices, about going this way or that. But, at this moment in time, what we may need more of is attending to the multiple threads in which mathematics education is entangled and how the choices that seem on offer are already the consequence of a set of assumptions and commitments—and to think, what could things look like before the crossroads?

Acknowledgments

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Endnotes

[i] Sinclair and Jackiw (2005) describe three ways of technology evolution in mathematics education. The first wave focused on learners' interactions with technology (such as Papert's research with Logo); the second wave shifted from programming languages to technologies that were more transparently related to the school mathematics curriculum, such as graphing calculators, computer algebra systems and dynamic geometry environments; the third wave was concerned with technologies that attended to the social context of the mathematics classroom.

[ii] But this should not necessarily be seen as a positive development for mathematics education. Paying software programmes were maintained and came along with teacher support and, frequently, curriculum materials; they could be expected to be developed by professional software designers, and to last for long periods of time.

[iii] As far as I am aware, there are very few examples of digital technologies that have been designed for other groups of students who have been identified as under-achieving, based on gender,

race or socio-economic status. One exception, which dates back to the 1990s, was Klawe's E-GEMS project (see Inkpen et al., 1995), which was targeted specifically for girls. A small number of researchers have also explored the use of digital technologies with bilingual learners, who also face particular challenges in the mathematics classroom (see Ng, 2016).

[iv] Goodwin and Highfield (2013) distinguish three types of digital technologies: instructive, manipulable and constructive. Sinclair and Baccaglini-Frank (2014) describe each as follows:

> Instructive digital technologies tend to promote procedural learning, relying on evaluative feedback and repetitive interactions with imposed representations. Manipulable digital technologies enable the imposed representations to be manipulated so as to engage students in discovery and experimentation. […] Finally, constructive digital technologies are ones in which learners create their own representations, which are often the goal of the activity, thereby promoting mathematical modeling and what Noss and Hoyles (1996) characterize as expressive uses of technology. Goodwin and Highfield argue that while instructive technologies may be well-suited for procedural learning, manipulable and constructive technologies better support conceptual learning.

[v] That is it possible to do this strikes me as quite interesting, but coherent with the view that mathematics—and thus the learning of mathematics—can be separated from its technologies.

[vi] And example of this can be found in the New York State Common Core Mathematics Curriculum.

[vii] See, for example: https://www.teachingchannel.org/videos/teaching-stem-strategies and https://www.teachingchannel.org/videos/stem-lesson-ideas-heat-loss-project. An analysis of these STEM videos is currently in preparation (Bakos et al, in preparation).

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Chapter 2

Curriculum and Related Factors

Research Reports

Brief Research Reports

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Posters

WORKING COLLECTIVELY TO DESIGN ONLINE TEACHER EDUCATION CURRICULUM: HOW DO TEACHER EDUCATORS MANAGE TO DO IT?

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This paper is part of a three-year inquiry that supports and investigates the work of groups of mathematics teacher educators using technological tools to design and implement multimedia practice-based teacher education curriculum materials. This paper describes the kinds of activities, interactions, and tools used by mathematics teacher educators to engage in such work. Using Engeström's Activity Theory as a framework, we organize our observations of the groups' work sessions, noting differences across the groups' objectives and ways of organizing the division of labor and tools for engaging in the work. Our results suggest the activity of collective curriculum development amongst teacher educators can take on at least three distinct types of interactions. We present these types of interactions as "caricatures" (Lambdin & Preston, 1995), using data from all of the groups to represent composite descriptions.

Keywords: Curriculum, Instructional Activities and Practices, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development, Technology

Introduction

We share data from an ongoing NSF project that engages groups of mathematics teacher educators in collective work using technological tools to design and implement online practice-based teacher education curriculum materials. The work within that project can be broadly framed as part of the larger efforts to reimagine mathematics teacher education through the development of a common curriculum (Ball and Forzani, 2011) centered on practice-based experiences for enabling novices to learn to teach *in, from*, and *for* practice (Lampert, 2010). The efforts to reimagine mathematics teacher education may tread some of the same terrain as the well-studied efforts to reform K-12 mathematics through the design of better curriculum (e.g. Lappan & Phillips, 2009) and professional development for supporting teachers to use those materials (Remillard, Herbel-Eisenmann, & Lloyd, 2009). Both efforts attempt to address deficiencies in the current systems by reimagining, in some measure, what happens in instructional settings (whether in K-12 or higher education); and both treat curriculum as a lever to do that.

To understand how teacher educators may use the curriculum of teacher education to make teacher education practice-based, it is useful to consider the different ways in which teachers use curriculum in K-12 settings: While there is a tradition in which curriculum developers create materials, teach them to teachers, and then teachers implement them with fidelity as a goal, that is by no means the only use. Reviewing the literature of curriculum use studies, Remillard (2005) describes three kinds curriculum use studies and their corresponding perspectives. The first set of studies takes the perspective of curriculum as a "fixed entity" and takes for granted that the teacher serves as a "conduit for the curriculum". The second set of studies takes the perspective of the curriculum as a more or less stable starting point from to which the teacher makes adaptations in ways that may be more or less faithful to the curriculum design. The third set of studies takes the perspective that the teacher is positioned as an active interpreter of the written curriculum and "author" of the enacted curriculum (Doyle,1992). Ball and Cohen (1996) suggests a reconceptualization of curriculum as a site first for teacher learning and then a resource for student learning. This suggestion fits well into this third perspective as such an approach gives teachers an opportunity to work collaboratively with curricula materials in order to decide how they will use them to solve the problems of improvement.

In this paper, we aim to describe and explain how groups of teacher educators organize their collective work around the task of designing and implementing technology-mediated practice-based curricular materials for teacher education. Research on curriculum use suggests that there could be a host of ways that the field addresses the larger problems of developing common practice-based curricular materials for teacher education; and each of these approaches comes with different kinds of affordances and constraints for the work at hand. We wonder about the various ways in which mathematics teacher educators might elect to organize themselves around the task of developing technology-mediated practice-based teacher education materials and what sort of affordances and constraints can be found across the variety of organizational choices. In this paper, we describe and explain three ways in which 12 groups of mathematics teacher educators engaged in the activity of collectively developing and using technology-mediated practice-based materials for teacher education. To do this, we use methods from activity theory, noting differences across the groups' objectives, division of labor, and tool usage. To illustrate these differences, we borrow a practice from Lambdin and Preston (1995, p. 130) and create "caricatures" of groups to describe these differences, where a caricature represents a composite description by combining information from all 12 groups.

Methods

Setting

In this paper, we present our findings regarding the types of interactions between 12 Fellows and their Inquiry Group Members (IGMs) across a two-and-a-half year timeframe from May, 2014 to November, 2016. The Fellows come from research institutions (Doctoral institutions with the moderate, higher, and highest levels of research activity) and serve in a variety of positions (Assistant, Associate Professor, and Full Professor as well as Lecturers). Next, each Fellow formed their own inquiry group that included one to seven members from a variety of institutions and geographic locations. The Fellows assembled inquiry groups for the purposes of developing technology-mediated mathematics teacher education curriculum materials.

To develop these materials, the Fellows and their Inquiry Group Members had access to the tools and capabilities within the Lesson*Sketch* platform (www.lessonsketch.org). Lesson*Sketch* provides teacher educators with a suite of online tools for composing and interacting with multimedia representations of practice. ***Depict*** offers users a drag-and-drop environment allowing users to easily represent scenes from a classroom in the form of a storyboard. ***Annotate*** allows users to make time-stamped comments on a variety of media files, such as video, audio, or storyboards. ***Plan*** offers users a drag-and-drop environment for authoring agendas for interactive experiences for clients, integrating multimedia tools for both producing and interacting with representations of practice with more traditional course planning tools such as multiple choice and open-ended question generators. In addition to those tools, Lesson*Sketch* also has accompanying capabilities for enabling users to manage and study client interactions with the experiences. One such capability is the ***Experience Manager*** that allows users to distribute online experiences to clients by either assigning the experience directly to clients in Lesson*Sketch* or by providing them with an access code or an email link. The second capability, ***Reports,*** allows users to collect data about clients' activities within such experiences, including both user contributions (such as responses to questions or pins on a video) and behaviors (such as time spent on an activity).

For the first year of the project, the Fellows worked on drafting an instructional module(s) for one of their own courses. The Fellows' modules (like the Fellows' teaching assignments) were

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varied, with some designed for pure mathematics coursework, others for mathematics methods coursework, and still others for general education coursework. During the first year, the Fellows met together with the project group for two face-to-face meetings and participated in monthly online meetings across the year to check in with one another. During the second year of the project, the Fellows recruited Inquiry Group Members to help implement and/or construct modules. During this time the Inquiry Group Members met together with their Fellow and with the project team for face-to-face for work sessions on three occasions and held their own meetings throughout the year (either virtually or face-to-face on their own schedule). The Fellows continued to meet virtually, with one another, every month.

Data Collection

We collected a variety of data to document the ways inquiry groups organized themselves to collectively develop and/or enact curricular materials. For this research, we documented each group's work in several ways. During the year one face-to-face meetings, we observed the Fellows' interactions with one another and the project staff, collecting audio recordings of whole group discussions and taking field notes during their work sessions. During the year two face-to-face meetings, we observed the Fellow's interactions with the Inquiry Group Members and with project staff, taking field notes about the ideas exchanged and the roles various group members were taking on. Across both years, we conducted and recorded monthly interviews with Fellows using video conferencing software, to support their progress. In the Fall of 2016, we surveyed Inquiry Group Members using adapted versions of the Concerns Based Adoption Model (George, Hall, & Stiegelbauer, 2006) and Team Climate (Anderson & West, 1998) Surveys. We used this survey to investigate the group distinctions as well as some of our observations about differences in possible group characteristics (state of the modules when the IGM joined the group, agency, similarity of professional goals) more thoroughly. We used Inquiry Group Members' responses to the survey to verify the nature of each group's activity (e.g., whether or not the primary activity and ways of working—*implementation, collective construction,* or *independent construction*—we had observed were compatible with the primary activity and ways of working the group members identified) as well as to confirm some key characteristics that were difficult to fully perceive from observation alone. Lastly, we collected system-use data to understand whether and how different groups used the various tools and capabilities within the Lesson*Sketch* platform.

Data Analysis

To begin describing the inquiry group interactions in a systematic way, we analyzed the data using Engeström's (1987) activity theory, and its related mediational triangle (Figure 1). Activity theory was developed to model goal- (or object-) oriented behavior as activity systems, accounting for the collective nature of human activities as interactions between distinct elements.

While all the inquiry groups could be described as comprising the same type of *subjects* (mathematics teacher educators) working on behalf of the same type of *community* (fellow mathematics teacher educators) for the same *outcome* (namely educating future or current mathematics teachers), our observations of their activities suggested several important differences across groups within the *object*, *division of labor*, *rules*, and *tool* components of the mediational triangle. First, we noticed differences in "what" the inquiry groups were collectively focused on doing together, that is a difference in the groups' *objects (or goals)*. Avowedly what they all had to do related to an instructional module. Groups seemed to primarily be focused either on constructing modules (either collectively or individually) or implementing a module created by the Fellow or some other group member. Based on these differences, we categorized the groups' activity systems according to one of two objects: *construction* or *implementation.*

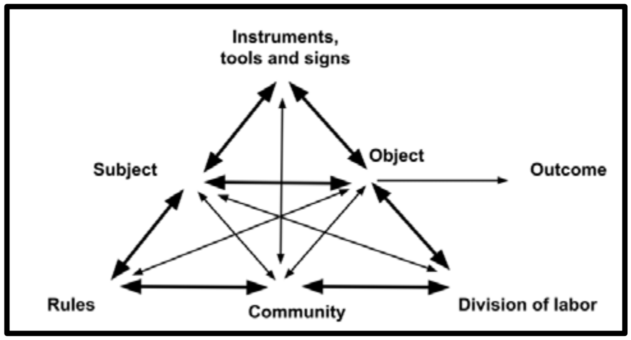

Figure 1. Engeström's mediational triangle (Engeström, 1987)**.**

Second, while all of the groups with an *implementation* objective seemed to use the same *division of labor*, namely the Fellow played the role of "curriculum writer" while the Inquiry Group Members implemented that curriculum in their own settings, we noticed differences in how those groups with *construction* objectives divided the labor. Some of those groups took on the task of constructing a module(s) in such as way that the Fellow and the Inquiry Group Members worked together to develop a single set of materials; other groups took on the work so that the Fellow played the role of "lead innovator"—developing his or her materials first—and the Inquiry Group Members each followed suit by patterning their own materials after the Fellow's work, but not necessarily in ways that would allow for the materials to be implemented together. Based on these differences in the *division of labor* we categorized the construction groups into two different types: *collective construction* or *independent construction*.

Third, we noticed some important broad similarities in the ways the tools mediated the work of the Fellows and Inquiry Group Members. To begin, the ***Depict***, ***Annotate,*** and ***Plan*** tools were primarily used for their authoring capabilities. While the ***Plan*** tool was created for authoring experiences, there are many other ways in which ***Depict*** and ***Annotate*** could be used. While the ***Depict*** tool can be used to author content for experiences (e.g., develop storyboards for students to interact with), it can also be used to provide feedback to students' contributions (e.g., to provide a visual interpretation of a student's vague narrative account of a classroom event and ask whether it happened in that way). Similarly, while the ***Annotate*** tool can be used to author content for experiences (e.g., identify moments of a video for students to comment on), it can also be used to provide feedback to students about their contributions within an experience. For the most part, however, we observed the Fellows and their Inquiry Group Members using ***Depict*** and ***Annotate*** to *author module content*. Thus, for the purposes of this work, we classified the use of ***Depict***, ***Annotate,*** and ***Plan*** as mediating primarily the authoring of modules; while the capabilities within ***Experience Manager*** mediating the *distribution of modules for review* prior to implementation as well as *distribution of modules for implementation* with students; and the capabilities of ***Report*** for *analyzing aspects of the module use*.

Finally we noticed some important differences across groups in terms of the ways in which they used the different capabilities (*Authoring* Modules, *Review* and *Distribution* Modules for Implementation*,* and *Analyzing* Module Use*)* in the Lesson*Sketch* system to mediate their collective work. We suspected that there would be meaningful differences in the ways in which these groups used the tools and capabilities to mediate their collective work, but since much of their tool usage happened when we were not directly observing them we could not be certain which tools and capabilities they were accessing without a closer examination of system data.

Results and Discussion

In this section, we present our overall findings by developing caricatures or composites of the Inquiry Groups' work based on their group structure, the conditions that seemed to characterize the group's activity system, and the ways in which the tools and capabilities afforded by the Lesson*Sketch* platform seemed to mediate that activity. By taking the inquiry group to represent the unit of analysis, we emphasize that the caricatures do not reflect the work of individuals but the larger activity system. The caricatures were created during the data analysis process as opposed to being a priori to the analysis.

The categorization of the groups or subgroups as engaged in *implementation, collective construction,* or *independent construction* activities came fairly easily from the observations as described above. Our observations were confirmed in the survey responses from the Inquiry Group Members. Those working in *implementation* groups describing their work primarily in terms of *using*, *piloting*, or *suggesting revisions* to the module created by the Fellow and those in the *construction* groups describing their work primarily in terms of *building*, *designing*, or *creating* module(s). Inquiry Group Members engaged in *independent construction* activities indicated relatively more concern about the personal consequences of the project including logistics and the time involved in the activities of the project than those engaged in *collective construction* activities. To represent these composites in more memorable ways, we use the metaphor of different ways of having a dinner party: (1) Hosting; (2) Potluck; (3) Cooking Club.

Hosting. One way to organize a dinner party is for the host to prepare a single meal for the guests. While customs may differ, this kind of organization usually calls for the bulk of the meal preparation to take place prior to guests' arrival. Similarly, those groups with an *implementation* objective commenced after the Fellow had drafted a version of the module that was ready for distribution and the primary focus of the group was to implement a common set of teacher education modules. These groups tended to be large (~5 members) and the members held similar professional goals, usually in the form of a common course or a common approach to teacher education. Coming back to the metaphor of a host preparing a meal for guests, the host needs to consider ahead of time the match between the dish prepared and the kinds of foods the guests are accustomed to eating. The host could ensure this match by preparing a dish common enough to be palatable to all of the guests or by selecting guests amenable to the kind of dish that will be served. We see evidence of the Fellows in these groups using both strategies, both designing the module around common themes in the field as well as identifying Inquiry Group Members according to similar perspectives on teacher education. Once these groups gathered, their activities were highly structured, with the Fellow providing the module, the Inquiry Group Members enacting it with their students and providing data back to the group to inform the Fellow's revision of the module. These clearly defined roles seemed to come with fairly hierarchical structures that positioned the Inquiry Group Members to enact the module without revisions, as to provide the cleanest data back to the Fellow. While some exceptions were made, these negotiations happened privately between the Fellow and the individual Inquiry Group Member. This kind of hierarchical structure guiding the *division of labor* is somewhat unsurprising if one considers the metaphor of a host preparing a single meal for many guests. Modifications to the meal just prior to the serving could be quite difficult for a host to accommodate and such modifications could jeopardize the more primary activities of the evening, such as sharing a meal together or gathering feedback about a dish. The kinds of comments Inquiry Group Members made following implementation of a module were summative—focused tweaking small elements of the module. Again, in light of the metaphor, this is perhaps unsurprising given the kinds of access guests at a dinner party have to the actual production of the meal. The structure of this type of Group could be observed in the use of the technological tools, the Fellows in these groups (compared with Fellows from the other two groups), were the heaviest users of the *review, distribution,* and *analysis* tools; while their Inquiry Group Members (compared with Inquiry Group Members from the other two composites), tended not to use many tools.

Potluck. A second way to organize a dinner party is for everyone to bring a single dish to share with others at the host's home and collectively the individual contributions make up the meal, sometimes called a potluck. Those groups engaged in an *independent construction* activity, commenced after the Fellow had drafted a version of the module and the focus of these groups' activity was on both providing feedback on the Fellow's modules and using the Fellow's module as inspiration for each member to make their own. These groups also tended to be large (~5 members) and its members held different professional goals; some joining because the work might offer research opportunities while other members joining to learn how to use technology to construct materials for their own courses. Like the *Potluck* model for dinner parties, these groups handled their collective work by dividing and conquering, with members carrying out their roles in fairly disconnected ways, working parallel to or in tandem to one another's efforts sometimes unaware of the various work of other members. Unlike the *Hosting* model for organizing dinner parties, *Potluck* models do not require a host to ensure that the prepared dishes match the kind of foods guests might be interested in consuming. For one, a guest's own dish can provide some assurance for such a match, but also the wide variety of dishes to choose from ensures that guests will find something they are amenable to eating. Similarly, the Fellows in these groups were not observed needing to make any sort of accommodations or negotiations regarding implementation of modules, nor did the group make any sort of official bid that any materials would be implemented, leaving it mostly up to the Inquiry Group Members to decide what, if anything, they would like to try out in their own contexts. That said, like the participants at a potluck who sometimes seek out recipes for particular dishes brought by guests, Inquiry Group Members' knowledge was seen to be a resource for offering ideas for their own module and revisions for the Fellow's module. Both the Fellows and their Inquiry Group Members were the heaviest users of the authoring capabilities (when compared with their counterparts across the other two groups).

Cooking Club. A third way to organize a dinner party is for guests and host to plan and cook a meal together as with cooking clubs or progressive dinners. Distinct from the *Hosting* or *Potluck* models for organizing a dinner party, the *Cooking Club* model for dinner party organization does not require for the bulk of meal preparation to happen prior to commencing the activity. Instead, the host takes on the responsibility to send invitations or perhaps make provisions for supplies for the meal; and the guests for such events arrive with anticipation for taking part in the cooking. Similarly, groups engaged in *collective construction* activities commenced at a time when the module was still in the form of a vision and not yet drafted in any concrete way and the primary focus of the group was to create a common set of teacher education modules. These groups tended to be smaller (1 to 3 members) and members held similar professional goals usually in the form of a common course or a common approach to teacher education. Unlike the *Hosting* model for organizing dinner parties, *Cooking Club* models do not place exclusive responsibility on the host for ensuring that the prepared meal matches the kind of foods guests might be interested in consuming, because the decision on the meal to be prepared is shared by the group. Similarly, the Fellows in these groups were not observed needing to make any sort of accommodations or negotiations regarding the eventual implementation of the collectively developed module. Perhaps like a *Cooking Club* that has been gathering for some time, these groups' exchanges were characterized by an insider language, where group members seemed to have a shared understanding for the meaning and value of particular constructs (making it sometimes difficult for an outsider, such as the researchers, to follow the conversations). This way of using language seemed to, at least in some ways, make the Inquiry Group Members' knowledge readily accessible for use by the group to design and revise the materials. The activity of *collectively constructing* modules seemed to promote the sharing of practical knowledge within these groups that would often times move fluidly between bursts of creation followed by more theoretical conversations. We liken this kind of sharing of knowledge through collective action to the gathering of individuals around a counter to collectively dice a meal's ingredients and who may learn simply by carrying out the practice of dicing near others who are also dicing, but might also stop to clarify the distinctions between the practices of chopping, dicing, and mincing. The Fellows in these groups, while not the highest users in any of the categories of *authoring*, *reviewing, distributing*, or *analyzing*, these Fellows maintain fairly high uses across all capabilities; while their Inquiry Group Members were the only frequent users of the *reviewing* capabilities (compared with Inquiry Group Members from the other two composites), they were also users of all capabilities.

Significance of the Research

In the above findings, we presented three different caricatures representing the ways in which we observed teacher educators organize themselves around the activity of constructing mathematics teacher education curriculum materials. In the presentation of these three caricatures, we see two important differences between the larger body of literature on curriculum use and this work. The first difference stems from the fact that these groups were designing digital materials that can be easily edited which is distinct from the more canonical use of curriculum in which materials are less amenable to such edits. Related to this difference, we take as critical the finding regarding the ways in which the digital tools seemed to mediate various kinds of activities related to the design and use of online curriculum material. The second difference stems from the fact that these groups were comprised of teacher educators, rather than K-12 teachers. Distinct from K-12 curriculum use, we note that the "status" of curricular materials in this project is far from "fixed". The mathematics teacher educators featured worked closely (or perhaps they were themselves) with curriculum writers. Related to this difference, we take as critical the finding that teacher educators not officially "charged" with the writing of the materials (as was the case for the Inquiry Group Members) can be positioned in very different ways within the work of developing and implementing curricular materials for teacher education. Perhaps most importantly, we see this work as laying the groundwork to begin asking questions about what each of these various models of activity affords to the work of designing and implementing teacher education with digital curricular materials.

Endnote

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MIDDLE SCHOOL MATHEMATICS TEACHERS' PERCEPTIONS OF THE STANDARDS FOR MATHEMATICAL PRACTICE EMBEDDED IN CURRICULAR RESOURCES

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This study examines the perceptions of the Standards for Mathematical Practice (SMPs) held by 34 middle school mathematics teachers (MSMTs) as evidenced by their interactions with seven lessons drawn from thinking device (TD) and delivery mechanism (DM) curriculum types. MSMTs' perceptions of the SMPs consistent with their wording in the Common Core State Standards for Mathematics (CCSSM) included a flexible definition of precision beyond calculation. However, MSMTs also possessed a number of perceptions of the SMPs that were at odds with the wording of these standards in the CCSSM. For instance, they considered a curriculum resource's imperative for students to use a tool to be an indication of SMP 5. MSMTs whose district-adopted curriculum was categorized as TD had significantly less invalid SMP justifications than teachers using DM curricula t(34) = 2.41, p = .022.

Keywords: Curriculum, Standards, Middle School Education, Instructional Activities and Practices

The majority of students in the US are situated within educational systems linked to the Common Core State Standards for Mathematics (CCSSM). As we enter the seventh year of CCSSM implementation we find ourselves at an educational crossroads. These crossroads consist of reflecting on what we have learned thus far with regard to CCSSM implementation and determining where we need to go from here to realize these ambitious standards. CCSSM research has investigated the standards themselves (Schmidt & Houang, 2012), examined elementary level mathematics textbooks' alignment to the CCSSM, ascertained teachers' perceptions of the CCSSM (Davis et al., 2014), and described ways that teachers make the Standards for Mathematical Practice (SMPs) explicit to students during classroom instruction (Selling, 2016).

Opfer, Kaufman, and Thomas (2016) investigated the perceptions of a nationally representative sample of K-12 public school teachers in the US and found that teachers reported spending less time on SMP 7 (structure) or SMP 3 (constructing arguments) than other SMPs. In contrast, other research (Davis et al., under review) suggests that teachers consider the SMPs to be components of each lesson that they construct. Opfer and colleagues also found that teachers at the elementary level were more likely to misunderstand SMP 4 (modeling) than secondary teachers. Heck and colleagues (2011) noted that a group of mathematics educators and policy researchers they surveyed were concerned about the separation of the SMPs from content in the CCSSM as well as a lack of clear descriptions of what a trajectory in learning SMPs might look like across grades. An important component that is missing from this research involving teachers' perceptions of the CCSSM is the mediating presence of their district-adopted curricular resources. Mathematics teachers frequently use textbooks (Banilower et al., 2013) and other research that we have completed as part of our larger study suggests that teachers interpret textbook materials vis-à-vis the CCSSM (Roth McDuffie et al., 2017). Given this setting, we were especially interested in the perceptions of teachers using different types of curricula as textbooks that have been referred to as standards-based were created from documents that contained processes similar to the SMP. Hence, this study was designed to answer two research questions.

1. What perceptions do a group of middle school mathematics teachers (MSMTs) hold with regard to the SMPs as revealed through their work with two types of curricular resources?
2. How do the SMP perceptions of a group of MSMTs differ by district-adopted curriculum type?

Frameworks

Teachers' Interactions with Curricular Resources

We take the perspective that teachers' curriculum use involves what Remillard (2005) describes as participation with the textbook. That is, we consider teachers to be active interpreters of their curricular resources. Moreover, we consider these interpretations to be governed by teachers' personal resources (e.g., beliefs), the contexts in which they work, orientation, professional identity, students, and curriculum (Stein, Remillard, & Smith, 2007). Additionally, teachers themselves are transformed by their work with curricular resources (Remillard). We situate teachers' work with their curricular resources within Stein and colleagues temporal phases of curriculum use. In particular we use their terminology *intended curriculum* to denote the lesson plans that teachers create from their curricular resources or *written curriculum*. We consider curricular resources to encompass all of the materials associated with a program (e.g., assessment resources) in print or digital forms.

Types of Curriculum Programs

In earlier work (Choppin, McDuffie, Drake, & Davis, 2015) we conceptualized curricular resources based upon monologic and dialogic communication functions. Curricula were categorized as following a *delivery mechanism* (DM) if they serve a monologic function where content is viewed from the perspective of an expert and delivered to novices. Curricula were categorized as *thinking device* (TD) if they serve a dialogic function where the goal involves soliciting the thinking of novices and using this knowledge to move novices towards more complex thinking levels. We use the terminology *TD teachers* and *DM teachers* to denote teachers whose districts have adopted TD curricula types and DM curricula types, respectively, and we place the curriculum type after the teacher's pseudonym in the results section.

Methods

This study is a component of a larger study examining how MSMTs interact with their curricular resources in the context of the CCSSM. This component of the larger study used staged lesson plans (SLPs) to reveal MSMTs' interactions with their curricular resources vis-à-vis the CCSSM. In an SLP, teachers who have used one type of curricular resource for at least one year were given one week to create an intended curriculum from a different type of curricular resource. The SLP was designed to reveal how MSMTs used their district-adopted curricular resources by asking them to plan from a different type of curriculum resource. During the SLPs MSMTs were asked a series of questions involving the CCSSM content standards and SMPs, the SLP curricular resources as well as the district-adopted curricular resources, and the intended curriculum. Participants were purposefully drawn from school districts that had adopted both TD and DM curricula in print and digital forms from both rural and urban middle school settings. A total of 62 different MSMTs working in four different CCSSM states completed 75 SLPs in two waves during the 2013-2014 and 2014-2015 school years. In the initial wave of SLPs, MSMTs were asked to create an intended curriculum from a SLP curriculum resource that was of a different type than the district-adopted curricular resource. In the second wave of SLPs, a selection of MSMTs was asked to create an intended curriculum from three different lessons from a TD curriculum (described in more detail below). All teachers were asked which SMPs were addressed in the respective intended curricula that they created from these curricular resources. However, only 34 of these teachers were asked specifically to point out in their intended or written curricula where SMPs occurred. The 37 SLPs completed by these teachers form the data set at the center of this study. A total of 21 and 13 MSMTs had been using a DM and TD curriculum for at least one year, respectively.

SLP Curricular Resources

The curricular resources used in the SLPs were drawn from three different curriculum resources, two of which were considered to be TD, and one DM. The two categorized as TD were *Connected Mathematics 3* (CM) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2014) and *College Preparatory Mathematics* (CPM) (Kysh, Dietiker, Sallee, Hamada, & Hoey, 2013). The DM curriculum was *Glencoe Mathematics* (Glencoe) (Carter, Cuevas, Day, & Malloy, 2013). MSMTs who taught grades 6 were provided with a lesson involving proportional reasoning: (Glencoe – P) (Carter et al., 2013, pgs. 14-27); (CM – P) (Lappan et al., 2014, pgs. 18-23); *Core Connections: Course 1* (CPM1) (Kysh, Dietiker, Sallee, Hamada, & Hoey, 2013a, pgs. 224-227); or *Connections Course 2* (CPM2) (Kysh, Dietiker, Sallee, Hamada, & Hoey, 2013b, pgs. 743-745). MSMT who taught grade 8 were provided with a lesson involving linear functions (Glencoe – LF) (Carter et al., 2013, pgs. 267-277); (CM – LF) (Lappan et al., 2014, pgs. 5-11); or *Core Connections Course 3* (CPM3) (Kysh, Dietiker, Sallee, Hamada, & Hoey, 2013c, pgs. 308-311). We chose roughly equal numbers of grade 7 teachers to work with proportional reasoning and linear functions lessons. For each SLP curricular resource, MSMTs were provided with the student textbook, lesson planning resources, unit planning resources, assessment resources, and a copy of the CCSSM. The number of MSMTs working with each SLP curriculum resource by district-adopted curriculum type appears in Table 1.

Table 1: Participants by Curriculum Type and SLP Curriculum Resource

	District-Adopted Curriculum Resource	
SLP Curriculum Resource	DM	TD
Glencoe – P	–	3
Glencoe – LF	–	5
CM – P	9	–
CM – LF	11	–
CPM1	2	–
CPM2	–	4
CPM3	1	2

Analysis

The data in this study were analyzed through iterative cycles (Miles, Huberman, & Saldaña, 2014). In an initial cycle of analysis we coded data with a variety of broad codes connected to a larger project. In this study, we focused on data coded as 01-MP (mathematical practices), 0-Curric (descriptions of intended curricula or written curricula), and 04-Adapting (adaptations made to the written curricula). Data coded as 01-MP were subsequently coded for each of the SMPs based upon a word or words associated with that SMP. For example, the word *persevere* led us to categorize these data as SMP 1. Interview excerpts including language that was ambiguous (e.g., explore) or potentially could pertain to more than one of the SMPs (e.g., explain) was excluded from analysis. Next, a combination of in-vivo and descriptive coding was used on data associated with each SMP. Last, we examined the codes within each SMP for themes shedding light on MSMTs' perceptions. We determined the validity of each MSMT's justification for the presence of a SMP in the curricular resources or intended curriculum by comparing the teacher's justification to the written description for each of the SMPs in the CCSSM using techniques similar to Opfer and colleagues (2016) as well as our own previous work (Davis et al., under review) in excerpts coded as 0-Curric. Data coded as 04-Adapting were used to better understand the adaptations made with respect to the SMPs. Our analyses of the written description of the SMPs in the CCSSM led to the identification of SMP 1 and SMP 3 in both the CM – P and CM – LF SLP materials and we calculated the percentage of DM teachers who identified these SMPs in these materials. As the number of invalid SMP justifications made by the 34 teachers in our study met the assumptions of an independent samples *t*-test we used a two-tailed test to examine the significance of the differences in invalid justifications between TD and DM teachers with an alpha level of .05. Additionally, we calculated the percentage of valid SMPs out of the total SMPs noted for DM and TD teachers.

Results

SMP Perceptions of MSMTs Regardless of Curriculum Type

A common theme running through the majority of MSMTs' responses in the first SMP was multiple approaches. MSMTs perceived that both perseverance and sense making required multiple approaches. Additionally, the majority of teachers noted that problems that required students to make sense of them and preserve in solving them were complex in some way. As Davidson (TD) put it in her SLP, "If they're going to persevere in solving something, it better be something that is going to challenge their thinking in some way."

Only six teachers (all DM) mentioned that their SLP curricular resources contained components of SMP 2. A common theme among responses with regard to this SMP was real-world contexts. That is, some teachers stated that for this SMP to be present students needed to consider the mathematics embedded in a real-world situation. Contextualization and decontextualization were both mentioned by MSMTs in reference to SMP 2.

For many MSMTs, SMP 3 embodied the development of arguments and the careful examination of the arguments produced by others. Instead of arguments, however, MSMTs often stated that students would "discuss," "share," or come to a "consensus" about different ways to solve a problem such as determining which mixture was "most orangey" in the CM – P lesson.

Three of the SLP curricular resources (Glencoe – LF, CM – LF, and CPM3) specifically directed students to create tables, graphs, and equations for a variety of real-world contexts involving linear functions. An example of one set of questions from CM – LF appears below.

Name	Walking Rate
Alana	1 meter per second
Gilberto	2 meters per second
Leanne	2.5 meters per second

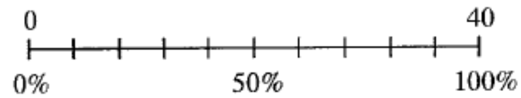
A. 1. Make a table showing the distance walked by each student for the first ten seconds. How does the walking rate affect the data?
2. Graph the time and distance on the same coordinate axes. Use a different color for each student's data. How does the walking rate affect the graph?
3. Write an equation that gives the relationship between the time t and the distance d walked for each student. How is the walking rate represented in the equation?
(Lappan et al., 2014, pgs. 6-7).

MSMTs considered questions in the textbook lessons such as the one above to be instances of SMP 4. Christiansen (DM) justified the presence of mathematical modeling in the set of questions above in the following way: “They’re making a table as a model. They’re making a graph to model the situation. They’re also going to be asked to write an equation for each situation. That would be a type of model.”

None of the MSMT discussed other aspects of modeling such as moving from the mathematical model back to the real-world context or the assumptions that needed to be made in creating the mathematical model for some real-world situation. For Dietrich (DM) and other MSMTs the presence of what they deemed to be a real-world situation was a necessary and sufficient condition for SMP 4 to occur in the lesson.

Consider the problem appearing in the CPM1 (p. 226) curricular resource.

> With your team, you will use the percent ruler shown at right to examine a sample of 40 raisins and peanuts.



> Copy the percent ruler onto your paper. Then use it to determine how many raisins would make 50% of the sample. How many raisins are in 10% of the sample? (Kysh et al., 2013a, p. 226).

MSMTs in our study felt that imperatives asking students to use such tools in the curricular resources was evidence that students were gaining proficiency with SMP 5 as seen in Tyler’s (DM) statement: “You have to use appropriate tools strategically in this one because you’re working with a percent ruler.” Other teachers pointed to the use of tools such as graphing calculators to check their work in creating the graph by hand. Thus, for these teachers they considered such tools as being used strategically by students. Other teachers considered the open-ended use of tools to be a prerequisite for the presence of SMP 5 in curricular resources.

Despite previous research suggesting that many teachers feel that SMP 6 appears in each mathematics lesson they prepare (Davis et al., under review) only 15 out of 34 MSMTs listed SMP 6 as appearing in the SLP curricular resources they were given or their intended curricula. Altogether MSMTs’ mentioned precision with regard to communication, accuracy, measurements, labels, reading/creating graphs, calculations, and gathering data.

Several MSMTs described the presence of SMP 7 in their SLP curricular resources. For instance, Dietrich (DM) wanted to draw students’ attentions to the structure of percent being compared to 100 in the CPM1 lesson. Similar to the CCSSM architects, MSMTs connected structure with pattern identification (SMP 8). This is perhaps best seen in Christiansen’s (DM) examination of the CM – LF curricular resource where she perceived the curricular resources as providing opportunities for students to identify the structure of constant slope within a linear function as embodied in tables, graphs, and equations.

SMP 8 expects students to look for and express regularity in repeated reasoning. MSMTs noted that students would have opportunities to identify the y-intercept and slope of linear functions appearing in tables, graphs, and equations either in the written or intended curriculum.

SMPs Perceptions by MSMTs’ District-Adopted Curriculum

We found that 63% of SMP justifications made by teachers using a DM curriculum type and 85% of SMP justifications made by teachers using a TD curriculum type were valid. There was a statistically significant difference in the number of invalid SMP justifications for TD and DM curriculum types $t(34) = 2.41$, $p = .022$. Regardless of curriculum type MSMTs struggled with correctly justifying SMP 4 and SMP 5. For TD teachers these were the only two SMPs in which they had incorrect justifications. In SMP 1, DM teachers only drew attention to making sense of problems and not to perseverance. Several DM teachers confused complexity with abstraction in SMP 2. For instance, one of the lessons (CM – P) asked students to determine which of four different orange juice mixtures was most orangey and least orangey. Martin (DM) stated that this problem involved SMP 2 and justified the practice in this way, "For number two, with reasoning abstractly and quantitatively, now we're getting into, are they going to think outside the box on certain things as far as 'How am I going to get to what's most orangey or least orangey?'" Interestingly, none of the TD teachers identified SMP 2 in the SLP curricular resources. In SMP 3, DM teachers described checking answers for correctness as an engagement in the development of an argument. Both TD and DM teachers incorrectly justified SMP 4. TD teachers focused on the presence of multiple representations of a function (e.g., graph), but did not connect these to real-world contexts. DM teachers' incorrect justifications in SMP 4 involve the presence of multiple representations or a real-world context. DM and TD teachers both incorrectly asserted that the presence of tools such as a table was evidence of SMP 5. In SMP 6, one DM teacher (Shaw) stated that when students were learning a new method for solving a problem, students did not need to be precise in their work. In SMP 7 and 8 there was not sufficient detail to determine the validity of the justifications of the same two DM teachers (Cartwright and Tyler). Additionally, only 45% of the DM teachers identified SMP 1 and only 35% of DM teachers identified SMP 3 across the CM – P and CM – LF lessons.

A total of eight TD MSMTs were engaged in an SLP for the Glencoe – P or Glencoe – LF lessons. In all of these cases, the teachers saw no indication of the SMPs in the lessons. Granville (TD) summed up her evaluation of the Glencoe – P materials in the following way, "I mean what they say is that the aspects of mathematical thinking, practices 1, 3, and 4 are emphasized in every lesson. I just had a really hard time imagining the way this seemed to play out that they were doing any kind of engaging in any of the practices." Consequently, all eight of the teachers made significant adaptations to their DM curricular resources. The eight TD teachers began their intended curricula by taking problems from the DM curricular resource that were presented as being solved in one way (e.g., table) and providing them to students without an expected solution method. They felt that without presenting a particular method, these problems better embodied SMP 1 as they were less leading, more investigative and would be complex for students to solve. These TD teachers also provided students with a variety of different tools for students to use to solve these problems, which they felt was better connected to SMP 5 due to the fact that students had to choose which tool they would use to solve the problem. Another common theme in the intended curricula among these teachers was the use of cooperative groups whereby students would be expected to solve the problems together and engage in argumentation as they justified their solution methods, thereby engaging in SMP 3.

Discussion and Implications

This study examined a group of MSMTs' perceptions of the eight SMPs as well as how those perceptions differ by MSMTs' district-adopted curriculum type. On the one hand, our findings suggest that the MSMTs we sampled are able to correctly identify SMPs 1, 2, 3, 6, 7, and 8 in a variety of curricular resources or their intended curricula. On the other hand, MSMTs' perceptions about SMPs 4 and 5 were problematic. Teachers using both DM and TD curriculum types did not identify the connection between a real-world context and the mathematical representation or the need to translate from the mathematical representation back to the real-world context. They also did not mention other aspects of modeling such as determining what aspects of the real-world situation should be included in the model and which should be discarded. The MSMTs we interviewed tended to focus on just one action associated with modeling. In SMP 5, MSMTs considered the mere presence of a tool such as a percent ruler to be sufficient for students to gain proficiency with this practice. That is, they were concerned less about what tools were appropriate for a given situation and what it meant to use those tools strategically.

We assert that MSMTs' difficulties with SMP 4 and 5 are connected to the issue of learning trajectories involving these practices. It is not only important to determine what a particular SMP looks like at a particular grade level (Heck et al., 2011), it is important to understand what types of knowledge and skills comprise an SMP, when those should best be taught, how those skills are sequenced, and what curricular resources embodying these activities look like. For example, we would expect that an initial step in developing competency in SMP 5 would involve understanding how to use particular tools. However, as several MSMTs in our study noted in their intended curricula, students need to be asked to complete tasks where they must choose which tool is best suited for the task and justify that use of tools. As we stand at these CCSSM educational crossroads, an important next step in supporting teachers in bringing these standards to life involves providing professional development for teachers in these two SMPs, articulating a set of competencies associated with these skills, and developing curricular resources that embody these skills set within a reasoned trajectory.

We found that the DM teachers in our sample were less successful in identifying SMPs in their curricular resources or their intended curricula than the TD teachers we interviewed. This suggests that the understanding of SMPs exhibited by the DM teachers in our sample is different from the understanding of the SMPs held by TD teachers we interviewed. Consequently, we would expect that DM teachers' classrooms would provide fewer opportunities for students to engage in the SMPs for two reasons. First, DM teachers may experience difficulty in ascertaining when these practices occur in their curricular resources. Second, as the TD teachers demonstrated when they engaged with DM curricular resources, these materials may simply not provide students with many opportunities to engage in the SMPs.

TD teachers engaged in SLPs involving DM curricular resources demonstrated that invigorating these materials with SMPs is not an easy task. In the eight cases where TD teachers were engaged in this work, their intended curricula bore only a slight resemblance to the DM curricular resources from which they were drawn. If educational policy advocates wish to take the development of students' SMP proficiency seriously or as the CCSSM implores, connect content to the SMPs, our study suggests that not all types of curricular resources are created equal. Indeed, bringing the SMPs to life in the classroom may require that we seriously consider the adoption of TD curriculum resources and the concomitant professional development they require as we reflect on our location at the CCSSM crossroads and where we go from here.

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MIDDLE SCHOOL MATHEMATICS TEACHERS' USE OF CCSSM AND CURRICULUM RESOURCES IN PLANNING LESSONS

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As part of a larger study, we report findings on teachers' use of the Common Core State Standards for Mathematics (CCSSM) and teacher resources (TR) that were included with teachers' published curriculum programs. We analyzed 147 lesson planning interviews with 20 middle school teachers to understand how teachers interpreted and enacted the CCSSM while working with their curriculum materials. We investigated teachers' noticing of CCSSM and features of TR in planning lessons. Regardless of curriculum, teachers perceived that the lessons were designed to address the CCSSM. Findings for patterns among curriculum type, teacher orientation, and teachers' noticing are presented. Implications for curricular policy and design are discussed.

Keywords: Curriculum, Curriculum Analysis, Instructional Activities and Practices, Middle School Education.

The purpose of this study was to explore patterns related to teachers' orientations to instruction (Remillard & Bryans, 2004), teachers' uses of district-adopted curriculum programs (i.e., *the designated curriculum* [Remillard & Heck, 2014]), and specific curricular features teachers noticed (Jacobs et al. 2010, 2011) as they used teacher resources to plan lessons (i.e., *the intended curriculum*). These lessons – and the designated curriculum – were ostensibly aligned with the Common Core State Standards for Mathematics (CCSSM) (i.e., *the official curriculum*) (Remillard & Heck, 2014). The CCSSM (CCSSI, 2010) were initially adopted by 45 states plus the District of Columbia, and, despite a rollback in some states, the CCSSM or CCSSM-based standards are still in place in most states. Thus, the CCSSM-adopting states share a relatively common articulation of content and the progression of content across the grades. This provides researchers an opportunity to consider how districts and teachers interpret standards and to understand the role of curriculum materials in the process of enacting those standards.

When asked to compare the CCSSM with prior standards, teachers interpreted the CCSSM as requiring a greater emphasis on problem solving, discovery, communication, and conceptually-driven instruction (Roth McDuffie et al., 2015). Although teachers expressed a relatively strong view of these CCSSM features, prior research on teachers' enactments of similar recommendations in the National Council of Teachers of Mathematics Standards documents (NCTM 1989) showed that even reform-minded teachers did not tend to implement the recommendations beyond superficial features (Coburn et al., 2016; Spillane & Zeuli, 1999).

Framework

Our framework draws on three complementary perspectives: orientations toward teaching and learning mathematics, teachers' professional noticing, and types of curriculum programs. Each perspective is described briefly below (also see Roth McDuffie et al., 2017).

Orientations toward Teaching and Learning Mathematics

We see teachers as designers as they work with and enact curriculum across a range of classrooms contexts (Brown, 2009; Remillard & Heck, 2014). Productive enactments and adaptations of curriculum materials, desired outcomes of the design process, are responsive to local contexts and involve teachers noticing students' mathematical thinking in relation to curriculum resources (Choppin, 2011). However, most adaptations of high cognitive demand tasks cause the cognitive demand to decline to procedural routines (Stein et al., 1996). Thus, how teachers use materials can limit learning opportunities for students; however, others have pointed as well to curriculum materials as limiting factors (Stein et al., 1996). Thus, *which* curriculum materials are designated for use and *how* teachers enact materials can both affect student learning and achievement (Stein et al., 2007; Tarr et al., 2008). In regard to teachers' use of materials, teachers' orientations toward curriculum materials influence how the materials are enacted (Remillard & Bryans, 2004). Remillard and Bryans describe teachers' *orientation toward curriculum materials* and its relationship to learning as,

> A set of perspectives and dispositions about mathematics, teaching, learning, and curriculum that together influence how a teacher engages and interacts with a particular set of curriculum materials and consequently the curriculum enacted in the classroom and the subsequent opportunities for student and teacher learning. (p. 364)

To classify teachers' orientations, we turned to Munter, Stein, and Smith's (2015) two instructional models of instruction, *dialogic* and *direct*. Munter and colleagues' characterizations of primary instructional patterns in US mathematics classrooms represent a consensus view from a group of expert stakeholders, and they describe nine characteristics associated with each model. *Dialogic instruction* entails teachers providing students with opportunities to: wrestle with big ideas, assert and justify claims, and engage in carefully designed, high cognitive demand tasks (cf., Stein et al. 1996). Teachers engage in practices including orchestrating rich class discussions, introducing representations that can be used repeatedly in different situations, and sequencing activities in ways that position students as autonomous learners (Munter et al., 2015). Dialogic instruction is consistent with visions for effective teaching and learning espoused by NCTM (NCTM, 2014) and seminal research in mathematics education (e.g., NRC, 2005; Stein et al., 2007). Although both dialogic and direct instruction reflect a commitment to students' understanding of mathematics, *direct instruction* aligns with an acquisition approach (Sfard, 1998). Teachers maintain primary intellectual authority (along with the textbook) by: presenting an objective for a lesson, demonstrating how to complete problems, scaffolding students' practice, and evaluating to correct students. To engage students, teachers maintain a brisk pace, invite unison responses, and praise correct responses (Munter et al., 2015).

Given that meaningful and authentic problem solving, sense-making, and explaining and justifying solutions are emphasized in the CCSSM's Standards for Mathematical Practice (MPs), then it seems that the CCSSM align with a dialogic model. Yet, the CCSSM are ambivalent on pedagogical approaches (McCallum, 2012). On one hand, the MPs align with the characteristics and goals of dialogic instruction; on the other hand, due to the major gaps in empirically developed learning trajectories in key middle grade topics (Daro, Mosher, & Corcoran, 2011), the middle grade content standards are based on the logic of the discipline as much as they are framed by developmental and reasoning-focused approaches. Thus, the CCSSM leave room for teachers to attend to, interpret, and enact the content standards and MPs in various ways.

In contextualizing research on teaching in a broader system, we turned to Remillard and Heck's (2014) model describing a system for curriculum policy, design, and enactment, as described above, with a focus on: *official curriculum* (e.g., CCSSM and/or other policy documents); the *designated* *curriculum* (plans and curriculum materials authorized by local educational authorities) and *teacher-intended curriculum* (interpretations and decisions in planning). We considered how teachers used and worked between the CCSSM as an official curriculum and their designated curriculum to develop teacher-intended curriculum.

Teacher Noticing in Teaching and Learning Mathematics

An emerging body of research on mathematics teachers' *noticing* supported us in studying how teachers construct an intended curriculum and then enact curriculum (Jacobs et al., 2010, 2011; Mason, 2011). Although researchers have framed noticing in slightly different ways, a commonality is that noticing involves not only the attention that teachers give to classroom actions and interactions, but also teachers' reflections, reasoning, decisions and actions. Jacobs and colleagues defined *professional noticing of children's mathematics thinking* as consisting of a set of three interrelated skills: attending, interpreting, and deciding how to respond (Jacobs et al., 2010, 2011). Jacobs and colleagues argued that *deciding to respond* should be included as part of noticing because it is linked to the other skills of professional noticing (attending and interpreting) "during teachers' in-the-moment decision making." Jacobs and colleagues (2011) view the three skills of attending, interpreting, and deciding to respond as "inextricably intertwined" (p. 99), and we share this view. In forming our analytical framework to investigate teachers' work with curriculum, we adapted research on teacher noticing (Jacobs et al., 2010, 2011) to include curriculum as an object of noticing. Other researchers independently have begun to use a framing of curricular noticing in studying prospective teachers as they learn to work with curriculum materials (*c.f.,* Males et al., 2015).

Types of Curricular Programs

We conceptualize curricula according to monologic and dialogic communication functions (Wertsch & Toma, 1995). We characterize curriculum programs as *delivery mechanism* (DM), if they are designed from the monologic function, in that the content is developed from the perspective of expert performance, to be delivered to novices. In contrast, *Thinking Device* (TD) curriculum programs emphasize the dialogic function so that the primary goal is to elicit student thinking and to provoke interactions that generate understanding. In previous work for the larger study, the curriculum programs used by the participating teachers were analyzed and classified according to these two types (Choppin et al., 2016). The above perspectives framed the study and served as the foundation for our analytic frameworks, as described in the next section.

Methods

From our larger data set, we purposefully selected 20 teachers using four different curriculum programs, with two TD programs and two DM programs. We applied qualitative methods of analytic induction and constant comparison (Bogdan & Biklen, 2007; Miles, Huberman, & Saldaña, 2014) to identify patterns and themes regarding teachers' use of the CCSSM and the teacher resources (TR) that are provided in teachers' designated curriculum materials in planning lessons. The research questions driving the study were: (1) In planning lessons, what do teachers notice in CCSSM and in TR?; and (2) How do types of curriculum materials and teachers' orientations relate to teachers' noticing during planning?

Data Sources

From our larger project data, we selected four districts with curriculum programs of different types. From these districts, we selected 20 teachers who participated for at least one year, so that we had a representation of each of the middle grades (grades 6 to 8) and teaching experience (from first-year to over 20 years). All participating districts and teachers stated that they were implementing the CCSSM in their instruction. Data sources included 147 interviews: pre- and post-lesson interviews that focused on teachers' planning with their designated TR, and interviews as teachers planned a lesson with materials that were different from their designated curriculum (using contrasting resources). We collected data over three academic years from Fall 2012 (start of Year 1) to Spring 2015 (end of Year 3), and districts participated in either two or three years of the project, with three to four interviews conducted each year with each teacher participant (see Table 1). The classification shown for each curriculum is based on prior analysis (Choppin et al., 2016).

Table 1: Teachers, Designated Curricula, and Curriculum Type

District	Teachers (with # of Interviews per Teacher)	Designated Curriculum Program (by Year of Study)	Curriculum Type
Anna	Anderson (6), Cartwright (3), Dietrich (7), Martin (3), Shaw (6)	Digits (Fennell, 2010) (Y2, Y3)	DM
Chester	Allen (9), Granville (6), Menard (7), Pless (11)	Connect Mathematics Project (CMP, Lappan et al., 2014), CMP2 (Y1), CMP3 (Y2,Y3)	TD
Denton	Amedon (4), Blackburn (12), Gagnon (10), Gates (9), Hastings (7), Leonard (12), Sprague (6)	Glencoe (Carter et al., 2013) (Y1, Y2), CMP3 (Y3)	DM (Y1, Y2) TD (Y3)
Sanders	Boris (8), Gryder (6), Pearle (8), Ross (7)	CPM Mathematics (Kysh et al., 2013), (Y2, Y3)	TD

Data Analysis

We analyzed data through iterative cycles (Miles, Huberman, & Saldaña, 2014). Initially, using qualitative data analysis software, we coded data with a set of broad codes related to the larger project. For this study, we focused on data coded as "teacher resources" and "planning." We then ran reports to gather all data with these codes for the 20 teachers. We conducted finer level coding of these reports for instances of: (1) evidence of dialogic or direct orientations, applying Munter et al.'s (2014) nine characteristics; and (2) curricular noticing of the CCSSM (e.g., content standards, mathematical practices) and features of the TR (e.g., lesson structure, suggested questions, example problems, student approaches). We generated analytic memos for the participants to describe patterns and conjectures and compile data associated with these patterns. To examine patterns across participants, we created a matrix with rows for each participant and columns for foci of noticing and orientations, as described in the codes above. Within each cell we recorded findings for each teacher and then examined patterns and differences by curriculum program and by curriculum type.

Results

We categorized teachers into one of four categories based on teachers' designated materials and orientations evidenced in planning (see Table 2). For a few teachers, identifying orientation was not as clear as for most. For example, Pearle predominately demonstrated a dialogic orientation when she explained that she focused on "big problems and not just your memorization or your simple computation, like the math that I grew up [doing]" and on writing to support thinking. However, Pearle planned to introduce new vocabulary by presenting it to students at the beginning of the lesson (a direct orientation). In these cases, we classified based on the predominant orientation, with no more than one characteristic aligning with the other orientation. Teachers' orientations were consistent in planning with both their designated materials and with the contrasting materials provided in the interview. We identified two primary patterns for orientation and type of designated materials: TD materials paired with dialogic orientations and DM materials paired with direct orientations. That is, for 17 of the 20 teachers, their orientations aligned with the design of their designated curriculum materials. For the remaining three teachers (who demonstrated a direct orientation and were using TD materials), they had previously used Glencoe (DM) and were in the first year of using CMP3 (TD). Their comments and planning indicated that they noticed ways CMP3 was different from Glencoe, but they continued to remain at the center of the lesson, hold authority for content, and prioritize procedures. For example, Gates stated, "I'm …struggling with [CMP because] kids do not get to the standard algorithm…As I say to [my students], I need you to do 145 divided by 7 and just do it with the old standard [algorithm.]"

Table 2: Teachers Categorized by Orientation and Their Designated Curriculum

District	Thinking Device Materials (TD)	Delivery Mechanism Materials (DM)
Dialogic	Allen, Amedon, Boris, Granville, Gryder, Leonard*, Menard, Pearle, Pless, Sprague* (10 Teachers)	(0 Teachers)
Direct	Gagnon*, Gates*, Hastings* (3 Teachers)	Anderson, Blackburn, Cartwright, Dietrich, Gagnon*, Gates*, Hastings*, Leonard*, Martin, Shaw, Sprague* (11 Teachers)

*Note: Denton teachers changed from Glencoe to CMP during the study. Teachers marked with an * appear in two categories, based on the materials they used that year.

Next, we analyzed patterns for curricular noticing in teachers' planning with their designated TR and with contrasting materials (see Table 3). Within each cell, italicized phrases are the topics of noticing, and text that follows represents the primary and consistent themes for each form of curricular noticing (i.e., how teachers attended, interpreted, and decided to respond) within that category. As much as possible, we incorporated teachers' phrasing and terms to represent the theme (e.g., "big ideas", "key questions", "investigations", "inquiry-based", "talk through", "key steps"). In three of the four categories, teachers interpreted the TR as aligning with CCSSM; however, teachers with a dialogic orientation interpreted DM materials as not addressing the CCSSM. These teachers were planning with contrasting TR (Glencoe), rather than their designated TR (CMP3). Thus, all teachers viewed their designated TR as aligned with CCSSM, and yet their interpretations and decisions with the CCSSM and TR in lesson planning varied, as shown in the other cells.

As an example, Allen (dialogic orientation) evidenced her noticing of *CMP3*'s TR features and planned to provide an initial, informal exposure to ratio as a way to develop understanding:

> [I want students to] understand what the numbers are, what they're there for, what they're being used as.…Do they know [what each part of the ratio refers to]?….I always feel like the [first] investigation, it's really just that informal exposure….I think being able to recognize different types of comparisons, what they might look like, how you might get them, I'm just kind of starting to set the stage for [understanding that] isn't [just one] type of comparison.

In contrast, Gates (direct orientation) noticed *Glencoe's* TR features by focusing on the steps she planned to demonstrate and how students will practice these steps.

> For example, 2+y=3, we all use the strategy of bringing down a railroad track and then doing whatever you do to the side…. So I have a process where I usually have them do a few problems with me,…and if I feel like they're okay, I have them do a few problems with a partner, and then if they're doing well, … I let [students work] independent[ly].

Table 3: Noticing Patterns in Planning with Designated vs. Contrasting Curriculum

Teacher Orientation	Curriculum Type	
	Thinking Device Materials (TD)	Delivery Mechanism Materials (DM)
Dialogic Teachers	*CCSSM and TR Alignment*: Interpreted materials as aligned with CCSSM	*CCSSM and TR alignment*: Interpreted materials as not aligned with CCSSM
	MPs: Attended to MPs in CCSSM and decided to feature these through open problems and investigations.	*MPs*: Attended to lack of focus on MPs, interpreted as limiting students' opportunity to learn, and decided not to not use or substantially adapt TR.
	TR Feature, Problems and Homework: Attended to and worked problems as students might to anticipate their thinking, strategies, and confusions (interpreting and deciding). Selected problems to align with big and with MPs (interpreting and responding).	*TR Feature, Problems and Homework*: Attended to problems and homework, interpreted as focused on skills and as not deep enough to induce reasoning, conjectures, and explaining. Decided to adapt or replace or only use in limited ways for practice.
	TR Feature, Lesson and Participation Structures: Attended to Launch-Explore-Summary (L-E-S) structure. For each phase considered key questions and approaches to engage students in productive struggle, communicating, and justifying (interpreted). Decided to launch the lesson with key questions and contexts, how to use cooperative groups, and how to facilitate a summary discussion.	*TR Feature, Lesson and Participation Structures*: Attended to the role the curriculum materials and the teacher played in presenting ("telling") students what steps to use to solve problems, with time for students to practice similar problems. Interpreted the heavy focus on whole group and practice as limiting students' development of understandings and engagement in MPs. Decided to substantially adapt or replace approaches from TR.
Direct Teachers	*CCSSM and TR alignment*: Interpreted materials as aligned with CCSSM	*CCSSM and TR alignment*: Interpreted materials as aligned with CCSSM
	MPs: Attended to MPs, interpreted as different from designated materials, decided not to use the TR approaches related to mathematical practices due to perceived time needed and/or needing to cover "basics" first.	*MPs*: If attended to CCSSM, attended to content standards (not MPs), interpreted as topics to be covered, and decided to cover standards by following TR (showing procedures and providing time for practice).
	TR Features Problems and Homework: Attended to inquiry-based approaches, interpreted as "overwhelming" for planning and students, and decided problems were beyond their students' capabilities. Decided to adapt or replace with practice problems.	*TR Feature, Problems and Homework*: Attended to problem sets as a first-step in planning, selected problems based to match students' current skills and to practice new content (interpreting and deciding).
	TR Features, Lesson and Participation Structures: Attended to the L-E-S	*TR Features, Lesson and Participation Structures*: Attended to examples to

	structure. Interpreted that students need more direct instruction and practice, viewed investigations as too challenging for students and requiring too much time. Decided to scaffold and model problems first and supplement to ensure that students had skills, procedures, and practice needed before attempting investigations.	model and problems to assign for individual seatwork and/or homework. Decided on examples, what to model, how to talk through the problem solving process, key steps to emphasize, and key cautions to share. Decided on errors to look for when students were practicing problems and ways to correct or prevent these errors.

Discussion and Implications

A growing body of evidence indicates that characteristics of curriculum impact teaching and learning (e.g., Stein, Remillard, & Smith, 2007; Tarr et al., 2008). Indeed, we found that teachers' orientations matched the type of curriculum they were using in most cases. For the three teachers whose direct orientation was different from the approach of their TD materials, they attended to differences in the curriculum approaches, but then discussed how they were "struggling" to plan lessons as TR suggested, and often supplemented with practice problems from past DM resources. This pattern and other findings above indicate that a TD curriculum can support teachers' dialogic orientations in planning and incorporating CCSSM (and especially the MPs). However, similar to past reform efforts, the CCSSM and curriculum materials can be interpreted and enacted in multiple ways (Coburn, Hill, & Spillane, 2016; Remillard, 2005; Spillane & Zueli, 1999). Teachers also might attend to differences and then decide to plan based on their past practices or past materials. Teachers need support (e.g., professional development, coaching) and time to enact TD lessons in ways that are consistent with goals for dialogic instruction. This study is a next step in understanding specific ways teachers notice and interact with different types of TR and with CCSSM. This can inform both curriculum developers in designing curriculum and teacher educators in preparing teachers to enact ambitious practices.

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THE CHANGING EXPECTATIONS FOR THE READING OF GEOMETRIC DIAGRAMS

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Students studying geometry at the secondary level are expected to read diagrams in different ways than those in elementary school. In this paper, we present an analysis of the changes in diagrammatic expectations by comparing the geometric diagrams found in Grade 1 U.S. textbooks with those in U.S. high school geometry textbooks. This work included developing and using a coding scheme that recognizes dimensions of reading a diagram geometrically, including the type of object represented, use of deduction, use of mental redrawing, interpretation of markings, and the necessity of the diagram. The way in which elementary and secondary students are expected to interpret diagrams was shown to change along several of these dimensions, posing potential learning barriers for students. We end our paper with a discussion of what our results mean for the learning of geometry.

Keywords: Curriculum, Geometry and Geometrical and Spatial Thinking, Elementary School Education, High School Education

An identical task with the same geometric diagram can be found at different grade levels with different expectations for interpreting the diagram (Dietiker & Brakoniecki, 2014). For example, in elementary school, a diagram of a quadrilateral with 4 apparent right angles is supposed to be identified as a rectangle, whereas in high school, the same diagram is expected to be interpreted as a quadrilateral that is not necessarily a rectangle. How are students expected to read information from geometric diagrams in mathematical tasks, specifically those found in textbooks? And how might these expectations change? In a study of the expectations of textbooks with respect to how students read geometric diagrams, Dietiker and Brakoniecki (2014) expand on Pimm's (1995) notion of *reading geometrically* and propose multiple dimensions of reading geometric diagrams. These dimensions, gleaned from analyzing the geometric tasks in multiple elementary and secondary textbooks (including traditional and reform curricula from multiple countries), represent distinct aspects of geometric diagrams that students are expected to pay attention to and interpret as they negotiate the meanings of mathematical tasks.

In this present paper, we report on our continuing analysis of textbooks to reveal how the expectation of diagrammatic reading changes as students progress through school. In particular, we compared the geometric diagrams found in Grade 1 U.S. textbooks with the diagrams of U.S. high school geometry textbooks in order to learn how different the expectations are. This work included developing and using a coding scheme that recognizes the dimensions of reading geometrically, which are described in detail in this paper.

We end our paper with a discussion of what our results mean for the mathematical learning of geometry. With evidence that students are expected to develop sophisticated ways to negotiate meaning from diagrams, we argue that within each of these dimensions, educators can craft opportunities for students to develop strategies for reading geometric diagrams to ease the transition from elementary to secondary school.

Framework

This study examines the mathematical content with regard to geometric diagrams within the *textbook curriculum*. The textbook curriculum is specifically limited to comprehensive written curricular materials that are published for use by teachers and students. Although the textbook curriculum has an impact on curriculum as enacted in classrooms, this analysis is limited to the content as it is interpreted by readers (i.e., the researchers) of the texts. For this study, *problems* include all textbook prompts (whether interrogatives or not), such as tasks, activities, and questions for which an expected response from a student is provided in the teacher edition, although withheld from students. Thus, worked examples (i.e., tasks that are completely solved within the student text materials) are not framed as problems.

An *expectation* of a problem is framed as a limiting condition with regard to a student's response of a question or task. For example, if an assumption from a diagram is necessary (such as interpreting an unmarked angle in a geometric diagram as a right angle) to get the expected answer provided in the textbook, then we argue that this assumption is a *diagrammatic expectation*. In any geometric diagram, there are many potential assumptions that could be made. We limit our definition of expectations to those that are required based on the given answers in the teacher textbook.

Methods

In order to learn how the expectations for reading geometric diagrams differ from elementary to high school, the teacher and student materials from four U.S. textbook series were selected for analysis, including two from first grade and two from high school. These grade levels were selected in order to demonstrate the change in expectations of diagram interpretation that students experience. The two elementary textbooks include the University of Illinois at Chicago's *Math Trailblazers* (2008, "MT") and the University of Chicago's *Everyday Mathematics* (2007, "EM"). Within these textbooks, we considered diagrams in the problems in all chapters focused on geometry, including topics such as shapes, volume, and symmetry. The two high school textbooks include the *CME Project Geometry* (2009, "CME") and *Prentice Hall Mathematics Geometry* (2004, "PH"). In these textbooks, we analyzed all diagrams for problems and questions in Chapter 1 in order to learn about the assumed expectations of geometric reading at the start of a formal geometry course in high school. In all textbook portions that were analyzed, we eliminated from analysis any problems for which the teacher edition listed an incorrect answer.

Due to the fact that the purpose of this work is to establish the expectations for how students interact with diagrams, only the diagrams that are part of a student task were analyzed. This does not include diagrams included in exposition or in worked examples, as students do not have to interpret or make decisions about these diagrams. Additionally, because we did not have the assessment materials for all four curricula, we restricted our analysis to lesson materials focused on learning new content.

Methods of Analysis

To describe the reading expectations of the geometric diagrams, we developed five overarching codes. The first describes how the reader is expected to interpret the diagram *as* something (e.g., a real life object or a representation of a set of objects). When analyzing our interpretation of geometric diagrams *as* something, we recognized multiple distinguishable characteristics that became sub-codes. Some tasks included geometric diagrams that were meant to be interpreted *as drawn* (as indicated in the task statement and answer). For example, a task that asks a reader to measure the diagram to make statements about the geometric object is expecting this reader to interact with the diagram as the geometric object under study. Another example is a task in which a student is expected to indicate (by drawing) a line of symmetry for a geometric object depicted in a diagram.

In analyzing those tasks that require a reader to interpret the geometric diagram as drawn, we recognized that some require an assumption of either a metrical or topological relationship condition by the reader. For these tasks, there is a positive consequence for making assumptions based on the diagram, and having skepticism toward the diagram is disadvantageous. For example, in the task shown in Figure 1, a reader needs to assume that points E, C, and B are collinear in order to get these answers (displayed in pink) correct.

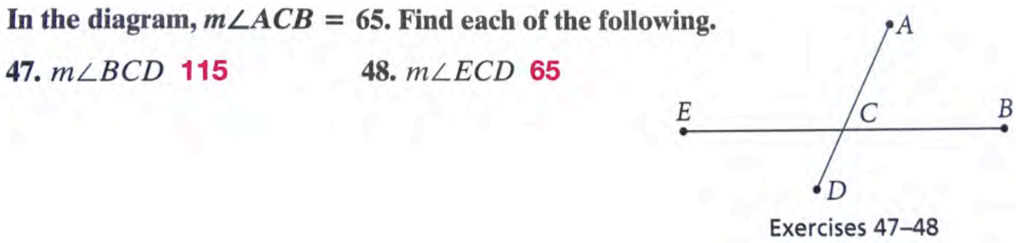
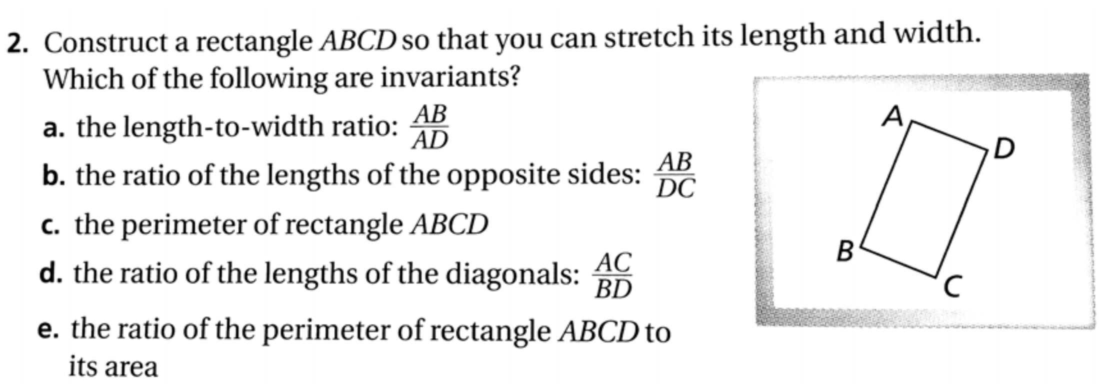


Figure 1. An example of *As Drawn With Necessary Assumption* from PH (2004, P. 31).

However, not all geometric diagrams are positioned by the text to be taken as drawn. Others are positioned in such a way that they are *representations* of an abstract geometric object (or a set of objects) and thus, a student is expected to <u>not</u> make assumptions of the geometric object based on the diagram. In these cases, a reader may be expected to read the diagram as a representation of a particular geometric abstraction when given a diagram that is not necessarily accurate. For example, for the diagram in Figure 2, which accompanies a prompt for students to determine the largest rectangle in the image, a reader would have a negative consequence if they assumed the angles of the rectangular characters were as depicted (which are not drawn as right angles because of the 3D orientation).

Figure 2. Example of *Representing with Assumptions* from MT (2008, p. 187).

Reading as *representing* also includes geometric diagrams that represent multiple geometric objects (read as a generality). That is, in some tasks, a reader is expected to recognize that a geometric diagram is a single representation of a *multiplicity*. Along with these, we note that some of these require a reader to make at least one additional assumption. The geometric diagram in the task in Figure 3 is an example of a diagram representing a multiplicity since a reader is expected to interpret the diagram as one of many.



2. Construct a rectangle *ABCD* so that you can stretch its length and width. Which of the following are invariants?

a. the length-to-width ratio: $\frac{AB}{AD}$

b. the ratio of the lengths of the opposite sides: $\frac{AB}{DC}$

c. the perimeter of rectangle *ABCD*

d. the ratio of the lengths of the diagonals: $\frac{AC}{BD}$

e. the ratio of the perimeter of rectangle *ABCD* to its area

Figure 3. Example of *Representing as Multiple* from CME (2009, p. 54).

Other tasks do not explicitly include what we commonly consider to be geometric diagrams, but instead include an image of a real world object (such as the soda can in Figure 4) with the expectation that it will be *interpreted as a geometric object* (i.e., a cylinder).

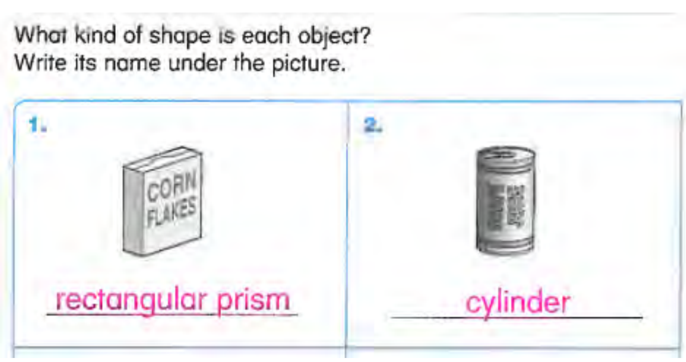


Figure 4. Example of *Real World Object* from EM (2007, p. 147).

Beyond the representations of geometric objects, we identified some geometric diagrams that are *not* representations of geometric objects. Some of these contain information that renders a geometric object as impossible or contradictory. For example, if a diagram of a triangle were marked with angle measures that do not sum to 180°, we interpreted that diagram as a *misrepresentation*. In addition, we coded geometric diagrams for which there is no expectation that a reader interprets the objects as geometric as *non-geometric*. For example, in a pattern problem with a string of triangles and squares, the students are not expected to interpret the objects as geometric. In fact, the use of the geometric shapes could easily be replaced with diagrams of flowers and firetrucks with no effect on the task.

The sub-codes for distinguishing how an object in a problem is to be interpreted and their interrelationships are represented in Figure 5.

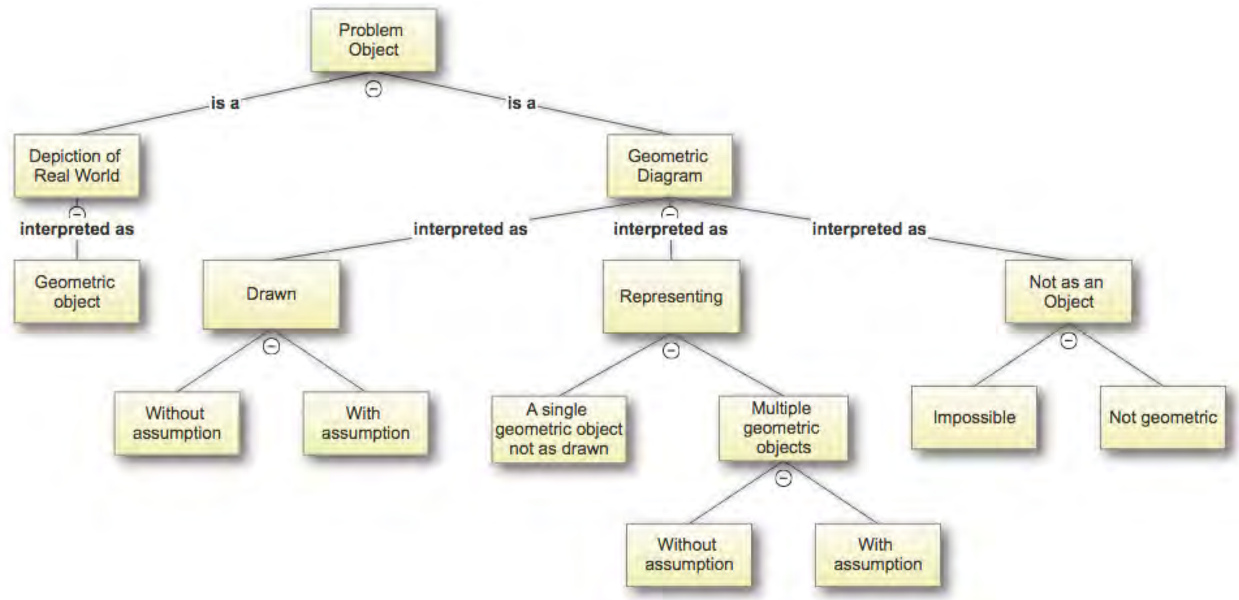


Figure 5. Diagram of sub-codes and their interrelationships for *Interpreting as*.

In addition to coding for the interpretation of the diagram, we coded four other dimensions of reading geometrically: whether *deductive reasoning* from the diagram is required to solve the task, whether the reader needs to *mentally redraw* the diagram to answer the task, whether the reader needs to *interpret conventional mathematical markings* to solve the task (e.g., reading the labels for the points in Figure 1), and whether *reading the diagram* is necessary to answer the task (e.g., the diagram in Figure 3 is supplementary while that in Figure 1 is necessary to solve the task).

Using this coding scheme, the three researchers analyzed each diagram from the selected portions of textbooks for the expectations of reading geometrically. These researchers include two mathematics educators and one doctoral student, of which two have high school teaching experience and one has extensive textbook design experience. Each code represents a consensus of all three researchers.

We suspected that there was a relationship between the intended grade of the textbook (elementary or secondary) and the various categories described above (the expected interpretation of the geometric object, whether deduction was necessary, whether mental redrawing was necessary, whether markings needed to be interpreted, and whether the diagram was necessary to the problem at all). To test the grade level's independence on each of these categories, we performed a Fisher's Exact Test (Fisher, 1922) between each grade level category and each of the above listed task categories to test the hypothesis that each of these categories was independent of the grade level of the textbook. Observed differences were statistically significant for $p < 0.01$.

Findings

The frequency of each type of diagrammatic expectation for textbooks of each grade level is reflected in Table 1.

When comparing how the textbooks expect students to *interpret as*, there was a statistically significant difference between the distribution of categories for elementary and high school texts ($p<0.001$). This enables us to assume that there is some dependence between the grade level of the textbook and the method of interpreting the diagram as an object. While the Fisher Exactness Tests indicates a likely dependence between categories, it does not identify specifically where the dependence exists. Thus, what follows is a summary of the more striking differences found in our coding results, highlighting where these differences likely exist.

Table 1: Frequency of Geometric Diagram Expectations

Task Expectation	Sub-Code	Elementary (n=61)	Secondary (n=156)	Fisher Exact P
Interpreting as	Real world as geometric object	5 (8.2%)	10 (6.4%)	0.000*
	Drawn	10 (16.4%)	51 (32.7%)	
	Drawn with assumption	36 (59.0%)	35 (22.4%)	
	Representation of single object	1 (1.7%)	4 (2.6%)	
	Representation of multiple	0 (0.0%)	45 (28.9%)	
	Representation with assumption	1 (1.7%)	6 (3.9%)	
	Impossible/contradictory	0 (0.0%)	1 (0.6%)	
	Non-geometric	8 (13.1%)	4 (2.6%)	
Using deduction	Required	0 (0.0%)	7 (4.5%)	0.195
	Not required	61 (100.0%)	149 (95.5%)	
Mentally redrawing	Required	0 (0.0%)	16 (10.3%)	0.007*
	Not required	61 (100.0%)	140 (89.7%)	
Interpreting conventional markings	Necessary	0 (0.0%)	81 (51.9%)	0.000*
	Supplementary	0 (0.0%)	26 (16.7%)	
	No markings	61 (100.0%)	49 (31.4%)	
Reading the diagram	Necessary	59 (96.7%)	123 (78.9%)	
	Supplementary	2 (3.3%)	33 (21.2%)	0.001*

Note. *Significant to $p < .01$.

Among the 61 diagrams of the elementary school textbooks and the 156 diagrams in the secondary textbooks, the most commonly expected interpretation of elementary textbook diagrams was *as drawn with assumptions*, with 59% of the diagrams in elementary. This means that a majority of the diagrams in elementary textbooks require students to interact with the diagram as the geometric object and that the student needs to make assumptions about measurements (such as a perceived right angle or a relationship between lengths) or properties (such as whether sides are parallel) based on how the diagram looks. Interestingly, high school textbooks also contain diagrams with this expectation, although they occur less frequently (22.4%). Instead, the most common expectation in the secondary diagrams is to interpret a diagram *as drawn* but without assumptions, which occurs in 32.7% of that grade level's diagrams, in contrast to 16.4%, as found in elementary textbooks.

Another noticeable difference between the grade levels' expected interpretations of diagrams was found for diagrams that *represent multiple* objects. No elementary school problems required students to interpret a diagram as a representation of multiple objects. In contrast, this was the second most frequently expected interpretation of the secondary diagrams, required for 45 (28.9%) of them. This suggests that high school texts expect students to know how to interpret a geometric object in a diagram as a general (rather than particular) representation at the start of a formal geometry course.

The geometric diagrams that *did not require the interpretation as a geometric object* were more found more often in elementary textbooks (13%) than secondary textbooks (3%). In addition, although there were relatively few instances of interpreting a diagram as *a single representation* overall, with only 5 diagrams in total, this occurred more often in the secondary textbooks (4). Although we expected to find more instances of diagrams depicting *real world* objects as geometric objects in elementary textbooks, the frequency of these diagrams was surprisingly similar in both grade levels (8.2% in elementary, compared to 6.4% in secondary).

Among the other categories of analysis, several also showed significant differences between elementary and secondary problems. A statistically significant difference was found when considering whether or not the diagrams needed to be *mentally redrawn* to solve the task ($p<0.01$). In the elementary textbooks, this expectation was not found. However, of the diagrams in the secondary textbooks, approximately 10% required a reader to visually manipulate a geometric object in order to solve the problem. Examples of these problems included tasks that require students to visualize what would happen to a geometry object if a vertex were dragged or how a diagram might change if a particular edge varied in length.

Another statistically significant difference was found when we compared the need to *interpret markings* of elementary diagrams versus those in high school diagrams ($p<0.001$). In the two elementary school textbooks, not a single diagram included any markings (right angle, congruent segment length, point name marking, etc.). This is in contrast to the high school textbooks' diagrams, of which almost two-thirds (68.6%) contain conventional markings. Of these, the majority required the interpretation of markings to solve the task (75.7% of those with markings, or 51.9% of all secondary diagrams). The remaining 16.7% of the secondary diagrams that contained conventional markings included a text prompt that supplied the information conveyed by these markings, rendering the markings in the diagram supplementary.

Lastly, we found a statistically significant difference ($p<0.001$) between the grade levels as to whether a student is expected to *read a diagram*. The diagrams in the elementary textbooks were almost always necessary to solve the task (96.7% of the time), in contrast to the high school texts which more frequently included diagrams that were supplementary to the task (21.2% of the time).

In contrast to the statistical differences described above, there was not a significant difference between the elementary and secondary diagrams regarding *using deduction* to solve a problem based on a diagram. None of the diagrams in the elementary textbooks require deduction and less than 5% of the diagrams in the high school tasks do so. In the elementary texts and opening chapters of the high school texts, it is almost never necessary for students to deduce a piece of information about a diagram which then needs to be used to learn additional information about that same geometric object.

Discussion and Implications

In this paper, we provide evidence that at some point in the transition from elementary school to the beginning of high school, there is a shift in expectations of how students are expected to read diagrams. As they start school, students are typically expected to make geometric assumptions based on how a diagram appears without being explicitly told about relationships that are necessary to solve a problem. By the time these students enter high school, they are expected to be able to reason about an object using only the information they are explicitly told and to not make assumptions based on how a diagram appears. This change in geometric interpretation of a diagram is consistent with van Hiele's (1959) description of sophistication of geometric understanding; younger children are expected to interpret geometric diagrams as a whole and only later begin to recognize the properties of geometric objects and their interrelationships. If a student does not recognize that a right angle is a property of a square, for instance, then marking right angles of diagrams of squares is pointless. Thus, it is sensible that textbooks for young children would contain the expectation that geometric diagrams be interpreted based on features as drawn (i.e., it looks like a square, therefore it must be a square).

However, we found it surprising that high school students are still expected to interpret geometric diagrams as drawn. Since high school geometry includes formal proofs, for which students are typically expected to reason only from given statements, it appears that students are expected to recognize and distinguish when they are able to make assumptions based on a diagram and when they are not. Even when students are not expected to assume metrical properties (e.g., an angle is a right angle just because it looks like a right angle), the students are expected to assume topological properties (e.g., if it looks like the figure is closed, it is). We wonder how students learn to distinguish when it is "okay" to make assumptions from diagrams and when it is not.

Interestingly, there was one aspect of reasoning with geometric diagrams that was not shown to be statistically significantly different from elementary to high school, which was whether diagrams required deductive reasoning. We expect that had we analyzed subsequent chapters in the high school textbooks, especially chapters in which students are asked to prove properties of geometric figures, that there would be more diagrams that require students to deduce new information about a geometric object from a diagram. Thus, based on this analysis, this shift may occur within the geometry course in high school.

Among all these dimensions in which reasoning about diagrams is expected to change, we wonder how aware curriculum authors and teachers are of these changes, and in what ways (if at all) these changes are communicated to students. We suspect that some of students' difficulty with geometry may be at least partly due to an inability to successfully navigate the implicit expectations of reading of geometric diagrams and we believe that helping students recognize the multiple roles that diagrams can play in geometry and mathematics is critical for their success.

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A CURRICULUM-BASED HYPOTHETICAL LEARNING TRAJECTORY FOR MIDDLE SCHOOL ALGEBRA

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Research on learning trajectories in mathematics has grown considerably over the past decade. In this paper, we contribute to this body of research and present a curriculum-based hypothetical learning trajectory for middle school algebra. In doing so, we make the visible the process by which we developed this initial hypothetical learning trajectory, highlighting the considerations, decisions, and challenges we faced as part of this work.

Keywords: Learning Trajectories, Algebra, Curriculum Analysis

Background

Learning Trajectories in School Mathematics

A variety of definitions of the LT construct exist in the research literature, with substantial differences in focus and intent (see e.g., Clements & Sarama, 2004; Corcoran, Mosher, & Rogat, 2009; Confrey, 2008; Simon, 1995). According to the Consortium for Policy Research in Education (CPRE) Report on Learning Progressions for Mathematics (Daro et al., 2011), LTs are empirically grounded and testable hypotheses about how, with appropriate instruction, students' understanding of, and ability to use, core concepts and explanations and related practices grow and become more sophisticated over time (National Research Council, 2009). These hypotheses describe the pathways students are likely to follow to develop mastery of core concepts. Specifically, in our work we ascribe to the definition of LT proposed by Confrey et al. (2008): A researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representations, articulation, and reflection, towards increasingly complex concepts over time. We further ascribe to the idea of a conceptual corridor (Confrey et al., 2009), which incorporates the possibility of multiple pathways toward learning, as well as attention to the landmarks and obstacles that students typically encounter along those pathways.

Approaches to Learning Trajectory Development in Mathematics

In the mathematics education community, LT researchers differ in how they conceptualize LTs, including the grain size of descriptions of levels of student thinking, how students move among the levels, and ways in which LTs are validated (Confrey, Maloney, Nguyen & Rupp, 2014). Such conceptual differences also include different approaches to LT development. For example, Clements and Sarama (2014) begin with mathematical goals or big ideas that are derived from both empirical research and expertise from mathematicians. From the mathematical goals, they develop cognitive models of student thinking based on theoretical and empirical models, and then move to developing tasks that are designed to promote student learning at different levels. Similarly, Lehrer, Kim, Ayers and Wilson (2014) begin with descriptions of student thinking that are based on empirical studies, extant research and mathematical expertise, which are coordinated into different levels of knowledge. From these descriptions, Lehrer et al. (2014) develop means of supporting changes among knowledge levels, to support student thinking, including instructional activities. Still, Confrey et al. (2014) begin with a synthesis of relevant research together with clinical interviews with students and teaching experiments in classrooms. While such approaches offer different insights into pathways of student thinking, such approaches often do not include the tools and resources teachers use everyday to plan for and enact instruction, such as curriculum materials, as a central component of the LT conceptualization. Indeed, teachers regularly use curriculum materials to determine the mathematics they are going to teach, which has considerable implications for pathways of student thinking.

An Alternative Approach to Learning Trajectory Development

In our work as part of the iFAST Project, we use mathematics curriculum materials as the starting point for articulating a hypothetical LT. Briefly, the iFAST Project is a multi-year project focused on articulating LTs in middle school algebra to inform the design of LT-based professional development for teachers. Thus, two main components of our work on the project involves understanding students' learning pathways within middle school algebra, and enhancing teachers' understanding of LTs to inform their use of effective assessment practices in the classroom. A central premise underlying our work is that high quality formative assessment practices depend on teachers having a clear sense of learning goals, student LTs, criteria for locating students along the trajectories, sharing this information with students, and using it to inform instructional decisions.

The development of proficiency in algebra holds a unique role in students' success in mathematics, serving as a gatekeeper to more advanced mathematics and affecting mathematics achievement in high school and beyond. The Common Core State Standards for Mathematics has reconfigured the sequencing of algebra content across grade levels, introducing it in Grades 6 and 7 with a major focus in Grade 8, and calls for students to learn algebra earlier and to more advanced levels than has traditionally been the case. As a result, whether or not middle school mathematics teachers are teaching a course designated as Algebra 1, they are being held accountable for all students' learning of rigorous content related to strands in algebraic functions and equation-solving. In the iFAST Project, our learning trajectory work is centered on linear functions and linear equations topics in middle school algebra.

We focus on the Connected Mathematics Project 3 (CMP3) curriculum as it is widely used and the treatment of linear functions and equations topics is consistent with other functions-based curricula in the U.S. As our main focus is understanding students' learning pathways within CMP3, we needed first to generate a map tracing the hypothetical learning opportunities of algebra concepts embedded in the curriculum. Of course, this is only a hypothetical description as the actual learning opportunities students encounter are mediated by multiple other factors (e.g., school, teacher, implementation, etc.). Thus, our approach stands in contrast with other approaches in least two ways: it is specific to CMP3 and we generated a hypothetical learning map first. In the sections that follow, we describe the process we developed to generate such a map and make visible the process by which we developed a curriculum-based hypothetical LT for middle school algebra.

Articulating a Hypothetical Learning Trajectory

Initial Considerations

As we embarked on understanding what students seem to learn and what are the remaining obstacles, the need to understand what learning pathway was intended for students to follow in the curriculum became evident. In order to map the opportunities provided by the curriculum to learn about specific algebra concepts we had to devise a process to understand and represent them. We wanted to produce a map of such opportunities as presented in the curriculum, thus we decided to start working with the unit goals provided by the curriculum materials.

Our initial intent was to map all linear functions and equations related topics within CMP3. This task proved to be too ambitious. As such, we narrowed our focus to what we considered to be a high leverage topic within the linear functions and equations domain. We focus on the transition from proportional to non-proportional linear functions, with a particular emphasis on rule writing. Thus, we selected units (i.e., focus units) that focus on proportionality, functions, proportional functions and linear functions. Within CMP3, we selected the following units: Grade 6 – Comparing Bits and Pieces, Grade 6 – Variables and Patterns, Grade 7 – Comparing and Scaling, Grade 7, Moving Straight Ahead, Grade 8 – Thinking with Mathematical Models, and Grade 8 – Say It With Symbols.

The first challenge we faced in creating this map was to come up with a useful grain size to describe the new content that students were offered an opportunity to learn about. We first considered working directly with the goals corresponding to focus units, but a map spanning across Grades 6-8 turned out to be too busy and difficult to be used by other researchers and teachers. Therefore, we decided to group unit goals that could be linked by a common mathematical theme.

Generating a Curricular Map

This following process is presented in a distilled way but it took several attempts for it to emerge to address limitations and implement desired improvements. We will try to note some of these back and forth in the process of coming up with the final process of clustering. The first challenge we faced in creating this map was to come up with a useful grain size to describe the new content that students were offered an opportunity to learn about. We first considered working directly with the goals corresponding to focus units but a map spanning throughout the three years grades 6-8 turned out to be too busy and difficult to be used by other researchers and teachers. Therefore, we decided to group unit goals that could be linked by a common theme. We referred to this process as "clustering". A preceding crucial step was to assign to each lesson problem unit goals.

Clustering process. One focus unit at a time, three researchers first independently clustered goals into clusters; second, researchers identified discrepancies and discussed the clusters by looking both at the representative lesson problems and the goals as stated in the curriculum until an agreement was reached. We iterated this process throughout all focus units.

External validation of clustering process. Once the researchers had completed the clustering process for all focus units, we convened a group of external reviewers (comprised of mathematics education researchers with familiarity with algebra and CMP) who conducted an external validation of the clustering process. This work entailed: (1) assessing the relationship between lesson problems and assigned goals, (2) assessing how the goals were group together forming a cluster, and (3) assessing whether the lesson problems selected were considered representative. External reviewers agreed with most of our work, they provided some minor suggestions for us to consider. One of the main contributions of this round of feedback pushed our team to think about the representative problems in two different lights. One as an exemplary lesson problem where students have the opportunity to learn and laser in a concept vs. lesson problems that would afford the most learning opportunities in the limited time an after-school professional development workshop affords, for example.

Connecting Clusters Across Grade Levels

Thus far in the process, we had goals, clusters and representative lesson problems at the unit level without an explicit connection across units and grades. In connecting the clusters among themselves two unforeseen processes unfolded: (1) in order to be able to express successive refinement in a trajectory we found it necessary to refine the language of the cluster, and (2) we revised some of the clusters and re-grouped them. Several elements were used in coming up with the curricular map across grades. The main organizing element is time, clusters are organized in columns according to a specific grade moving from 6th grade on the left to 8th grade on the right. Within a specific column (grade), we followed the order of the units chronologically according to the curriculum. To organize clusters and decide how they relate to each other we made decisions by looking at the representative problems, the goals and clusters; we pay specific attention to successive refinement of a same concept.

Discussion & Conclusion

Generating a curriculum based hypothetical trajectories map was a necessary step in our approach, as we set out to describe students' levels of understanding on proportional and linear functions taking into account what learning opportunities were provided by the curriculum chosen by the district. We think that this might provide a more nuanced picture as it not only takes into account the common core standards and extant research but also the opportunities students had to learn from the chosen curriculum. The ultimate goal of the project is to generate a CMP3 curricular map with levels of understanding (i.e., ranging for most proficient to incorrect) for each cluster. We have designed a set of three instruments to do this: end of unit assessments, end of year assessment, and a pre and post-assessment. These instruments differ on multiple aspects from each other (e.g., constructed response/multiple choice, content specific/non-specific to a unit/grade, etc.) but a detailed description of those exceeds the scope of this paper. By putting in relation the CMP3 HLT Map together with the multiple data sources we will be able to identify, describe and illustrate both levels of understanding as expected by the layout of the curriculum as well as "out of trajectory" levels of learning. The "out of the trajectory" learning might occur given the spiral nature of cmp3 where concepts are revisited at several later instances across those three years. In putting forward this process, we hope other researchers can replicate it by researchers to generate a map for other curriculums.

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ALIGNING RESEARCH AND PARENT PERSPECTIVES OF MULTIPLE STRATEGIES

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In this research report, we examine the perceptions that parents have regarding the use of multiple strategies in their children's classrooms, and then align their perspectives with the research literature. The themes of adaptivity and flexibility of strategies, conceptual and procedural knowledge, disposition and identity, implementation of multiple strategies, and the lack of effectiveness for all students were found in the literature as well as within the parent comments. Some of the parents produced counter-examples from the literature based on their experiences with their own children. By understanding the varying notions that parents (and the literature) have about the use of multiple strategies in mathematics, we can begin to create a way to have productive conversations with parents about curriculum reform.

Keywords: Curriculum, Elementary School Education, Number Concepts and Operations, Problem Solving

Introduction

> But the fact is there isn't great research behind it. And the other fact is that … students who are coming out of learning these strategies—the math scores have weakened. Because I'm all for doing a different strategy if you have proof that that strategy is even better than what we've been doing all along. However, at the end of the day the scores indicate that, no, it hasn't been the best approach to teaching math.

The comment above describes a tension expressed by many parents regarding current pedagogical practices that emphasize multiple strategies for problem solving and computation. The general curriculum expectations are for students to have opportunities to develop procedural flexibility and conceptual understanding through the process of selecting and using appropriate strategies in novel situations (e.g., National Council of Teachers of Mathematics, 2000).

The benefits of a multiple strategy approach have often not been accessible to or understood by parents. While parents, such as the mother quoted above, may reject that this pedagogical approach is best, the literature itself is not universally supportive of multiple strategies. In this paper, we investigate parents' perceptions of the value of alternative strategies in supporting children's learning of mathematics and how these perceptions are aligned with research. Our ultimate goal in doing this analysis is to identify fruitful ways to engage in productive conversations about mathematics curriculum reform with parents.

This report relates to the conference theme of research "as an intersection" by creating a path for researchers, teachers, and parents to be on the same road or conversation. It also addresses the conference theme of research "as a barrier" by examining the discourse used by parents surrounding the use of multiple strategies in mathematics classrooms and the impacts that these perceptions have on the parents' thoughts and feelings about how their children are learning.

Theoretical Framework

Within the literature, the rationale for teaching students multiple strategies is multifold. First, it has long been believed that such an approach supports the development of strategy adaptivity; that is, the ability to flexibly and creatively apply or generate an appropriate solution strategy to solve a given mathematics problem (Hatano, 1982). Second, teaching students multiple strategies is intended to enhance conceptual and procedural knowledge of number (Rittle-Johnson & Schneider, 2014). An awareness of various solution strategies is often associated with deeper conceptual understanding of how these strategies are used (e.g., Verschaffel et al., 2009). Also, students who develop and use procedures flexibly are more capable of using and adapting existing procedures when faced with unfamiliar problems (e.g., Blote et al., 2001). A third rationale in the literature is that multiple strategies cultivate appropriate attitudes towards math. This potential influence on disposition has, for example, been documented in Boaler and Selling's (2017) longitudinal study of two contrasting mathematics teaching practices (project-based vs. traditional) and their impact on students' identity and expertise in mathematics during their adulthood.

Is teaching multiple strategies feasible and valuable for all students? First, some research studies suggest that a multiple strategies approach may exacerbate the difficulties of low achieving students. Poorer working memory or other learning difficulties suggest that the goal of developing student adaptivity might be of limited value. Auer, Hickendorff, and Putten (2016) found that "lower ability students made counter-adaptive choices between the two strategies" by choosing a strategy that led to inaccuracies (p. 52). Second, exposing students to multiple strategies early may promote positive dispositions, but delaying exposure to multiple strategies may lead to greater adaptivity (Rittle-Johnson & Schneider, 2014). Their study indicated that flexibility in use of procedures was higher than the students in the delayed-exposure condition. Third, as Silver et al. (2005) note, although it is well accepted that students need experiences solving problems in more than one way, it is difficult to operationalize. The possible obstacles math teachers face include the actual and perceived limitations in teachers' mathematical knowledge, limited instructional time, restrictive conceptions of student ability, and a lack of opportunity to develop instructional routines related to teaching multiple solutions.

The parents in our study expressed a range of benefits and also disadvantages of their own and their children's experiences with a multiple strategies pedagogy. Parent perspectives are important because their involvement in children's educational experiences has far-reaching benefits such as improving achievement, increasing motivation, and reducing anxiety (Pattall, Cooper, & Robinson, 2008). Understanding parent perspectives and how they are or are not aligned to research allow researchers and teacher educators to refrain from dismissing criticisms as obstacles, and reconsider the validity of parent perspectives—even if they are based on a sample size of 1: their child.

Mode of Inquiry

Forty parents from urban and rural communities participated in our study. They completed a demographic questionnaire and participated in one of ten focus groups taking place in their respective communities. Focus groups, each approximately two hours long, were used as generative sites of data collection with the knowledge that differing parent perspectives required participants to explain their perspective to others, thus allowing us to notice differences in their experiences. The focus groups were structured to prompt parents to address their observations of their child's learning and the curriculum; compare their own mathematics schooling with their children's; describe their interactions when helping their child with mathematics at home; provide expectations about their child's mathematics learning; and describe communication received with the school and teacher about mathematics curriculum. Follow-up interviews were conducted with a subset of the parents

(15) in the focus groups to allow them to expand on and further clarify their comments from the focus group.

A common topic of discussion identified in the focus groups were parents' perceptions related to multiple strategies and alternative algorithms (see McGarvey & McFeetors, 2016). In the focus groups and interviews, the parent participants referred to a multiple strategies approach over 400 times. In this paper, we focus specifically on this large sample of comments using a thematic analysis (Braun & Clarke, 2006) of the focus group data using individual utterances as a unit of analysis. Through a constant comparative approach in sorting statements made by the parents in the focus groups, we identified key assertions in relation to the participants' perceptions of the role of multiple strategies in their children's learning of problem solving and computation. In the results, we explicate the ways in which parents make sense of alternative algorithms in their children's learning.

Findings

In Table 1, we highlight the range of themes we identified related to parents' perspectives of multiple strategies along with sample comments.

Table 1: Parent Perspectives on Multiple Strategies

Theme	**Assertions from Parents**
Adaptivity and flexibility of strategies	The answer is because once problems get more complicated, you might not know them at first sight. So you have to learn all the different strategies, that's why I think they teach all the different strategies because once it's more complex, it's like algebra and things like that. You need to know different ways to get the answers, ways that makes sense to you. Wouldn't you rather be really good at one than kind of mediocre at five?
Conceptual and procedural knowledge	They wanted them to understand the concept of all of the strategies and then pick the ones that worked better for them.
Disposition and identity	My middle son with ADHD has his problems. He would never be able to memorize …. It takes him a little while because he's slower and needs to focus, but he can use the strategies that he likes, which is not the standard algorithm, but he can figure it out. If she has to use the 15 different methods, then she sort of spirals out [of control].
Implementation of multiple strategies	There's this disconnect that they [teachers] think that they have to teach all of those ways and they [children] have to do them all of those ways. And we have to give it to them over and over and over again. Even if it doesn't work for them. I wish the teacher had had a better answer to that question. And I wish the teacher had known why she was teaching it those ways. I don't know if she knows.
Not effective for all	You have all of these strategies and you know having your child draw 90

students (both low and high achieving)	ticks on, you know paper, that's where you're wasting your children's time … in four or five different strategies you never allow them to master a particular method. She's not a drawer, she hates art. We were focusing more time on drawing pizza cones and all these things that had to do for these worksheets that were coming home, and coming up with a strategy name, that we weren't actually doing math. If there's, you know, 75% of children learn the best how we learned it growing up, why change it? And if those 25% or 20% or 15 or 10% are having troubles, then let's implement those strategies for those children instead of for broad all.

Conclusion

We note that parents did not necessarily view a multiple strategies approach from only one perspective. Instead, they often included multiple perspectives within one statement. In addition, although we provided here both the benefits and challenges to a multiple strategies approach as described in the literature, parents often provided counter-examples based on their perspectives that are important not to ignore.

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ANALYSIS OF MATHEMATICS CURRICULUM MATERIALS TO ASCERTAIN STUDENTS' POTENTIAL TO DEVELOP AGENCY, AUTONOMY AND IDENTITY

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This brief research report outlines a study of College Preparatory Mathematics (CPM) student curriculum materials to determine how language features of mathematical tasks position students to develop aspects of agency, autonomy and identity as learners of mathematics. I make a case for why I think these three attributes are important for mathematics learning, and I use analytic tools from Systemic Functional Linguistics to explore opportunities students have to develop agency, autonomy and identity when working on mathematical tasks in CPM textbooks.

Keywords: Curriculum Analysis, Middle School Education

Purpose of Study

The study investigates ways that language used in selected College Preparatory Mathematics (hereafter, CPM) mathematical tasks positions students to experience aspects of autonomy, agency and a sense of identity as learners and doers of mathematics. Agency, autonomy and identity are important because research (e.g. Lester & Cai, 2016) seems to indicate that students should be given more opportunities to rely more on themselves as they work on challenging mathematical tasks. Challenging mathematical tasks often have a high level of cognitive demand (Stein et al., 2000) and are often non-routine. Whether students of mathematics operate individually or with other learners in groups as they work to solve non-routine problems during lessons, it can be argued that they are required to assume greater control of the problem-solving process, independent of their teacher. By taking greater control of their learning, students are in essence being autonomous and engaging in agentic learning. I am interested in the CPM curriculum because it is an instance of problem-based or problem-centered learning. This form of learning often involves learning mathematics by working on complex and open-ended problems which are typically ill-structured so as to allow for several possible approaches and answers (Hmelo-Silver, 2004). Problem-based learning can provide learners of mathematics with opportunities to develop agency, autonomy and identity as doers of mathematics. Studying language used in mathematical tasks in CPM textbooks can inform research on how students are positioned to develop agency, autonomy and identity as learners of mathematics when engaged in problem-based or problem-centered learning.

The following research question guides my analysis of CPM curriculum materials:

> In what ways are language features of mathematical tasks in CPM student textbooks likely to support the development of aspects of students' agency, autonomy and identity?

Theoretical Framework

Agency is the first component of my theoretical framework. Lipponen and Kumplainen (2011) state that "agency can be defined as the capacity to initiate purposeful action that implies will, autonomy, freedom, and choice" (p. 812). There is an action component to agency, which may be afforded or constrained by conditions determining an individual's ability to act. Cobb et al. (2009) further distinguish agency into conceptual and disciplinary varieties. Learners experience conceptual agency when they are allowed to come up with their own methods and understandings during learning while they experience disciplinary agency when they are able to engage in mathematical work that involves known procedures.

Autonomy is the second component of my theoretical framework. It can be thought of as “a capacity to take control of one’s own learning” (Benson, 2011, p.58). For Benson, control refers to control of cognitive processes, of learning content and of learning management. In my analysis of mathematical tasks from CPM curriculum materials, my focus is primarily on the capacity for students to take control of cognitive processes either as they work individually or collaboratively in groups. Taking control of cognitive processes depends on affordances and constraints determined by language choices used to guide students’ to work on mathematical tasks. As these affordances and constraints also determine the capacity to act, the notions of agency and autonomy are intimately intertwined. The difference between these two concepts is quite subtle as to make the two terms have practically one and the same meaning. According to Benson (2007), “…agency can perhaps be viewed as a point of origin for the development of autonomy, while identity may be viewed as one of its more important outcomes.” (p. 30).

The third component of my theoretical framework is the notion of identity. Cobb et al. (2009) differentiate between students’ normative identity and students’ personal identities. They define normative identity as “both the general and the specifically mathematical obligations that delineate the role of an effective student in a particular classroom” (p. 43). The normative identity can be shared across students with different personal identities. For my conceptual framework, I am primarily interested in opportunities indicated in mathematics texts for students to develop normative identity as doers of mathematics.

Methods

Data consists of selected mathematical tasks for classwork from the most recent versions of student texts for the Core Connections (CC) Integrated I, II, & III CPM curriculum materials that address the subject of functions and graphs. Functions and graphs are a ubiquitous topic in mathematics, encountered by students over multiple years. Learners of mathematics see functions and graphs in numerous ways across the mathematics curriculum of high school, and this topic has importance in the future study of mathematics at the college level and beyond. As such, if functions and graphs are taught and learned in a way that can promote students’ agency and autonomy then as students encounter functions and graphs again in the study of higher mathematics, they could activate and draw on their developed ability to be agentic and autonomous. The CC Integrated textbooks are designed to have each chapter include a number of sections. Each section encompasses several lessons. For instance, chapter one of CC Integrated I, ‘Functions’, is organized into three sections, all of which deal with the topic of functions.

This brief research report analyzes instances of communication and interaction between textbook authors and readers. These communications and interactions can be seen as forms of guidance transferred from the textbook writers to students when directing students on how to work on mathematical tasks. The guidance takes the form of textbook writers giving some information or demanding some action of readers. Information given may be in the form of mathematics theorems, definitions or some other form of theory. Information given may also be in the form of statements of particular affordances or constraints of a mathematics task. Information demanded may be in the form of answer responses required of students.

These forms of exchange (giving or demanding) are represented grammatically by three kinds of clause moods: declaratives (usually statements), interrogatives (usually questions) and imperatives (usually commands) made on the part of the textbook writers. Zolkower & Shreyar (2007) define a clause as “the smallest possible group of words within a text that has meaning” (p. 182). Clauses as instances of communication serve as the unit of analysis for this study. In particular, I am interested in analyzing imperative clauses appearing in mathematical tasks. They are an aspect of the interpersonal meta-function, an aspect of Systemic Functional Linguistics which involves interactions (Thompson, 2013). In this case, the interactions are between textbook authors and students. Imperative clauses task students to take various actions. In these actions, there can be opportunities for students to experience agency and autonomy in their learning. I analyze two kinds of imperative clauses: inclusive and exclusive (Herbel-Eisenmann, 2007). Inclusive imperatives can position students as thinkers and demand that students' responses reflect their thinking. These imperatives feature verbs such as "explain", "describe", "justify", and "predict". They can be found in the following typical example clauses taken directly from CPM textbook data "explain how you found your answer", "describe additional features in your own words", and "be ready to justify your statements". Exclusive imperatives on the other hand direct students to carry out actions that tend to be less about what students think and more about executing actions related to established mathematics procedures. Examples of verbs found in exclusive imperative clauses are "calculate", "draw", "make", "use", "plot", and "write". Some example clauses from CPM textbook data are "use your calculator", "draw a slope triangle", "sketch the graph of $h(x) = (x + 3)^2 + 4$", "make a table" and "write an equation".

Through the study of imperative clauses, it is possible to point out opportunities for agency and autonomy students are given to rely on their own thinking and creative resources as doers of mathematics. The degree to which students are directed through inclusive imperatives versus exclusive ones can also shed light on how textbooks position students to develop identities as mathematics thinkers.

Results

The following results are based on analysis of clauses in 245 mathematical tasks from 57 lessons on functions across the three textbooks.

Table 1: Verbs in Inclusive and Exclusive Imperatives

Inclusive imperatives	Frequency	Exclusive imperatives	Frequency
Explain	56	Write	47
Describe	23	Use	33
Justify	19	Sketch	29
Predict	6	Draw	23
		Graph/Plot	17
		Label	15
		Solve	10
		Calculate	9
		Make	9
Totals	**104**		**192**

Discussion

The results in Table 1 generally align with those obtained for inclusive and exclusive imperatives in Herbel-Eisenmann (2007). The results show that the CPM texts feature both inclusive and exclusive imperatives. Table 1 shows a wider spread and more instances of exclusive imperatives than of the inclusive kind. For the analyzed CPM mathematical tasks, it may be inferred from these results that students are being positioned by the text in these mathematical tasks to carry out a greater variety of actions that pertain to established procedures in mathematics. On the other hand, the relatively high incidence of the verb "explain" shown in Table 1 draws attention to inclusive imperatives. "Explain", and other verbs such as "describe", "justify" and "predict", all of which featured in analyzed inclusive imperatives can give opportunities for students to develop conceptual agency, autonomy and their normative identity as they engage in thinking about and doing mathematics while working on mathematical tasks during lesson time. As such, this finding emerging from analyzed inclusive clauses can be seen as a language feature of mathematical tasks on functions in CPM student textbooks that positions students to express their conceptual agency, autonomy in their thinking and which can subsequently influence students' development of normative identity as a doer of mathematics.

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CONCEPTUAL DEFICITS IN CURRICULAR INTRODUCTIONS TO MULTIPLICATION

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This paper presents evidence that the U.S. curricular introduction to multiplication, and more broadly, "multiplicative relationships," is fundamentally additive and therefore conceptually deficient. Conceptualizing multiplication as the replication of equal-size groups of discrete objects and repeated addition provides insufficient support for understanding the full range of multiplicative relationships, numerically and quantitatively. The additive approach fails to extend sensibly to (a) many multiplicative situations (e.g., Cartesian product, area and volume measure, and scaling) and (b) the multiplication and division of fractions, decimals, and integers. The paper summarizes evidence from three levels of curriculum: The Common Core, elementary mathematics curricula, and mathematics texts for elementary teachers. The analysis points to the need for innovative curriculum development to support multiplicative thinking.

Keywords: Curriculum Analysis, Number Concepts and Operations, Elementary School Education

Research Objective

This analysis was undertaken to carefully examine how U.S. mathematics curricula introduce the operation of multiplication in the elementary grades. Specifically, it examined how elementary mathematics curricula—at three different levels of articulation—align with or depart from additive conceptions of multiplication. Quantitatively, additive conceptions focus on situations with equal-sized groups of discrete (countable) objects. Numerically, multiplication is presented as repeated addition. Such introductions create challenges for teachers and students when later work in mathematics and science presents situations and multiplication and division in non-whole number systems that are difficult to understand in additive terms (Davis & Simmt, 2006). If additive conceptions are currently dominant, new curricular approaches will be needed to provide more flexible and extendable foundations for teaching and learning multiplicative relationships. This study was motivated by a prior study of U.S. curricular presentations of area measurement that revealed a gap between additive, array-based conceptions and the multiplicative combination of length and width (Smith, Males, & Gonulates, 2016).

Theoretical Perspective

The analysis was guided by three perspectives: (1) the conceptual analysis of multiplicative situations, (2) Thompson's theory of quantitative reasoning, and (3) a general orientation toward learning as a constructive process. Following Vergnaud's (1983) analysis of the "multiplicative conceptual field," researchers have offered different typologies of multiplicative situations that are conceptually distinct and not easily reduced one to another. Greer's (1992) influential analysis identifies four: equal groups, area, multiplicative comparison, and Cartesian product. Understanding multiplication requires seeing how the operation applies differently to each type.

Situations, typically presented in curricula as "word problems," are structured by quantities and quantitative relationships (Thompson, 1994). Quantities are measurable attributes of objects or relationships. The theory of quantitative reasoning (QR) distinguishes discrete from continuous quantities, additive from multiplicative relationships, and operations of comparison from combination. For example, rectangular area can be seen either in additive terms as a discrete collection of squares and in continuous terms as the multiplicative combination of length and width.

QR emphasizes that mathematical thinking depends as much on the quantitative understanding of situations as on the numerical procedures used to compute numerical values.

If understanding multiplication involves both quantitative and numerical insights and if different types of situations relate discrete and continuous quantities in different multiplicative ways, then learning necessarily involves significant conceptual growth and change. If learning is a constructive process, students' initial understandings of multiplication must eventually adapt to and accommodate the wider range of situations, quantities, and multiplicative relationships. Initial conceptual foundations must be rich and flexible enough to accommodate that growth.

Method

The analysis examined three nested levels of elementary mathematics curricula. First, all content standards that referred to multiplication in *the Common Core State Standards for Mathematics* (*CCSSM*) (NGA & CCSSO, 2010) were located and analyzed, with special attention to grades 2 to 5. The analysis also examined the relevant passages of three *CCSSM* progressions documents. Second, grade 2 through 5 student and teacher materials in three elementary textbook series were analyzed: *Everyday Mathematics*, 4th edition (The University of Chicago School Mathematics Project, 2016) ("*EM4*"); *Engage New York* (New York State Department of Education, n.d.) (*"ENY"*) and *enVisionMATH Common Core* (2012) ("*enVision*"). Each was written or revised after the publication of the *CCSSM.* Third, four mathematics textbooks written for pre-service teachers and selected based on colleagues' recommendations were analyzed; the lead authors were Bassarear (2012), Beckmann (2014), Billstein (2013), and Sowder (2012). All parts of each text that dealt with multiplication and division and multiplicative situations were analyzed. At each curricular level, the analysis focused on: (a) the first multiplicative situation (or meaning) presented, (b) the presence or absence of an explicit definition of multiplication, (c) the relative frequency of discrete and continuous quantities in problems, and (d) the treatment of other types of multiplicative situations (after the first). The author carried the analysis.

Results

Multiplication in the *CCSSM*

The *CCSSM* presents three types of multiplicative situations in grades 3 through 5 (equal-groups, area, and comparison), with strong emphasis on equal-sized groups and discrete quantity. Rectangular area is presented as a type of discrete array (where the objects are squares). Multiplicative comparison is presented and then extended to scaling, in both cases as a numerical operation. Cartesian products are never mentioned. Overall, the meaning of multiplication is anchored in equal-sized groups of discrete objects. Area and volume measures are counts of squares and cubes, not the multiplicative combinations of lengths.

The *CCSSM* does not explicitly define multiplication but does explicit tie the meaning of multiplication to groups of discrete quantities, "Interpret products of whole numbers, e.g., interpret 5 x 7 as the total number of objects in 5 groups of 7 objects each" (Standard 1 3.OA). Area measure is linked to discrete arrays; "area involves arrays that have been pushed together" (*CCSSM*, Table 2, p. 89). The *CCSSM* does not describe area as a continuous quantity—the amount of space enclosed in a boundary. Multiplicative comparison is introduced in Grade 4 in numerical terms, "Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 x 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5" (Standard 1, 4.OA). Numerical multiplication is extended to fractions and decimals in grade 5.

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Multiplication in Three Elementary Curricula

All three textbook series follow the *CCSSM's* presentation of multiplication relatively closely. But *ENY's* treatment of multiplicative content is extremely close to the *CCSSM*, so the analysis above applies equally and directly to that series. Counts of *ENY* word problems show much greater attention to discrete than continuous quantities.

enVision's presentation also emphasizes equal groups and illustrates numerical products as repeated sums of the same addend. Topic 4 in grade 3 concerns "Meanings for multiplication." Lesson 1, "Multiplication as repeated addition," states that multiplication "is an operation that gives you the total number when you join equal groups" (p. 101). Lesson 2 adds that "multiplication can also be used to find the total in an array" (p. 103). Lesson 4 introduces multiplicative comparison, but only for discrete quantities ("times as many"). Word problems in Topic 4 exclusively involve discrete quantities. In Topic 6, Cartesian products of two small sets of objects are represented as arrays. Topic 14 ("Area") defines area measure as "the number of square units needed to cover a region" (p. 342)—explicitly a discrete meaning. Grade 4 continues the focus on repeated addition of equal groups of discrete objects, reminding students to use their addition knowledge, "You have learned addition facts. Now you will use them to help you learn to multiply" (p. 6).

EM4 lays the foundation for multiplication in grade 2. Unit 8, "Geometry and Arrays," focuses on equal group situations and arrays, which are closely linked. After students partition rectangles into squares, teachers are directed to explain multiplication as an operation that involves "finding the number of objects in equal groups or rows" and that "an array is one way to represent equal groups" (Teacher Lesson Guide [TLG], p. 739). The text also states to teachers that "[u]sing equal groups language helps children build the conceptual foundation for multiplication (TLG, p. 746). All quantities in Unit 8 are discrete. In grade 3, Unit 1 focuses on numerical methods for finding products in equal groups situations and emphasizes skip counting and repeated addition. Unit 4 on area measure moves quickly from counting individual squares in rectangles to multiplying the number of squares in each row by the number of rows. Despite some efforts to move beyond rectangular area as pushed-together arrays, *EM4's* focus in grades 2 and 3 is on equal groups, arrays, and discrete quantity. The series introduces multiplicative comparison in grade 4 and scaling in grade 5, but never presents Cartesian products.

Multiplication in Elementary Pre-Service Texts

Three of the four texts explicitly define multiplication. Billstein and Bassarear define multiplication in numerical terms as repeated addition; Beckmann does so as the replication of equal groups. Sowder does not provide a definition but initially "views" multiplication as repeated addition. Beckmann presents all of Greer's four types of situations and adds arrays of discrete quantities. Her text interprets all five types as equal groups of discrete quantities. For example, the elements of Cartesian products are presented in an ordered list and then arranged into equal groups. Billstein presents three of Greer's types, excluding only comparison. Sowder excludes equal groups and adds an "operator view"—the only text to do so. Bassarear presents only Cartesian product and comparison.

All four texts explicitly acknowledge some concern with repeated addition. Beckmann admits that groups are not always salient in multiplicative situations; Bassarear states that pre-service teachers should look for other models when situations don't fit repeated addition well. Billstein states that repeated addition can lead to "misunderstanding," so other models should be introduced. Sowder indicates that the sole focus on repeated addition will limit children's thinking and could lead to the misconception that multiplication makes numbers bigger.

All texts extend multiplication to non-whole numbers in different ways. Billstein and Beckmann extend their definitions to integers and fractions (via repeated addition and equal groups respectively) but abandon them for decimals. Sowder and Bassarear use repeated addition for most cases of integers but shift to “area models” for fractions and decimals (Bassarear) or fractions alone (Sowder). Sowder treats decimals as whole numbers. In each text, the authors’ additive conceptions fail to extend to decimal multiplication or to some cases of integer multiplication, but these limitations are not addressed. Overall, despite variation and stated concerns, three of the four texts explicitly define multiplication as repeated addition or the replication of equal groups. Only one (Sowder) moves away from an additive conception.

Discussion

These results show remarkable consistency in how multiplication is introduced and developed in U.S. elementary mathematics curricula. The clear focus is on equal groups of discrete objects and repeated addition as the means of computing numerical products. But the foundation of equal groups does not generally extend to the multiplication of non-whole numbers and is a poor fit to important types of multiplicative situations. Introducing multiplication via equal-sized groups of discrete objects eases the task of initial teaching and learning, but it does so at substantial cost to later learning. We need innovative curriculum development work to articulate essentially multiplicative approaches to the operation of multiplication and the nature of multiplicative relationships beginning in the elementary years.

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DISCOURSE ANALYSIS OF A SOUTH AFRICAN OPENLY LICENSED MATHEMATICS TEXTBOOK

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South African curriculum reform efforts have focused on making education more equitable. One way to measure this goal is to examine the voice of a curriculum. Thus, I use a discourse analytic framework to evaluate selected chapters from the textbook Everything Maths for Grade 12. The textbook provides access into the standard language of academic mathematics, but is not entirely consistent with the ontology of mathematics described in South African national curriculum documents. The role of the reader is more often that of a scribbler than a thinker. The textbook could be improved by adopting a more personal tone and requiring a deeper level of thought.

Keywords: Curriculum Analysis, High School Education, Equity and Diversity, Classroom Discourse

Mathematics curricula may not be sufficient to ensure equitable education without trained teachers to enact that curriculum, but it is a good place to start (Boaler, 2002). Mary Schleppegrell (2012) discusses how discourse analysis can help us to understand equity issues in mathematics classrooms. Of particular focus is how participants are positioned with respect to authority. For instance, when the author(s) tell the reader what they found in a previous exploration, this can be read as an attempt to control the reader's thinking (Herbel-Eisenmann, 2007). By presenting results from a discourse analysis of a South African textbook, I show how it contributes to and detracts from an equitable classroom, and the extent to which it is aligned with statements about mathematics and equity in national curriculum documents.

Social Justice and National Curriculum Documents

The Curriculum and Assessment Policy (CAPS; DoBE, 2011) is the national curriculum document that governs the textbook under review in this study. It contains explicit discussions of social justice issues. CAPS (DoBE, 2011) states one of its goals as "to promote accessibility of Mathematical content to all learners" (p. 8) and that students should "participate as responsible citizens in the life of local, national and global communities" (p. 9).

CAPS (DoBE, 2011) describes mathematics as follows:

> Mathematics is a language that makes use of symbols and notations for describing numerical, geometric and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves.

The description of mathematics as a human activity makes it appropriate to deploy a discourse analysis to analyze how well the textbook developed to suit these goals meets these aims and promotes the ideals of equity and social justice found throughout this document. In particular, the textbook should represent mathematics as an evolving human endeavor, position students as capable of mathematical reasoning, and not position itself as an infallible authority.

Discourse Analytic Framework

The discourse analytic framework used for this study was developed by Morgan (1996) and based on the linguistic work of Halliday (1978). The framework was operationalized by Herbel-Eisenmann (2007). This framework considers three functions of the text: the interpersonal, ideational, and textual features (Halliday, 1978). The interpersonal function is the focus of this analysis. It involves looking at how personal relationships are established in a text and where power is situated in those relationships (Morgan, 1996).

One way this framework is operationalized is by looking at any personal (first- or second-person) pronouns in the text (Herbel-Eisenmann, 2007). Using first-person pronouns acknowledges the existence of the authors (although the use of *we* can serve several functions; see Pimm, 1987), while using second-person pronouns acknowledges the existence of the readers. Another way the interpersonal function is analyzed is by looking at the questions and imperatives used in addressing the reader (Herbel-Eisenmann, 2007). The imperatives are classified as either inclusive or exclusive. Rotman (1988) defines the inclusive imperatives ("'consider', 'define', 'prove' and their synonyms", p. 9) as those which draw the speaker into the text and demand cognitive engagement. Exclusive imperatives, which Rotman (1988) classifies as all other mathematical verbs, are those that ask the reader to perform actions that are already meaningful in the shared world of the text. This would include situations such as asking the reader to solve an equation for an unknown variable.

Methods

Background of the Textbooks

For this study, the material for analysis was the South African Grade 12 *Everything Maths* (Siyavula, 2014) textbook from the *Everything Maths & Science* series, which grew out of a prior openly licensed textbook series called *Free High School Science Texts* (FHSST; 2008). Siyavula is a technology company focused on "building an integrated learning experience, drawing on the benefits of open content and adaptive practice for mastery in Maths and Science" (Siyavula, 2015).

Textbook Selection

This textbook series was chosen because it provides a much needed resource – free science and mathematics textbooks – to many schools that would otherwise lack this resource. It is available in both English and Afrikaans. The English texts are the ones analyzed here. The chapters analyzed are those on Finance, Trigonometry, and Euclidean Geometry. Finance was chosen because of its focus on applications, particularly applications that may well be relevant to students' lives. Trigonometry was chosen because of its flexibility – it can be used for rote calculations, for proofs, and for applications. Euclidean Geometry was chosen as the most likely chapter where students would be expected to write proofs and/or explain their answers. These were chosen because proofs and applications contain the most words, and thus the most material for discourse analysis. Additionally, since the type of imperatives is included in the analysis, those chapters with proofs should contain the largest sample of inclusive imperatives.

Analysis

The Finance, Trigonometry, and Euclidean Geometry chapters of the Grade 12 *Everything Maths* text (Siyavula, 2014) were analyzed using the framework developed by Herbel-Eisenmann (2007). This involved examining the voice of the texts using personal pronouns (e.g., I, we, you), and questions and imperatives. Specifically, the analysis included identifying personal pronoun usage (first and second person, singular and plural) and the number of questions and inclusive and exclusive imperatives. Inclusive imperatives are those that ask a reader to think more deeply about a concept and construct their own meaning, while exclusive imperatives ask a reader to execute a specific task in a well-defined manner.

Results

Counting by sentences, the Finance chapter comprised 37% of the text analyzed, the Euclidean Geometry chapter 31%, and the Trigonometry chapter 32%. The exercises constitute the largest portion, with 43.3%. The exposition and worked examples are then similar in size, with 26.9% and 29.8% of the text, respectively. These proportions should be kept in mind when reviewing the rest of the results, as these unequal proportions mean that not every section is weighted evenly.

Pronouns

Table 1: Personal Pronoun Usage in *Everything Maths Grade 12*

Section of Text	1st person singular	1st person plural	2nd person
Exposition	4	65	29
Worked Examples	0	64	0
Exercises	0	2	2
Total	4	131	31

The *Everything Maths* chapters chosen for analysis contain 168 total personal pronouns (see Table 1). Of these, 74 are in the Finance chapter and 70 are from the Trigonometry chapter, with only 22 (all first-person plural) from the Euclidean Geometry chapter. All of the first-person singular pronouns are in the Finance chapter. The majority (59%) of the personal pronouns are concentrated in the exposition, but the exposition only represents 30% of the text, making this tendency more pronounced. The exercises contain a scant 2% of the personal pronouns.

Imperatives

Table 2: Questions and Imperatives in *Everything Maths Grade 12*

Section of Text	Questions	Exclusive Imperatives	Inclusive Imperatives
Exposition	9	28	3
Worked Examples	7	165	30
Exercises	54	171	61
Total	70	364	94

The *Everything Maths* selections contain about six and a half times as many imperatives as questions (see Table 2). There are not quite four times as many exclusive imperatives as inclusive imperatives. The majority of the questions are from the Finance chapter, while the majority of the inclusive imperatives are from the Trigonometry and Euclidean Geometry chapters. The Trigonometry chapter has almost as many exclusive imperatives as the other two chapters combined. The questions and imperatives are concentrated in the worked examples and exercises, with 38% of them in the worked examples (30% of the total sentences) and 54% in the exercises (43% of the total sentences). Only 8% of the questions and imperatives are found in the exposition (27% of the total sentences).

Discussion

Pronouns

The most consistent personal pronoun used in this text is the first person plural (e.g., we, our, ourselves). Pimm (1987) commented on the ambiguous use of the term *we* in mathematical discourses, noting that it can be used to denote the author(s), the authors together with the reader, or the mathematical community. Ultimately, however, this convention is characterized by the vagueness of its referent. Pimm noted that the effect on him of reading such a text was as follows:

> [C]hoices had been made, ostensibly on my behalf, without me being involved. The least that is required is my passive acquiescence in what follows. In accepting the provided goals and methods, I am persuaded to agree to the author's attempts to absorb me into the action. (p. 72-3)

Thus, while the use of this personal pronoun could be interpreted as emphasizing the human nature of mathematics, it may actually be a way to exert authority.

In this textbook, the use of *we* certainly seems to indicate some use of formality, at least when it is the only personal pronoun present. In the Finance chapter, the use of *we* was balanced by other personal pronouns, particularly *you*. This contributes a tone that feels more personal. On the other hand, in the chapters on Trigonometry and Euclidean Geometry, *we* drastically outnumbers the instances of *you*, the only other personal pronoun. When not balanced by other personal pronouns, *we* denotes a much more formal tone. Thus, *Everything Maths* has a consistently formal tone. This is not particularly consistent with the statements in the national curriculum document characterizing mathematics as a human, social endeavor.

Questions and Imperatives

Everything Maths contained many more imperatives than questions, with the imperatives sometimes labelled as questions. This gives the text a more authoritative voice (Herbel-Eisenmann, 2007), especially when combined with the exclusive use of *we* as a personal pronoun (Pimm, 1987). It also contains many more exclusive imperatives than inclusive imperatives. This means that the student/reader is positioned primarily as a *scribbler*, rather than a *thinker* (Rotman, 1988). While both roles are necessary to become a mathematician, a better balance between the two should be achieved. The Finance chapters in particular positioned the reader almost exclusively as a scribbler.

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HOW KOREAN ELEMENTARY MATHEMATICS TEXTBOOKS DEVELOP INVERSE RELATIONS: CONCRETNESS FADING

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Despite of the significance of inverse relations in elementary mathematics, how student might be supported to learn the relations remains largely unknown. By examining the presentation of the inverse relations in a representative Korean elementary mathematics textbook series, this study explores how the Korean textbooks may facilitate the transition from concrete to abstract representations to promote student learning of the relations. A total of 138 instances of the inverse relations were identified. Findings revealed several important aspects of the Korean textbook series. First, inverse relations were contextualized in real-world situations through the method of storytelling within introductory problems. In addition, the use of multiple visual representations in concrete representations facilitated students' sense-making of computational procedures in abstract representations, which aligns with the process of concreteness fading.

Keywords: Algebra and Algebraic Thinking, Elementary School Education

Introduction

Inverse relations among fundamental mathematical ideas have been recognized as a critical foundation for school mathematics. However, students consistently have difficulties in understanding the inverse relations for two reasons: its abstract nature, and its irrelevance to their lives. Given the key role of textbooks as the primary resource of learning, this study aims to narrow such research gap by exploring representational transition in textbooks through an examination of the inverse relation between addition and subtraction in a representative Korean textbook series. From curriculum perspectives, the purpose of study is to examine learning opportunities that are presented in Korean textbooks. Since international textbook analysis examines how curriculums in high-achieving countries provide students with opportunities to learn, investigating Korean math textbook is expected to contribute to the development of students' understanding of the relations (Son & Senk, 2010; Son, Watanabe, & Lo, 2017). The study is centered on the following questions: (1) How are introductory, guided, and practice problems interweaved to support the learning of inverse relations in Korean textbooks?; (2) How are concrete and abstract representations connected to present inverse relations?; and (3) How is the learning of inverse relations spaced over time?

Theoretical Perspectives

Inverse relations are one of the most important relationships in elementary mathematics because reasoning with inverse relations plays a key role in the development of mathematical thinking and computational effectiveness. With such importance, the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) have emphasized that elementary students are expected to learn about relationships between the four operations. An understanding of inverse relations is necessary to comprehend the four basic arithmetic operations and to develop early algebraic thinking (Nunes, Bryant, & Watson, 2009). The knowledge of inverse relations can help to simplify students' computational difficulties. Inverse relations are useful to flexibly compute [e.g., 81–79 = ( ) can be thought of 79 + ( ) = 81 (Ding, 2016; Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009), check computations (e.g., verifying that 27–12 = 15 by computing 15 + 12 =

27; Baroody, 1987), and solve difficult word problems such as initial unknown change problems in which a quantity may increase but the solution may involve subtraction.

Despite of its significance to all aspects of early mathematics, previous research has shown that many students tend to fail to gain a formal understanding of inverse relations (Torbeyns et al., 2009; Ding, 2016). According to prior research (e.g., De Smedt et al., 2010; Torbeyns et al, 2009), formal instruction on inverse relations is lack of the learning of abstract knowledge in concrete contexts. The previous research has been proved the concreteness fading, “the process of successively decreasing the concreteness of a simulation with the intent of eventually attaining a relatively idealized and decontextualized representation that is still clearly connected to the physical situation that it models” (as cited in Ding & Li, 2014, p. 105), to be an effective method for learning scientific principles and mathematical rules (Goldston & Son, 2005).

Methods

We chose to examine Korean textbook series published by the Ministry of Education, Science, and Technology (MEST). The MEST textbook series held the teaching of each topic in one chapter containing several interrelated lessons. In each lesson, there are three steps: introductory, guided, and practice problem. All introductory, guided, and practice problems in the textbook series were examined as an instance of inverse relations. We also consider each identified instance as either a concrete or an abstract representation based on its overall nature. Furthermore, since we consider that concreteness and abstractness are viewed as relative in this study, for each instance, we coded the levels of concreteness and abstraction. For concrete representations, we differentiated several types of problem formats including (a) word problem with visual representation, (b) visual representation and (c) word problem. From (a) to (c), there was a decreasing level of concreteness because the visual information became less involved. For abstract representations such as computation problems, we differentiated the types of numbers. We viewed one-digit computation as relatively less abstract than two- and three-digit computation because the former might be more familiar to students and multidigit computation might require regrouping, which could involve more cognitive loads. After coding, we first counted the frequency of all the problem types. Then, we counted the frequency of instances for the inverse relation between addition and subtraction under the three contexts: introductory, guided, and practice problems, respectively. We conducted the detailed analyses at three tiers: (1) only introductory problems, (2) the total problems across chapters over grades, and (3) only the problems related to inverse relation. We examined the common patterns emerged from introductory problems. We also examined the representational changes from introductory to practice problems across chapters over grades. Furthermore, we examined only the problems that formally and informally instruct the inverse relation between addition and subtraction.

Summary of Selected Findings

A closer examination of the Korean textbooks along three dimensions revealed several noticeable pedagogical techniques that seemed to facilitate the transition from concrete to abstract representations.

The Method of Storytelling

The Korean textbooks presented 462 problems across chapters from grade 1 to 2 (see Table 1). Among 462 problems, there were a total of 72 introductory, 53 guided, and 337 practice problems. Among 72 introductory problems that were situated in word problem with visual representation, there were common patterns in presenting each problem: (1) presenting a word problem with accompanying pictures; (2) involving the manipulation of concrete objects (e.g., counting by using actual stickers, drawing shapes); (3) presenting number sentences that embody the relation between addition and subtraction. All the introductory problems (n = 72) were presented in word problems with pictorial representation (see Figure 1).

Figure 1. Examples of pictures used for storytelling.

The situation in each word problem was interrelated with each other under the same theme. For example, the situations in the word problems in lesson 11, 12, and 13 of chapter 3 in grade 2 were connected to each other: playing the game of go, baking cookies, and earning reward stickers. These three situations could be traced back to the real-life situation described in lesson 1 where a second grader were writing what happened today at his school in his journal. With using a total 6-page of pictures, the method of storytelling is usually implemented. The method of storytelling can be categorized into three parts: a) discussion before storytelling; b) teacher's reading-aloud; c) making connections between the situation in the story and students' real-life situation. Through storytelling in lesson 1, the introductory problems across other lessons are intertwined and seem to be able to facilitate making connections between students' informal knowledge (and the formal knowledge of addition and subtraction. Therefore, the method of storytelling may promote the specificity of the concreteness of introductory problems.

Table 1: The Frequency of Total Problems Across Chapters from Grade 1 to 2 in KM

Grade (volume)		Introductory Problem		Guided Problem		Practice Problem		Totals per row
	Chapter	Concrete	Abstract	Concrete	Abstract	Concrete	Abstract	
1 (v. 1)	3	18	0	3	0	48	20	89
1 (v. 2)	3	14	0	8	4	28	36	90
1 (v. 2)	5	13	0	12	1	20	32	78
2 (v. 1)	3	14	0	7	5	32	72	130
2 (v. 2)	2	11	2	12	1	34	15	75
Totals per column		70	2	42	11	162	175	462

The Use of Multiple Representations

The Korean textbooks presented 459 problems including the problems directly related to inverse relation between addition and subtraction from grade 1 to 2. Among 459 problems, there were a total of 277 concrete and 182 abstract representations. Among 277 concrete representations, there were a total of 67 word problems with visual representation, 100 visual representation problems, and 23 word problems. In each grade, similar patterns were found: a) concrete representations outnumbered abstract representations; b) the proportion of the problems with visual representations was bigger than the ones without visual representations, which indicates a trend of fading concreteness within concrete representations.

The Connection Between Informal and Formal Learning of Inverse Relations

The Korean textbooks presented 138 instances of the inverse relation between addition and subtraction from grade 1 to 2. The formal instruction of the relation occurred in grade 1 (n =107). Among 138 instances, there were a total of 21 introductory problems, 14 guided problems, and 103 practice problems. Several patterns of representation uses were revealed. First, all the introductory problems were presented in concrete contexts, which were word problem with visual representation. Second, the majority of guided problems (13 out of 14) were presented in concrete contexts, which were pictorial representation. Third, practice problems involved both concrete (n = 41) and abstract (n = 62) representations.

Discussion and Implications

This study contributes to the current literature because it illustrates how formal instructional resources such as textbooks may support students' learning of inverse relations. The underlying assumption of this research is that improved curriculum quality improves student learning and teacher learning, directly and indirectly. Given Korean's students' high achievement in international assessment, the Korean approaches may suggest pedagogical supports for students' meaningful and explicit learning of the inverse relations in elementary school. We found that, inverse relations were contextualized in real-world situations through the method of storytelling. In addition, the use of multiple visual representations in concrete representations facilitated students' sense-making of computational procedures in abstract representations, which aligns with the process of concreteness fading. The findings about Korean textbooks' insights and pedagogical techniques may provide rich information for textbook designers and classroom teachers to refer back to.

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NAVIGATING PROJECT-BASED LEARNING: ONE TEACHER'S DEVELOPMENT OF A PBL CURRICULUM WITH A FOCUS ON SOCIAL JUSTICE

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Project-based learning (PBL) is a method of instruction that can be harnessed to address issues of social (in)justice in mathematics classrooms. Using a descriptive and exploratory case study, I investigated how one teacher (re)develops the curriculum for a PBL project with a focus on social justice and the dilemma she encountered in the process. Using Remillard's (1999) framework, results indicated that PBL with a social justice lens requires careful time to plan and there are multiple dilemmas that this teacher encountered in the process.

Keywords: Curriculum, Instructional activities and practices, Equity and Diversity

Project-based learning (PBL) is a powerful pedagogical approach to use when incorporating issues of (in)equity and social justice in one's teaching. *Project-based learning* (PBL) is an innovative form of instruction that can provide students with the necessary and vital skills needed for the 21st century (Bell, 2010; Buck Institute for Education, 2003; Meyer, Turner, & Spencer, 1997). The Buck Institute for Education (BIE), an organization dedicated to helping teachers enact PBL, define PBL as "a systematic teaching method that engages students in learning knowledge and skills through an extended inquiry process structured around complex, authentic questions and carefully designed products and tasks" (BIE, 2003, p. 4). Some of the challenges noted by researchers are time to plan and design the project, alignment to standards, support for student learning, and assessment of students' learning (Krajcik, 1998; Marx, Blumenfeld, Krajcik, & Soloway, 1997; Mchugh, 2015).

Although researchers have argued for the importance of teachers planning projects that attend to important facets of PBL such as the driving question, content aligned to standards, and authenticity of the project (Markham, Larmer, & Ravitz, 2003; Marx, Blumenfeld, Krajcik, & Soloway, 1997), research has not examined in detail how teachers anticipate and achieve these goals. Research has focused on teachers' implementation of PBL (Blumenfeld et al., 1991; Krajcik, 1998), but few have focused on the planning and design stage of PBL with an emphasis on social justice and alignment to standards. A focus on the design and planning of projects in PBL is analogous to curriculum development, thus, research is needed in documenting how teachers develop curriculum for projects.

With regard to the types of projects students should tackle in PBL, the BIE, stated: "projects provide students with empowering opportunities to make a difference, by solving real problems and addressing real issues" (Buck Institute for Education, 2003). Yet, few studies of PBL have shown examples of students in PBL classrooms working on projects that address real issues of social significance (see Mchugh, 2015). Thus, as Thomas (2000) stated, "PBL practitioners are in a position of having to construct a unique instructional model almost completely on their own without guidance, texts, resource materials, or support" (p. 35). Research on *teaching mathematics for social justice* (TMfSJ) thus far has shown a need for teachers to become curriculum developers because supplementary materials are needed to address the help them achieve the goals of TMfSJ (Brantlinger, 2013; Gutstein, 2003). However, thus far, these projects have been an addendum to the classes which is different from PBL. To that end, the purpose of this study was to explore how a mathematics teacher developed the curriculum in one PBL project focused on social justice along with any dilemmas experienced in the process.

Conceptual Framework

Remillard (1999) presented a framework for examining two teachers' curriculum development. In this study, Remillard presented three stages of curriculum development which are the design arena, the construction arena, and the mapping arena. The design arena "involves selecting and designing tasks for students" (Remillard, 1999, p. 322). The construction arena involves "enacting these tasks in the classroom and responding to students' encounters with them" (Remillard, 1999, p. 322). The mapping arena "involves making choices that determine the organization and content of the curriculum" (Remillard, 1999, p. 322). According to Remillard, these arenas are not linear or isolated but are instead interrelated (see Figure 1). Although Remillard's study focused exclusively on the teachers' curriculum development with respect to a specific textbook, I found an adaptation of the framework worthwhile for this study.

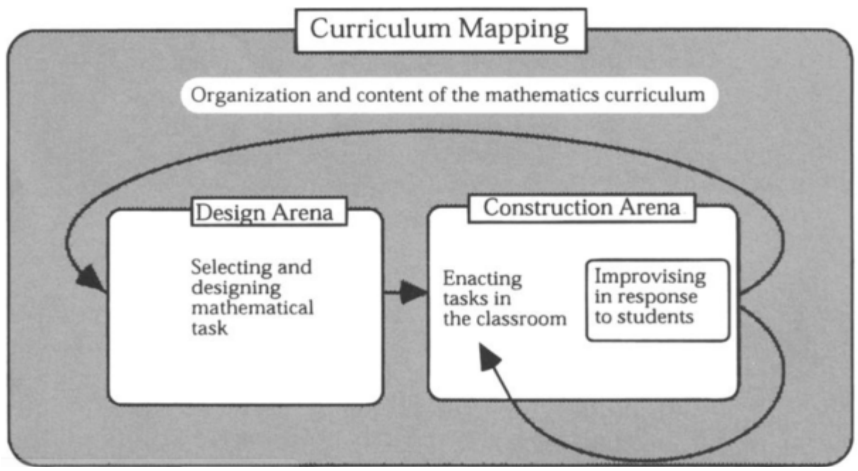

Figure 1. Remillard's (1999) Overview of the Three Arenas.

Using Remillard's (1999) three arenas (design arena, the construction arena, and the mapping arena) to frame this project, the two research questions that guided this study were:

1. Across the three arenas, how does a teacher develop the curriculum for a project in PBL while incorporating social justice and attending to the mathematics standards?
2. What dilemmas does the teacher encounter during the project development?

Data Collection and Analysis

I used a qualitative, descriptive and explanatory single case study method (Stake, 1995; Yin, 2003). Miles and Huberman (1994) define a case as "a phenomenon of some sort occurring in a bounded context" (p. 25). The bounded context in this case is this specific teacher's process across all three arenas which focuses not only on teaching mathematics using PBL, but also incorporating social justice goals and mathematics standards into her teaching. I used purposeful sampling in identifying a teacher who has taught PBL Geometry, and who has social justice goals in her teaching. A component of case study method is the use of multiple data sources to increase credibility (Yin, 2003). Therefore, I collected data from multiple sources – interviews, observations, field notes, physical artifacts – to allow triangulation in the analysis process to make the results of the study trustworthy.

Participant

Ms. Tara is a white woman in her late twenties who teaches both in Prosper High School (PHS) and Prosper New Tech High (PNTH) – "a school within a school." PHS is a visual performing arts high school located in a small urban school district in a Midwestern city. PNTH is a magnet school located inside PHS. It is a part of the national New Tech Network of schools whose focus is to "transform schools into innovative learning environments" (New Tech Network, n.d.). PNTH launched in the fall of 2014 with seventh and eighth grades and expanded to ninth and tenth grades in the 2015-2016 school year. During the study, there were about 400 students in PNTH and these students all took core classes using an integrated PBL approach. Although demographics information was not available for PNTH, I share information for PHS to provide some additional context. PHS student population was about 43% African American, 24% White, 18% Hispanic, 10% Asian, and 5% identified with races. In addition, over 60% of the students at PHS qualified for free and reduced lunch.

Ms. Tara was in her fifth year of teaching and her third year of teaching using PBL exclusively. At the time of this study, Ms. Tara taught ninth grade geometry in PHS and co-taught algebra and physics in PNTH. Given that my focus is exclusively on the mathematics, I decided to work with her on the Geometry course, as it was not co-taught with a teacher in a different discipline.

Data Collection

I worked with Ms. Tara from conception to completion of a PBL project (lasting three weeks) with an intentional social justice component. I collected data for this study in February 2017 using three types of data: interviews, physical artifacts, and field notes from classroom observations. Data collection was split into two phases: planning phase (design and mapping arena) and enactment phase (construction arena). In each of the data collection sources, I ensured that I focused on the specific project and asked Ms. Tara to zoom out to capture elements of the mapping arena.

Data Analysis

In answering the first research question aimed I organized my findings in the planning phase and the enactment phase. In the design and mapping arena, I used data from the initial interview and the planning observation and interview to inform my description and in the construction arena, my observation, field notes and trigger interviews formed the corpus of this data. I copied the transcriptions from these data sources from the InqScribe software into a word document and read every line while coding using DA (design arena), MA (design arena), and CA (construction arena) to designate the three arenas. For the second question, I sought to examine the dilemmas I observed in the planning phase and in the enactment phase of the project. I define dilemma to represent difficult choices Ms. Tara had to make given two alternatives. One way I achieved this was comparing what Ms. Tara said in the initial interview to what she stated in the trigger or final interview. I also compared what she stated in the planning interview with what I observed in the class and in my field notes. In other instances, Ms. Tara's statements indicated clear instances of dilemmas. I transcribed the data verbatim and read through the initial interview first and coded key themes that appeared salient based on my literature review and in the data. After arriving at my codes, I split the dilemmas in the planning phase and the enactment phase. Furthermore, some dilemmas were subsumed into another given their close overlap. The labeling of the dilemmas were paraphrased from Ms. Tara's own words.

Findings and Discussion

Students worked on the wheelchair ramp project that required them to determine whether the four wheelchair ramps in their school were compliant with the Americans with Disabilities Act (ADA). Ms. Tara's curriculum development for a project mostly comprised looking at the big picture. That is, she determined the project students would embark on, the mathematics content standards required, and wrote the entry letter. From my understanding at this point, I felt that the engagement between Ms. Tara and her students in the construction arena would determine the next stages of curriculum development. In the construction arena, the anticipated co-development of the project did not occur because Ms. Tara used a more teacher-centered approach resulting from dilemma she experienced.

In the planning phase, some of the dilemmas she encountered were project authenticity, project ownership, and project scaling. In the enactment phase, she faced dilemmas in the type of teaching style she employed, comparing students' ability in PNTH with those in PHS, access to technology, and not centering her social justice goal. My findings showed that the construction arena did not elicit curriculum development from the interaction between Ms. Tara and her students because her students were not given the freedom to drive their learning as is required in PBL and TMfSJ. The extensive dilemmas give rise to a need for PBL scholars to examine in more detail and perhaps over multiple projects, how teachers navigate the PBL design process especially with a focus on social justice. In conclusion, PBL is a complex endeavor that requires careful planning to enable students to engage in rich, meaningful, and complex questions. Moreover, given that authenticity is of paramount importance, teachers may not be able to develop the curriculum for projects ahead of consulting with their students. More research is needed in understanding teachers' curriculum development for PBL and to support them by assuaging the dilemmas they encounter in this process.

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PARENTS' RESPONSES TO COMMUNICATION ON CURRICULUM REFORM

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In Canada, parents' growing concerns about the "new math" are drawing public attention. Rather than dismiss such concerns, understanding parents' perspectives and garnering their support is essential to ongoing curriculum reform and children's success. In this paper, we present results of a phenomenographic study examining parents' conceptualization of the current mathematics curriculum. We focus specifically on parents' responses to school-to-home communication regarding mathematics curriculum reform: (a) seeking out further information, (b) accepting communication as given, and (c) resisting information disseminated. We examine parents' perceived communication through a postmodernist framework, that is to consider communication as (im)possible, inevitably political, and subjectless.

Keywords: Elementary School Education, Curriculum, Policy Matters

> The new math curriculum that my kids are learning? Honestly, I don't quite know what is all about. There has been little communication why they changed something that had worked perfectly fine into something that vague. (Parent participant)

The comment above describes the tension expressed by many parents regarding the recent mathematics curriculum change and the lack of communication throughout the reform process. In Canada, the tug-of-war between "new math" and "back to the basics" has gathered tremendous societal attention. Frustrated parents have joined coalitions (e.g., WISE Math) and launched petitions (e.g., Tran-Davies, 2013) in an effort to oppose curriculum reforms and to advocate returning to basic skills, standard algorithms, and mastery through memorization.

To identify the challenges of mathematics curriculum reform, we built on our previous work of reframing public opposition into collective concerns (McGarvey & McFeetors, 2015) to investigate parents' perceptions of curriculum change. Parent focus groups revealed that parents' opposition to curriculum reform stems from a lack of information about the curriculum and questionable student progress. This phenomenon urged us to examine communication as a condition of parents' curriculum perceptions. We previously illustrated perceived sources of mathematics curriculum-related communication described by parents (McFeetors, McGarvey, Yin, & Pinnegar, 2016). In this paper, we ask: How can we understand the range of ways parents respond to communication about mathematics curriculum reform? Our goal is to identify fruitful ways to engage in productive conversation about mathematics curriculum reform with parents.

Perspective: Related Literature & Theoretical Framework

Home-school communication is commonly viewed as a verbal or written interaction between parents and teachers through various channels or shared activities (Dyson, 2001). Researchers recognize home-school communication as a key form of parent involvement (Epstein, 1987) with benefits for students' learning and motivation (Civil & Bernier, 2006; Pattall, Cooper & Robinson, 2008). However, current practices are criticized for their "school centric" orientation (Jackson & Remillard, 2005, p. 59). Especially in the context of curriculum reform, the school-led one-way communication cannot address parents' questions about aspects of the current mathematics curriculum that are inconsistent with their own schooling experience (Bartlo & Sitomer, 2008). Even researchers tend to narrowly define the scope of communication as a planned event initiated by the school institution. For instance, Epstein (1996) specifies communicating as the process where schools are "sending home report cards" (p. 215).

We problematize this one-dimensional understanding of school-parent communication by following a postmodernist framework (Mumby, 1997) and consider communication as a fluid process where information leaks through predetermined channels. Moreover, the fluidity also reflects in the agents of communication: differentiation may not be made between sender and receiver of information. Rather, communicative processes mobilize the institutionalized subjectivity and the participants have agency to construct information and form new identities (Mumby, 1997). In other words, parents may perceive communication in ways that are not intended by schools and form their identities accordingly. The postmodernist framework requires a methodology open to the facets of parents' perspectives about communication.

Mode of Inquiry

We used phenomenography (Marton, 1986) to inquire into the range of parents' perspectives on how mathematics curriculum reform has been explained to them. Phenomenography aids in exploring "the qualitatively different ways in which people experience or think about various phenomena" (Marton, 1986, p. 31). Researchers draw phenomena into participants' awareness (Gurwitsch, 1964) to characterize and understand their perspectives. The variation in participants' perceptions aids researchers in generating categories which emerge from the data, emphasizing the meaning of participants' perspectives through related categories to provide a mapping of the field of inquiry.

Data Collection

Forty parents from urban and rural communities in a Western Canadian province participated. They were recruited through school council groups and parent networks. Using a volunteer process enabled us to listen to parents who are active in their school communities, reflective of vocal parents in the media. All participants completed a demographic questionnaire and participated in a focus group. Two-hour focus groups generated data of differing parent perspectives because participants were required to explain their perspective to others. We focused on their children's and their own experiences of learning mathematics, mathematical interactions at home, and communication received from teachers and schools about curriculum. Individual follow-up interviews were conducted with fifteen parents to elicit further specific examples and to seek clarification in understanding perspectives shared in the focus group.

Data Analysis

We used a constant comparative approach to identify parents' statements which were qualitatively different perceptions expressed about communicating mathematics curriculum reform. Phenomenographically, perception of experience can be analyzed through structural and referential aspects of participants' awareness. Two phases of analysis proceeded with individual coding, group comparison, and some re-coding. Our first sorting phase focused on structural aspects and yielded four categories of school-to-home communication forms parents received. Our second sorting phase focused on referential aspects and examined the messages parents determined within the communication forms. From these ascertained messages, we developed three categories of parents' responses to school-to-home communication which we report below.

Results: Presenting & Interpreting Categories of Description

While our previously-reported categories of communication forms (McFeetors et al., 2016) helped identify what parents counted as communication channels, our subsequent interpretation of parents' data indicated this understanding would be insufficient in developing more effective ways of communicating *with* parents about curriculum reform. In fact, the assumption that school-to-home communication (Epstein, 1996) is effective as a one-directional process has been brought into question by our participants' perspectives. Our results offer an understanding of the ways in which parents respond to the variety of communications, giving us insight into the meaning of curriculum reform parents were determining from available information.

Overall, parents' responses to communication about curriculum reform can be organized into three categories which include eight subcategories summarized in Table 1. These categories capture the broad range of parents' stances and actions resulting from the communication flowing from schools to home. Parents demonstrated through articulating their previous and current perspectives that they would freely move among subcategories depending on the context and sometimes inhabit two different subcategories simultaneously. Below we explain each category with an illuminating example from data of one subcategory. In the presentation, all subcategories will be explicated with multiple data excerpts.

Table 1: Parents' Responses to Communication about Mathematics Curriculum Reform

Seeking out further understanding	Accepting communication as given	Resisting information disseminated
• Cobbling together messages • Constructing rationales • Gathering more information • Interacting with children	• Satisfied • Concerned	• Discrediting information and sources • Dismissing information as meaningless

Seeking Out Further Understanding. Many parents wanted to support their children, and so they actively sought to develop an understanding of approaches to teaching and learning mathematics as well as mathematics content in reform-based curriculum by connecting more information. For example, Lindsey illustrates *gathering more information* when she reported, "I went to different math seminars for parents … explained some of the different things that teachers are doing in class and how to make things fun for our kids when you're doing math. And that has helped a lot for my son." She *interacted with her son* to examine new approaches and also shared these ideas with friends who in turn used her information in conjunction with formal messages from school to *cobble together messages* of how children are currently learning.

Accepting Communication as Given. A smaller proportion of parents were located in this category, perhaps because parents willing to participate in research tended toward a more active stance. However, parents who perceived *satisfaction* in amount and quality of communication attended focus groups to, in one parent's words, "make sure there's a different perspective … than the media plays things out." Maggie demonstrates confidence in her understanding of reform-based curriculum. Sarah demonstrated similar confidence, but was displeased with changes because multiple methods caused "a lot of confusion." Parents who were static in their response demonstrate an important feature of categories, that differing opinions on curriculum reform could lead to similar responses toward communication.

Resisting Information Disseminated. Parents in this category were active in their response to communication. As they received school-to-home information, they pushed back against the messages they did not see as relevant to their children's mathematics learning. Strident parents in this category *discredited information and sources*. For instance, Olga generalized without specific examples that “a lot of evidence shows...that inquiry, discovery-based learning is not the most effective way to teach” and disparaged “some educational consultants thought that this would be the best way to go.” Even though the curriculum states, “students investigate a variety of strategies” (Alberta Education, 2007/2016) the parent convinced a teacher, “the new curriculum says that you can choose one method to teach my daughter.” Communication that was assumed to be direct information was interpreted by an agentic parent in ways not intended.

Discussion

Curriculum reforms are crossroads where schools, teachers, parents, policy makers, and other education stakeholders at once face promising opportunities and unpredictable challenges in determining the future of education. A reconfiguration of home-school communication can offer parents a stance grounded in deeper understandings of curriculum reform and realize their agency to engage with their children’s school education. The research findings, therefore, can open up further possibilities of dialogue with parents and engender their support for curriculum reforms addressing the diversity of children learning mathematics.

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ANALYSIS OF STUDENT WORK COMPARISON TASKS EMBEDDED IN 7TH GRADE CCSSM-ALIGNED MATHEMATICS CURRICULA

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Keywords: Curriculum, Curriculum Analysis, Middle School Education

Research in mathematics education has shown the promise of teachers' examination of student work for improving teachers' understanding of students' thinking (Silver & Suh, 2014) and for assessing and improving teaching (Boston, 2014). Less research has focused on students' examination of student work as a way for students to make sense of the mathematics (Rittle-Johnson, Star, & Durkin, 2012). However, recent work (Gilbertson et al., 2016), has explored different sources of student work (student-generated, teacher-generated, and curriculum-generated) and specifically considered student work embedded in curriculum materials as a context for student learning.

Our research is guided by the question: *What types of mathematical learning opportunities are provided for students in tasks that require the reader to compare multiple instances of student work?* Based on an existing analytic framework (Gilbertson et al., 2016) for the criteria and nature of student work, this poster will present findings from a document analysis of curriculum-generated student work in CCSSM-aligned curricula with a specific focus on student work tasks that require the student reader to make comparisons among multiple examples of students' conjectures and/or strategies. Even within the same type of problem, specifically in this case, comparing student strategies, different mathematical learning opportunities are present that would require different types of engagement from the student.

In our work, we consider students' analysis of curriculum-generated student work as a "potential change in route" for research on the role of student work from the typical positioning as a professional development or professional learning tool for teachers to a learning tool for students that promotes student understanding. Our findings report on the features of the problem tasks (e.g. location in text, types of representations, whether correctness is known) and how these features relate to intended mathematical understanding (e.g. conceptual, procedural, levels of cognitive demand). This work has implications for the current learning opportunities afforded by student work in curriculum materials and possibilities for enrichment of existing tasks, informing future curriculum and assessment research and development work.

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AT THE INTERSECTION OF CURRICULUM USE AND PLANNING

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Curriculum materials are integral to the teaching and learning of mathematics and have a direct influence on what teachers actually plan for and enact in their classrooms (Brown & Edelson, 2003). Through a semi-structured think aloud interview, we study teachers' use of curriculum through the Curricular Noticing Framework (Males, Earnest, Amador, & Dietiker, 2015) which describes what teachers attend to, how they interpret what they attend to, and how they decide to respond to the curriculum materials. We discuss how each teacher moves through these three phases in an interrelated fashion and ultimately use this framework to gain deeper insight into the teacher planning process for a hypothetical lesson on slope.

The four teachers in this study were part of a cohort of Noyce Fellows (two master teachers and two novice teachers) funded by the National Science Foundation, which targets the needs of recruiting and retaining high-quality teachers to teach in high-need schools. All had experience teaching slope, but had never used the curriculum materials provided in the interview.

Findings suggest teachers engaged in all three curricular noticing phases when planning. However, each teacher shifted among phases in varying ways, as illustrated by Figure 1.

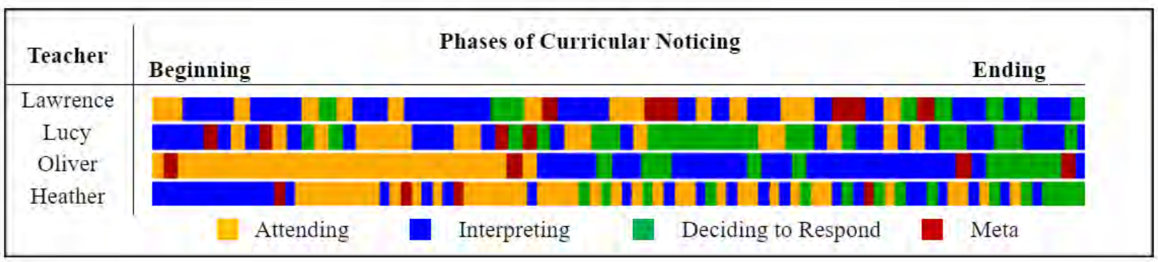

Figure 1. Phases of curricular noticing engaged in by teachers during planning session.

Even when teachers decided to respond in the same way, their interactions with the curriculum materials differed in the attending and interpreting phases. Further, we found teachers' attention to the materials was influenced by their initial interpretations, particularly when these were negative. This resulted in some teachers not engaging with a substantial portion of the materials and potentially missing opportunities afforded by the materials.

This study provides insight into how different teachers approach the same curriculum materials and produce a plan to enact in the classroom. Understanding the process of how teachers plan using curriculum materials has implications for teacher education programs, collaborative lesson planning, and curriculum development.

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EQUITY IN A COLLEGE READINESS MATH MODELLING PROGRAM: LIMITATIONS AND OPPORTUNITIES

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Keywords: Equity and Diversity, Post-Secondary Education, Teacher Education-Inservice/Professional Development

As young adults in the United States transition from secondary to postsecondary educations, they traverse a high stakes crossroad in terms of access, preparation, and enduring financial risk. Students from non-dominant racial groups and from low income backgrounds tend to enter less-prestigious universities and they tend to take—frequently using student loans—more remedial classes that do not earn credits towards graduation compared to White and higher income students (Complete College America, 2012; Venezia & Jaeger, 2013). University algebra classes are among the most highly enrolled undergraduate courses and many students fail, withdraw or receive poor grades in these classes, requiring re-enrolment (Haver, et al, 2007). University algebra present significant barriers to students as they access and pay for a university degree.

At our university, secondary students can enroll concurrently in our university's algebraic math modeling course through a program called the Entry-Point College in the Schools (CIS) program. The program is intended to serve primarily students from groups that are under-represented at universities in terms of race, ethnicity or language status; and students with middle-range academic performance. The prospect of reducing college costs and improving academic preparation means that CIS program has grown tremendously over the last five years, from two schools serving around 30 students to over 30 schools serving 800 students.

Despite the program's popularity, its goals are only partly achieved. This poster uses several modes of reflection to report on this equity-focused CIS course in algebraic math modelling. We use Harouni's 2015 commentary on the political economy of mathematics to highlight these dilemmas in terms of the interaction of secondary and tertiary educational institutions. Harouni suggests that the content and pedagogy of mathematics education at a particular time and place are highly influenced by the purposes mathematics serves for students and by the institutional structures in which it is taught. First, we describe the legal and institutional structures that allow concurrent enrollment in secondary and tertiary courses. Then we discuss the equity objective through program enrolment data. We also offer teachers' perspectives to reveal the opportunities and tensions that they experience within the program. Tensions include whether students reduce their university course load, whether teachers are comfortable aligning their practices with university standards, and whether the program fulfils its equity goals (Pazich & Teranishi, 2014).

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GENERATIVE UNIT ASSESSMENT: RE-VISIONING ASSESSMENT FOR GROWTH

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Assessment "is a process whose primary purpose is to gather data that support the teaching and learning of mathematics" (NCTM, 2014, p. 89). With a focus on assessment as process rather than product, Shepard (2000) calls for a reform in classroom assessment practices that mirrors the shifts in instructional practice and goals for education. In response, we explored the use of a performance assessment [PA] as an end-of-unit assessment which remains underutilized compared to traditional tests. The **purpose** of this project was to examine: What is the impact of designing and implementing a PA to assess students' learning at the end of a unit?

We used collaborative action research as the **mode of inquiry**. A grade 4/5 teacher and her 25 students in a suburban school participated over 4 months. The teacher collaborated in designing a geometry unit integrating two commercial games (Equilibrio and Quartex) with a blend of activity and didactic instruction, culminating in a PA where students created new puzzles for Equilibrio and identified mathematical elements. Data included: bi-weekly student work products, photographs, field notes, completed PA with screencasts, and interviews with pairs of students and the teacher. Analysis produced coding for two categories: teacher and students' perspectives on PA; and, characteristics of assessment practices that are generative.

Results indicate that using a PA was an effective unit test replacement because it was a generative unit assessment sponsoring teacher- and student-growth. The teacher noted, "I have totally changed my approach to teaching math…to focus on thinking." The students perceived higher cognitive demand as "it was bigger than just doing a unit test," yet greatly valued "a chance to show what I was thinking" and "it's better than just writing out a piece of paper your answers because you get to use your own imagination and create something." We developed four features of **generative unit assessment**, with teacher data in italics: 1) teacher's deepening awareness of students' math thinking (*lots of students shocked me at how much they knew* and *helped me understand their thinking*); 2) students' developing consolidated understanding (*they had to use the word symmetry and applying that into their tower helped them understand better* and *even in the summative assessment they were still learning*); 3) students' emerging productive disposition (*they were really excited to do math…anxiety was removed* and *they asked if I could always do it like this* and *it felt like they wanted to show their learning rather than get a [mark]*); and, 4) students' improving use of math processes (*I could see from their first tower to their second…their problem solving was something I focused on*). Student data will also be included.

In **discussion**, our interpretation of the PA experienced surpassed summative evidence leveraged to inform future teaching and learning (Harlen, 2005) to a *generative* process of growth. While an important facet of the study is a synergy between assessment and instruction, we find ourselves at a crossroads where classroom practices need to critique a means-end view of data and recognize the human endeavor of learning—where *generative assessment* occasions growth for all involved as critically important purpose of attending to students' learning.

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EXAMINING STORYLINES OF EMERGENT BILINGUALS IN ALGEBRA TEXTBOOKS

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Stereotypes of mathematical learners permeate society through storylines (or narratives) that maintain the status of whiteness (Nasir, 2016). Stereotypes typically reserved for emergent bilinguals (EBs) include an ill-preparedness for school mathematics instruction and a need of remediation and support. When these storylines manifest in the classroom they impact EBs' access to high-quality mathematics instruction, the availability of mathematical identities, and reify a marginalized status (Battey & Leyva, 2016; de Araujo, Smith, & Sakow, 2016). To understand the storylines of EBs exhibited in curriculum resources, we used positioning theory (Harré & van Langenhove, 1999) to examine the instructional supports of three high school algebra textbook teacher's guides. The decision to focus on algebra was due to its position as a "gatekeeper for [U.S.] citizenship" (Moses & Cobb, 2001, p. 14). The texts selected were from the three largest publishers and aligned with the Common Core State Standards.

Preliminary findings indicate a storyline of EBs that predominately aligns with stereotypes that reproduce racial hierarchies in white ideology. Mathematically, the storyline situates EBs in need of remediation and "below level." Specifically, the suggested accommodations center on the provision of students' engagement in repetitious, skill-based exercises. Such positionings reinforce an inferiority status, perpetuates whiteness through racial hierarchies of ability by signifying who is "good" at mathematics, and restricts the development of a mathematical identity (Battey & Leyva, 2016). Linguistically, the storylines focus on EBs' status as English learners and, in some instances, assume fluency in a first language, however the benefits of bi/multilingualism are absent. EBs are also positioned as students who lack specific mathematical vocabulary in English and benefit from learning these terms devoid of context. Such instructional approaches run counter to research, which values language-in-use to develop fluency in mathematical discourse (Moschkovich, 2002).

Although our findings may be unsurprising on the surface, they call attention to the ways teacher's guides perpetuate stereotypes of EBs on a societal scale and restrict access to high-quality mathematics instruction. In this way, such curricula can reinforce white ideology in U.S. institutions through its maintenance of the status quo (Battey & Leyva, 2016).

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TEACHERS' EXPERIENCES IN INTERDISCIPLINARY UNIT DESIGN: THE ABSENCE AND PRESENCE OF MATHEMATICS

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Keywords: Curriculum, Teacher Education-Inservice/Professional Development, Middle School Education

Engaging students in authentic learning experiences is widely heralded among mathematics education communities (NCTM, 2014) but designing curriculum that centers on authentic work, is meaningful for all students, and maintains the integrity of the subject matter is not a trivial task, particularly for teachers who may have limited curriculum design expertise. Despite the role that teachers play as vehicles between curriculum and student learning, only a few studies include teachers as integral participants in curriculum design (Skilbeck, 1984). While some teachers have embraced roles as co designers of curriculum, others have refrained from making any modifications out of "respect for the expertise of the textbook authors" and perceived external curriculum developers as "experts" regarding the instructional design for their classes (Even & Olsher, 2014, p. 346). It seems that many teachers do not feel as though they have the autonomy and/or competency to develop meaningful, standards-based curriculum.

The purpose of this study was to explore the experiences of mathematics teachers as they designed authentic, interdisciplinary curriculum units and uncover teachers' conceptions of the role of standards-based mathematics curriculum within this context. The study sought to answer the following two research questions: 1. In what ways is mathematics absent and present in teachers' curriculum design work? 2. What are teachers' experiences and conceptions of designing mathematics curriculum?

In this exploratory study, the researchers worked with seven teachers at a middle school in a southeastern state, where 74.3% of students are economically disadvantaged, on designing interdisciplinary curriculum. Data was collected between September 2015 and May 2016 and includes, audio-recorded pre- and post-interviews, video-recorded bi-weekly planning meetings, teacher and student artifacts, and video-recorded classroom observations and teacher workshop days. Using Actor Network Theory (Fenwick & Edwards, 2010) as our theoretical lens together with qualitative methods, we considered the networks associated with teachers' perceptions of and choices about curriculum design. Findings indicate that teachers' initial perceptions of mathematics and its fit within an interdisciplinary unit changed during the curriculum design, implementation and reflection cycle. Preliminary mathematical ideas dissipated during the design phase and were re-imagined following unit implementation. Teachers also reported positive experiences in designing curriculum noting the investment of their time and collaborative synergy manifested in student engagement and ownership of their learning.

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THE CONTINUING MARGINALIZATION OF MATHEMATICS IN PREKINDERGARTEN: A POLICY PERSPECTIVE

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Keywords: Early Childhood Education, Policy Matters

Although research has shown that mathematics is at least as important as literacy to young children's long-term academic success (Duncan et al, 2007), early mathematics continues to receive less attention than literacy in most prekindergarten programs (National Research Council, 2009). The goal of this study is to explore this lack of attention to early mathematics from a policy perspective. The research question framing the study was: How do states describe key content in literacy and mathematics for prekindergarten children and their teachers? Drawing on methods for content analysis, my research team and I examined the early learning standards for prekindergarten for all 50 states as well as any competency standards for prekindergarten teachers. These standards were organized in different ways for each state, with certification bands ranging from *Birth to Age 5* to *Prekindergarten to Grade 12*. We engaged in both quantitative and qualitative analyses of these standards, although for the sake of space, only quantitative data are reported here.

Overall, analysis showed that while a growing number of states attend to mathematics in their early learning standards for prekindergarten, most still give mathematics far less attention than literacy. For example, while only 3 states identified fewer than 20 subtopics that should be addressed with prekindergarteners in relation to literacy, 13 states specified fewer than 20 subtopics related to mathematics. While simply having more standards is not necessarily better, these data suggest that many states provided more detailed portrayals of early literacy than early mathematics. A similar relationship was found in teacher competency standards with only 4 states offering no subtopics in their literacy standards for teachers, but with 12 offering no subtopics in mathematics. Further, 13 states had no required coursework for prekindergarten teachers related to mathematics, while only 6 states had no requirements for coursework related to literacy. For both mathematics and literacy, attention to teachers' content knowledge was highly related to the structure of the certification band, with bands that ended in the elementary years generally requiring more attention to content than bands that ended with prekindergarten or kindergarten. For example, in looking at certification bands across the 50 states (n=64 because some states had multiple certification bands that include prekindergarten), 7 of the certification bands that include elementary grade levels required teachers to take courses related to mathematics, while none of the bands that stopped at prekindergarten or kindergarten required such courses. This study suggests that the documented lack of attention to mathematics in prekindergarten is not simply the result of resistance or lack of interest by teachers, but is, at least in part, systemic, resulting from a lack of attention to mathematics in state standards.

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THROUGH THE EYES OF A PST: USING EYE TRACKING TO EXAMINE PLANNING WITH CURRICULUM MATERIALS

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Planning is one of the first out-of-classroom activities that prospective teachers (PSTs) engage in when learning to teach that is linked to improving teaching (Morris, Hiebert, & Spitzer, 2009). Furthermore, the task of planning can be difficult, particularly when using curriculum materials. Despite the complexities involved in understanding how to use curriculum materials to plan, this is an often-neglected topic in teacher education (Drake, Land, & Tyminski, 2014). We draw on the *Curricular Noticing Framework* (Males, Earnest, Dietiker, & Amador, 2015) to describe what elements PSTs attend to in CPM and PEI when planning a lesson.

Across PSTs, results indicate attention across teacher and student materials was balanced with most PSTs attending only slightly more to student materials. However, notable differences existed in the *ways* in which PSTs attended to each set of materials. Figure 1 shows a representative sample of one PSTs attention for a portion of each text. What this indicates is that this PST, like others, attended in a more sporadic manner to the PEI text than to the CPM text.

When interacting with an element of CPM, he seemed to gaze at that element and read it from start to finish. However, with PEI he moves from one element to the next, skimming the text.

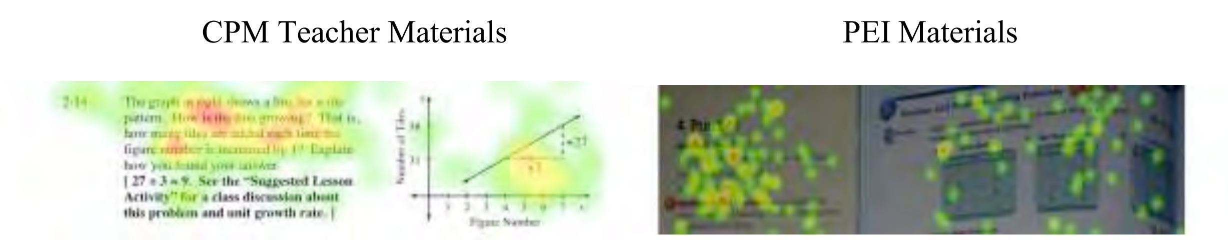

Figure 1. Attention across CPM and PEI materials.

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Chapter 3

Early Algebra, Algebra, and Number Concepts

Research Reports

Brief Research Reports

Posters

A QUESTIONING FRAMEWORK FOR SUPPORTING FRACTION MULTIPLICATION UNDERSTANDING

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This research examined the role of the teacher in supporting students to make sense of fraction multiplication when using a problem solving approach. Using a qualitative approach, the teaching of four skillful experienced sixth-grade teachers was examined as they implemented a problem-based unit on fraction multiplication. This paper will present a questioning framework used by teachers that supported students' conceptualization in this domain and highlight resulting implications for teacher practice.

Keywords: Number Concepts and Operations; Rational Numbers; Instructional Activities and Practices; Problem Solving

This research is situated in the context of fraction operations with a specific focus on fraction multiplication. It is founded upon arguments that more attention and effort should be paid to unpacking the professional work that teachers do in classrooms (Grossman et al., 2009). The fraction operation research literature has documented that students can invent procedures for operating with fractions (Kamii & Warrington, 1999). In the domain of fraction multiplication, it has been established that students bring initial knowledge that can serve as a starting point for algorithm development (Mack, 2001). In an effort to better understand teacher practice in the area of fraction multiplication the following research questions were posed: What are key conceptual obstacles students encounter when engaged in a problem solving, rather than a procedural approach to understanding fraction × fraction multiplication? How do teachers use questioning and discursive practices to support students to make sense of what fraction × fraction multiplication is an enactment of?

How to engage students in productive mathematical discussions is challenging (Stein, Engle, Smith & Hughes, 2008). One challenge a teacher faces when working to cultivate a teaching practice where mathematics is learned through problem solving, is how to support learners without taking over or reducing the level of mathematical work students should engage in. In my work with teachers, they have commented that they do not know how to guide students when they are struggling. They find themselves explaining what to do rather than redirecting students in a way that helps them think and reason. This research was conducted to inform the creation of professional development materials to use with teachers interested in developing a practice where students are engaged in mathematical reasoning and problem solving as part of learning about fraction operations. The questioning framework for fraction multiplication that is presented here was one tool developed to support practicing teachers.

Theoretical Framework

Gravemeijer and van Galen (2003) emphasize that instead of concretizing algorithms for students, teachers can use an emergent approach where students are positioned to invent algorithms. They describe this guided reinvention process as one that starts with carefully chosen contextual problems where students model a mathematical situation. With this approach, students solve problems through modeling that leads them to reason with numbers in particular ways. Ideas for operating with numbers can emerge from work that focuses on learning to reason with numbers and exploring what is involved when numbers are manipulated in particular ways.

According to Mack (2001), students bring informal knowledge related to partitioning fractions that can support making sense of fraction multiplication. Through modeling students can develop mental images and ideas that will support their understanding of what fraction multiplication is an enactment of. Important areas to develop include fractions as operators, developing meaning for finding *parts of parts of a whole*, and developing flexibility about what is the unit. Flexibility with the unit is especially important because the unit shifts when multiplication is enacted. In additive situations the numbers (ex: 1/2 + 3/4) represent actual quantities such as 3/4 of a pound and 1/2 of a pound. In a multiplication situation one of the numbers represents a quantity. The other number is an operator. For example, when 2/3 × 1/4 is enacted the goal is to determine what 2/3 of the quantity 1/4 is. In a scenario where someone wants to plant 2/3 of 1/4 of a garden with beans there are multiple levels of partitioning taking place. Initially, a whole garden in partitioned so that 1/4 of the whole garden can be represented. Next, one must partition the 1/4 of a garden into thirds and identify 2/3 of the 1/4 in order to know what *part of the part of the whole* garden is planted with beans. Finally, in order to determine how much of the whole garden is used for planting beans, the part of the part of the whole identified for planting beans must be expressed as what fraction of the whole garden is used for beans. While it might be tempting to provide students with the shortcut that "of" means multiply, the actual enactment of multiplication with fractional numbers is much more complex.

Armstrong and Bezuk (1995) offer that in order to make sense of fraction multiplication students need partitioning experiences that lead to the analysis of relationships between partitions and the whole. From an instructional point of view it is important for students to have the opportunity to encounter and make sense of "part of a part of a whole". From a mathematical point of view, students need opportunities to explore through modeling what is happening when the operation of multiplication is enacted. This research aimed to understand the teacher's role in supporting the development of this understanding by engaging students in a problem-solving approach to fraction × fraction multiplication rather than an approach focused on demonstration of a procedure.

Methodology

The setting used for this study were the classrooms of four sixth-grade teachers and their students. Each of the teachers used the Connected Mathematics Project (CMP) II instructional unit *Bits and Pieces II: Using Fraction Operations* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006a) as their primary curriculum. The unit uses a guided-reinvention approach to developing meaning for fraction operations. It allows algorithms to arise through student engagement with both contextual and number-based situations. In this setting assumptions can be made about the tasks used and about the fraction-related concepts that were developed prior to, and during the unit on fraction operations. In the timeline for the sixth graders who were part of this study, students came to the fraction operations unit with previous experiences that supported their understanding of fractions as quantities and their ability to model fractions. Prior to implementing the *Bits and Pieces II* unit, the *Bits and Pieces I: Understanding Fractions, Decimals and Percents* (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006b) unit was also implemented.

The four teachers were skillful experienced teachers. The teachers had between 6 and 16 years of experience teaching with CMP. The researcher had prior opportunities to interact with two of the teachers in their classrooms. These interactions provided information on how the teachers organized their learning environment and engaged students to reason with mathematical ideas. The teachers had a strong understanding of the mathematics they taught and their students as learners of that mathematics. These teachers engaged their students in conversations about their mathematical work as they engaged in problem solving and reasoning. The other two teachers were identified by contacting fellow mathematics educators, researchers, and district-level personnel known by the researcher to have a history of working with teachers in CMP classrooms. They were provided the criteria described above and asked to recommend, if they could, a teacher who strongly met all of the criteria. The directions explained that a teacher must meet all criteria and to not make a nomination for the sake of nominating.

This study used a qualitative design. During the teaching of the Bits and Pieces II unit, classroom lessons were videotaped each day during the five to six weeks it took to cover the unit. In addition, teachers wore an audio recorder during each lesson. The video recorder was used to record small group discussions. When visiting the classrooms of the teachers, the researcher engaged in participant observation. This included observing, taking field notes, interacting with students during small group work time, and meeting with the teacher after the lesson to seek their perspective on the lesson and students' mathematical progress. During the part of the instructional unit that focused on fraction × fraction multiplication, the researcher visited each teacher's class during at least one day of the three to four-day lesson sequence. When the unit concluded, the researcher brought all four teachers together to examine selected student work, videos of their teaching, and discuss patterns emerging in the data.

Data analysis was guided by Erickson's (1986) interpretive methods and participant observational fieldwork. The multiple data sources allowed for triangulation. The video and audio data were transcribed and analyzed for emerging themes in relation to the research question. Questions that framed the data analysis included *"What approaches to solving problems emerged in discussions as students shared their reasoning?"*, *"How did teachers respond to students?"*, *"How did teachers direct the mathematical focus of these discussions?"*, and *"What approach did the teacher take when students struggled mathematically?"* When the researcher met with the four teachers, emerging themes along with relevant classroom video clips from lessons were reviewed. It was during this process of data reduction and collaborating with the teachers that the researcher began to identify data that answered the research questions. From this analysis a questioning framework was developed that captured interactions teachers had with students when using an emergent approach to fraction × fraction multiplication. The questioning framework is presented in Figure 1. The questioning framework was linked with issues referred to as "sticky points". The sticky points emerged and became articulated during the researcher's discussions with teachers about their interactions with students. Sticky points are common areas where students struggle mathematically to make sense of the enactment of fraction × fraction multiplication. These are also documented in the literature (ex.: Mack, 2001; Armstrong & Bezuk, 1995). The sticky points are one's that typically or expectedly emerge when instruction uses students' ideas as the starting point. The questions that form the questioning framework were apparent in the classroom teachers' dialogue with students as the teachers attempted to move students through these sticky points toward valid mathematical ideas and understandings.

Results

The CMP curriculum introduces students to fraction multiplication using the context of selling pans of brownies at a school event. A typical problem might ask the following: *What fraction of a pan of brownies will I have if I buy ¾ of a pan that is ½ full.* This context leads to development of an area model. Student were given a labsheet with squares (brownie pans) that they used to model the problem by developing a drawn visual representation of what happens when buying ¾ of ½ of a pan of brownies. Through drawing the students begin to develop a mental image of what fraction multiplication is an enactment of and reason about what each fraction represents in the process. As students draw models to enact and solve the problem there are common sticky spots that arise. The teachers anticipated and used these as opportunities to build new ideas related to understanding fraction multiplication. Figure 1 contains the questioning framework that emerged from the analysis of classroom data described in the methodology section. It is reflective of the mathematical interactions the teachers had with students while working to model and solve brownie pan problems as part of making sense of what fraction × fraction multiplication entailed.

1. What is the problem asking you to do?
2. Tell me about your picture. What does it show?
3. How much are you starting with? How can you show that?
4. What is the problem asking you to find? How can you show that?
5. How much of what you started with do you need?
6. How much of the whole pan do you need?
7. How many pieces are in a whole pan?
8. How many of those pieces do you end up needing?
9. Do you have more than you started with or less? Why does that make sense?
10. What number sentence would write to show what the problem is asking you to do mathematically?

Figure 1. Questioning framework for fraction × fraction multiplication.

As in most any class, students will need support based on where they are in their overall fraction understanding—some more than others. Rather than show students what to do, teachers posed questions to focus their attention on particular mathematical ideas while students worked to develop a picture that modeled what was happening in the brownie pan scenarios they were presented with. See figure 2 as one possible model that a student might develop. Asking students what they are starting with by posing question 3 (also see Fig. 2a), which in the ¾ of ½ scenario is the second fraction ½, and asking students why this makes sense, is used to establish which fraction is the starting quantity. In the problem context, ½ of a pan is an actual quantity. The fraction ¾, which is the operator, is focused on when asking question 4 (also see Fig. 2b). While a teacher could tell students what fraction they should draw first when making their model, the expectation of teachers observed was that students engage in reasoning and problem solving. They wanted students to figure out what they were being asked and work accordingly using what they knew about fractions and partitioning. This is why the teachers asked questions 1 through 4. These questions helped with one of the common sticky points—which fraction do I start with and which fraction am I operating with when modeling a fraction times fraction situation. Often, students want to start with the first fraction written in the problem statement. They want to begin by partitioning and shading the brownie pan to show ¾ of a pan of brownies. However, they need to begin by showing that there is half of a pan of brownies to start with. Questions 3 and 4 when used together draw out or direct reasoning toward what each of the fractions in the multiplication problem represents visually. Students need to understand what each fraction represents—one is a starting quantity and one is an operator. Questions that ask students to read (and reread) the problem context support their ability to process and reason about what the problem is describing as well as asking.

A second sticky point was what represents the unit being partitioned and named. When the problem starts, the unit is the whole brownie pan. You start with half of a whole pan of brownies. When asked to find ¾ of ½ of the pan, the unit or whole that is partitioned is half of the pan. When asked to find ¾ of ½, students may not be aware that there is a shift to a new unit and they mark or partition the full brownie pan. In other words, they find ¾ of the whole pan rather than ¾ of ½ of the pan. Also, while a student may correctly partition the half pan into fourths, they may not be able to articulate why it makes sense to do this. Together questions 3 and 4 draw and focus student reasoning on what each of the fractions represent when the problem is enacted. Question 5 (also see fig. 2b) focuses on what part of the part of the whole do you need. With ¾ of ½ of the pan, question 5 leads students to articulate that they need ¾ of ½ of the whole pan. When students can articulate this, they might then realize they need to shade ¾ of ½ of the pan. Or, perhaps a teacher might respond, “If you need ¾ of ½ of the pan, how could you show that in your picture?” At this point most student realize that they need to partition ½ of the pan into four equal parts and shade three of the parts. Question 6 then aims to get students to articulate that they need ¾ of ½ of the whole pan. In some cases this led the teacher to prompt the student to write beside or under their picture “¾ of ½ of a pan”. In other words, students were prompted to express and record that the brownie pan scenario could be captured or expressed as “¾ of ½ of a whole pan”. This will eventually support question 10 (also fig. 2d) that asks what number sentence could you write to show what the problem is asking you to do mathematically.

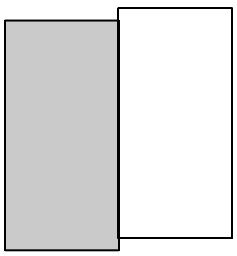
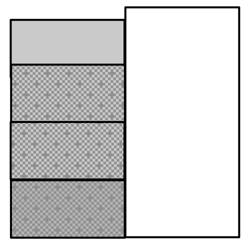
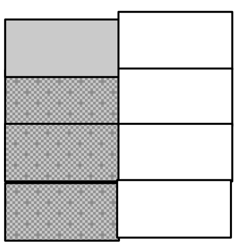
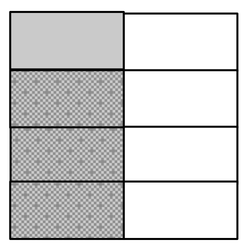
2a.  **Q3: How much are you starting with?** **[½ of a pan of brownies]**	**2b.**  **Q4: What is the problem asking you to find?** **Q5: How much of what you started with do you need?** **[¾ of ½ of the pan of brownies]**
2c.  **Q6: How much of a whole pan do you need?** **Q7: How many pieces are in a whole pan?** **Q8: How many of those pieces do you end up needing?** **[3/8 of the whole pan of brownies; 8; 3]**	**2d.**  **Q10: What number sentence could you write?** **¾ of ½ of a full pan is 3/8** **or** **$\frac{3}{4} \times \frac{1}{2} = 3/8$**

Figure 2. Possible model representing ¾ of ½ of a pan of brownies.

A third sticky point involved expressing the solution based on what the problem is asking. Questions 7 and 8 (also see fig. 2c) are designed to direct attention to what the problem is asking—what part of the part of a whole pan of brownies would get if you bought ¾ of ½ of a pan of brownies. Often students say the solution is ¾. While it is true that ¾ of ½ of the pan is shaded, the solution is expressed as the portion of the original unit or the part of one whole pan. This requires a second shifting of units from finding part of ½ of a pan back to finding a part of one whole pan of brownies. The solution 3/8 is the part of the whole pan of brownies that are bought. Question 7 directs students to determine how many 1/8 pieces are in the whole pan. Question 8 focuses on how many 1/8 pieces are being bought. In addition to these questions, teachers might also ask students to reread the problem and to determine what they were asked to find. In the CMP curriculum the brownie pan problems ask, "What fraction of a *whole* pan of brownies is bought?"

As students experience the shifts across units, and because they are developing a visual model to reason with, the teachers focused on what was happening in relation to a fourth sticky point. Drawing from their work with whole numbers, students often think that multiplication leads to a product that is larger than the factors being multiplied together. Question 9 prompts students to look at their brownie pan picture, (or array model when they begin to use symbolic rather context-based problems) and consider that when they multiply by a faction that the solution is less than then what they started with. This sets up an opportunity to discuss what is happening when multiplying fractions and why fraction × fraction multiplication leads a solution where you have less than what you started with.

Returning to question 10, which may not be posed until after modeling and discussion of multiple problems, students are prompted to attach symbolism to the situation and their models. In the data this question led to discussions about why, for example with ¾ of ½, that the solution is expressed in eighths, and why there would be "three" eighths. From discussion of the ideas related to their models and what fraction × fraction multiplication is an enactment of, the algorithm "multiply numerators, multiply denominators" began to emerge. While the initial problems posed used the brownie pan context, students also worked with non-contextual fraction × fraction multiplication problems. While the idea that students could think of multiplication as finding a fraction "of" a fraction was addressed, the teachers focused discussions on finding a "part of a part" and used the brownie context to give meaning to this when instructional tasks shifted from contextual to symbolic. As students responded to questions from the framework they engaged in the thought processes associated with the enactment of fraction × fraction multiplication. The concept of fraction × fraction multiplication as finding a part of a part was supported.

Discussion and Significance

NCTM (2014) argues the importance of letting students engage in productive struggle. Often teachers are concerned that by not demonstrating up front to student what to do to solve a problem it will lead to confusion among students. How to support students and not "tell" is challenging. Providing teachers with problem contexts such as the brownie pan scenario is important in supporting a change in practice. However, a good problem alone does not help teachers develop ways to help students when they are stuck or get students started solving problems without telling them how. Good problem contexts do not ensure a teacher will have a way to support student problem solving. The questioning framework in coordination with "part of a part" problem contexts like the brownie pan problems is a potential tool for helping teachers shift away from a practice based on telling. While it sits outside the scope of research reported here, this framework in conjunction with video case analysis was used with teachers in a professional development setting. These teachers were working to develop their practice to use an emergent approach to fraction operations. Preliminary data analysis suggests that the questioning framework was an important support for teachers working to engage students in fraction multiplication algorithm development based on a problem solving approach.

Just as students benefit from learning to persist, teachers also need to learn to work through students' moments of uncertainty. Making sense of student reasoning while at the same time making instructional decisions about how to reply or what to ask in discussions with students, and how to guide without taking over student thinking is complex (Stein, Engle, Smith & Hughes, 2008). Facilitating discussions with students about what is happening as they work on problems such as those presented here may be daunting to teachers who are new to or working to develop a practice where students engage in problem solving and reasoning. The questioning framework described here, especially when paired with analysis of student work and classroom video cases in a professional development setting, can offer teachers a plan for listening to and responding to their students. It provides a pathway for supporting student reasoning so that teachers do not feel they have to resort to telling.

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DESIGN RESEARCH ON PERSONALIZED PROBLEM POSING IN ALGEBRA

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Algebra is an area of pressing national concern around issues of equity and access in education. Recent theories and research suggest that personalization of instruction can allow students to activate their funds of knowledge and can elicit interest in the content to be learned. This paper examines the results of a large-scale teaching experiment where 8th grade students posed, solved, and shared algebra problems related to their out of school interests in topics like sports, video games, and social networking. Results suggest that the teaching experiment improved both learning of and interest in algebra compared to "business as usual" instruction, particularly for those students who were struggling. Theoretical and practical implications are discussed.

Keywords: Middle School Education, Design Experiments, Algebra and Algebraic Thinking

Algebra is a gatekeeper to higher-level mathematics, with significant implications for both equity in education and students' economic attainment (Moses & Cobb, 2001). Failure rates in algebra continue to be high, especially among low-income students and students of color (Allensworth, Nomi, Montgomery, & Lee, 2009). Students' interest in learning math declines over adolescence generally (Frenzel, Gotez, Pekrun, & Watt, 2010), and during algebra courses specifically (McCoy, 2005). Concepts from algebra are not seen as being connected to students' worlds, including their home and community activities (Chazan, 1999). Math curricula are often not designed to be relevant to students from diverse backgrounds (Ladsen-Billings, 1995).

Exploring ways to connect math to students' lives, experiences, and funds of knowledge is critical to making algebra both accessible and captivating. All students bring to the classroom mathematical funds of knowledge (Civil, 2007; Moll & Gonzalez, 1994), ways of reasoning quantitatively from their home and community. Students draw upon rich algebraic ways of reasoning when pursuing their out-of-school interests in areas like sports and video games (Walkington, Sherman, & Howell, 2014). If these funds of knowledge can be brought into the classroom, they may allow students to better access and understand mathematical ideas (Boaler, 1994; Walkington, 2013).

This paper reports a study where 8th grade students pose their own personalized "algebra stories." Personalization refers to the instructional approach of making connections between students' interests in topics like shopping, music, and social networking, and instructional content they will be learning in school (Cordova & Lepper, 1996; Walkington, 2013).

Theoretical Framework

The theory behind personalization draws upon two major ideas – interest as a motivational variable, and mathematical funds of knowledge. Interest is the psychological state of engaging and the predisposition to re-engage with objects, events, or ideas (Hidi & Renninger, 2006). Higher levels of interest have been associated directly with improved performance and learning (Potvin & Hasni, 2014). Higher interest is also connected to important mediators of learning like attention, engagement, persistence, perceived competence, and use of learning strategies (Kim, Jiang, & Song, 2015; Linnenbrink-Garcia, Patall, & Messersmith, 2013), and with motivational variables like self-efficacy, self-regulation, and achievement goals (Harackiewicz et al., 2008). Personalizing instruction by connecting it to students' out-of-school interests may thus elicit their interest for the content to be learned, allowing for increased engagement and motivation.

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Students also bring to the classroom funds of knowledge from their home and community lives that are historically-accumulated and culturally-developed (Civil, 2007); students' out-of-school interests are one dimension of these funds of knowledge. Prior research has explored the creation of instructional school units around children's experience with money and with home-based knowledge of gardening and construction (Civil, 2007). Interviews with families have revealed that they use mathematical practices while cooking, sewing, engaging in construction, and scheduling (Gonzalez, Andrade, Civil, and Moll, 2005). Using students' funds of knowledge can increase their legitimate participation in the classroom (Barton & Tan, 2009). Thus personalizing instruction to students' out-of-school interests may allow students to draw upon their prior knowledge of using quantities and numbers in everyday life in useful ways, allowing them to better understand and access the mathematical content to be learned.

Problem-posing – the activity of having students author mathematical tasks – "improves students' problem-solving skills, attitudes, and confidence in mathematics, and contributes to a broader understanding of mathematical concepts and the development of mathematical thinking" (Singer, Ellerton, & Cai, 2013, p. 2). Learning to pose a mathematically valid story problem is a challenge for students who must come to appreciate the importance of problem features. When posing a story problem, students must first avoid making common errors like posing non-mathematical, trivial, or unsolvable questions (Silver & Cai, 1996). They next must select units of measure and include realistic quantities that relate to one another in a known fashion (Silver & Cai, 2005). Research on personalization has thus far focused on problem-solving instead of problem-posing – the present study extends this research.

Prior research on personalizing mathematics instruction to students' out-of-school interests in topic like sports or movies has found that this approach elicits interest (Hogheim & Reber, 2015), and can promote learning (Cordova & Lepper, 1996; Walkington, 2013). However, effects are small, and producing banks of personalized problems is difficult for curriculum developers. In the present study, we enlist the students as the authors of their own algebra stories. In this way, learning becomes "personalized" as the students themselves write and solve problems based on their out-of-school interests in topics like sports, social networking, and video games. We examine the effects a 4-day teaching experiment which implemented personalized problem posing, sharing, solving. The research questions are: (1) How does participation in the teaching experiment impact students' understanding of algebraic concepts? and (2) How does participation in the teaching experiment impact students' interest in and self-efficacy for algebra?

Method

Procedure and Participants

This paper describes the fourth phase in a five-phase design-based research program (Brown, 1992; Collins, Joseph, & Bielaczyc, 2004). In design research, educational researchers "engineer" learning interventions and theories, with continuous adjustment and experimentation, to allow evidence-based claims to be made. The initial phases of the design research involved interviews and a small-scale pull-out teaching experiment where students posed, solved, and shared personalized problems. We then applied a teaching experiment methodology to four intact classes of 8th grade students to further develop and refine hypotheses. We follow the definition of a teaching experiment in Steffe and Thompson (2000) where students' mathematical development is tracked over time as emerging hypotheses about the "mathematics of students" arise and are tested. We set out to tackle a widely-acknowledged issue at our site – students' struggle to solve algebra story problems - and coordinated pragmatic and theory-based concerns as we determined "in the moment" and after each session how to guide learning.

The procedure for the teaching experiment was as follows. During a pre-test, students indicated which topics they were interested in: sports, video games, social networking, shopping, food/cooking, cell phones, computers, part-time jobs, and reading/writing. For students in the experimental group, selections were used to place them into groups of 3-5 students who all shared one of these interests. During each class, groups would solve algebra problems about topics the group was interested in like sports or cell phones. The problems they solved were written by the researchers, but were almost always based on problems that students in the classes had previously written. After solving a personalized problem, groups would be asked to write their own problem with a similar type of linear function (e.g., no intercept, negative slope, system) that corresponded to their group's shared interest. They would solve their own problem, and sometimes would trade problems with another group. The class would discuss both the problems students solved and the problems they posed. Learning was personalized in that students were writing and solving problems based on their out-of-school interests.

We also employed a comparison group. The two teachers participating in the teaching experiment taught approximately half of their class sections using "business as usual" instruction, and pre- and post-assessments were distributed to both the 4 classes participating in the teaching experiment and the 3 classes receiving "business as usual" instruction. Although comparing a teaching experiment delivered by a research team to a single teacher implementing their normal instruction may not be balanced, our purpose was to simply explore what the possibilities and limits of this approach might be. The problem-posing intervention may actually be more effective when delivered primarily by the classroom teacher, as teachers have far greater familiarity with their students and the curriculum.

Participants included 171 students (94 experimental and 77 control) in 7 classes of two teachers. Two of the classes (45 students; 1 class in experimental and 1 class in control) were 8th grade Algebra I classes where more advanced students were placed. The other 5 classes (126 students; 3 classes in experimental and 2 classes in control) were regular 8th grade math classes. Students were enrolled in a middle school in a large metropolitan area. Participants were 56% female, 90% Hispanic, 4% African-American, 4% Caucasian, and 2% Other race/ethnicities, with 91% Economically Disadvantaged (ED) and 39% Limited English Proficient (LEP). Eight students (all in the experimental group) had a special classification where they were immigrants who had been in the country for less than a year and spoke only or mainly Spanish.

Measures and Analysis

All participants took a pre-test that measured their knowledge of linear functions. There were 2 forms of the pre-test, which were randomly distributed within each class. Each form contained 3 algebra story problems, and then an additional prompt where students were asked to pose their own story problem. This fourth item was included because we were wondering whether students' willingness to pose a problem at pretest would interact with the degree of benefit they received from the teaching experiment. The post-test contained identical items, with one exception – for students not in Algebra I, instead of the prompt asking them to pose a problem, they were instead asked to solve a problem that involved direct variation (i.e., a directly proportional relationship with no intercept term). Because of how the teaching experiment unfolded, far more time than anticipated was spent on direct variation, so it seemed important at post to measure students' understanding. Pre-tests and post-test items were identical across the experimental group and the control group. Items were drawn from released items on algebra assessments like the state standardized test and the Smarter Balanced assessment.

On the first page of their pre- and post-test, all students were given an 11-item questionnaire. The first 8 items were from the situational interest scale in Linnenbrink-Garcia et al. (2010) (example item: "I enjoy the subject of algebra."), and the final 3 items were self-efficacy items written based on Bandura (2006; example item: "I feel confident in my ability to do algebra."). Cronbach's alphas for each scale were between 0.88 and 0.90, suggesting good reliability.

Gains from pre- to post-test were analyzed using mixed effects logistic regression models. The outcome variable was a 0/1 indicator of whether each student got each problem on their post-test correct. Random effects were added for student ID and problem ID. This analysis method was used because it could handle that different students got different forms of the test, and it allowed for there to be a different "difficulty level modifier" (modelled as a random effect) for each individual problem. Fixed effects included a 0/1 variable for condition (control or experimental group), pre-test score (with each part of each problem on the pre-test being counted as 1 point), and which course the student was enrolled in (8^{th} grade math or Algebra I).

Because random assignment was conducted at the level of a classroom, we knew that there could be significant pre-existing differences between students in different class periods. For example, in our sample, special education students tended to be in certain periods, as did students in our subgroup of recent immigrants to the U.S. For this reason, we sought to include as many additional predictors to compensate for pre-existing differences between class periods as possible – including gender, ED, LEP, Talented and Gifted (TAG), and Special Education (SPED) status, students' score on the mid-year standardized mathematics test administered by their district that took place shortly before the teaching experiment, students' initial level of situational interest in and self-efficacy for mathematics, and whether the student was a recent immigrant. We only, however, retained fixed effects that were significantly predictive in the models. In addition, on the pre-test, the final question asked all students to try to pose an algebra problem about their interests. Scoring this problem as "correct" was problematic and therefore it was not included in the calculation of the pre-test score. Instead, we created a 0/1 indicator variable that simply showed whether the student had attempted to pose a problem. We were particularly interested in whether students' willingness to pose a problem at pre-test would moderate the effectiveness of the teaching experiment. Models were initially fit without interaction terms (Model 1) and then all two-way interactions with Condition were subsequently tested (Model 2). For the situational interest and self-efficacy measures, each student's 1-5 ratings for each scale was averaged, and used as the outcome in a linear regression model. Similar fixed effect predictors were tested for inclusion, including Condition, average rating on the pre-questionnaire, and grade level. *D*-type effect sizes were calculated using the method outlined in Chinn (2000); in Cohen (1988), effect sizes of 0.2, 0.5, and 0.8 are considered small, medium, and large, respectively.

Results

Table 1 shows how the experimental and control groups compared on pre-measures and the post-test. While they were very comparable in terms of the measures of interest and self-efficacy, the experimental group had directionally lower scores on both the pre-test and the mid-year district standardized assessment.

Table 1: Comparison of Experimental and Control Group

	Control Group Avg (SD) (*N*=77)	Experimental Group Avg (SD) (*N*=94)
Situational Interest	3.07 (0.87)	3.04 (0.85)
Self-Efficacy	3.00 (0.94)	3.06 (0.98)
Pre-test Score	22.84% (28.05)	20.46% (30.03)
Mid-Year Standardized Test	49.10% (15.40)	44.83% (17.78%)
Post Test	31.28% (29.59)	34.37% (29.31)

The latter difference neared significance in a one-tailed *t*-test ($p = .053$). As mentioned previously, these differences are not surprising given the clumsy nature of random assignment at the classroom level. For this reason, it is clear that post-test differences between groups should be interpreted using statistical methods that take into account relevant covariates, like our regression models.

Performance on Algebra Post-Test

Results for the regression analyses predicting performance on the post-test items are given in Table 2. Model 1, the main effects model, shows that being in the personalization condition significantly enhanced post-test performance, with a small-to-medium effect size calculated at d=0.35 (Odds=1.87). Other factors that enhanced post-test performance included higher scores on the pre-test and the district standardized test, being in Algebra I, not being a recent immigrant, and attempting to pose a problem on the pre-test. The latter main effect is somewhat surprising, given that it also had a small-to-medium effect size calculated at d=0.39. This variable might be indicating students' level of proficiency with the English language, which may be important when solving algebra story problems.

Model 2, the interactions model, revealed an interaction between condition and course, where personalization was most beneficial for students not in Algebra I with a medium effect size d=0.53 (Odds = 2.60). There was not a significant effect of personalization on learning for students in Algebra I (B=0.954-1.025=-0.071). The effect of personalization on post-test performance is being driven by the subgroup of students who are most in need of assistance – those placed into the lowest mathematics track. These students were considerably more likely to solve post-test problems correctly if they participated in the teaching experiment, compared to business-as-usual.

Table 2: Mixed Effects Logistic Regression Models Showing Post-Test Performance

	Main Effects Model (Model 1)				**Interactions Model (Model 2)**			
Random Effects:	**Variance**				**Variance**			
Student ID	1.146				1.084			
Problem ID	2.580				2.583			
Fixed Effects:	**B(SE)**	**Odds**	**95% CI Odds**	**Sig.**	**B(SE)**	**Odds**	**95% CI Odds**	**Sig.**
(Intercept)	-4.885 (0.576)	0.008	(0.002, 0.024)	***	-5.157 (0.595)	0.006	(0.002, 0.019)	***
Mid-Year Test Linear Functions	0.046 (0.008)	1.047	(1.031, 1.063)	***	0.047 (0.008)	1.048	(1.032, 1.064)	***
Pre-Test	0.018 (0.006)	1.018	(1.007, 1.03)	**	0.017 (0.006)	1.018	(1.007, 1.029)	**
8th Grade Math	(ref.)				(ref.)			
Algebra 1	1.688 (0.325)	5.410	(2.847, 10.279)	***	2.184 (0.409)	8.880	(3.958, 19.923)	***
Wrote Story Pre	0.699 (0.263)	2.013	(1.198, 3.382)	**	0.811 (0.266)	2.250	(1.332, 3.801)	**
Recent Immigrant	-2.846 (0.952)	0.058	(0.009, 0.38)	**	-2.959 (0.948)	0.052	(0.008, 0.337)	**
Control Condition	(ref.)				(ref.)			
Personalized Condition	0.628 (0.237)	1.873	(1.173, 2.992)	**	0.954 (0.289)	2.595	(1.466, 4.595)	***
Personalized Condition x Algebra 1					-1.025 (0.516)	0.359	(0.13, 0.993)	*

Note. * p<.05, ** p < .01, *** p < .001. (ref.) denotes the reference category to which effects are compared.

Ratings on Interest and Self-Efficacy Post-Questionnaire

Results for the regression analyses predicting interest ratings on the post-questionnaire are in Table 3. For the main effects model (Model 3), the only variables that predicted interest at post were interest rating on the pre-questionnaire and score on the district standardized test. However, in Model 4 which tested for interactions with condition, there was a statistically significant interaction between condition and students' tendency to write a story problem on the pre-test of algebra skill. For students who wrote a story problem at pre-test, there was no difference in interest between the experimental and control groups on ratings of interest (B=0.307-.489=-0.182, p=0.165). However, for students who did not write a story problem at pre-test (62.6% of students), there was a significant positive difference in interest ratings at post of 0.307 points (95% CI [0.101,0.514]), p=.004), favoring the personalization group. Receiving personalized instruction does seem to be associated with an increase in interest, but this effect is limited to students who prior to the teaching experiment had potentially weaker problem-writing skills. This again suggests the personalized problem-posing activities are benefitting struggling students. Regression results for the self-efficacy items (not shown) showed no significant effects for Condition or for the interaction of Condition with any of the other predictors (ps > 0.1).

Table 3: Linear Regression Models Showing Avg. Interest Ratings on the Post Questionnaire

	Main Effects Model (Model 3)			Interactions Model (Model 4)		
Fixed Effects:	**B(SE)**	**95% CI**	**Sig.**	**B(SE)**	**95% CI**	**Sig.**
(Intercept)	0.231 (0.164)	[0,0.643]		0.091 (0.178)	[-0.256,0.440]	
Mid-Year Test	0.007 (0.003)	[0.002,0.012]	**	0.009 (0.003)	[0.004, 0.105]	***
Avg. on Interest Pre-Questionnaire	0.749 (0.050)	[0.651,0.847]	***	0.723 (0.050)	[0.625,0.820]	***
Wrote Story Pre				0.332 (0.121)	[0.096,0.569]	**
Control Condition				(ref.)		
Personalized Condition				0.307 (0.105)	[0.101,0.514]	**
Personalized Condition x Wrote Story				-0.489 (0.171)	[-0.824, -0.154]	**

Note. * p<.05, ** p < .01, *** p < .001. (ref.) denotes the reference category to which effects are compared.

Problems Posed by Students

An analysis of what occurred during the 4 days of the teaching experiment, while important to this research as a whole, is beyond the scope of the current paper – here our research questions focus only on pre-/post- differences. However, we give some examples of problems written by students in Table 4 to provide some context for the quantitative results.

Table 4: Problems Posed by Students

Session of Teaching Experiment	Example of Problem Students Posed
Session 1	David is Instagram famous and every minute he gets 40 likes. Fill in the chart with the number of likes he will get in 4 minutes.
Session 2	Lucas is playing GTA Band every time he dies, he loses $40.00. Write a linear equation that shows the relationship between money and every time he dies.
Session 3	The Dallas Stars are destroying the Red Wings tonight. In the first period it was 11-2. If this keeps up for the next two periods, what will be the final score? Make a linear equation.
Session 4	Melanie had 60% of battery on her phone. She lost 10% every hour. Write a linear equation that shows the relationship between % of battery and hours.

Discussion and Significance

We contrasted an approach where students posed, solved, and shared problems related to their out-of-school interests to business-as-usual instruction in 8^{th} grade math classes. The control group experienced direct instruction where they solved problems on worksheets and discussed them as a class. An interesting facet of the comparison is that the control group tended to solve many more problems per class period (10-20 problems), while the experimental group focused in on posing and solving just a few. To an outside observer, the control group likely appeared to be more orderly and efficient. However, the experimental group learned more from the "messiness" involved with grappling with challenging mathematical ideas, and also in some cases saw increases in their interest in learning algebra. This is consistent with other studies on researcher-delivered (rather than student-generated) personalization (e.g., Walkington, 2013). However, compared to this prior research on personalization, here were see a slightly bigger effect size (medium instead of small), and this is also one of the first studies to test personalization with a more diverse student population. This study is also the first classroom study to put students at the center of the personalization process where they are posing their own problems based on their interests and experiences.

This study shows the potential of pedagogical approaches that make mathematics meaningful and relevant, conceptualize students as competent agents who can control their own learning, and that allow for rich mathematical discussions around challenging ideas (e.g., Moses & Cobb, 2001; Boaler, 2002). This study was carried out in an urban middle school in danger of not meeting state mathematics achievement benchmarks, with large class sizes and a diverse student population, many of whom did not speak English as their first language. Approaches that utilize and value the funds of knowledge that all students bring with them to the classroom can improve learning and interest and promote equity (e.g., Civil, 2007). Although funds of knowledge research has been critiqued for not employing multiple methods (Rios-Aguilar, Kiyama, Gravitt, & Moll, 2011) like quantitative analyses of effectiveness, this study expands the research base. An approach where students draw upon their own funds of knowledge, rather than rely solely on the teacher to make connections to their lived home and community experiences, could be significantly easier to scale and more authentic. This study also offers evidence that the activation of interest and student learning of mathematics go hand-in-hand (e.g., Hidi & Renninger, 2006; Mitchell, 1993). Challenging activities can increase the motivation of students struggling in the mathematics classroom, if proper supports (like funds of knowledge) are utilized. And finally, the results of this study inform the next iteration of our design-based research trajectory, the ultimate goal of which is to build an intervention that teachers can implement in different classroom contexts.

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DISENTANGLING THE RESEARCH LITERATURE ON "NUMBER SENSE": THREE CONSTRUCTS, ONE NAME

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We review research literature concerning "number sense" from several related fields. Whereas other authors have pointed to difficulty defining "number sense" or to some degree of inconsistency in the literature, we argue instead that this is a case of polysemy: There are 3 different constructs that go by the same name. In this article, we clarify the research literature concerning "number sense" by naming and defining these 3 constructs, identifying similarities and differences between them, and contrasting themes in each body of literature by drawing upon a sample of 124 research articles that focus on "number sense."

Keywords: Number Concepts and Operations

> What's in a name? That which we call a rose, by any other name, would smell as sweet. — Juliet in Romeo & Juliet, William Shakespeare

There has been increasing interest in "number sense" from researchers in fields including experimental psychology, mathematical cognition, special education, and mathematics education. As an illustration, in a search for research articles with "number sense" in the title, we found 13 articles published in the 1990s, 40 articles published in 2000–2009, and 71 articles published in 2010–2016. Yet there seem to be a wide variety of uses of the term *number sense*. For example, consider the following two article titles: "Relationships among computational performance, pictorial representation, symbolic representation and number sense of sixth-grade students in Taiwan" (Yang & Huang, 2004) and "Wild number sense in brood parasitic Brown-headed Cowbirds" (Low, Burns, & Hauber, 2009). As this contrasting pair illustrates, we find a wide range of uses of the term *number sense* in the research literature.

More broadly, researchers in the social and behavioral sciences have become concerned with impediments to progress resulting from confusion over constructs (Brown, 2015; Gintis, 2007; Larsen, Voronovich, Cook, & Pedro, 2013; Le, Schmidt, Harter, & Lauver, 2010; Shaffer, DeGeest, & Li, 2016). Two particular issues have the potential to plague the research literature: synonymy and polysemy. *Synonymy* refers to different terms having the same meaning. *Polysemy* refers to the same term being used in different ways. Larsen et al. (2013) argue that these issues result in a proliferation of constructs and meanings, leading to "reverse progress" (p. 1532) as less is known over time about relationships between constructs, relative to the number of constructs that appear in the literature. As we will demonstrate, we regard the varied uses of the term *number sense* in the research literature as a problematic case of polysemy.

Many authors have noted difficulties defining "number sense" or disparities in the definitions and descriptions found in the literature (e.g., Andrews & Sayers, 2015; Berch, 2005; Dunphy, 2007; Howell & Kemp, 2005; 2009; 2010; Lago & DiPerna, 2010). In a seminal article published 25 years ago, McIntosh, Reys, and Reys (1992) emphasized the need to clarify the meaning of "number sense" in order for related research to progress. More recent articles have pointed to differences in definitions and assumptions about "number sense" but have assumed that these reflect different "perspectives" concerning a single construct, rather than fundamentally different constructs with the same name (Berch, 2005; Andrews & Sayers, 2015).

The above critiques of the "number sense" literature have each been situated within a particular field. Our review, which is a product of collaboration between researchers in special education and mathematics education, is concerned with the need for greater clarity in the "number sense" literature across fields. In this article, we categorize the literature focusing on "number sense" based on researchers' definitions and assumptions. Our systematic review of a sample of 124 research articles leads to a clear conclusion that is responsive to the issues identified above: Instead of disagreement over a single construct, we find three distinct "number sense" constructs at play in the literature. We argue that this is a problematic case of polysemy and a microcosm of broader issues of construct confusion in the social and behavioral sciences. It is difficult for research concerning a particular "number sense" construct to advance so long as authors continue to attempt to draw upon literature concerning different constructs that go by the same name. These different constructs involve contrasting assumptions about the nature of "number sense" and they are embedded in traditions with distinct orientations and concerns.

Method

We conducted a literature review to answer the following research question: How is the term *number sense* used in research literature in the social and behavioral sciences? In particular, how is "number sense" defined, and what assumptions do researchers make about the nature of "number sense"? We searched five databases (Academic Search Complete, Education FullText, ERIC, JSTOR, and Psyc INFO) for research articles with "number sense" in their titles. It was important to control the scope of our review in this way, because we sought to identify how "number sense" was defined and analyzed and to identify characteristics of "number sense" research. This being our purpose, including all articles that made any mention of "number sense" would have muddied our results. The numbers of journal articles that mentioned "number sense" varied from 168 to 1,587 in the databases listed above, and many of these were not research studies or were studies that did not focus on "number sense." For the purposes of our review, the relevant studies were those in which "number sense" was central to the research. We found the inclusion of "number sense" in the title of the article to be a reasonable proxy for this centrality.

We searched for all such articles that met the criteria described above and that were published on or before December 31, 2016. We focused on research articles published in peer-reviewed journals. Thus, we filtered out practitioner articles, books and book reviews, and conference abstracts and proceedings. We also filtered out publications not written in English. A final count of 124 articles qualified for inclusion in our sample. We recognize that some high-quality articles concerning "number sense" may not be included in our sample as a result of these requirements. Our purpose was not to provide comprehensive reviews of the literature belonging to each "number sense" tradition. It was to identify and describe "number sense" constructs based on a sufficient sample of the research literature associated with each construct.

In addition to the sample of articles described above, we consulted seminal works and publications of historical significance that explicitly addressed "number sense." We identified these based on their being cited frequently in our sample of research articles and/or representing a synthesis of research related to "number sense" in a particular research tradition. We included in this category the works of Dehaene (1997/2011), Geary, Berch, and Koepke (2015), Sowder and Schappelle (1989), and Sowder (1992). Consulting these sources provided us greater access and insights into the history of "number sense" research and enabled us to answer questions concerning definitions, assumptions, findings, and themes in cases in which a consensus could not be identified within our sample of articles.

We initially read a selection articles from our sample, focusing on authors' interpretations of "number sense" and the apparent origins of those interpretations (based on citations and use of key constructs), as well as the populations studied and methods used. We proceeded using open coding to define distinct "number sense" constructs. We refined our definitions through constant comparative analysis as we reviewed additional articles (Corbin & Strauss, 2008). Once we had reached a saturation point, we settled on three constructs: (a) *innate number sense*, (b) *early number sense*, and (c) *mature number sense*.

We grouped the bodies of literature related to each of the three constructs into distinct *research traditions*. We each took primary responsibility for reading and summarizing the literature belonging to one of the traditions. We developed these summaries iteratively with feedback from one another. We then compared our summaries based on key concepts, analytic approaches, findings, and themes. Questions that arose concerning similarities and differences between traditions or lack of clarity regarding any terms led us to return to our sample of articles and/or our set of seminal publications for answers. This process, too, was iterative. We refined our summaries of the aspects of each tradition in order to focus more clearly on the similarities and differences between traditions. Length constraints for this manuscript do not permit us to report on the research traditions in any detail; however, the themes that we identified helped to focus our descriptions of the "number sense" constructs and related concepts from each tradition.

Findings: Three "Number Sense" Constructs

We describe the three "number sense" constructs, along with key concepts related to each of them, based on our review of the literature. We also highlight similarities and differences in assumptions that distinguish these constructs.

Overview of the Three "Number Sense" Constructs

Innate number sense (INS) is believed to be an inborn set of neurological abilities that is common to humans and some animals. Thus, INS research involves infants, children, adults, and non-human animals (e.g., Libertus & Brannon, 2009; Halberda & Feigerson, 2008; Low et al., 2009). This construct concerns perception and discrimination of magnitudes, rather than explicit knowledge of number words or symbols. Much of the research with humans involves observing brain activity while participants perform tasks such as determining which of two sets consists of more items (e.g., Dehaene, 2001; Libertus & Brannon, 2009; Stoianov & Zorzo, 2012). Dehaene (1997/2011) believes that most people are born with an equal endowment of number sense and, therefore, INS is not predictive of success in learning mathematics. Dehaene's use of the term *the* number *sense* (with the definite article *the* and emphasis on the word *number*) is indicative of the view of INS as an innate sense, which is related to visual and auditory perception.

Early number sense (ENS), in contrast to INS, includes learned skills that involve explicit number knowledge, such as counting items using number words and comparing numbers represented symbolically as numerals. Some researchers believe that ENS builds upon the more basic INS (Andrew & Sayers, 2015; Aunio et al., 2005; Geary, et al., 2015). Levels of ENS skills vary from person to person and are influenced by education and experiences in early childhood (Cheung & McBride-Chang, 2015; Dunphy, 2006). ENS is regarded as an important predictor of success in school mathematics (Dyson, Jordan, & Glutting, 2011; Locuniak & Jordan, 2008; Jordan, Kaplan, Locuniak, & Ramineni, 2007). Accordingly, ENS skills are well aligned with school mathematics, especially in the early childhood years (preschool to Grade 2). Typically, studies of ENS involve young children or students with disabilities. ENS research does not belong to a single field. It is conducted primarily by researchers in mathematics education, special education, and cognitive psychology.

We use the term *mature number sense* (MNS) to distinguish the "number sense" construct that features prominently in the mathematics education research literature. MNS encompasses multidigit and rational number sense, and studies focus primarily on middle-grades (i.e., upper elementary and middle school) students and preservice teachers. Like ENS, the MNS construct refers to something learned, rather than innate. In contrast to ENS, MNS is typically described in terms of *components*, which refer to conceptual structures and habits of mind, rather than skills (e.g., McIntosh et al., 1992; Reys & Yang, 1998). For example, MNS is associated with flexibility in mental computation (Markovits & Sowder, 1994). Furthermore, whereas ENS is well aligned with school mathematics, MNS is often contrasted with school mathematics: Students who competently perform computations using standard algorithms may not exhibit characteristics of MNS, such as flexibility (Reys & Yang, 1998; Reys et al., 1999).

Key Concepts in "Number Sense" Research Traditions

Having provided an overview of the three constructs, we delve deeper into related concepts that appear in the three corresponding research traditions.

Key concepts in INS research. According to Dehaene (2001), "Number sense is a short-hand for our ability to quickly understand, approximate, and manipulate numerical quantities" (p.16). INS is considered part of an evolutionary process related to neurological abilities. Specifically, three neurological abilities are associated with INS: perceptual subitization, magnitude discrimination, and the use of a mental number line. *Perceptual subitization* can be defined as rapidly or immediately identifying numerosities of sets consisting of up to three or four items (Clements, 1999; Dehaene, 2001). Any numbers that are beyond four are then approximated with less precision (Clark & Grossman, 2007). *Magnitude discrimination* consists of indicating the difference in cardinality between two sets of items (presented visually or auditorily) (Dehaene, 2001). The *mental number line* is a mental approximation of magnitude based on a continuous number line believed to be present in an individual's mind (Clark & Grossman, 2007). The use of a mental number is inferred from the ability to "quickly decide that 9 is larger than 5, that 3 falls in the middle of 2 and 4, or that 12 + 15 cannot equal 96" (Dehaene, 2001, p. 16). (Although comparisons of numerals require explicit number knowledge, they are taken as evidence of a mental number line in humans who have developed such knowledge.)

Key concepts in ENS research. ENS is conceptualized and studied as a set of skills. In a prominent example, Jordan and colleagues (2006) focused on "assessed skills that have been validated by research and are relevant to the mathematics curriculum in primary school" (p. 154). Six main skills are focal in ENS research: number recognition, counting, number patterns, number comparison, number operations, and estimation. *Number recognition* requires children to associate the number symbols with the vocabulary and meaning of numbers. *Counting* includes ordinality, cardinality, and counting backward or forward starting with an arbitrary number. *Number patterns* is the ability to copy a given pattern or identify a missing number in a sequence. *Number comparison* refers to awareness of the magnitude of given numbers and the ability to make comparisons between different magnitudes. *Number operations* involves the ability to perform simple calculations of sums and differences within 10 or 20. *Estimation* refers to magnitude estimation of symbolic and non-symbolic quantities, including the use of a number line to identify the approximate location of a number (Andrews & Sayers, 2015; Berch, 2005;

Baroody et al., 2012; Ivrendi, 2011; Jordan et al., 2006; Howell & Kemp, 2010; Malofeeva, Day, Saco, Young, & Ciancio, 2004; McGuire, Kinzie, & Berch, 2012).

Key concepts in MNS research. McIntosh and colleagues (1992) provided a definition of MNS that has often been cited or paraphrased:

> Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing, and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity. (p. 3)

The above description seems to capture the gist of the term *number sense*, as it is commonly used in the mathematics education community. It focuses on habits of mind and ways of behaving mathematically that are considered desirable, such as flexibly manipulating numbers.

Despite this holistic definition, MNS is typically partitioned into up to six components: *understanding of the meaning and size of numbers* (e.g., to compare fractions), *understanding and use of equivalent representations of numbers* (e.g., to write rational numbers in different ways), *understanding the meaning and effect of operations* (e.g., to reason about the effect of dividing by a number between 0 and 1), *understanding and use of equivalent expressions* (e.g., to compare expressions involving different numbers and/or operations), *flexible computing and counting strategies for mental computation, written computation, and calculator use* (e.g., to select strategies and perform mental computation), *and measurement benchmarks* (e.g., to estimate the height of an object) (Reys et al., 1999; p. 62). Many studies use assessments designed to measure specified components of number sense (e.g., Yang & Lin, 2015).

Similarities and Differences

In summary, INS is regarded as innate and equally distributed among normal people at birth. It is also found in some animals. INS consists of a set of basic neurological abilities, which do not account for success in learning mathematics. ENS, by contrast, is regarded as learned. It is unequally distributed among people and is not found in animals. ENS is typically conceptualized as consisting of a set of skills, and these skills are well aligned with primary-grades mathematics. MNS is also learned and unequally distributed among people. It is typically described as consisting of a set of components, which include conceptual understandings and habits of mind. In contrast to ENS, MNS is often described as being at odds with students' typical experiences in school mathematics and the mathematical knowledge and orientation that result.

Discussion and Conclusion

Whereas other authors have observed differences in definitions or interpretations of "number sense," they have assumed that this confusion surrounds a single construct (Andrews & Sayers, 2015; Berch, 2005). For example, Berch (2005) referred to "*the* concept of number sense" (p. 333, emphasis added) and Andrews and Sayers (2015) described "number sense" as "*a* poorly-defined construct" (p. 257, emphasis added). These previous observations were made from the perspective of researchers in the ENS tradition, and their purpose was not to clarify the "number sense" literature more broadly. Whereas INS and MNS research represent two extremes, ENS research lies in between them, and particular studies may lean closer to one side or the other. Thus, it is ENS researchers who face the most potential for confusion in attempting to navigate the muddled "number sense" literature. It is no surprise, then, that ENS researchers have taken the lead in recent efforts to clarify what "number sense" means in order to facilitate progress in ENS research.

Our systematic review led to the identification of three "number sense" constructs. To the best of our knowledge, our review of the "number sense" literature is the first of its kind. It had the express purpose of analyzing the use of the term *number sense* in research literature across fields in order to disentangle the meanings and assumptions associated with the term. The contrasting features of the constructs underscore the need to clearly distinguish them.

The results presented above contribute to the literature by clarifying distinctions between three constructs that have gone by the same name. The widespread use of the term *number sense* to refer to three distinct constructs belonging to different traditions has led to some confusion and has not gone unnoticed in the literature. By systematically coding the articles in our data set according to the authors' assumptions about "number sense" and their methods of investigation, we were able to clarify the nature of the construct within each tradition.

The differences that we identified in definitions and assumptions are not a trivial observation about the literature. We noted in our review many instances of inappropriate citations across research traditions. For example, some MNS articles cite INS research to support their claims about "number sense" despite the fact that their research concerns a different construct. To be clear, authors working in one "number sense" tradition should not be citing authors working in another tradition, unless there is explicit acknowledgment of the differences between traditions and unless there is a particular reason for the citation. We suggest that researchers use terms such as INS, ENS, or MNS to distinguish the "number sense" construct that they are investigating. Although "number sense" is a catchy term that rolls off the tongue, its loose usage across related research traditions has led to confusion and impediments to progress.

In conclusion, we find in the "number sense" literature a problematic case of polysemy. As Larsen et al. (2013) state, "The task of integrating research by connecting synonymous constructs and parsing polysemous constructs is an urgent one if behavioral science is to advance" (p. 1533). We agree, and we add that the same issue applies to research in the social sciences. In particular, in order to propel progress in "number sense" research in a variety of fields, there is a need to clarify the construct under investigation within each tradition.

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EXPLORING THE STRUCTURE OF EQUIVALENCE ITEMS IN AN ASSESSMENT OF ELEMENTARY GRADES

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This study is focused on the structure of equivalence problem to probe the evolution from operational to relational view of students' understanding of equals sign. We propose a modified construct map which incorporates the intermediate levels in such a transition which were previously ignored. Our findings suggest that the structure of number sentences (place value and the position of answer box) has undeniably significant role in developing students' conception of equivalence. In addition, the designed and validated example presented here could potentially serve as a tool for better assessment of understanding of equivalence.

Keywords: Number Concepts and Operations, Algebra and Algebraic Thinking, Elementary School Education, Assessment and Evaluation.

Background / Overview

There is a consistent and increasing focus by educational researchers on the development of elementary grade algebraic reasoning, specifically in regard to the use of open number sentences (e.g. 4 + ☐ = 5 + 7) (Falkner, Levi, & Carpenter, 1999; McNeil & Alibali, 2004; Molina & Ambrose, 2008; McNeil, Fyfe, Petersen, Dunwiddie & Berletic-Shipley, 2011). Such open number sentences often include expressions on both sides of the equation and are often introduced as arithmetical equations where students are tasked to find the unknown or missing number in place of a blank or empty box. This affords students the opportunity to explore the underlying structure of an arithmetical equation and improve their understanding of the meaning of symbols and operations. Various researchers (Mc Neil & Alibali, 2004; McNeil et al., 2006; Sherman & Bisanz, 2009; Powell, 2014) suggest that many elementary students are introduced to only traditional arithmetic equations (i.e., a + b = c). These studies suggest that such *operations equals answer* type equations encourage the operational view and may hinder students' development of a relational view of the equals sign.

In an operational view of the equal signs students carry the notion that the equals sign means *makes*, *produces the answer*, *find the total*, or as an indication *to do something* such as computation (Behr, Erlwanger & Nichols, 1976; Kieran, 1981; Seo & Ginsburg, 2003; Knuth, Stephens, McNeil & Alibali, 2006; McNeil et al., 2006; Jacobs et al, 2007). Students holding a relational view consider the equals sign as a mathematical symbol which represents the sameness of the expressions or quantities on either side of an equation (Kieran, 1981; Baroody & Ginsburg, 1983; Falkner et al., 1999; Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Blanton, Levi, Crites, & Dougherty, 2011). A vast majority of research suggests that a relational understanding of equivalence is a first step towards early algebraization (Falkner et al., 1999; Carpenter & Levi, 2000; Blanton & Kaput, 2005; Jacobs et al., 2007; Byrd, McNeil, Chesney, & Matthews, 2015). Students holding a more operational view tend not to develop the conceptual understanding of arithmetic and other more advanced mathematics such as Algebra (Kiren, 1981; Knuth et al., 2006; Jones, Inglis, Gilmore, & Dowens, 2012; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). Thus, the traditional way of introducing equal sign ($a + b = c$) tends to focus predominately on step-by-step computation to find the answer rather. By contrast, explorations of the underlying structure or number relations between and within the expressions appears to require alternative forms of equations.

The research to date generally interprets the structure of equations as open number sentences ($a + b = c + \square$) vs traditional arithmetic equations ($a + b = \square$). However, it is unclear whether and to what degree other mathematical structure within such equations interacts with children's conceptions of equivalence. Namely, place-value is an aspect of mathematical structure that children engage concurrently with their developing conceptions of equivalence. The aim of this study is to investigate the role of number structure, specifically place-value with whole numbers, in students' conception of mathematical equivalence. To facilitate this purpose, we examined third grade students' responses to a conceptions of equivalence assessment using psychometric analysis.

Theoretical Framework

Numeric Structure in Number Sentences

Traditionally, most studies on equivalence define two basic categories of students' conceptions of the equals sign: operational and relational. More recently, Rittle-Johnson et al. (2011) elaborated on this dichotomy (operational vs relational) and expanded it into four levels ranging from rigid operational to comparative relational. Students at Level 1, the rigid operational level, are expected to successfully solve the traditional format (i.e., a + b = c). Students at Level 2, the flexible operational level, maintain an operational view of the equal sign with some flexibility to correctly solve and accept the atypical or "backwards" equations (i.e., c = a + b and a = a) as valid. At Level 3, the basic relational level, children successfully solve, evaluate, and encode equation structures with operations on both sides of the equal sign (such as a + b = c + d or a + b - c = d + e). Finally, children identified at Level 4, the comparative relational level, show a more nuanced understanding of the equals sign. These students can correctly solve and evaluate equations by comparing the expressions on both sides of equal sign. Students at this level use compensatory strategies. For example, in solving 37 + 24 = 36 + □, such students may recognize that 36 is 1 less than 37 and use this knowledge to determine that the unknown number must be one more than 24 (Carpenter et al. 2003).

More recently, Singh & Kosko (2015, 2016a) observed other possible levels in the continuum of conception of equivalence. Therefore, we argue that further modifications to the field's models for the ways students consider equivalence are needed. Specifically, students who can successfully solve $a + b = c + d$ types of equations are currently evaluated as holding a *basic relational* conception of the equals sign. However, Rittle-Johnson et al (2011) suggest that the construct is continuous, which allows for the possibility of other sub-constructs between consecutive levels.

Singh and Kosko (2015) conducted a teaching experiment with a variety of equivalence problems and found that some students demonstrated a *pseudo-relational conception* (PRC) of equivalence. Specifically, student with a PRC can solve problems of the form $a + b = c + \square$ when the numbers involved allow them to regroup 10's and 1's in an obvious manner. For example, in problems like 34 + 25 = 50 + □**,** such students first regroup 10's from 34 and 25 (30 + 20 = 50), which is visually available on right hand side. Thus, these students are then able to add the ones (4 + 5 = 9) to find the solution. At first glance, it may seem like these students hold a relational view of equivalence. However, when closely comparing their strategies in other equations of the same structure but different numeric structure, it was apparent that the place-value structure of such equations allowed students to use different strategies than working with other $a + b = c + \square$ types of equations. For example, in solving 15 + 24 = 20 + □, a student successful with the prior example failed to provide the correct solution when using their regrouping of 10's and 1's strategy. This suggests that the mathematical structure numbers in an equation, such as that of place-value, plays an important role in students' conceptions of equivalence. This was verified in another study by Singh & Kosko (2016a) in which the authors found that some students can successfully solve problems like 4 + 5 + 8 = □ + 8 by finding the answer 9 (i.e., 4 + 5), whereas the same students demonstrated a different conception in solving very similar problem like 7 + 6 + 4 = 7 + □☐. These students used an *adding all then subtracting* strategy to solve the problem 7 + 6 + 4 = 7 + □☐. Specifically, students first added 7 + 6 + 4 and then the 7 on the right-hand side to get 24. They then subtracted 7 to obtain the answer 17. Singh and Kosko (2016a) suggest that the use of this strategy may have more to do with the position of the missing value box than students' operational or relational view of the equal sign.

Conceptual Model for Conception of Equivalence

To define the way an equivalence problem is presented to students, different researchers have used different terminologies, or the same terminology with different meanings. Molina & Ambrose (2008) used the term structure in reference to the structure of mathematics operations. Later, Molina, Castro, & Castro (2009) used the term structure in the same sense as used by Kieran (1989), describing the surface structure of arithmetic and algebraic expressions. Recently. Stephens et al. (2013) used the term "equation structure" as opposed to focusing on probing their computational fluency. This study uses a broad definition of structure of number sentences which includes an emphasis on place value and the position of the missing value box(es). Thus, the definition differs significantly from the meaning of structure used in prior studies.

Using our definition of structure of number sentence, and based both on our previous findings (Singh & Kosko, 2015; Singh & Kosko, 2016a), and ongoing work with elementary students, we suggest six levels of conception of equivalence along a continuum (Figure 1). The construct map in Figure 1 includes levels from basic operational (least sophisticated) to full relational (most sophisticated). A student at the basic operational level can successfully solve traditional number sentences (a + b = c) with various positions of unknown or box such as 6 + 7 = ☐ or 5 + ☐ = 9, while students at the flexible operational level can successfully solve less typical number sentences (e.g. 19 = ☐ + 3; 24 = 10 + ☐; ☐ = 5 + 7). However, both types of conceptions include student strategies that rely on an operational view of equals.

We argue that the transition from flexible operational to basic relational is not always smooth and is accompanied by the existence of *pseudo-relational level*. Students at this level are able to solve number sentences which have operations on both sides (such as a + b = c + d), but only in cases where the number sentences can be solved by using regrouping of ones and tens addends.

Similar to the pseudo-relational level, students at the basic relational level can successfully solve number sentences which have operations on both sides (a + b = c + d) with the position of the box directly after or before the equals sign. Such students can confirm the sameness of expressions on both sides of the equals sign through computation.

Prior to a full relational conception, we suggest some students demonstrate what we describe as an advanced basic relational level. This level is characterized by students who can successfully solving number sentences with operations on both sides (a + b = c + d), but such students may use either computation or compensatory strategies. Finally, at the full relational level, students can successfully solve number sentences which have operations on both sides (such as a + b = c + d) and two unknowns either both on same side or one on each side of equal sign by relying predominately on compensatory strategies.

Direction of Increasing sophistication in "Mathematical Equivalence"

Students	Responses to Items
Full Relational	Anticipates successful use of compensatory strategies to solve item types, e.g. □ + 28 = 46 + ○;
Advance Basic Relational (□ at the end of RHS of =)	Anticipates confirmation of the sameness of expressions on both sides of equal sign either using computation or compensatory strategies in items e.g. 15 + 27 = 18 + □
Basic Relational	Anticipates confirmation of the sameness of expressions on both sides of equal sign with computation in items e.g. 13 + □ = 24 + 8; 27+ 16 = □ + 35.
Pseudo-relational (with regrouping	Anticipates to solve specific items e.g. 22 + 15 = 30 + □ by regrouping of ones and tens of addends. Students get first addend in right side of equal sign by regrouping tens of both addends of left side and find the unknown by simply regrouping ones.
Flexible-Operational	Anticipate to solve atypical number sentences such as operations on right e.g. 19 = □ + 3; 24 = 10 + □; □ = 5 + 7 or 12 = □.
Basic Operational	Only successfully solve traditional number sentences (a + b = c) with various positions of unknown or box such as 6 + 7 = □ or 5 + □ = 9.

Direction of Decreasing Sophistication in "Mathematical Equivalence"

Figure 1. Construct map for knowledge of equivalence.

The different levels of students' conception of equivalence are included in the construct map shown in figure 1. As discussed above, prior studies (Singh & Kosko, 2015,2016a) describe students' transition from operational to relational conception of equivalence is not always smooth. Rather, students' holding an operational view may do things that resemble, but do not comprise, a relational view of equivalence (i.e. *pseudo-relational*) when solving some specific types of equations. The construct map in figure 1 illustrates that as students move in a continuum they should engage in different levels of conception of equivalence.

Methods

Sample

Data were collected in Fall 2016 from 157 third grade students (49.7% male; 50.3% female) in a suburban school district in a Midwestern U.S. state. Students were enrolled in one of eight classrooms across four schools in the district. The district includes a predominately white student population (74%), with a significant portion of economically disadvantaged students (40%).

Test Development and Item Design:

Our previous work on equivalence indicates some observable gaps in Rittle-Johnson's established framework (Singh & Kosko, 2015, 2016a). In order to address these gaps, we designed a new assessment utilizing the aforementioned construct map. An initial version of the assessment was piloted with fourth and fifth grade students (n = 157) with 33 items. The overall reliability (Cronbach's alpha = 0.92) of the initial test was sufficient. However, the item discrimination for several items was not sufficient. Also, the infit mean square statistics for more than a quarter of items indicated that the initial construct map lacked unidimentionality. To improve these shortcomings in item design we revised some items for better fit. First, items with insufficient fit statistics were removed or revised. Many of these items illustrated what may be different constructs related to but not identifiable as conception of equivalence. Next, true/false items were removed since these items were found to have significant structural differences than missing-value addends, and such items also tended to have too low of difficulty to provide sufficient information for the assessment. The revised instrument included 22 items across six sub-constructs along a continuum (thus, these subconstructs are theorized to be hierarchical). Figure 1 shows the revised construct map with example items for the six subconstructs.

The revised instrument was used in the analysis of Fall 2016 data. Raw data was inputted into digital files before dichotomously coding student responses (0 = incorrect answer; 1 = correct answer). This allowed for examining raw response distributions, as well as analyzing the dichotomous data via a Rasch model. The significant feature of the Rasch model is its ability to transform ordinal data into equal-interval scales (Bond and Fox, 2015). The item difficulties in the Rasch model can be determined, by the process of item calibration, independently of the distribution of persons' abilities in the data and the measurement of person's traits (i.e. abilities) is independent of test items used to measure that trait. Another useful feature of the Rasch Model is that it facilitates the process of constructing measurement variables. In other words, the model is derived independently of data, tests are then constructed to fit the model, and then the data are used to see if they conform to the requirements of the model.

Results

The equivalence assessment was found to have sufficient internal reliability (α = 0.92). Crocker & Algina (1986) suggest that a Cronbach's alpha of 0.90 or higher is sufficient for cognitive assessments. Test statistics for the Rasch model indicate sufficient item reliability (0.92) and person reliability (0.89). To examine the unidimentionality of the assessment, infit and outfit statistics were calculated. The item difficulties range from -3.14 to + 2.12, which is considered as a good practice to have a range of difficulty among items in an assessment. In this assessment, we hypothesized that the item difficulty should increase from lower (i.e. operational) to higher (i.e. advance relational) levels and item difficulty appeared to increase as expected. The infit statistics are weighted and provide more weight to the performance of person whose ability is closer to the item difficulty level whereas outfit statistics is not weighted and as a result more sensitive to outlying scores. This is the reason that investigators give more attention to any small irregularities in infit scores than large outfit scores (Bond and Fox, 2015) The average item mean-square infit statistics is 0.90 and average mean-square outfit statistics is 0.77, which is considered sufficient. Contrasting the overall sufficient item fit statistics, item 8 (12 = □) was found to have a relatively high mean square infit statistic (1.45), which indicates more randomness than expected (figure 2). We had observed similar results for such items with our pilot of grade 4 and 5 students, and data from grade 3 students indicates that this particular item format (a =□ □) may need further study in regards to the concept of equivalence. We decided to remove this item from our final equivalence assessment, given the continued poor fit across samples. All remaining items appeared to have sufficient fit (Bond & fox 2015), with point-biserial statistics ranging between 0.65 to 0.79. To examine how items hypothesized to target specific levels along the continuum in the construct map, a Wright map was constructed (figure 2).

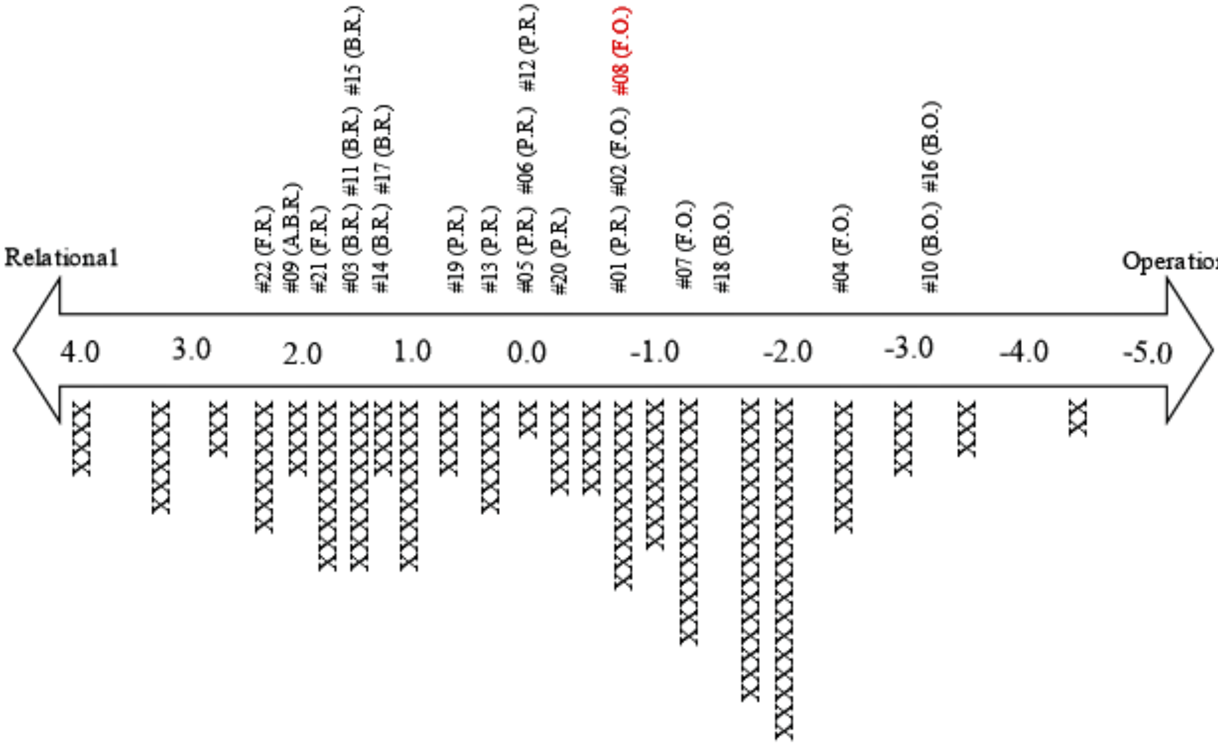

Figure 2. Wright map for equivalence assessment.

The Wright map shown in figure 2 was produced by ConstructMap 4.6.0 (Kennedy et al., 2010). Targeted levels of sophistication (see Figure 1) are abbreviated after each item number (*B.O.=Basic Operational; F.O.=Flexible Operational; P.R.=Pseudo-Relational; B.R.=Basic Relational; A.B.R.=Advanced Basic Relational; and F.R. Full Relational*). The location of items in the Wright map (Figure 2) aligns well with the hypothesized level of sophistication on the construct map (Figure 1). However, certain items (i.e., 8, 2, 18, & 21) visually appear at the same level as items at lower or higher hypothesized levels. Examination of the items' delta statistics and associated confidence intervals indicates that those items hypothesized as more relational than operational do indeed have higher delta statistics.

Item 21 (☐ + 28 = 46 + ○) had a lower delta statistic than predicted, and therefore appears lower on the Wright map than expected. After a close inspection of students' raw responses, it was observed that some students put a zero in the circular blank position "○." However, the the instructions for this item stated that students should "find a number bigger than 10 to write in the ☐…[and] any other number to write in the ○ that makes the problem true". By allowing for the possibility of students to use zero, it may have reduced the difficulty of the item. Specifically, the item was meant to engage students in composing and decomposing number in relation to equivalence. Thus, instructions for these items may need revision in future assessments.

Discussion and Conclusion

Through our findings, we establish that the structure of number sentences, particularly in regard to the role of place value, has a significant role in students' conception of equivalence. The results of our statistical analysis indicate that students may rely on more visually obvious aspects of place value in solving equations. This may appear to be relational at face value, but is not as sophisticated a conception of equivalence as other similarly formatted number sentences. Motivated by our findings from previous work (Singh & Kosko, 2015, 2016a) we incorporated new intermediate levels in Rittle-Johnson and colleagues' (2011) framework of students' conception of equivalence. Our suggested, modified construct map appropriately incorporates students' transition from basic operational to full relational understanding by considering these additional transitional stages. Furthermore, the designed and validated assessment described here can serve as a tool for researchers and practitioners interested in students' conceptions of equivalence.

Our results provide useful guidelines for instructors and curriculum designers. Specifically, our findings suggest more attention be paid to the role of place value in the teaching and learning of equivalence. Furthermore, there is a need for more careful examination of students' understanding of equivalence in regard to various mathematical concepts. Future research is needed to confirm and extend the findings presented here. For example, prior research has found relationships between students' conception of equivalence and multiplicative reasoning (Singh & Kosko, 2016b). Given the connection identified here regarding place value, a better understanding of how equivalence interrelates with children's developing number sense is highly needed. Additionally, findings presented here provide evidence that the structure of items similar to a = □ is in need of further study, given that understanding the *Reflexive Property of Equivalence* is crucial for students' success in future advanced mathematic. We expect that by conducting such research, the field may better understand why such items do not consistently align students' conceptions of equivalence.

This study found that place value appears to be inherently tied with conception of equivalence. These results are highly significant and need additional study. There appear to be other connections between various concepts and equivalence research (Singh & Kosko, 2016b), potentially indicating that students' unit coordination may relate to their coordination between expressions. Such interrelationships warrant detailed investigation.

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LEARNING INTEGER ADDITION: IS LATER BETTER?

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We investigate thirty-three second and fifth-grade students' solution strategies on integer addition problems before and after analyzing contrasting cases with integer addition and participating in a lesson on integers. The students took a pretest, participated in two small group sessions and a short lesson, and took a posttest. Even though the results reveal significant gains for both grades from pretest to posttest, second graders gained significantly higher than fifth graders. In this paper, we explore students' treatment of the negative sign and describe this gain difference.

Keywords: Number Concepts and Operations; Elementary School Education; Cognition

When students encounter new concepts, they try to apply their prior knowledge in an effort to make sense of the new information. Consequently, children's negative integer knowledge builds upon their whole number understanding (Bofferding, 2014). The transition from working with whole number concepts to interpreting new number classes appropriately requires substantial time and considerable conceptual change (Vosniadou, Vamvakoussi, & Skopeliti, 2008). However, there is a noteworthy time gap between when children learn about whole number concepts and the introduction of negative numbers. While whole numbers are introduced at an early age, negative numbers and operations are currently not taught until sixth and seventh grade (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Introducing negative numbers in upper elementary and middle school grades could serve as a barrier to student learning. As a result of this large gap, students may interpret negative numbers in various ways given their whole number knowledge, which may influence their solution strategies in problems involving negative numbers. For instance, students may treat the negative sign as a subtraction sign (e.g., solving -4 + 7 as 7 – 4) (Bofferding, 2010). In fact, Murray (1985) found that students continued to explain that you could not take larger numbers away from smaller ones (e.g., 5 – 8) even though they could solve other negative integer problems.

Even first graders can move toward having more formal mental models of integers, integer order, and integer values by engaging in instructional activities that help them focus on the meaning of the negative sign and evaluate the numbers in relation to each other (Bofferding, 2014). Similarly, instructional activities can help students make specific connections between the results of operating with negative and positive numbers. One way to encourage these connections is via the analysis of two carefully chosen, contrasting integer problems shown as worked examples. Studies have identified contrasting cases as a powerful instructional tool that has promising results in students' developmental knowledge in the mathematics classroom (e.g., Rittle-Johnson & Star, 2007; 2009; 2011). As one of the promising ways to help students conceptualize negative number operations, this method can help students leverage their prior knowledge of whole number addition to making sense of negative number addition problem types.

However, it is not clear to what extent analyzing contrasting cases of integer addition problems could benefit younger students compared to older students. Although younger elementary students can reason productively about integers, they may need more intense experiences in order to facilitate their conceptual change. On the other hand, older students may be less willing to modify their conceptions. In this paper, we analyze the performance of younger and older elementary students before and after they compare sets of integer addition problems. Further, we explore how their interpretations of negative number addition change.

Specifically, we ask the following research questions:

1. How do second and fifth-grade students perform in solving integer addition problems before and after analyzing contrasting cases involving negative numbers?
2. How do students' interpretations of the negative sign and addition operations change before and after analyzing the contrasts?

Theoretical Framework

Interpretation of the Minus Sign

Students engaged in working with algebraic equations experience difficulty when manipulating negative numbers due to the different perceived meanings of minus signs (Vlassis, 2004). The three primary meanings of the minus sign are unary, binary, and symmetric and also play a role in students' interpretations of integer addition and subtraction problems. The binary meaning of the minus sign corresponds to the subtraction operation (Vlassis, 2004), which students with a whole number mental model will interpret to mean getting less or smaller (Vosniadou, Vamvakoussi, & Skopeliti, 2008). Students with only a binary understanding of the minus sign might ignore negatives or treat them as a subtraction sign (Bofferding, 2010). For example, -8 + 6 can be turned into a subtraction problem as 6 – 8 or 8 – 6 and might be answered incorrectly or correctly. The symmetric, or opposite, meaning of the minus sign indicates an operation of multiplying by -1. In other words, the symmetric meaning illuminates the opposite positions of negative and positive numbers (Vlassis, 2004). Students rely on the symmetric meaning when they add integers as if they were positive and make the answer negative (Bofferding, 2010). For instance, -2 + 5 might be solved as 2 + 5, with students adding the negative sign at the end, and answering -7. Finally, the unary meaning of the minus sign involves seeing the negative sign as attached to a numeral, a negative number (Vlassis, 2004). Students with a strong unary understanding will often start at a negative number and count towards or away from the negative direction (Bofferding, 2010).

Conceptual Change

In prior research, students' difficulties in interpreting minus signs were primarily reflected in situations involving successive signs in arithmetic operations. From a conceptual change lens, this is unsurprising. Until students encounter negative numbers, they would not see two successive minus signs. Their unawareness of the multiple roles of the minus sign resulted in the mistreatment of the signs and signaled partial conceptions (Vlassis, 2008). In terms of conceptual change, students were trying to make sense of new information in light of their prior knowledge. During this process, they likely formed synthetic mental models (Vosniadou & Brewer, 1992), conceptions that blend their prior understanding with new hypotheses about the meaning of the new signs. Different studies have explored students' various ways of reasoning in their arithmetic solution strategies. They framed how students' understanding of the multiple meanings of the minus sign correspond to their integer arithmetic solutions (e.g., Bofferding, 2010, Lamb et al., 2012; Murray, 1985, Vlassis, 2008). Lamb, Bishop, Philipp, Whitacre, and Schappelle (2016) claimed that no single best model exists that could be applied in students' ways of reasoning. We build on this literature by using a conceptual change lens to investigate how students' conceptions of the minus sign – and consequently their strategies – change as they explore contrasting integer addition problems.

Methods

Participants and Settings

Both second and fifth-grade students engaged in this study. We chose to work with second graders because they represent students who typically have not had formal instruction in negative numbers but have whole number understanding. We chose fifth graders because they often have heard about negative numbers and would learn about them formally in the following school year. We recruited fifth-grade students from two public elementary schools in a rural Midwestern school district where 32% of students were classified as English-language learners (ELLs) and 75% qualified for free or reduced-price lunch. Based on their pretest scores, 17 of the 32 fifth graders who returned permission slips were selected for this analysis as they had not reached the ceiling in terms of their order and value integer mental models (formal mental model) (Bofferding, 2014). It was important not to include students with formal mental models so that we could compare their growth to the second graders' growth. Because we had already worked with second graders at the schools where we recruited the fifth graders, we recruited second graders from one classroom at a different school. The public elementary school was in a small city within a Midwestern school district where 14% of students were classified as ELLs and 57% qualified for free or reduced-price lunch. Overall, 16 second-grade students from the recruited class participated.

Design and Materials

Students from both grades completed a pretest, participated in two small-group sessions, engaged in a whole-class lesson, and finally took a posttest.

Pretest and posttest. Test items were designed to evaluate students' knowledge related to integers and were identical on both pretest and posttest. Both tests took approximately 30 minutes to finish for students in both grades. Questions focused on ordering integers, integer value comparisons, integer addition, directed magnitude comparisons, and transfer problems with addition. After both tests, we interviewed some of the students to learn more about their answers and strategies and to clarify students' insights.

Small-group sessions. During the sessions, students analyzed sets of contrasting integer addition problems in groups of two to four. They analyzed worked examples of correct and incorrect solutions to integer addition problems within four different illustration contexts including (a) a gingerbread boy starting at zero and moving on a number path situated on a hill (translation model [see Wessman-Enzinger & Mooney, 2014]), (b) an ant starting at the first number in the addition problem and moving below and above ground next to a vertical number line, (c) a chip model (counterbalance model [see Wessman-Enzinger & Mooney, 2014]), and (d) a folding number line (see Tsang, Blair, Bofferding, & Schwartz, 2015). Each session took approximately 20 minutes.

In their first session, students analyzed integer addition problems with two positive numbers in comparison with adding a positive number to a negative number (e.g., $3 + 5 = 8$ versus $-3 + 5 = 2$). This contrast can help emphasize that regardless of the starting number, adding a positive number always corresponds to a movement towards the positive direction (or up). At the end of the session, students solved eight integer addition problems, four where they added two positive numbers and four where they added positive numbers to negative numbers.

In the second session, students compared addition problems with two negative numbers to addition problems with adding a negative number to a positive number (e.g., $-2 + -5 = -7$ versus $2 + -5 = -3$). This comparison can help students to realize that adding a negative number to either a positive or a negative number always results in a movement in the negative direction (or down). At the end of the session, students solved eight integer addition problems, four where they added two negative numbers and four where they added negative numbers to positive numbers. The collective

16 problems solved at the end of their both sessions composed what we call a midtest and helped us understand how they would answer the problems immediately after making comparisons with them.

Whole-class instruction. The whole-class lesson, led by one of the researchers and given separately to fifth and second graders, had two parts: interactive instruction and a game. Instruction began by helping students think about moving on an integer number path to solving integer addition problems. Therefore, adding a positive number represented moving up on the number path (going more in the positive direction), and adding a negative number indicated a downward movement on the number path (moving more in the negative direction). Students helped explain how they would move the gingerbread boy along the number path to solve example integer addition problems. As students' understanding built upon the directional movements of adding two integers, the researcher-teacher presented an additive inverse problem (e.g., -2 + 2) to introduce the "zero pair" concept. As demonstrated, zero pairs occurred when the two movements added on the number path ended at zero. Students later engaged in a game where they had to make zero pairs using positive and negative one cards.

Analysis

To address the first research question about students' performance, we ran a repeated measures ANOVA grouped by grade level on the 17 arithmetic items presented to students on the pretest and posttest. Next, in order to determine students' progress before, during, and after the study more thoroughly, we determined students' average percent correct on six common problems (-9 + 2, -8 + 8, 6 + -8, 9 + -9, -1 + -7, and -2 + -2) that were given to students on the pretest, midtest, and posttest. Students' responses to items of a similar problem type (e.g., -9 + 2 and -8 + 8) provided some insight into their interpretations of the minus signs and helped paint a picture of their conceptual change. We sought further clarification of the meanings students attributed to the minus signs through the interview data.

Results

Overview of Students' Arithmetic Gains

The results of the repeated measures ANOVA indicate a significant main effect of test, $F(1, 31) = 62.32$, $p<.001$. Students from both grades gained significantly from pretest to posttest. Table 1 shows that even though the fifth-grade students' average score on the pretest arithmetic items was higher than second graders, they scored a lower average on the posttest arithmetic items than second-grade students. In fact, based on the repeated measures ANOVA, the difference between the grade level gains is significant, with second graders making greater gains on average than the fifth graders, $F(1, 31) = 4.51$, $p<.001$.

Table 1: Students' Mean Scores (And Average Percent Correct) On Integer Addition Problems

Grade Level /Tests	Pre-test ($M \pm SD$) (average percent correct) (17 items)	Mid-test ($M \pm SD$) (average percent correct) (12 items)	Post-test ($M \pm SD$) (average percent correct) (17 items)
Second Grade (N = 16)	5.1 ± 6.2 30%	7 ± 3.8 58%	13.5 ± 2.8 79%
Fifth Grade (n = 17)	7 ± 3.9 41%	6.9 ± 2.8 58%	11.8 ± 3.8 69%

Students' Interpretation of the Minus Sign

We explored how student's interpretation of the minus sign might support the significant difference between the two grade level gains. We, therefore, investigated students' responses to six integer addition problems common to the three tests. Students' percentage correct on each problem was calculated for each group (see Table 2). On average, fifth-grade students performed better than second graders on four of the six common integer addition problems (67%) on the pretest, while second graders scored higher on all posttest common items (with one tie).

Table 2: Students' Average Percent Correct on Common Integer Addition Problems

Common integer addition		-9 + 2	-8 + 8	9 + -9	6 + -8	-1 + -7	-2 + -2
Second Grade	Pretest	38%	38%	19%	19%	31%	25%
	Mid test	44%	75%	44%	56%	81%	75%
	Post test	75%	94%	94%	69%	88%	69%
Fifth Grade	Pretest	12%	29%	41%	29%	71%	59%
	Mid test	35%	41%	47%	29%	88%	82%
	Post test	65%	82%	82%	65%	88%	47%

Note: Shaded cells indicate questions where that grade level did better than the other.

Fifth graders' pretest. Fifth graders frequently gave answers consistent with a strategy of adding a negative sign to the sum of two integers' absolute values. If students only used this strategy on the problems where it would lead to correct answers, we would expect them to use it roughly 29% of the time. Among all of the 17 integer addition problems, they had these types of answers for 46% of the problems on average. Because of this, not only did fifth graders have a higher average performance on -1 + -7 and -2 + -2 compared to second graders on the pretest, but their average scores were the highest on these compared to the other problems (9 + -9, 6 + -8, -9 + 2, and -8 + 8) where that strategy would not work. A fifth-grader, Jennifer, in describing her answer said, "They're both negative[s], so it is going to be [a] negative."

Another consistent strategy applied by fifth graders was treating the negative sign as a subtraction sign, the binary meaning of the minus sign. Across the problems fifth graders provided answers consistent with subtracting one number from the other and getting positive answers for 19% of the 17 problems on average. In six cases, students provided answers illustrating the common misconception that you always subtract a smaller number from a bigger number (e.g., solving 4 + -5 = 1). Interpreting the minus sign as a subtraction sign could help them in solving 5 + -2 = 3, which 7 fifth graders did. However, only one of them also solved 7 + -3 = 4, because they were more inclined to add the numbers. Additionally, their tendency to subtract in the treatment of the negative numbers could not help them to correctly answer problems such as -9 + 2, for which counting up or down from the negative requires unary meaning of the minus sign.

Second graders' pretest. Unlike the fifth graders, second graders did not often provide answers consistent with adding a negative sign to the sum of two integers' absolute values in the problems.

Their answers only reflected this strategy on 9% of the 17 problems on average. Also, their answers were consistent with subtracting to get a positive number only 9% of the time on average as well, which seemed more likely when the negative was first, given their better performance on -8 + 8 versus 9 + -9. When solving 6 + -8, lower number of possible answers on using subtraction operation with integers (e.g., 6 + -8 = 2 or -2 or 0) suggest they applied the binary meaning of the minus sign less often. When solving 6 + -8, Sandra who answered two demonstrated a conception of starting with the larger absolute value, "I did eight, seven, six, five, four, three, two and that is how I got it, and negative, if negative plus a positive number, you go backward to the negative." However, Edward, a second grader used the binary meaning to get a negative answer, reasoning, "Because six minus eight is negative two [be]cause it has to go below zero," also demonstrating an awareness of the unary meaning given to his answer.

Starting at the negative number and counting up or down the other number (e.g., -9 + 2 counting up -8 and -7 or counting down -10 and -11) leverages the unary meaning of the minus sign. Second-grade students' slightly higher average scores on -9 + 2 are consistent with them interpreting negative numbers as corresponding to a point on the number line, the unary meaning of the minus sign. Tara explained her solution for -9 + 2, "So I counted on my fingers negative eight, negative seven, and it is negative seven." Timmy said, "If it is a negative number plus [a] regular number, it would be less of the negatives. So, negative nine plus two equals negative seven."

Midtest. Students' gains for the six problems on the midtest show their development in interpreting multiple meanings of the minus sign. Second graders still averaged higher for -9 + 2 and -8 + 8 compared to fifth graders. Interestingly, even though fifth graders started higher in 9 + -9 and 6 + -8 on the pretest, second graders had a higher average correct for 6 + -8 on the midtest. Fifth-grade students' responses appeared to be influenced by their prior knowledge and treating the minus sign with a symmetric meaning (answering -14). In fact, fifth-grade students' consistent responses over 12 integer addition problems illustrate they applied the symmetric meaning on average for 33% of problems; whereas, second graders did so for 26% on average.

Second graders' posttest**.** Both grades' highest average was in -8 + 8 and 9 + -9, which demonstrates the significance of the zero pairs introduction in the lesson. Tara said, "I know that negative nine plus nine equals zero because negative means below zero and it is nine below zero, so it would be just zero. [Be]cause it is nine more than the negative." Timmy described a zero pair problem, "If you have a negative number plus a normal number, it would be zero. If you have normal nine plus a negative nine, it would be zero."

Surprisingly, students' average scores were lower on -2 + -2 after the midtest. Since zero was the most usual incorrect answer, one explanation could be that they identified the two same numbers and a negative sign in the problem, associated it with the zero pairs problem, and answered zero. Another second-grade student, Anthony, said, "So, I looked at it and I thought it's zero for a second. But I thought that would be negative two plus two. So, I thought negative two plus negative two would be negative four." Based on their responses, they used the symmetric meaning 15% of the time on average.

Second graders' responses to 6 + -8 did not result in -14 anymore, which reveals that they abandoned the symmetric meaning of the minus sign for this problem and treated the negative sign in a way that could provide the correct answer (unary meaning starting at -8 or binary meaning instead). Tara's unary interpretation is exemplified, "So, I know that since that is a lower number [referring to 6] than this one [referring to 8], then the answer would be still in the negatives. So, I started at negative eight and went up six." Another student, Kathryn, who interpreted the negative sign with a binary meaning said, "I did six and took away eight." The researcher asked, "So, when you ran out of six, what happened after that?" She continued, "I got into the negatives." Also, students explained their responses with zero pairs. Anthony mentioned zero pairs when solving 6 + -8, "I noticed six plus eight is 14 but six plus negative eight, when we did the zero problems [zero pairs] if it was six plus negative six, he [gingerbread man] would be going up and then go down. So, I noticed that so I did it." Further, when they answered -9 + 2 = -11, it was not necessarily because they focused on a symmetric meaning of the minus sign. Anthony started at -9 and answered -11 stating, "So, it would be nine plus whatever. It is still be going up but if you do [plus] a negative, it would have to go down." This student started to understand integer addition as involving directional movements that can be both upward or downward but was still trying to make sense of how the direction interacted with the numbers.

Fifth graders' posttest. Fifth graders scored slightly lower on -8 + 8 and 9 + -9, often answering -16 or -18 respectively, either due to overuse of the symmetric meaning or using the unary meaning and moving in the wrong direction. Further, their responses to the problem 6 + -8 involved -14, unlike second graders. Similar to second graders, they scored lower on -2 + -2, often associated with overgeneralizing the zero pairs concept. Though both grades averaged the same in -1 + -7, fifth graders did not improve from the midtest. Overall, fifth graders' responses highlighted the symmetric meaning for an average of 30% of 17 integer problems. This compared to 15% average among second graders provides initial evidence that fifth-grade students' stronger prior conception limited their ability to make use of the contrasting cases and change their thinking based on the instruction compared to the second graders.

Discussion

Our results provide insights into students' interpretations of minus signs and solution strategies for integer addition problems before and after making comparisons between them. Similar to the results of the fifth graders presented here, Murray (1985) found that adding two negatives was easy for upper elementary students compared to other integer addition problems. The results here suggest that this may be due to older students' tendency to focus on the symmetric meaning of the minus sign. Given this, one interpretation of our data is that if students naturally focus on the symmetric meaning and this interpretation persists without challenge (as could be the case with the fifth graders who relied on the symmetric interpretation heavily), they have more difficulty shifting away from it. Therefore, the second graders may have been more open to changing their conceptions about numbers, minus signs, and operations because they had not had much prior opportunity to think about them and establish strong initial conceptions. The data shows fifth graders' stronger prior conception about the negative sign influenced their future reasoning; however, second graders changed their interpretation of the minus sign throughout the study. Their willingness to apply the new information in their solution strategies helped them to improve higher.

Students experience a barrier to learning negative numbers when they have time to establish strong preconceptions without the opportunity to address them, a problem exacerbated by the location of integer standards in upper grades. In order to overcome students' difficulties in conceptualizing the negative numbers, we should provide opportunities for them to develop their integer number sense knowledge from at least second grade. Contrasting problem types (Rittle-Johnson & Star, 2009) using different conceptual models (Tsang, Blair, Bofferding, & Schwartz, 2015; Wessman-Enzinger & Mooney, 2014) helped students to improve their understanding. By capturing students' reasoning, discoveries, and interpretation through these activities, we can provide additional tasks more in-line with their needs to facilitate their learning and exploration process (Behrend & Mohs, 2005/6) throughout the following grade levels.

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OBJETO MENTAL FRACCIÓN DE ALUMNOS DE SECUNDARIA CON PROBLEMAS DE ABSENTISMO ESCOLAR
MENTAL OBJECT FOR FRACTIONS OF MIDDLE SCHOOL STUDENTS WITH ABSENTEEISM PROBLEMS

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En este documento se describen características del objeto mental fracción de estudiantes, de 12 a 14 años de edad, de una secundaria pública, quienes tienen problemas de absentismo escolar y bajo rendimiento académico. Para ello, se diseñó un cuestionario como parte de un proyecto de investigación cuyo objetivo general es contribuir a la construcción de mejores objetos mentales fracción a través de una secuencia de enseñanza. El diseño de los ítems del cuestionario está estructurado de acuerdo con los resultados de una fenomenología didáctica de las fracciones y el contenido curricular propuesto para los últimos años de primaria. Los resultados indican que los alumnos tienen mayor éxito al responder cuestiones relacionadas con fenómenos de partición de figuras geométricas, mientras que tienen menor éxito para usar las fracciones en la recta numérica.

Palabras clave: Números Racionales, Educación Primaria, Cognición

Planteamiento del Problema y Objetivos

La enseñanza y el aprendizaje de las fracciones siguen siendo tema de interés en el campo de la investigación en matemática educativa, pese a la gran cantidad de estudios que ya se han hecho durante las últimas décadas. Algunas de las principales razones de esa preocupación se vinculan con el hecho de que las fracciones forman parte integral del curriculum matemático y porque de acuerdo con Siegler, Duncan, Davis-Kean et al. (2012) el conocimiento de esos números es uno de los predictores del desempeño en matemáticas de alumnos egresados de primaria hasta el bachillerato. Los resultados de investigaciones recientes versan sobre dificultades que los estudiantes siguen enfrentando cuando resuelven tareas o problemas que implican el uso de las fracciones (ver por ejemplo a Ni y Zhou, 2010 y Petit, Laird y Marsden, 2010) a pesar de los cambios que se han instrumentado en su enseñanza.

Por lo anterior se pretende contribuir a la construcción de un mejor objeto mental fracción de los estudiantes a partir de la educación básica. Los resultados que se exponen en este documento son de un estudio piloto hecho con el propósito de caracterizar el objeto mental fracción de los estudiantes con problemas de bajo desempeño escolar al finalizar la escuela primaria. El estudio piloto forma parte de una investigación más amplia y esos resultados se toman en consideración al estructurar un modelo de enseñanza con el cual se favorezcan procesos de construcción de mejores objetos mentales fracción de alumnos de ese tipo de comunidades.

Referente Teórico

Para el desarrollo de la investigación general se tomó como marco teórico y metodológico a los Modelos Teóricos Locales (MTLs) desarrollados por Filloy, Rojano, Puig y Rubio (1999), en donde el objeto de estudio es visto desde la interrelación de cuatro componentes que se construyen, el componente de competencia formal, el de enseñanza, el de cognición y el de comunicación.

En esta parte del proyecto se pone énfasis en los resultados de la construcción de los componentes formal y de enseñanza, ya que esas conclusiones se usaron principalmente para diseñar los ítems del cuestionario aplicado. Los resultados de la construcción del componente formal sirven como referente teórico para considerar en el diseño distintos fenómenos en los cuales aparecen las fracciones, mientras que los resultados del componente de enseñanza permiten seleccionar los contenidos de las fracciones que se evalúan en el cuestionario.

Para la construcción del componente formal del MTL se realizó una fenomenología didáctica de las fracciones, tomando como base las ideas de Freudenthal (1983), Kieren (1976; 1988; 1992) y Behr, Harel, Post, y Lesh (1992). De acuerdo con Freudenthal (1983, pp. 28-33), hacer una fenomenología es describir un concepto en su relación con los fenómenos para los cuales es el medio de organización. Una fenomenología didáctica rica ayuda a proveer a los estudiantes de una amplia variedad de ejemplos de fenómenos para construir mejores objetos mentales, entendiendo a un objeto mental como el conjunto de ideas sobre un concepto matemático (el objeto pensado) que han elaborado los alumnos y que precede a la adquisición del concepto.

Para hacer la fenomenología didáctica de las fracciones se han considerado fenómenos de las fracciones que aparecen tanto en el lenguaje cotidiano, como en la propia matemática. Las fracciones en el lenguaje cotidiano se utilizan principalmente para describir o comparar cantidades, valores de magnitud, razones o procesos cíclicos o periódicos. Los aspectos de las fracciones que se distinguen son: la fracción como fracturador, como comparador, como operador, como medidora y como número racional, ver Figura 1.

La fracción como fracturador se refiere al proceso de producir fracciones (fracturar), por medio del cual se relacionan las partes con un todo. Esto podría surgir de hacer una partición para hacer un reparto equitativo, una distribución o simplemente dividir cantidades o magnitudes con o sin resto. En el proceso de producir fracciones a partir de la relación de un todo y sus partes, el todo puede ser discreto o continuo, definido o indefinido, estructurado o carente de estructura. La parte también tiene sus variantes, mismas que se detallan en la Figura 1.

Según Freudenthal (1983) las fracciones también surgen de una comparación, la cual puede ser directa o indirecta. Cuando la comparación es directa, es decir, los objetos que se comparan se consideran o piensan como si uno fuera parte del otro, entonces esto se reduce a la fracción como fracturador. En cambio, cuando un tercer objeto media entre los objetos que se comparan, entonces la comparación es indirecta. En este último caso se establece una relación razón entre los valores de magnitud o entre los propios objetos que se comparan. En el proceso de establecer la relación razón se utiliza la fracción como medidora, ya que se puede emplear una medida no convencional o convencional para determinar valores de magnitud y así establecer la relación razón entre ambos objetos. La fracción como medidora surge también en la medición de segmentos sobre la recta numérica o como un valor que precede a una unidad de medida. Es importante mencionar que para identificar las fracciones que preceden a una unidad de medida, en el proceso fue necesario usar las fracciones en su aspecto de operador fracturante.

Se puede distinguir otro aspecto de la fracción, la fracción como operador, considerado como el inverso del operador multiplicación, es decir, el operador fracción actúa en el puro dominio del número. Se extiende esta fenomenología a un ámbito más abstracto y formal de la matemática, donde se identifica a las fracciones como elementos de clases de equivalencias del campo de cocientes que define al conjunto de los números racionales y sus propiedades.

Una explicación complementaria de los diversos aspectos de las fracciones se encuentra en Valenzuela, Figueras, Arnau y Gutiérrez-Soto (2016).

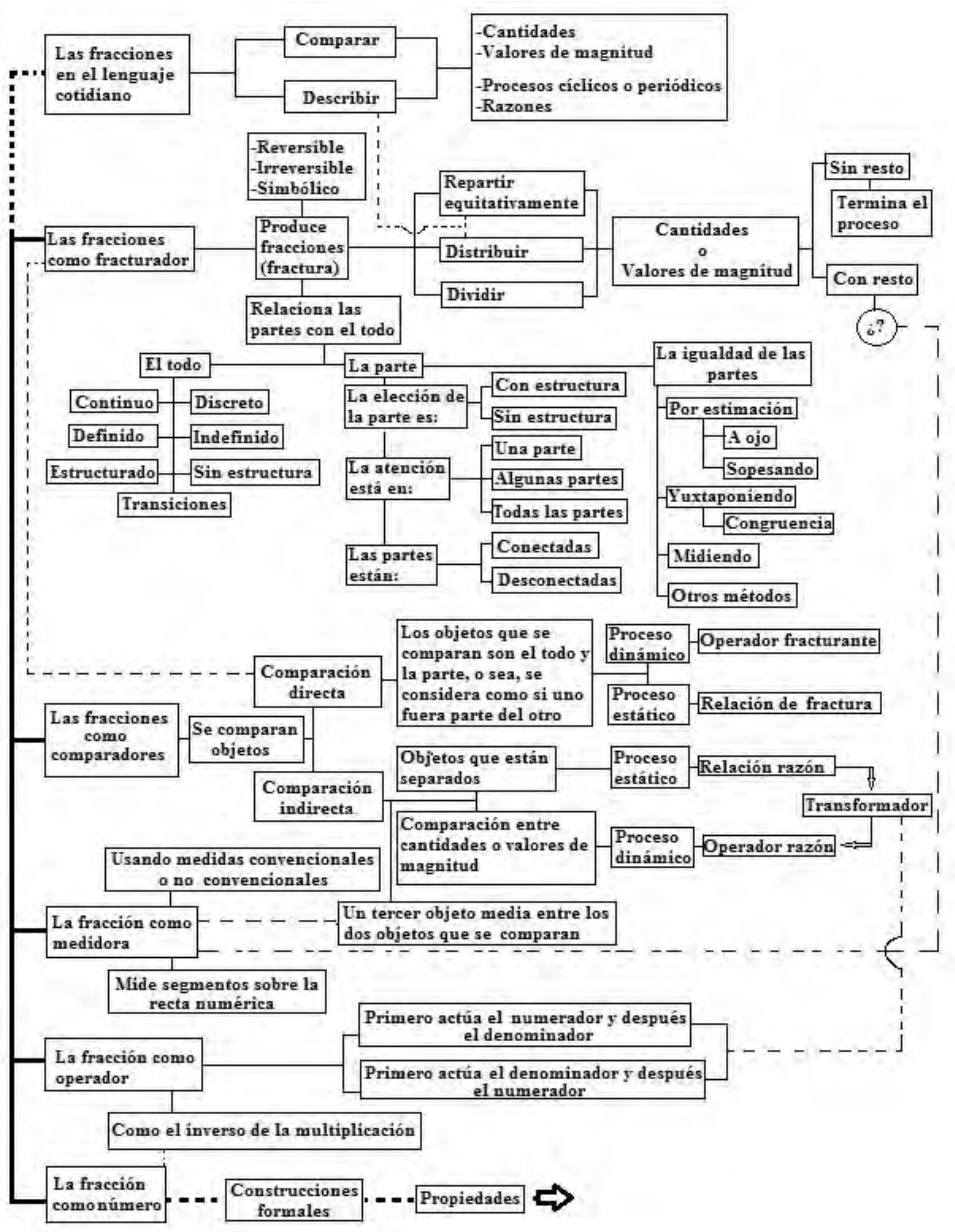

Figura 1. Usos y aspectos de las fracciones.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Diseño del Cuestionario

El cuestionario tiene seis ítems. El ítem uno está compuesto por ocho incisos, en los que se muestra en una representación de una fracción usando figuras geométricas para que el alumno escriba de forma simbólica la fracción correspondiente. En este ítem la fracción aparece como fracturador, específicamente se debe establecer una relación de fractura.

- En los incisos (a) y (c) el todo es continuo, definido y estructurado. Las partes están conectadas, su igualdad se determina por congruencia de áreas y la elección de las partes (coloreadas) es contigua. El inciso (d) cumple con estas condiciones excepto que la elección de las partes (partes coloreadas) no es contigua. Como el todo aparece partido en partes iguales se dice que se establece una relación de fractura. En el inciso (a) el todo es un rectángulo y en los incisos (c) y (d) círculos representan el todo.
- En los incisos (b), (e), (f) y (h) el todo es continuo, definido y estructurado. Las partes están conectadas, pero se definen dos unidades fraccionarias, el alumno debe completar o imaginar una partición con una unidad fraccionaria menor, por lo que la fracción podría actuar como un operador fracturante. En el inciso (h) la congruencia de las partes no es fácil de identificar. En los incisos (b) y (h) el todo está representado por cuadrados y en los incisos (e) y (f) por círculos.
- La representación del inciso (g) puede generar una transición de un todo continuo a uno discreto o viceversa, lo que puede causar una fracción propia o impropia, depende de cómo se interprete al todo.

El ítem dos tiene cuatro incisos, en cada uno se muestra una figura geométrica para que el alumno represente una fracción dada. En este caso, como los alumnos son quienes hacen la partición, la fracción aparece como un operador fracturante.

En el ítem tres se presentan dos situaciones con dos incisos cada uno, en los que aparece un todo discreto (bolas de colores), definido y estructurado de acuerdo con el color de las bolas. En el inciso (a) de cada situación se debe identificar la relación de fractura, mientras que en los incisos (b) es necesario comparar la cantidad de bolas de un color con respecto a la del otro color, por lo que la fracción aparece como una relación razón.

Sobre la recta numérica se deben representar 10 fracciones, ya sea como punto o como segmentos sobre la recta numérica; esta tarea constituye el ítem cuatro. Cinco fracciones son propias y cinco impropias; todas son menores que tres. Se han propuesto fracciones con denominadores distintos para que el alumno haga distintas particiones “a ojo”. En este ítem la fracción aparece como número en la recta numérica, pero también como medidora, se puede considerar como una unidad de medida de los segmentos sobre la recta numérica, que depende del número de partes entre las que se divide el segmento unidad, en este proceso la fracción aparece también como fracturador.

Para responder el ítem cinco los alumnos deben hacer una clasificación de fracciones propias e impropias, incisos (d) y (e) respectivamente. En los incisos (a) y (b) se piden dos ejemplos de fracciones ubicadas en un intervalo limitado por números enteros, (0,1) y (3,4) respectivamente, y en el inciso (c) el intervalo queda limitado por dos fracciones (7/8, 8/9).

El ítem seis corresponde a la resolución de un problema en el que aparecen distintos aspectos de las fracciones. En este caso la fracción se usa para describir una cantidad, como fracturador, comparador y como operador.

Población y Método

El estudio piloto se llevó a cabo con estudiantes de un instituto público de educación secundaria que se ubica en la ciudad de Valencia, España. 35 alumnos respondieron el cuestionario, 23 de ellos son de primer año y 12 de segundo, pero estos últimos asisten a un taller de regularización donde se trabajan contenidos matemáticos de primer curso. El rendimiento académico de todos los alumnos se considera bajo, de acuerdo con los criterios de evaluación que sigue el profesor de matemáticas. Además, los estudiantes tienen graves problemas de absentismo escolar.

El cuestionario se aplicó a los alumnos de primer grado de secundaria en dos sesiones de 45 minutos, los estudiantes de segundo grado lo resolvieron en una sesión. El cuestionario fue aplicado por el profesor del curso, se resolvió de manera individual y no se proporcionó ayuda.

Para realizar la caracterización de los objetos mentales de los estudiantes se han codificado las respuestas. Las correctas se han etiquetado con un 1 y las incorrectas con 0, en esta última categoría se han incluido aquellas respuestas que los estudiantes dejaron en blanco. Los resultados están organizados por cada ítem de acuerdo con las características del diseño, y también se organizan en una tabla de frecuencias en la cual se indica el porcentaje de éxito.

Resultados

Los resultados generales que se muestran en la Tabla 1 indican que los alumnos tienen mayor éxito al responder cuestiones que se relacionan con la representación simbólica de fracciones a partir de una representación gráfica, considerando modelos continuos y definidos (aspectos evaluados por medio del ítem 1). Los educandos tienen menor éxito para identificar fracciones entre dos números y clasificar fracciones como propias o impropias (aspectos evaluados mediante el ítem 5), así como para representar fracciones como puntos o segmentos en la recta numérica (aspectos evaluados a través del ítem 4). Se observa un descenso porcentual en cuanto al éxito obtenido que va del 58.21% en el ítem 1 hasta el 5.71% en el ítem 4.

Tabla 1: Resultados Globales por Ítem

Ítem 1	Ítem 2	Ítem 3	Ítem 4	Ítem 5	Ítem 6
163/280	54/140	48/140	20/350	16/175	19/140
58.21%	38.57%	34.29%	5.71%	9.14%	13.57%

Como se describió en el diseño del cuestionario, los ítems están formados por varios incisos, por esta razón varía el número de reactivos que se evalúan por ítem. Los resultados obtenidos para el ítem 1 se muestran en la Tabla 2. La información está organizada de acuerdo con las características del aspecto de la fracción como fracturador que se consideraron en el diseño del cuestionario.

Tabla 2: Resultados del Ítem Uno

<table>
<tr><td colspan="8">Representación simbólica de fracciones a partir de una representación gráfica
(Modelos continuos)</td></tr>
<tr><td colspan="7">Fracciones propias</td><td>Fracciones impropias</td></tr>
<tr><td colspan="3">Una unidad fraccionaria</td><td colspan="4">Dos unidades fraccionarias</td><td>Una unidad fraccionaria</td></tr>
<tr><td colspan="2">Elección de las partes contiguas</td><td>Elección de las partes no contiguas</td><td colspan="4">Elección de las partes contiguas</td><td>Elección de las partes contiguas</td></tr>
<tr><td>a)</td><td>c)</td><td>d)</td><td>b)</td><td>e)</td><td>f)</td><td>h)</td><td>g)</td></tr>
<tr><td>27/35</td><td>30/35</td><td>26/35</td><td>20/35</td><td>17/35</td><td>10/35</td><td>14/35</td><td>19/35</td></tr>
<tr><td>77.14%</td><td>85.71%</td><td>74.29%</td><td>57.14%</td><td>48.57%</td><td>28.57%</td><td>40.00%</td><td>54.29%</td></tr>
</table>

Los resultados indican que los alumnos tienen mayor éxito para establecer una relación de fractura en modelos continuos donde la partición tiene solo una unidad fraccionaria (incisos a, c y d), principalmente cuando se utilizan círculos (inciso c). Esto indica que el objeto mental de los estudiantes está más relacionado con este tipo de fenómenos, pero cuando se presentan fenómenos de partición donde hay dos unidades fraccionarias (incisos b, e, f y h) el porcentaje de éxito de respuestas correctas disminuye. Un error común que cometieron los estudiantes al responder el inciso (f) que evalúa este aspecto, está relacionado con el conteo de partes sin tener en cuenta la congruencia del área de las partes. Un ejemplo se muestra en la Figura 2.

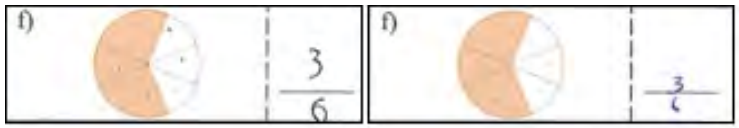

Figura 2. Respuestas de dos alumnos donde se desatiende la congruencia de las partes.

En la Figura 2 se observa el rastro de puntos que dejaron los estudiantes con el bolígrafo, lo que indica el conteo de las partes sin tomar en cuenta la congruencia. Otro error cometido al responder este inciso es el cambio de numerador por el denominador al momento de establecer la relación de fractura.

En la Tabla 1 se observa un descenso porcentual entre el ítem uno y el ítem dos de aproximadamente el 20% de éxito en las respuestas dadas por los alumnos. Esto indica que los fenómenos donde se usa la fracción como un operador fracturante requieren ser considerados en la enseñanza, particularmente cuando se tiene que representar una fracción impropia y se usan figuras geométricas menos usuales. Esto ayudaría a construir un mejor objeto mental fracción.

Los modelos discretos donde se usa la fracción como fracturador y como comparador, específicamente como una relación razón, fueron evaluados en el ítem tres y sus resultados se muestran en la Tabla 3. Con respecto a los ítems anteriores también se observa un descenso porcentual en el número de respuestas correctas. Se afirma que el objeto mental de los estudiantes está vinculado con los fenómenos donde se relaciona la parte con el todo, tanto en modelos continuos como discretos. Sin embargo, se requiere proponer para la enseñanza, más problemas donde la fracción actúe como comparador, ya que para resolver problemas relacionados con este tipo de fenómenos hay un porcentaje de éxito bajo.

Tabla 3: Resultados del Ítem Tres

Representación simbólica de fracciones a partir de una representación gráfica (modelos discretos)			
Distractor: estructurada respecto al tamaño (caja 1)		Distractor: no estructurada respecto al tamaño (caja 2)	
Fracturador	Comparador	Fracturador	Comparador
a)	b)	c)	d)
21/35	4/35	21/35	2/35
60.00%	11.43%	60.00%	5.71%

Representar fracciones en la recta numérica fue el ítem con menos éxito, ya que sólo 20 de 350 posibles respuestas fueron correctas. Las fracciones que se tenían que representar se muestran en la tercera fila de la Tabla 4.

La fracción 7/3 es la que pudieron representar correctamente más estudiantes, después 1/4 y 6/5. El hecho de que nadie haya representado correctamente la fracción 4/8 permite suponer que el objeto mental de los alumnos no les permite reconocer la equivalencia entre fracciones, porque incluso ningún alumno identificó que 2/12 y 1/6 son equivalentes.

Tabla 4: Resultados del Ítem Cuatro

Representación de fracciones en la recta numérica									
Fracciones propias					Fracciones impropias				
2/12	4/8	2/10	1/4	1/6	6/5	7/3	5/2	8/7	12/9
1/35	0/35	2/35	3/35	1/35	3/35	4/35	2/35	2/35	2/35
2.86%	0%	5.71%	8.57%	2.86%	8.57%	11.43%	5.71%	5.71%	5.71%

Uno de los errores comunes al resolver este ítem, se refiere al hecho de ubicar las fracciones en la recta numérica teniendo en cuenta solo el valor del numerador de la fracción pero sin hacer explícita una partición del segmento de recta, tal como se muestra en la Figura 3.

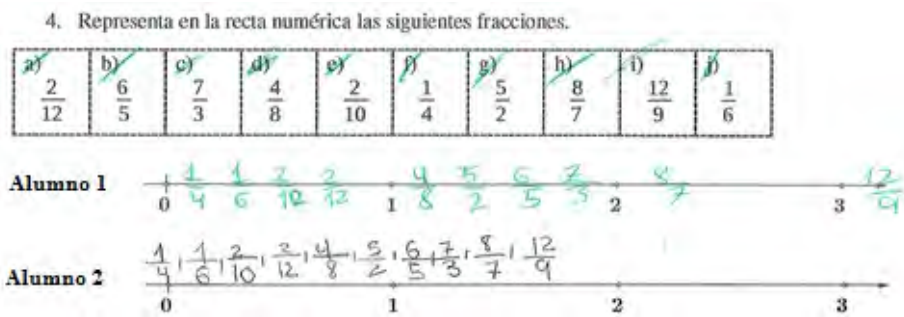

Figura 3. Representación de fracciones en la recta numérica.

Otro de los ítems de la prueba con menor porcentaje de éxito es el 5. En éste se evalúa la identificación de fracciones entre dos números enteros y entre dos fracciones, así como la clasificación de fracciones propias e impropias. Los resultados de esta evaluación permiten afirmar que los alumnos no reconocen características de las fracciones propias e impropias tomando en cuenta su relación con la unidad o la comparación entre numerador y denominador.

Los resultados de la evaluación del último ítem resultan de interés, ya que permiten confirmar que la resolución de problemas con datos numéricos en forma de fracción es otro tema en el cual los estudiantes tienen mayor dificultad.

De los resultados de los ítems propuestos en la prueba se puede afirmar que el objeto mental fracción de los estudiantes está vinculados principalmente con:

- Fenómenos de partición en donde el área de una figura geométrica, considerada como un todo continuo, definido y estructurado es dividido en partes iguales para establecer una relación de fractura, donde la igualdad de las partes se determina por congruencia de áreas. Pero cuando hay dos unidades fraccionarias los alumnos enfrentan dificultades.
- Otros fenómenos menos usuales pero que también se manifiestan se refieren a la división de un todo discreto para establecer una relación de fractura.
- Fenómenos de partición de un todo continuo y definido, donde la fracción actúa como operador fracturante, la elección de las partes es contigua y su igualdad se estima a ojo. Cuando hay variaciones en la estructura del todo los estudiantes enfrentan dificultades.

Conclusiones

Los resultados generales del test previenen que los estudiantes que participaron en la aplicación del cuestionario tienen un objeto mental fracción limitado, por lo que se requiere ofrecer una gama de actividades que promuevan la construcción de mejores objetos mentales, partiendo de fenómenos relacionados con los diferentes usos y aspectos de las fracciones que se ilustran en la Figura 1, y no limitarse a uno solo de sus aspectos durante la enseñanza de las fracciones. Ya que se hace notar que los estudiantes no transfieren de manera natural los conocimientos que tienen sobre un determinado aspecto de la fracción a otro. Por ejemplo, de fenómenos de partición donde se emplean modelos continuos como círculos o rectángulos no pueden pasar a una partición de segmentos en la recta para ubicar fracciones.

Aunado a lo anterior, se considera importante proponer actividades donde se aprovechen cada una de las particularidades que se detallan en el espécimen de fenomenología didáctica esbozado en este documento. Ya que a pesar de que los alumnos tienen objetos mentales sólidos relacionados con la fracción como fracturador, cuando se varía un poco para tomar en cuenta algunas particularidades, se observa que los estudiantes enfrentan dificultades. Respecto a los aspectos de la fracción como comparador, medidora, operador y número racional que se evaluaron en la prueba, los estudiantes mostraron un uso restringido de su conocimiento.

Los resultados también confirman que pese a que en el currículo de los últimos años de la educación primaria se propone explícitamente el estudio de las fracciones utilizando la recta numérica como recurso didáctico, así como el estudio de las características de las fracciones propias e impropias, los alumnos tienen poco éxito para resolver este tipo de tareas. Este resultado se atribuye al hecho de que estos aprendices han sido instruidos principalmente bajo el uso del modelo de áreas, ya que a pesar de que son alumnos de bajo rendimiento académico, mostraron tener conocimientos sólidos para representar fracciones propias cuando se usa dicho modelo, principalmente cuando se usan círculos o rectángulos.

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In this paper characterizations of mental objects for fractions of middle school students (from 12 to 14 years old) with absenteeism problems and low academic performance, are described. A test was designed as part of a research whose general purpose is to contribute in the building up of better *mental objects for fractions through a teaching sequence. Items for the test were structured according to the results of a didactic phenomenology of fractions and the curricular content proposed for the last years of elementary school. Results indicate that students have a greater success in issues related to continuous model partitions phenomena, whereas they are less successful using fractions on the number line. Students cannot transfer their knowledge from an area model to a linear model to identify or represent fractions.*

Keywords: Rational Numbers, Elementary School Education, Cognition

Problem Approach and Research Objectives

Teaching and learning of fractions continue to be a subject of interest within mathematics education research. The main reasons behind this concern are that fractions are an integral part of mathematics' curriculum, and according to Siegler, Duncan, Davis-Kean et al. (2012), fractions' knowledge has been characterized as one of the predictors of students' mathematics performance from secondary to higher education. Other research findings show difficulties that students face when solving tasks or problems involving the use of fractions (e.g., Ni & Zhou, 2010; Petit, Laird & Marsden, 2010) even though changes have been made in their teaching.

With this research, we want to contribute to the building up of a better mental object for fractions of students from primary education. The results presented in this document are from a pilot study done with the purpose of characterizing the mental object for fractions of students with low performance at the end of primary school. The pilot study is part of a broader research and these results are taken into account when structuring a teaching model that favor processes to constitute better mental objects for fractions of this type of students.

Theoretical Framework

For the development of the general research, Local Theoretical Models (LTMs) developed by Filloy, Rojano, Puig & Rubio (1999) were used as a theoretical and methodological framework. From the theoretical point of view, the LTMs serve to focus on the object of study through four interrelated components: formal competence models, teaching models, models for cognitive processes, and models of communication.

In this part of the research project, the emphasis is put on the results of the construction of the formal and the teaching models components. These results were mainly used to design the items that were evaluated in the test. Results of the building up of the formal component serve as a theoretical reference in designing the test in order to evaluate tasks related to different phenomena where fractions appear. Results of the teaching component construction enable the choice of specific fractions' contents evaluated in the test.

For the construction of the formal component of the LTM a didactic phenomenology of fractions was made, based on ideas of Freudenthal (1983), Kieren (1976; 1988; 1992) and Behr, Harel, Post & Lesh (1992). According to Freudenthal (1983, pp. 28-33), to make a phenomenology is to describe a concept in its relation to the phenomena for which it is a means of organization. A rich didactic phenomenology helps to provide students with a wide variety of examples of phenomena to constitute better mental objects, understanding a mental object as the set of ideas about a mathematical concept (the thought object) that the students have elaborated and which precedes concept attainment.

To carry out the didactic phenomenology of the fractions, phenomena that appear both in everyday language and in mathematics itself were considered. Fractions in everyday language are mainly used to describe or compare quantities, magnitude, ratios, and cyclic or periodic processes. Other aspects distinguished are: fraction as a fracturer, fraction as a comparer, fraction as an operator; fraction as a measurer, and fraction as a rational number, see Figure 1.

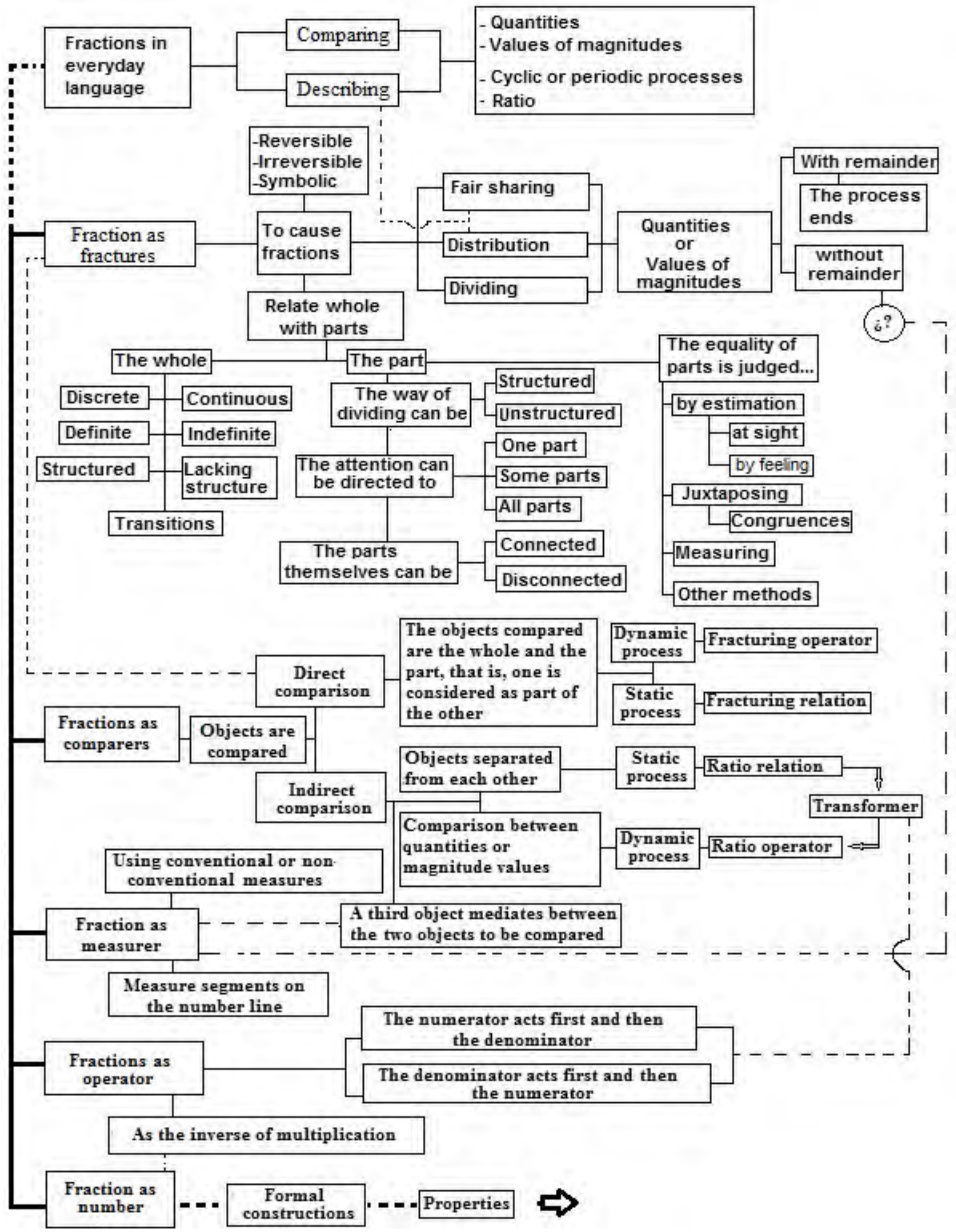

Figure 1. Uses and aspects of fractions.

Fraction as a fracturer refers to the process of producing fractions (fracturing), through which parts are related with a whole. This could arise from making a partition to make a fair sharing, a distribution or simply dividing quantities or values of magnitude with or without remainder. In the process of producing fractions in order to relate a whole with its parts, the whole can be discrete or continuous, definite or indefinite, structured or without structure. Parts also have its variants, which are detailed in Figure 1.

According to Freudenthal (1983) fractions also arise from a comparison, which may be direct or indirect. The comparison is direct when the objects being compared are considered or thought as one part of the other, in this case, the comparison is reduced to the aspect of fractions as a fracturer. In contrast, when a third object mediates between the objects being compared, an indirect comparison is carried out. In the latter case, a ratio relation between the values of magnitude or the objects that are being compared is established.

In the process of establishing the ratio relation, fractions are used as a measurer, because an unconventional or conventional measure can be used to determine magnitude values and establish the ratio relation between both objects. The fraction as measurer also arises in the measurement of segments on the number line or as a value that precedes a unit of measurement. It is important to mention that in order to identify the fractions that precede a magnitude value, in the process, it was necessary to use other aspects of fractions, for example as fracturing operator.

Another aspect of fractions can be distinguished: fraction as an operator. This aspect can be used as a fracturing operator, a ratio operator, and as a fraction operator. The last aspect is considered as the inverse of the multiplication operator, i.e, the fraction operator acts in the number's domain. This phenomenology can be extended to a more abstract and formal area of mathematics, where fractions are identified as elements of equivalence classes of the quotient field that defines the set of rational numbers and their properties.

A complementary explanation of the various aspects of fractions can be find in Valenzuela, Figueras, Arnau & Gutiérrez-Soto (2016).

Test Design

The test has six items. Item one has eight subsections. In each one a representation of a fraction is shown on a geometric figure in order that students write the corresponding fraction in a symbolic form. In this item the fraction appears as a fracturer, specifically, a fracturing relation must be established.

- In subsections (a) and (c) the whole is continuous, defined and structured. Parts are connected, their equality is determined by congruence of areas and the choice of the parts is contiguous. Subsection (d) has these characteristics except that the choice of parts (colored parts) is not contiguous. As the whole appears fractured in equal parts, a fracturing relation must be established. In subsection (a) a rectangle is use to represent the whole and in subsections (c) and (d) circles are chosen as wholes.
- Subsections (b), (e), (f) and (h) contain a continuous, defined, and structured whole. Parts are connected, but two fractional units are defined. Students must partition the whole or imagine a partition with only one fractional unit, so the fraction could act as a fracturing operator. In subsection (h) congruence of parts is not easy to identify. In subsections (b) and (h) squares represent wholes, and in subsections (e) and (f) circles are wholes.
- The graphical representation in subsection (g) can generate a transition from a continuous whole to a discrete one or vice versa, which can cause a proper or an improper fraction, depending on how the student interprets the whole.

The second item consists of four subsections. In each one, a geometric figure is shown so that students represent a given fraction. In this case, students produce a partition of the whole; for this reason, the fraction appears as a fracturing operator.

Two situations with two subsections are presented in item three, where a discrete whole (colored balls), defined, and structured according to the balls' colors is represented. In subsection (a) of each situation, the fracturing relation must be identified, while in subsection (b) it is necessary to compare the number of balls of one color with the number of balls of the other color, so that the fraction appears as comparer, specifically to identify a ratio relation.

In item four, ten fractions should be represented as a point or as segments on the number line. Five are proper fraction and the rest improper fractions. All of the fractions are less than three. Fractions with different denominators have been proposed to students to make different partitions "at sight". In this item, the fraction appears as a number on the number line, but also as a measure. It can be considered as a unit of measure of the number line segments that depends on the number of parts in which the unit segment is divided. In this process, the fraction also appears as a fracturing operator.

To answer the item five, students must make a classification of proper and improper fractions, subsection (d) and (e), respectively. In subsections (a) and (b), two examples of fractions in an interval limited by integers, (0, 1) and (3, 4) respectively, are requested, and in subsection (c) the interval is limited by two fractions (7/8, 8/9).

Item six is a problem in which different aspects of fractions appear. In this case, fractions are used to describe an amount, as a fracturer, a comparer, and as operator.

Setting and Participants

The pilot study was carried out with middle school students in Valencia, Spain. 35 students answered the test, 23 of them were studying in seventh grade, and 12 in eighth grade. The latter attended a remedial workshop where they worked on seventh grade mathematical contents. According to the evaluation criteria followed by the mathematics teacher, the academic performance of all students was considered low. The students had serious truancy problems.

The questionnaire was applied in two sessions of 45 minutes for students in seventh grade. The eighth grade students completed the test in one session. The mathematics teacher applied the test and the students solved it individually; no help was provided.

To characterize the mental objects for fractions that students had, answers were codified, using "1" for correct ones, and "0" for incorrect ones. In this last category, answers left blank were included. Results were organized by each item according to the characteristics of the design and are also in a table of frequencies that indicates the percentage of success.

Results

The general results showed in Table 1 indicate that students are more successful when answering items related to the symbolic representation of fractions, from a graphical representation, considering continuous and defined models, for example, the area model. These results show that students can establish a fracturing relation when working with this kind of representations. Students are less successful in identifying fractions between two numbers and classify fractions as proper or improper (aspects evaluated in item 5), as well as to represent fractions as points on the number line (aspects evaluated through item 4). There is a decreasing percentage of the success obtained that goes from 58.21% in item 1 to 5.71% in item 4.

Table 1: General Results by Item

Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
163/280	54/140	48/140	20/350	16/175	19/140
58.21%	38.57%	34.29%	5.71%	9.14%	13.57%

As described in the design of the questionnaire, test items consist of several subsections; for this reason, the number of answers evaluated per item varies. The results obtained for item 1 are shown in Table 2. The information is organized according to the characteristics of fractions as fracturer considered in the design of the test.

Table 2: Results of Item 1

Symbolic representation of fractions from a graphical representation (Continuous models)							
Proper fractions							Improper fractions
Defined Partition			Partially defined partition				Undefined partition
Choice of contiguous parts		Choice of no contiguous parts	Choice of contiguous parts				Choice of contiguous parts
a)	c)	d)	b)	e)	f)	h)	g)
27/35	30/35	26/35	20/35	17/35	10/35	14/35	19/35
77.14%	85.71%	74.29%	57.14%	48.57%	28.57%	40.0%	54.29%

Results reveal that students are more successful in establishing a fracturing relation in continuous models where the partition is defined (subsections a, c, and d), mainly when circles are used (subsection c). This indicates that the mental object that students have is related to this type of phenomena; but, when there are partition phenomena where the fractional unit is not completely defined (sections b, e, f, and h), the success of correct answers percentage decreases especially when two different fractional units are shown. A common mistake that students made when responding to subsection (f) is related to the counting of parts and to disregard the congruence of the area of the parts. An example of this is in Figure 2.

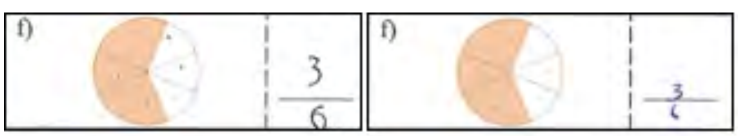

Figure 2. Answers from two students where the congruence of the parts is disregarded.

Figure 2 shows the trace of points that the students left with a pen, which indicates the counting of the parts carried out without considering the congruence of parts area. Another mistake that appears in this subsection is the change of the numerator by the denominator when establishing the fracturing relation.

There is a percentage decrease between items 1 and 2 of approximately 20% of success in the students' answers (see Table 1). The prior indicates that the tasks where the fraction is used as a fracturing operator need to be favored, in particular when an improper fraction has to be represented and when working with less common geometric figures, which would help to constitute a better mental object of the fraction.

The discrete models where the aspects of fractions as fracturer and comparator, specifically as a ratio relation, were evaluated in item 3 (see results in Table 3). In the previous items it is also observed a percentage decrease on the number of correct answers. Despite that the mental object students have is related to the phenomena where they establish a fracturing relation, in both, continuous and discrete models; however, it is necessary to propose more tasks for teaching, where the fraction acts as a comparator to solve problems related to this type of phenomena since there is a very low success rate of answers.

Table 3: Results of Item 3

Symbolic representation of fractions from a graphical representation (discrete models)			
Distractor: structured with respect to size of balls (box 1)		Distractor: non-structured with respect to size of balls (box 2)	
Fracturer	Comparator	Fracturer	Comparator
a)	b)	c)	d)
21/35	4/35	21/35	2/35
60.00%	11.43%	60.00%	5.71%

Representing fractions in the number line was the least successful item in the test since only 20 out of 350 possible answers were correct. The fractions to be represented are shown in the third row of Table 4. The fraction 7/3 was correctly represented by more students, then 1/4 and 6/5. The fact that no students had correctly represented fraction 4/8 allows supposing that the mental object students have does not allow them to recognize the fractions equivalence because no students identified that 2/12 and 1/6 are equivalent.

Table 4: Results of Item 4

Representation of fractions on the number line									
Proper fractions					Improper fractions				
2/12	4/8	2/10	1/4	1/6	6/5	7/3	5/2	8/7	12/9
1/35	0/35	2/35	3/35	1/35	3/35	4/35	2/35	2/35	2/35
2.86%	0%	5.71%	8.57%	2.86%	8.57%	11.43%	5.71%	5.71%	5.71%

One of the common errors in solving item 4, was locating the fractions in the number line taking into account only the numerator value of the fraction, but without making a partition of the line segment, as shown in Figure 3.

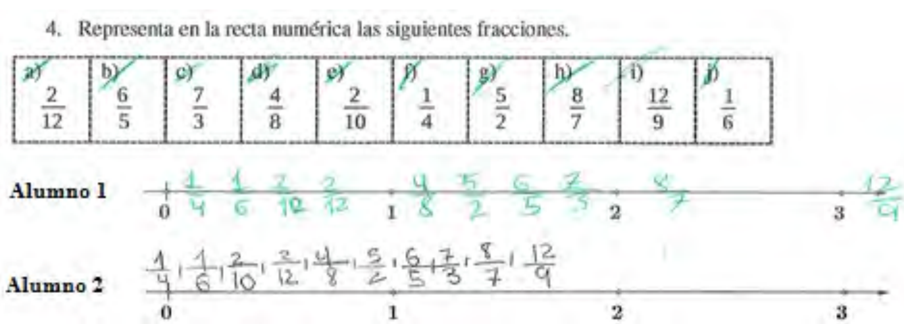

Figure 3. Representation of fractions on the number line.

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One of the other items with a lower percentage of success is item 5. The identification of fractions between two integers and two fractions, as well as the classification of proper and improper fractions, were evaluated. The results allow affirming that the students did not recognize characteristics of the proper and improper fractions taking into account their relation to the unit or the comparison between the numerator and denominator of the fraction.

The results of the last item's evaluation are interesting, those allow us confirming that the solving problem with numerical data in fraction form is another subject where students have greater difficulty since only 19 out of 140 possible answers were correct.

From the results of the items proposed in the test, it can be affirmed that the mental object fraction that students have is linked mainly to:

- Partition phenomena in which the area of a geometric figure, considered as a continuous, defined and structured whole, is divided into equal parts in order to establish a fracturing relation, and the equality of the parts is estimated by the congruence of areas. But, when the partition is not defined the students face difficulties.
- Other less usual but also manifest phenomena refer to the division of a discrete whole in order to establish a fracturing relation.
- Partition phenomena of a continuous and defined whole, where the fraction acts as a fracturing operator, the choice of the parts is contiguous and their equality is estimated "at sight". But, when there are variations in the structure or form of the whole, students face difficulties.

Conclusions

Students who participated in the test application have limited mental objects for fraction. It is necessary to offer a range of activities that will promote the construction of a better mental object; starting from phenomena related to different uses and aspects of the fractions illustrated in Figure 1, and not limited to only one of its aspects during the teaching of fractions. It is noted that students do not naturally transfer their knowledge about one particular aspect of the fraction to another one. For example, students' knowledge to solving problems related to partition phenomena where continuous models as circles or rectangles were used, do not use to make a segments partition on the number line in order to represent fractions.

In addition, it is considered important to propose activities that take advantage of each of the particularities that are detailed in the specimen of didactic phenomenology outlined in this paper. Although students showed solid mental objects related to the fraction as fracturer, it was observed that the students faced difficulties when the aspect of fracture takes account its particularities. Regarding the aspects of the fraction as comparator, measurer, operator, and rational numbers that were evaluated in the test, the students showed a very restricted use of knowledge.

The results also confirm that, although in the last years of elementary school curriculum it is explicitly proposed the study of fractions using the number line as a didactical resource, as well as the study of the characteristics of the proper and improper fractions, the students faced difficulties to solve these types of task. In fact, they did not show knowledge about it at all. This result is attributed to the fact that students have been instructed mainly under the use of the model area as a didactical resource. Although these students had low academic performance, they showed solid knowledge to represent their proper fractions when using this model, mainly when circles or rectangles were used.

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SECOND GRADERS' INTEGER ADDITION UNDERSTANDING: LEVERAGING CONTRASTING CASES

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In this study, we explore thirty-two second graders' performance on integer addition problems before and after analyzing contrasting cases involving integers. The students, as part of a larger study, completed a pretest, were randomly assigned to one of three intervention groups, and participated in two small-group sessions, one short whole-class lesson on integer addition, and a posttest. The group interventions differed in terms of which problems students compared in their small-group sessions. Based on students' solution strategies for integer addition problems and their treatment of negative signs, all three groups progressed in solving negative integer addition problems; however, students who initially contrasted adding two positives with adding a negative to a positive showed important differences, which we describe further.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education

Providing students with early access to integer learning is important. Recent standards do not require students to learn integer concepts until sixth grade and integer operations until seventh grade (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010); however, prior standards suggested that students should learn about integers as early as third grade (National Council of Teachers of Mathematics [NCTM], 2000). Moreover, current research indicates that young students can productively learn about and work with integers (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bofferding, 2014). Students who are unfamiliar with negative numbers or who over-rely on whole number concepts, will often ignore the negative signs when solving problems (e.g., $-4 + 5 = 9$) (Bofferding, 2010; Murray, 1985) or will encounter other obstacles (e.g., thinking addition always results in a larger number) (Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014). Often, students develop incorrect and resistant conceptions, such as the belief that you cannot subtract a larger number from a smaller one (Murray, 1985). It is therefore important to help students notice early on, not only the differences between positive and negative numbers, but how they affect operations.

Contrasting cases can be a powerful instructional tool for helping students focus on important structural features in problems and give students access to new problems or solution methods. In a study on learning algebra equations, students who compared alternative solution methods gained more in procedural knowledge and flexibility than those studying multiple methods sequentially (Rittle-Johnson & Star, 2007). In a subsequent study, students with limited prior knowledge in algebra benefited more from comparing problem types than solution methods (Rittle-Johnson, Star, & Durkin, 2009). In one study, first graders noticed and made use of negative signs more if they were in an instructional condition where they compared problems with and without negative signs (Bofferding, 2014). A feature of the contrasting cases used in Rittle-Johnson and Star's (2007) work is that they involved worked examples, powerful tools for helping students learn new information (Atkinson, Derry, Renkl, & Wortham, 2000).

Some textbooks introduce negative integer addition by first presenting the case of adding two negative numbers (Hake, 2007; Pearson Education, 2014) and then contrasting it with adding two positive numbers through worked examples. Providing students with the opportunity to compare the addition of two positive numbers with that of adding two negative numbers may encourage them to notice that the numeral increases in magnitude just like with positive number addition but with a

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negative sign before the answer. This problem type tends to be one of the easiest for those recently exposed to negatives (Murray, 1985). However, if adding two negatives makes intuitive sense to students, having them see a different problem type first might provide a more productive contrast. Therefore, we investigated the following question: When learning integer addition, what role does the sequence of problem contrasts play in second graders' (a) integer addition performance and (b) the development of solution strategies?

Conceptual Change Framework

Students' initial mental models for number are based on whole number understanding. As students learn about new numbers, they make sense of them in light of their prior understanding (Vosniadou, Vamvakoussi, & Skopeliti, 2008). With negatives, this can lead to a series of transition and synthetic mental models based on how students interpret the order and value of negatives (Bofferding, 2014).

Interpretations of the Minus Sign

Successful integer contrasting cases should help students interpret and make use of the meanings of the minus sign in productive ways. There are three meanings that students might ascribe to the minus signs in arithmetic problems (binary, symmetric, or unary), which play "a major role in the development of understanding and using negative numbers" (Vlassis, 2004, p. 471). The binary interpretation of the minus sign corresponds to the subtraction operation (Vlassis, 2004). Students with only a binary understanding of minus signs might ignore negatives or treat them as subtraction signs (e.g., solving 6 + -8 as 6 – 8 = 0 or 8 – 6 = 2) (Bofferding, 2010). The symmetric or opposite meaning indicates an operation of multiplying by -1 (switching from positive to negative or negative to positive) (Vlassis, 2004). Students rely on the symmetric meaning when they add integers as positive and make the answer negative (e.g., solving -2 + 5 as 2 + 5 and answering -7) (Bofferding, 2010). The unary meaning of the minus sign is that of the negative sign, designating negative numbers. Students with a strong unary understanding will often start at a negative number and count, either solving -2 + 3 by counting incorrectly, "negative three, negative four, negative five," or correctly, "negative one, zero, one" (Bofferding, 2010), depending on their conceptions of addition and integer values. Those with a strong binary and weaker unary understanding may still solve 6 + -8 as 6 – 8 but actually get -2.

Interpretations of Addition

When students add with positive numbers, they learn that counting up corresponds to an increase in a number's magnitude (Vosniadou, Vamvakoussi, & Skopeliti, 2008). However with negative numbers, students need to learn that adding a positive number corresponds to moving right on the number line (or up); whereas, adding a negative number corresponds to moving left on the number line (or down) (Bofferding, 2014). Stranger still, they need to understand that adding a positive or negative could result in either an increase or decrease in magnitude from the initial number (e.g., with -3 + 1, the final answer -2 has a smaller magnitude than -3 but for 3 + 1, the final answer 4 has a larger magnitude than 3). In this paper, we discuss how students' pattern of responses on negative addition problems changed following opportunities to analyze different sets of contrasting problems.

Methods

Participants and Design

Participants included 32 second graders (from a larger study with 109 second graders) from two rural, elementary schools in the Midwest (where 32.2% of students were English-language learners and 75.2% qualified for free or reduced-lunch). These students were chosen because when solving the integer addition problems on the pretest, they consistently answered by adding the absolute values of the numbers, ignoring the negative signs (e.g., -4 + 6 = 10 or -1 + -7 = 8) and sometimes answering random positive numbers. After completing a pretest, students were randomly assigned to one of three intervention groups. Students in each group analyzed different sets of contrasting integer addition problems over two sessions, engaged in one integer addition lesson, and finished by taking a posttest.

Data Sources

Pretest and posttest. Although students answered a range of integer problems (e.g., ordering, comparing values, missing addend problems), we focus the present analysis on 14 integer addition problems that students solved on both the pretest and posttest (see Table 1). These were presented to students on one page in their regular classes, and students were asked to solve them as best as they could. Within a few days after students took the pretest, we interviewed 20% of the original sample to learn more about how they solved the problems.

Table 1: Arithmetic Problems Given on the Pretest, Posttest, and Midtest

Negative + Negative	Positive + Negative			Negative + Positive		
-6 + -4	5 + -2	4 + -5	9 + -9*	-5 + 5	-2 + 0	-4 + 6
-1 + -7*	6 + -8	7 + -3	0 + -9	-9 + 2*	-3 + 1	-1 + 8
-4 + -3**, -2 + -2**	4 + -6**	8 + -3**		-1 + 3**	-8 + 8**	

**Indicates a problem also given on the midtest; **Indicates a problem only given on the midtest*

Small-group sessions. As mentioned, students were divided into three intervention groups (sequential, intuitive, and conflicted). Of the 32 students who ignored all negative signs on the pretest, 11 students were from the sequential group, 12 were from the intuitive group, and 9 were from the conflicted group. When studying the contrasts, students worked in small groups of 2-3 students from their same intervention groups. Students in the *sequential group* analyzed each type of addition problem separately and in contrast to similar problems. Students in the *intuitive group* first compared adding two positive numbers versus adding two negative numbers, an intuitive contrast. Students in the *conflicted group* compared adding two positives with adding a negative to a positive in session one (we called this group conflicted because addition usually makes a larger number). During the second sessions, students analyzed the two other types of problems they did not see in their first session (see Table 2).

Each group saw their contrasts within four different illustrated contexts: an inclined number path situated on a hill, a vertical number line showing ants moving below and above ground, positive and negative chips, and a folding number line (see Tsang, Blair, Bofferding, & Schwartz, 2015). Students discussed and wrote about the similarities and differences between the problems and pictures; analyzed incorrect answers based on research of students' common misconceptions (e.g., ignoring the negative sign); and determined how to use the illustrations to correctly solve the problems. At the end of each session, students solved 6 integer addition problems related to the problem types they explore during that session for a total of 12 problems (9 with negatives) across the two session. We refer these collective problems as the midtest in the analysis and results (see Table 1).

Table 2: Problem Types that each Group Solved During their Two Small-Group Sessions

	Sequential Group	Intuitive Group	Conflicted Group
Session 1	P + P vs. P + P, then N + N vs. N + N	P + P vs. N + N	P + P vs. P + N
Example:	3 + 5 vs. 4 + 4, then -3 + -5 vs. -4 + -4	3 + 5 vs. -3 + -5	3 + 5 vs. 3 + -5
Session 2	N + P vs. N + P, then P + N vs. P + N	N + P vs. P + N	N + N vs. N + P
Example:	-7 + 2 vs. -6 + 1, then 7 + -2 vs. 6 + -1	-7 + 2 vs. 7+-2	-7 + -2 vs. -7 + 2

Note: N = negative integer, P = positive integer

Whole-class instruction. Students participated in a 30-minute lesson focused on solving addition problems using a number path. Adding a negative number corresponded to going down or getting more negative and adding a positive number corresponded to going up the number path or getting more positive, leading to the introduction of additive inverses or "zero pairs" (e.g., -2 + 2). Students then played a card game using one stack of "1" cards, one stack of "-1" cards, and a die, where the goal was to collect cards in order to make zero pairs.

Analysis

To analyze students' integer addition performance, we marked each addition problem as either correct or incorrect and conducted a median test (a nonparametric test used for small sample sizes) on the pretest-posttest gain scores across groups. We did not include midtest results in the median test because the midtest did not have completely identical items.

In order to look for qualitative changes in students' solution strategies, we classified students' solutions to the integer addition problems on the tests according to their treatment of the negative signs and strategies, relying primarily on their response patterns for each negative integer problem type (positive plus positive, positive plus negative, and negative plus positive) and supplemented by interview data. First, we identified students who correctly answered all problems and designated them as *all correct, unary meaning*. If they had one incorrect within a problem type, we identified the type based on the codes and included it with the *all correct* code. For example, all of the students discussed here provided answers on the pretest consistent with ignoring the negative signs and adding the two numerals (e.g., -6 + -4 = 10; 6 + -8 = 14). Given the stability of their response pattern, we considered these students to be using this strategy even if we did not interview them; in some cases we did have interview data to confirm it. On the midtest and posttest, students sometimes provided responses that could have been coded in more than one way (*adds negative sign* vs. *directional* or *subtraction (negative)* vs. *directional)*. In these cases, we used their other responses within the problem type to infer which strategy they used and checked them with the transcripts when available. For example, students could get -6+-4 correct by either knowing that adding a negative means moving to the left on the number line or by adding 6 + 4 and making it negative. If students used the *adds negative sign* strategy on their other problems consistently, we assigned the same code to -6 + -4 = -10 (see Table 3).

Table 3: Strategies Second Graders Used on the Integer Addition Problem Types.

Strategies	Explanation (coded within each problem type)	Example: -4 + 1
Adds Negative Sign[3]	Student answers problems by adding the absolute value of the numbers and then making the answer negative.	4 + 1 = 5, so -5
Subtraction[2] (positive)	Student uses the negative sign as a subtraction sign and gets positive answers.	4 – 1 = 3 or 1 – 4 = 0
Subtraction[2, 3] (negative)	Student uses the negative sign as a subtraction sign and gets negative answers (or makes answers negative).	1 – 4 = -3 or 4 – 1 = 3, so -3
Ignores Negatives	Student ignores the negative signs.	4 + 1 = 5
Negative as Zero[1/2]	Student treats negatives as equivalent to zero.	0 + 1 = 1 or 4 – 4 + 1 = 1
Positive	Student answers with random positive numbers.	-4 + 1 = 10
Negative	Student answers with random negative numbers.	-4 + 1 = -8
Directional[1]	Students' answers are consistent with starting at the negative number and counting up or down the absolute value of the other number but not in a reliable direction.	-4 + 1 = -5 but -9 + 2 = -7
(Deviation)	Deviations to one of the above codes were noted when students' answers to a certain problem did not follow the pattern of the rest of the problems. This occurred particularly with the additive inverse problems (*zero pair*) that were part of instruction, problems where they had to *add zero*, or if a student *skipped* a problem.	

Meaning of minus sign: 1 = unary, 2 = binary, 3 = symmetric

Results

Performance on Integer Addition Problems

Overall, students in the conflicted and intuitive groups spent about 20 minutes in each of their sessions. Students in the sequential group spent about 40 minutes in each of their sessions because each problem type was explored in sequence instead of in comparison to a different problem type. Because the students discussed here ignored negatives on the pretest, they all started with scores of zero. Students in the conflicting contrast group had the highest median gain score from pretest to posttest (11.0) compared to those in the sequential group (8.0) and the intuitive group (4.5), X^2=5.869, p=.053. The conflicted contrasting group had significantly higher median scores than the intuitive contrast group. There was no significant difference between the conflicted and sequential groups; however, the conflicted group made higher gains with an intervention that was half as long as the sequential group. Table 4 presents additional information about the groups' performance on the pretest and posttest, as well as on the midtest. Not only did the conflicted group have the greatest median increase from pretest to posttest, but they also had the greatest increase in mean percent correct from pretest to midtest, from midtest to posttest, and from pretest to posttest.

Table 4: Groups' Performance on the Pretest, Midtest, and Posttest Addition Problems

Group	**Pretest (14 items) Mean (SD); %**	**Midtest (9 items) Mean (SD); %**	**Posttest (14 items) Mean (SD); %**
Sequential (n=11)	0 (0); 0%	1.7 (1.8); 19%	7.0 (4.9); 50%
Intuitive (n=12)	0 (0); 0%	1.8 (1.9); 20%	5.6 (4.0); 40%
Conflicted (n=9)	0 (0); 0%	2.3 (2.1); 26%	9.4 (3.6); 67%

Table 5 shows an additional breakdown of the problems by problem type on the midtest and posttest. Recall that the sequential and intuitive groups solved the N+N problem type (N=negative integer) at the end of their first session while students in the conflicted group solved the P+N problem type (P=positive integer). Across the three problem types, 5 out of 11 (45%) sequential group students, 6 out of 12 intuitive group students (50%), and 7 out of 9 conflicted group students (78%) correctly solved at least one problem on the midtest. On average, the groups did the best on problems related to the contrasts they saw in their first session.

Table 5: Students' Percent Correct for Each Problem Type on the Midtest and Posttest

Midtest	**N + N (3 items)** Average correct (% correct); Median	**N + P (3 items)** Average correct (% correct); Median	**P + N (3 items)** Average correct (% correct); Median
Sequential (n=11)	.82 (27%); 0.0	.36 (12%); 0.0	.55 (18%); 0.0
Intuitive (n=12)	**.92 (31%)**; 0.0	.42 (14%); 0.0	.42 (14%); 0.0
Conflicted (n=9)	.78 (26%); 0.0	**.56 (19%)**; 0.0	**1.0 (33%); 1.0**
Posttest	**N + N (2 items)**	**N + P (6 items)**	**P + N (6 items)**
Sequential (n=11)	1.0 (50%); 1.0	3.1 (52%); 4.0	2.9 (48%); 2.0
Intuitive (n=12)	1.1 (54%); 1.5	2.2 (36%); 1.0	2.3 (39%); 1.5
Conflicted (n=9)	**1.2 (61%)**; 1.0	**4.2 (70%); 5.0**	**4.0 (67%); 5.0**

Interpretations of the Negative Sign and Addition

Analysis of students' use of the negative sign and strategies for solving the integer addition problems provides additional insight into differences between the groups. Across the nine negative integer problems on the midtest, the sequential group had 5 out of 11 people (45%) continue to ignore the negatives when solving all of the problems. The intuitive group had 4 out of 12 people (33%) and the conflicted group had only 1 out of 9 people (11%) ignore all of the negatives. On the other hand, students in the conflicted group, who added negatives to positives in their first session, had more people interpret negative signs as subtraction signs (binary meaning) and subtract the numbers. The students in the other groups were more likely to make the answers negative if they did subtract (combination of symmetric and binary meanings) or add a negative sign after adding the absolute values (symmetric meaning). Finally, 75% of the students in the intuitive group, 55% of students in the sequential group, and only 22% of students in the conflicted group used a consistent strategy across all of the midtest problems.

On their first session recording sheets, many students in the sequential group ignored negatives. For example, one student ignored the negative signs when copying down two of the problems; another student wrote that the correct answer to -6 + -4 = 10, even though this was identified as wrong on the page. Students in the intuitive group paid much more attention to the negative signs, noticing when problems did or did not have them. Similarly, students in the conflicted group also noticed the negative signs, and, during follow-up interviews, one student indicated that it meant to “go back.”

On the posttest, on the problems where students had to add both a positive and negative number, 2 out of 11 students (18%) in the sequential group, 4 out of 12 students (33%) in the intuitive group, and 2 out of 9 students (22%) in the conflicted group had responses consistent with adding the absolute values and making the sum negative (symmetric meaning) for at least one problem type. On the other hand 2 out of 11 students (18%) in the sequential group, 2 out of 12 students (17%) in the intuitive group, and 5 out of 9 students (56%) in the conflicted group correctly answered all questions for each problem type (allowing for one mistake per type), such as N + P. These students got negative answers on problems when they couldn’t just add a negative sign, suggesting they had some unary understanding of the minus sign, accepting negatives as numbers in their own right.

Discussion

The results presented here provide insight into how students’ strategies and ways of thinking about the negative sign can be influenced when given access to integer problems through contrasting presentations. Although students with limited prior knowledge did well comparing problem types and when presented sequentially as in prior research (Rittle-Johnson, Star, & Durkin, 2009), their intervention took twice as long and the conflicted group still made higher gains. This data suggests students who initially do not acknowledge the negative sign might benefit most from analyzing contrasts where the initial focus is on the result of the operation (Positive + Positive vs. Positive + Negative) as opposed to the change in sign (Positive + Positive vs. Negative + Negative).

Based on the collective students’ gains by the midtest, exposure to the contrasting cases helped them change their thinking about the meaning of negatives and their use in addition problems. Further, their differences in strategy use suggest that depending on their contrasts, analyzing the problems gave them differential benefits in interpreting the problems. Students’ further gains from the midtest to the posttest suggest that the instruction made additional impact but that the conflicted group benefitted most, especially in terms of changes in their strategies.

Students in the sequential and intuitive groups were more likely to ignore the negative signs, suggesting that they had not developed a usable meaning for the negative signs after analyzing problems that included two negative numbers. Others in these groups were likely to add the absolute values of the numbers and then add the negative sign to the answer, suggesting they interpreted the negative signs as having a symmetric meaning (Vlassis, 2004). However, it is not clear that they understood a negative number as being an opposite of a positive number. A more accurate description may be that they saw the negative sign as a descriptor. By comparison, students in the conflicted group were more likely to interpret the negative sign as indicating a binary operation. Although they initially gave mostly positive answers, a focus on the negative sign as a subtraction sign reflects a shortcut often taught to students (i.e., adding a negative number is the same as subtracting a positive number). Interestingly, by the posttest, the conflicted group did better than the other groups overall, suggesting an improvement in identifying when the minus sign designates a negative number for beginning their count or when it designates an operation.

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THE INTERPLAY BETWEEN STUDENTS' UNDERSTANDINGS OF PROPORTIONAL AND FUNCTIONAL RELATIONSHIPS

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This research explores the interplay between students' understandings of proportional and functional relationships. Approximately 90 students participated in an early algebra intervention in Grades 3–5. Before the intervention and after each year of the intervention, we evaluated their understandings of proportional and functional relationships. Data revealed that among Grades 4 and 5 students who identified a correct function rule, a higher percentage were unsuccessful solving a proportional reasoning problem than those who were not able to identify a correct function rule. Namely, the data suggest that students' development of functional thinking may interfere with their development of understanding proportional relationships.

Keywords: Algebra and Algebraic Thinking, Elementary School Education

Decades of reform initiatives in teaching and learning algebra (e.g., National Council of Teachers of Mathematics, 2000, 2006) have brought about the "algebrafication" of elementary grades mathematics in which a number of core algebraic concepts are introduced into classroom curriculum and instruction (Kaput & Blanton, 2001). While research has documented the development of students' understanding of these concepts, what is less well understood is the interplay between concepts that, at face value, seem to be developmentally complementary to one another (e.g., functional thinking and proportional thinking). In what ways does learning particular algebraic concepts support or hinder the learning of other algebraic concepts? This question lies at the core of our study in which we examine the ways in which children's understandings of two concepts appear to interact and potentially constrain the development of their algebraic thinking. Specifically, we investigate the interplay between students' understanding of proportional relationships and students' understanding of relationships between quantities that co-vary in a non-proportional way (e.g., the functional relationship $y = 2x + 2$).

We chose to study these concepts because recent findings have shown that elementary students can reason about and describe relationships between co-varying and corresponding quantities (e.g., Blanton & Kaput, 2004; Schliemann et al., 2003) and, in fact, that even students in kindergarten and first grade can engage in this kind of thinking about co-varying and corresponding quantities (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015).

Our Early Algebra Intervention

This research is part of a three-year longitudinal study (viz., Blanton et al., 2017) whose overarching goal is to design, implement, and evaluate a Grades 3 – 5 early algebra intervention. We based the intervention on a synthesis of Kaput's (2008) analysis of algebra in terms of content strands and thinking practices (see Blanton, Stephens, et al., 2015 for an elaboration of the intervention). In particular, using Kaput's content analysis of algebra we frame the content of our

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intervention in terms of four fundamental thinking practices that characterize algebraic thinking: (1) *generalizing*, (2) *representing*, (3) *justifying,* and (4) *reasoning with* mathematical structure and relationships. We also identified several "big ideas" of algebra, that is, principles in a domain that are essential to developing an integrated understanding in that domain (Shin, Stevens, Short, & Krajcik, 2009) and that reflect content spaces in which the core practices of algebraic thinking (e.g., generalizing) can occur. The big ideas of algebra that comprised the early algebra intervention are as follows: (a) equivalence, expressions, equations, and inequalities; (b) generalized arithmetic; (c) functional thinking; and (d) variable.

One of the areas that becomes increasingly important as students transition into middle grades is proportional reasoning (NGA & CCSSO, 2010). Because of the connections between proportional reasoning and functional thinking, particularly as it relates to issues of rate of change and slope, we were interested in the interplay between students' functional thinking, developed as part of our Grades 3 – 5 early algebra intervention, and their early notions of proportional reasoning addressed in the regular curriculum. At face value, these two conceptual areas seem developmentally complementary to one another. Proportional reasoning involves generalizing two related quantities in which "the ratio of one quantity to the other is invariant" (Blanton, Stephens, et al., 2015, p. 43). Functional thinking involves "generalizing relationships between (two) covarying quantities and representing" those generalizations "using natural language, algebraic notation, tables, and graphs" (Blanton, Stephens, et al., 2015, p. 43). We view proportional reasoning as a subset of functional thinking because all proportional relationships can be described as functions, but not all functional relationships are proportional relationships. The purpose of this study is to investigate the interplay between students' functional thinking, developed as part of our Grades 3 – 5 early algebra intervention, and their early notions of proportional reasoning addressed in the regular curriculum.

Methods

We share data collected from a three-year longitudinal study in which we implemented and evaluated our early algebra intervention. To evaluate our early algebra intervention we assessed the algebraic thinking of students who participated in our intervention at several time points using a pretest and posttests.

Participants

At the pretest, participants included 103 Grade 3 students from a school in southeastern Massachusetts. The school's district is 8% non-white, 5% ELL students, and 20% low SES students. Due to attrition, 90 students participated in the entire early algebra intervention (i.e., participated in Grades 3 – 5).

Intervention

The Grades 3 – 5 intervention consisted of approximately 18 lessons per year and engaged students in the aforementioned algebraic thinking practices of generalizing, representing, justifying, and reasoning and the targeted big algebraic ideas. One member of our project team served as the classroom instructor for the intervention, beginning with the Grade 3 cohort and continuing with this cohort through the completion of Grade 5. The intervention was taught as part of students' regular mathematics instruction. The sequence of 18 lessons in each of Grades 3 – 5 included 6 lessons focusd on functional thinking. Functional thinking lessons were designed to get students to generate data, use function tables to organize data, identify functional relationships and represent in words and variables, and use these relationships to make inferences about function behavior. Lessons also included developing graphs to represent functions and interpreting functional behavior in graphs through quantitative and qualitative means. Functional thinking tasks focused primarily on linear functions, but also included quadratic and exponential functions. Proportional reasoning concepts were not explicitly taught in the intervention.

The instructional sequence was organized into Grades 3, 4, and 5. For each year of our intervention, we listed learning goals and organized them according to the associated big idea. The lessons were designed to address these learning goals. Each lesson began with a small-group discussion regarding a previously addressed learning goal, so that learning goals were revisited throughout the lessons. Then, a new learning goal was addressed through small-group problem solving and a whole-class discussion. Associated assessments were designed to test the effectiveness of the intervention by evaluating students' understandings of the big ideas and administered at the beginning of the intervention (Grade 3 pretest) and after each year of the intervention (Grade 3 posttest, Grade 4, and Grade 5).

Data Collection

Students who participated in the intervention were assessed at the beginning of Grade 3, and then again at the end of each year in Grades 3 – 5 using grade-level assessments designed by the project team. The same Grade 3 assessment was used as a pre/post measure in Grade 3, while the Grades 4 and 5 assessments included some identical items and some new items. Each assessment consisted of about 12 items (10 were multi-part open response, 2 were multiple choice). Here we focus on students' responses to two items that appeared on the assessments at each grade level.

The first item (see Figure 1), the *Caterpillar* task (adapted from NAEP, 2003), is designed to evaluate students' ability to reason proportionally. The second item, the *Brady* task (see Figure 2), was designed to evaluate students' understandings of functional relationships. Here, we focus in particular on part c2, which was designed to assess students' ability to generalize and represent a functional relationship using variables. Both of these items appeared on all four assessments given across Grades 3 – 5.

Data Analysis

Responses were scored using a coding scheme developed by the project team to capture both correctness of student responses as well as the types of strategies students used (Blanton, Stephens, et al., 2015). For the response to the *Caterpillar* task to be coded as correct, students must have provided a response of 30. If students also provided an explanation that demonstrated proportional reasoning, coders further identified the way that the student reasoned proportionally (i.e., as using calculations, tables, pictures, a unit rate or repeated addition). If students provided an incorrect answer, coders labeled the response with one of the incorrect strategies or "other." If no explanation or indication of strategy was provided, coders labeled the response "answer only." Here we focus on a specific incorrect strategy, incorrect linear relationship. Students who demonstrated this strategy wrote a response of "25" and typically explained that they used the linear relationship "$2x + 1$" to find their solution. Students found this solution because the relationship "$2x + 1$" results in the example provided, 5 caterpillars and 2 leaves.

A fourth-grade class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars? Explain how you got your answer.

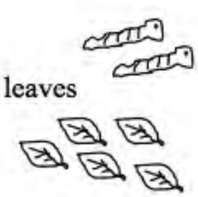

Figure 1. Proportional reasoning item (*Caterpillar* task).

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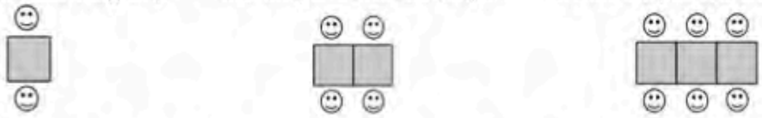
Brady is celebrating his birthday at school. He wants to make sure he has a seat for everyone. He has square desks.

He can seat 2 people at one desk in the following way: If he joins another desk to the first one, he can seat 4 people: If he joins another desk to the second one, he can seat 6 people:



a) Fill in the table below to show how many people Brady can seat at different numbers of desks.

Number of desks	Number of people
1	2
2	4
3	
4	
5	
6	
7	

b) Do you see any patterns in the table from part a? If so, describe them.

c) Think about the relationship between the number of desks and the number of people.
   1. Use words to write the rule that describes this relationship.
   2. Use variables (letters) to write the rule that describes this relationship.

Figure 2. Functional thinking item (*Brady* task).

When refining the coding scheme, we conducted iterative analyses of students' responses to these items. First, we identified strategies already documented in the research literature on children's algebraic thinking. For example, research shows that children may begin generalizing functional relationships by focusing on particular instances, demonstrating a recursive strategy (Blanton, Brizuela, et al., 2015). These external strategies served as a starting point for developing our coding scheme. If a response did not fit an external strategy, it was grouped with similar responses. We then identified patterns in these responses and developed new codes to capture these responses.

For the response to part c2 of the *Brady* task to be coded as correct, students must have written a function using variables to represent the relationship at hand (e.g., $y = 2x$). If students provided an incomplete (e.g., $2x$) or incorrect answer, coders labeled the response with the appropriate incorrect strategy. If a student's response could not be categorized using our coding scheme, coders labeled the response as "other."

Inter-rater reliability scores were computed for 20% of the items and at least 80% agreement was achieved between the coders. When coders disagreed, they discussed codes until agreement was obtained.

Results

In this section, we share results from the two written assessment items and focus on relationships we observed between students' understandings of functional and proportional relationships. For the *Caterpillar* task, we focus on one strategy in particular, the incorrect linear relationship strategy because we found an unexpected relationship between this strategy and another strategy. Students who used the incorrect linear relationship strategy incorrectly generalized a linear relationship between the number of caterpillars and the number of leaves (e.g., $2x + 1 = y$) were coded as using this strategy. For the *Brady* task, we also focus on one strategy, the correct function rule. Students who demonstrated this strategy correctly identified a function rule and represented it using variables in an equation.

We observed a trend in the ways that students who generalized functional relationships on part c2 of the *Brady* task reason about the *Caterpillar* task. In particular, the data revealed that among the students in Grades 4 and 5 who identified a correct function rule using variables to describe a generalized relationship between the two covarying quantities, a higher percentage of those students demonstrated the incorrect linear relationship strategy on the *Caterpillar* task than did the overall population of students. That is, the data suggest that in the context of the intervention, students' development of functional thinking and the development of their understandings of proportional relationships may be related. Although we are not certain of the nature of this relationship, these data suggest that some students' development of functional thinking may impede the development of their understandings of proportional relationships in the context of the intervention.

Table 1 shows the percentage (and number) of students who identified the correct function rule in response to the *Brady* task in Grades 3, 4, and 5. The number of students who identified the correct function rule is listed in parentheses. The data reveal that as students progressed through the intervention, they were better able to write the correct function rule using variables.

Table 1: Overall Student Performance on *Brady* Task

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Correct Function Rule	0.00% (0)	35% (36)	64% (61)	67% (60)

The percentages in Tables 2 – 4 were calculated using the number of students who used the correct function rule for each grade (as shown in Table 1). In other words, the denominator for each percentage in Tables 2 – 4 is the number of students (in parentheses) for the respective grade in Table 1. Table 2 shows how the subgroup of students—the 36 students—who identified the correct function rule at the Grade 3 posttest performed on the *Caterpillar* task at each time point. Table 3 shows how the subgroup of students—the 61 students—who identified the correct function rule at the Grade 4 test performed on the *Caterpillar* task at each time point. Table 4 shows how the subgroup of students—the 60 students—who identified the correct function rule at the Grade 5 test performed on the *Caterpillar* task at each time point.

Table 2: Percentage of Gr 3 Post Students who Provided a Correct Function Rule (*Brady* Task) and Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	0%	3%	7%	12%

Table 3: Percentage of Gr 4 Students who Provided a Correct Function Rule (*Brady* Task) and Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	0%	7%	13%	30%

Table 4: Percentage of Gr 5 Students who Provided a Correct Function Rule (*Brady* Task) and Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	0%	12%	30%	30%

Table 5 shows the percentage of all students who demonstrated the incorrect linear relationship strategy in Grades 3, 4, and 5. By comparing the performance of the subgroups on the *Caterpillar* task (Tables 2 – 4) to the overall performance of students on the *Caterpillar* task we noticed that students who identified the correct function rule were more likely to also demonstrate the incorrect linear relationship strategy in Grades 4 and 5 than was the general population of students.

The percentage of students identifying the correct function rule in response to the *Brady* task who demonstrated the incorrect linear relationship strategy in response to the *Caterpillar* task is less than the total percentage of students who demonstrated the incorrect linear relationship strategy in response to the *Caterpillar* task in Grade 3. We do not believe we can draw many conclusions from this due to the low overall performance on the *Brady* task in this grade. However, as success on the *Brady* task increases into Grades 4 and 5, we feel that more can be said about the interaction between students' strategy use on these items.

Table 5: Percent of Students who Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	2%	5%	9%	20%

Of the subgroup of students who identified the correct function rule in response to the *Brady* task in Grades 4 and 5, the percentage that also demonstrated the incorrect linear relationship strategy in response to the *Caterpillar* task is greater than the overall percentage of students who demonstrated the incorrect linear relationship strategy. Interestingly, the reverse relationship holds true for Grades 4 and 5 as well. That is, of the subgroup of students who demonstrated the incorrect linear relationship strategy in Grades 4 and 5, the percentage who also identified the correct function rule is greater than the overall percentage of students who identified the correct function rule.

Table 6: A Representative Student's Responses in Grades 4 and 5

	Incorrect Linear Relationship Resonse to *Caterpillar* Task	Correct Function Rule Response to *Brady* Task
Grade 4	X Y L c 5 2 25 12 the rule Well I saw it was 2×Y+1=X So I did 12×2+1=25	D×2=P D=Desks P=People
Grade 5	Cater. X | leaves Y 2 | 5 12 | 25 Rule: X·2+1=Y	X·2=Y

Table 6 shows one student's responses to both tasks at the Grades 4 and 5 assessments. This student was selected because his strategy gives an example of the combination we focus on in this paper. We chose to show the students' responses in both Grades 4 and 5 because we observed that several students demonstrated both strategies in both Grades 4 and 5. Specifically, 8 students demonstrated both these strategies in Grade 4. Of those 8 students, 6 of them also demonstrated both strategies in Grade 5. An additional 11 students demonstrated both strategies in Grade 5, totaling to 17 students.

Discussion

The results presented here highlight the complexity of the interrelated concepts involved in studying corresponding relationships in the elementary grades. It leads us to question the role of proportional reasoning in the regular curriculum and how that of functional thinking in our intervention (or even the regular curriculum, to the extent that functions are addressed in elementary grades) coincide.

Moreover, the findings reveal that the characteristics of corresponding relationships that are salient to students are not what we anticipated when designing the intervention. That is, the findings highlight that educators need to be cautious when drawing conclusions about what children know and how it is they come to know it. The findings show that while children may have knowledge of a particular concept, functional thinking in this case, they might misappropriate the concepts and tools in other situations. We can infer from the student's responses shown in Table 6 that this student, and based on the percentages shown in Tables 2 – 4 likely many students, chose to use one of the tools they were taught (e.g., a function table) to use when interpreting functions to represent the *Caterpillar* task. Students are taught function tables as a tool for interpreting functional relationships. Therefore, the fact that this student correctly responded to the *Brady* task and used a function table to interpret the *Caterpillar* task makes sense in the context of our intervention. The reason we did not observe students incorrectly responding to the *Brady* task and using the incorrect linear relationship strategy is because they did not have the tools (e.g., a function table) for interpreting functional relationships.

Lastly, we believe the data displayed in Table 6 highlight that when children come to know a concept in a certain way, they struggle to change the way they know that concept, especially in different contexts. This observation may indicate that students' thinking is entrenched from year to year because the context is relatively consistent. McNeil and Alibali (2004) noted that students resist adapting their understandings of the equal sign in different contexts and we view this finding as relevant to our interpretation of our findings. Similarly, our prior research (e.g., Strachota et al., 2016) on students' understanding of the equal sign and functional thinking have led us to consider how different contexts and co-developing big ideas might impede or support the development of children's algebraic reasoning. We believe these studies are a small slice of an increasingly important area of research in early algebra. In order to move forward in supporting students in developing understandings of algebraic concepts, we must better understand the interplay between concepts.

Due to the nature of our data (i.e., written assessments) we do not know with certainty what students might have been thinking when they demonstrated the incorrect linear relationship strategy. However, we can infer that students who demonstrated this strategy associated some aspect of the proportional relationship with the process of writing a function rule using variables. Moving forward, we hope to investigate what aspects of the task are salient to the students who use the incorrect linear relationship strategy and use these data to refine our instruction.

We acknowledge the limitations of this study specifically the small sample size of the subgroups and the fact that only two tasks are considered, but hope the findings will serve as the premise for future research that takes the same line of inquiry. Researchers have long advocated that algebra be developed as a longitudinal curricular strand. We agree with this perspective, and believe that our findings reveal the importance of continuously supporting students in developing understanding of core algebraic concepts associated with functional thinking and proportional reasoning. Further, we view the conceptual areas of proportional reasoning and functional thinking as interrelated, and recommend that early algebra curricula be designed to synthesize these core concepts of algebra, as well as all core concepts of algebra.

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THE PROCESSES AND PRODUCTS OF STUDENTS' GENERALIZING ACTIVITY

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Generalization has been a major focus of curriculum standards and research efforts in mathematics education. While researchers have documented many productive contexts for generalizing and the generalizations students make, less attention has been given to the processes of generalizing. Moreover, there has been less work done with high school students in advanced mathematical contexts. To address these issues we use a model of learning that enables us to make explicit the processes of generalizing. We exemplify this model of learning in the context of an interview study with high school students working on cubic relationships.

Keywords: Cognition, Learning Theory

Generalization has been a major focus of the Common Core State Standards as both a process and content standard. One reason for this focus in curriculum standards is that mathematics educators have identified that generalizing activity is a central means through which students construct new knowledge and as such should be a primary focus of school instruction (Davydov, 1972/1990). To date a majority of the research on generalization has taken place with elementary and early middle school students (grades K-7) in part because it is seen as a basis for and route to algebraic reasoning (e.g., Carraher, Martinez, & Schliemann, 2008). Furthermore, many of these studies have focused on patterns or functions that involve linear relationships, however, researchers have argued that there is a need for investigation of studying generalizing in situations that can involve non-linear relationships as well as studies that include older students (Amit & Neria, 2008). Dorfler (2008) has also identified that researchers studying generalization have focused their attention on developing contexts for generalization and characterizing the kinds and qualities of generalizations students make within these contexts. However, he identifies that there has been significantly less attention paid to the processes involved in generalizing (see Ellis, 2007 for an exception). To address these issues, we report on an interview study conducted with eight 10^{th}-12^{th} grade students who worked on establishing cubic relationships. We situate our work within a framework for studying learning where a central reason for this is so that we can specify particular processes involved in generalizing.

Theoretical Framework

Ellis (2007) differentiates between generalizing actions and reflection generalizations; a generalizing action is an action that precedes and may support a formal statement of generalization (i.e., is a process involved in generalizing) and a reflection generalization is a formal statement of generalization that a student expresses verbally or symbolically (i.e., a product of generalizing actions). We find this distinction useful in outlining our framework for learning which is based in scheme theory. A scheme is a repeatable way of operating that consists of three parts: an assimilatory mechanism, an activity, and a result (Piaget, 1970; Von Glasersfeld, 1995). The assimilatory mechanism involves a student in making an interpretation of a problem situation. The activity of a scheme involves the use of mental operations on imagined or perceptually present material, which transform the assimilated situation into a result. When a person re-processes the result of a scheme so that she can anticipate it prior to carrying out the activity, we call this an act of interiorization, and we consider the person to have constructed a concept. Concepts, then, are the results of schemes that are available to a person prior to operating and are what a person uses to assimilate situations.

For a scheme to be repeatable in different experiential situations means that its formation entailed generalizing actions (Ellis, 2007): A person uses the same scheme in two or more different experiential situations, which implies that the person has abstracted some similarity among the experiential situations because they trigger, vis-à-vis assimilation, the same scheme. We make the distinction between assimilation and *generalizing* assimilation to mark when a generalizing action happens in a particular experiential situation (Piaget, 1970); assimilation entails the re-activation of a scheme, but the abstraction of similarity among current and past situations is not a primary feature of this re-activation. Assimilation is *generalizing* when a primary feature of the re-activation of a scheme is the abstraction of similarity among the current and prior situations. We consider this abstraction of similarity to be a generalizing action that entails an act of learning because a person modifies the assimilatory mechanism of a scheme.

One marker that distinguishes situations involving assimilation and those involving generalizing assimilation is that in the latter case a person may experience a perturbation, a sense of cognitive dissonance. This perturbation may be expressed as uncertainty about what to do to solve a situation even though from the perspective of an observer a student has a scheme that could be used to solve the situation. The resolution of this kind of perturbation can occur through an abstraction of similarity between the current and prior situations in which a student has used his or her scheme. We provide one empirical example of this kind of learning later in the paper.

We consider a second type of learning to be within the realm of generalizing actions as well. We call this type of learning a functional accommodation; a functional accommodation differs from a generalizing assimilation in that a student makes a modification to the *activity* of her scheme in the context of its use (Steffe, 1991). We specifically consider a functional accommodation to entail a generalizing action when it enables a student to solve a broader range of situations. We provide three empirical examples of this kind of learning later in the paper.

The claim that a student has engaged in a generalizing assimilation or a functional accommodation means a researcher is inferring that some modification that was novel occurred in the context of a student using his or her scheme. We highlight that such processes are in the province of reflective abstraction where a reflective abstraction involves a projection of a novel way of operating from a lower to higher level along with the reorganization of the novel way of operating at the higher level (Piaget, 1970). Here we consider the "lower" and "higher level" to be defined by the fact that the scheme itself becomes more general in nature either because the assimilatory structure is broadened to include new experiential situations or because the change in activity of a scheme allows a student to solve a broader range of problems.

None of the acts of learning described to this point necessarily involve a student in being consciously aware of having made a modification to his or her scheme. This is a key reason that we consider them to be in the realm of what Ellis (2007) calls generalizing actions—actions that precede a formal statement of generalization. Further, we note that generalizing actions can occur in the absence of an actual formal statement of generalization yet they are an important part of documenting the *processes* involved in generalizing. We consider formal statements of generalization, what Ellis calls reflection generalizations, to be in the province of a reflected abstraction (Piaget, 1970)—a retroactive thematization of a way of operating that brings this way of operating to conscious awareness. We consider reflection generalization to be in the province of a reflected abstraction because to make a formal statement of generalization entails becoming to some extent consciously aware of how one is operating. Moreover, as Ellis's definition suggests, a reflection generalizations involve symbolizing—formal statements of generalization are made either with natural language or mathematical notation (both symbol systems).

Following Von Glassersfeld (1995), we view symbols as involving bi-directional relationships among a sound/graphic image, a person's re-presentations, and a concept (p. 131). The most important point of Von Glassersfeld's model for this paper is that when a person has constructed a symbol, any one of the three (a sound/graphic image, a person's re-presentations, and a concept) can call up any of the others. This observation means that a sound/graphic image (e.g., a verbal statement or the letter "*x*") can be used to call forth a concept where a concept is the operations of a scheme that no longer need to be implemented either mentally or materially in order for a person to consider them to be part of an experiential situation. We provide one empirical example of a reflection generalization that involves the processes of symbolizing.

Methods and Methodology

We provide empirical examples of generalizing actions and reflection generalizations from an interview study conducted with eight 10^{th}-12^{th} grade students. We conducted two hour long video recorded interviews that focused on problems like the *Card Problem.*

> *Card Problem.* You have the 2, 3, and King of Diamonds, a friend has the 2, 3, and King of Hearts, and another friend has the 2, 3, and King of Clubs. A three-card hand consists of one card from each person's hand (order does not matter). How many different three card hands are possible to make? How many three-card hands have no face cards, exactly one face card, exactly two face cards, and exactly three face cards?

The aim of this problem was for students to develop the equivalence that $3^3 = (2 + 1)^3 = 2^3 + 3 \bullet (2^2 \bullet 1) + 3 \bullet (2 \bullet 1^2) + 1^3$. We conjectured that this equivalence could grow out of reasoning that there were a total of 3^3 possible three-card hands, that this total could be quantified as $(2 + 1)^3$ because each person had 2 non-face card and 1 face card, and also that this total could be quantified as $2^3 + 3 \bullet (2^2 \bullet 1) + 3 \bullet (2 \bullet 1^2) + 1^3$ because: the number of three card hands with no face cards was 2^3; there were 3 ways to have one face card with each way having $(2^2 \bullet 1)$ three card hands, etc. During the interviews, students were encouraged to represent this reasoning using a 3-D array that represented all possible three card hands (Figure 1a), and to identify different regions of this array that represented three card hands that had no face cards (Figure lb, green region), one face card (Figure 1b, three blue regions, only two visible), two face cards (Figure 1b, three yellow regions), and three face cards (Figure 1b, red region). Students then worked toward a statement of generalization that $(x + 1)^3 = x^3 + 3 \bullet (x^2 \bullet 1) + 3 \bullet (x \bullet 1^2) + 1^3$. We regard the examples as "learning in process" because it was not possible to determine from two interviews the extent to which the modifications that students made were lasting.

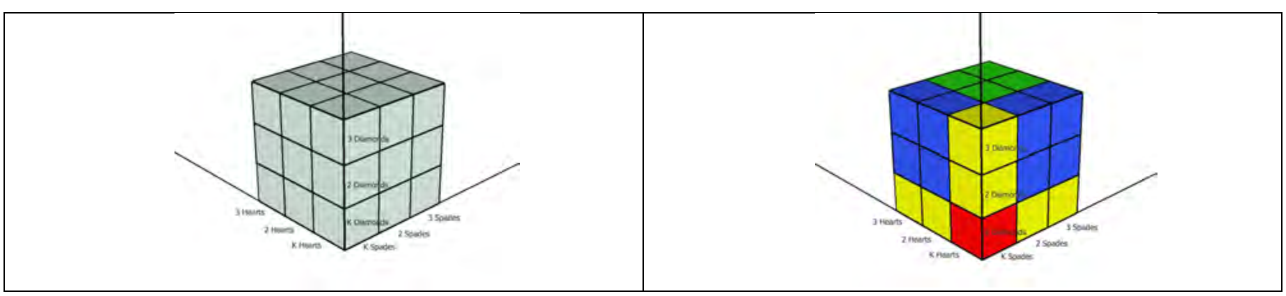

Figure 1a (left), 1b (right). 3-D array highlighting possible types of three-card hands[i]

Empirical Examples: Learning, Generalizing Actions, and Reflection Generalizations

Example one assimilation versus generalizing assimilation. We use Shante's solution of the Subway and Card Problem to illustrate generalizing assimilation. Shante solved the Subway Problem with relative ease she concluded that there should be 24 possible sandwiches.

Subway Problem: Subway has two kinds of bread, three kinds of cheese, and four kinds of meat. A sandwich is one bread, one cheese, and one meat. How many possible sandwiches can subway make?

The interviewer then asked her to determine the total number of possible three card hands in the Card Problem, which included making a number of sample three card hands with actual cards. When she was asked how many possible three card hands she could make, Shante responded quickly that there would be 27 three card hands because the answer would be “three times three times three.” However, when asked to explain the multiplication problem and to represent her solution, Shante expressed uncertainty.

I: Why is it that (three times three times three)?
S: Three cards, three hands and three kings? Wait. Okay. Can you say the question again? [The interviewer asks the question again, but Shante is still confused. The interviewer asks Shante to make a tree diagram]
….
S [responding to a question about making a tree diagram for the problem]: Okay. Wait a minute. Okay, now wait. What's going on here? Okay, so this would be a K and a three, and then the rest would branch out from that, right? [writes “K-3-2” on her paper.]
….
S [she has stated that the two in the “K-3-2” is the two of spades, and is responding to the interviewer’s question about what the other cards represent]: It will be … a K heart and a three heart … I think that's one hand.

Shante’s solution indicated that a critical difference for her in solving the Card Problem was determining what a valid three card hand was—in the Subway Problem it was clear to her that a sandwich could not consist of one bread and two cheeses. However, in the Card Problem it was unclear to her that a three card hand should not consist of one spade and two hearts, despite the fact that she had made three card hands with actual cards and these three card hands contained only one spade, one heart, and one diamond. The primary difference in these two situations seemed to be that each person had the same kind of object, cards, that were differentiated by suit whereas in the Subway Problem the objects were not of the same kind. This led to a perturbation for Shante, “this problem got me all the way messed up”, that she eventually resolved by relating the suits of the cards to the Subway Problem, “Could the spades be like the meats?....The diamonds could be like the cheeses?....Oh, the hearts would be like the bread!” Because Shante’s abstraction of similarity between the two situations was a primary feature of her solution of the problem, we considered her solution to involve a generalizing assimilation, which was an act of learning that entailed an adjustment to the assimilatory mechanism of her scheme.

Example two recursion of a scheme. Our second example is drawn from a 10th grade student in the interview study and allows us to examine a functional accommodation that involved recursion. Trevon was initially presented with a modification of the Subway Problem that involved four breads and six meats. Trevon said, “In my head I’m attributing one bread to each meat. … and if you do that, it would just be 4 times 6. [He subsequently made the beginning of a tree diagram].” Trevon was able to quickly complete the task and provide a justification for his way of operating, which we take as indication that he assimilated the situation using an extant scheme. The activity of his scheme involved multiple operations, but we highlight that it included a systematic use of his *pairing operation*, an operation that involves a student in creating the sandwich as a unit that contains two units (i.e, a bread and a meat), a pair. This operation can be the basis for establishing the identity that one times one is one, and we focus on it because it is a key feature of students’ solutions of combinatorics problems that differentiates them from other multiplicative situations.

Later in the interviews, Trevon was presented with the Card Problem where each person had three cards. He responded that the number of possible three card hands would be, “eighteen”, explaining, “there’s six combos [indicating hearts and diamonds combined] for each card and there’s three cards total [indicating the three spades]. So, that’s six times three.” Our interpretation of Trevon’s response was that he assimilated the Card Problem using the scheme he used to solve the Subway Problem. There were three spades that could be paired with each of six cards. The interviewer asked him to show the eighteen three card hands with the actual cards. To do so, he initially fixed a spade and simultaneously moved two cards, one heart and one diamond, to be paired with the spade. By doing so he essentially treated the heart and diamond cards as if they were like the meats in the Subway Problem, meaning he did not pair the heart with the diamond first to establish a heart-diamond pair. He experienced a perturbation, however, because he was trying to monitor which three card hands he had created and was not able to do so. His monitoring led him to introduce a novel way of operating: he paired each heart with each diamond, and took the result of this scheme, the nine heart-diamond pairs, as material to operate on with his scheme, pairing each heart-diamond pair with a spade.

We interpret this as a functional accommodation that entailed taking the result of his scheme as input for using his scheme again. For this reason, we consider the functional accommodation to have involved recursion; the recursive process did not involve his use of any novel operations. However, it involved more than just repeated use of the scheme—the result of the first instantiation of the scheme was embedded in the result of the second instantiation of the scheme. We consider this embedding process to be key to recursion. Moreover, we consider this to be a generalizing action because it broadened the class of problems Trevon could solve.

Example three embedding novel operations into the activity of a scheme. As our description of Shante’s activity in Example One suggested she had constructed a scheme for solving problems like the Subway Problem with breads, meats, and cheeses, and vis-à-vis a generalizing assimilation used her scheme to solve the Card Problem. She solved each problem representing the set of outcomes as either a list or as a tree diagram, and the operations she used to solve the problems were comparable to those outlined for Trevon in Example Two. The interviewer had the goal of having students represent the solution of these problems using multiple 2-D arrays, and then a single 3-D array. For the Subway Problem, Shante easily created two 2-D arrays (Figure 2a) to represent the problem. The interviewer asked her what was changing as she “moved horizontally along the x-axis” and she said, “the cheeses”, and responded similarly to a question about, “the meats”.

The interviewer then said that the goal was to think about how Shante could use her two-dimensional arrays to make a three-dimensional array, specifically asking, “what direction could you move for the breads to change?” Shante responded that she thought her listing of the breads in Figure 2a would “just be a title” for each array, and indicated that she was uncertain about what direction she might move in order for the breads to change. To further investigate this issue, the interviewer had Shante make two 2-D arrays using snap cubes (Figure 2b). After significant questioning, Shante figured out that she could stack one array on top of the other, and the interviewer had her make an array for a third bread (Figure 2c). At this point she still had “no idea” in which direction you’d move for the breads to change so the interviewer asked her to imagine where all the sandwiches with bread four, five, six, and seven would be, and for each she responded, “right above” all the sandwiches for the prior bread. The interviewer then asked her “if you had to draw a line to show where the breads would be represented (like the lines she had drawn for the meats and cheese), how would she draw it?” Shante responded, “It would be going up!”, and to show she put a pencil at the vertex of the bread and cheese axis in Figure 2c. From the perspective of mental operations, we consider Shante to have envisioned translating her 2-D array up one unit six times, and from this translation to have abstracted an axis on which the breads could be located that were similar to the axes for meats and cheeses.

Figure 2a (left), 2b (center), 2c (right).

To determine what other operations Shante might be using in her production of a 3-D array, the interviewer asked her to locate in Figure 2c where the sandwich with the second bread, second cheese, and third meat would be. Shante initially pointed to the top of the 3-D array where the sandwich with bread three, cheese two, meat three was located, and said, "It's like in this [pointing downward with her pencil], and then pointed to where the sandwich containing bread two, cheese one, and meat three was located, and said, "It's like right next to this one [pointing inward with her pencil towards where the cube was located]." She then clarified that by "in this" she meant "underneath" the cube that she had pointed to on top of the array.

Her identification that the cube for bread two, cheese two, meat three was "underneath" the cube that represented bread three, cheese two, and meat three, indicated that as part of translating the layers of her array upward she had also translated the referential system upward (i.e., the meat-cheese axes), and could use that to locate points in her 3-D array. The fact that she also identified the cube for bread two, cheese two, meat three as "next to" the cube for bread two, cheese one, and meat three, indicated that she could envision switching frames of reference; she used the meat-bread axes to locate the correct cube in the "cheese 1" plane, and then envisioned that the correct cube would be translated one unit inward on the 3-D array. We consider her switching frames of reference to be indication that she could mentally rotate the "meat-cheese axes" to become the "bread-meat axes".

We consider the operations of translation and rotation to have been *embedded* in the activity of her scheme for producing the set of outcomes. Because these operations were not part of her initial solution to the problem, we consider this to have been a functional accommodation to her scheme. We consider this functional accommodation to be a generalizing action because it allowed her to create a spatial structuring for a broader class of objects where her spatial structuring for 2-D arrays was embedded in her spatial structuring for 3-D arrays vis-à-vis the operations of translation and rotation.

Example four operating on the result of a scheme with operations external to the scheme. DeShay entered the interview study with a scheme for solving combinatorics problems like the Outfits and Subway Problem, and with relative ease could represent the set of outcomes using a 3-D array. We took this as indication that her scheme for solving such problems included the operations described in example two and three. Moreover, these operations seemed well established for DeShay, and so we considered that the result of these operations (the set of outcomes represented as a spatially structured 3-D array) was material that she could operate on with operations that were external to her scheme for producing them. To illustrate this issue, we provide data where DeShay was finding the regions of her 3-D array that corresponded to a three card hands that had a certain number of face cards, specifically the three card hands that contained one king where the king was the king of spades.

> D: Okay, so we have…Want to start with the king of spades, so it'd be these-by-these [points to the region in the 3-D array that represents the twenty-five three-card hands that contain the king of spades, Figure 3a], but it wouldn't be any in this row because that's the king of hearts (Figure 3b). So, it's just these four by these four. But it wouldn't be any -- or, these four by these five (Figure 3c). But it wouldn't be any of the very bottom because that's the king of diamonds (Figure 3d). So, it's this four by that four (Figure 3e).

To solve this problem DeShay took the *intersection* of the set of three card hands that contained the king of spades with the set of three card hands (Figure 3a) that contained the king of diamonds (Figure 3b), and *eliminated* this from the original set because those three card hands would contain both the king of spades and king of diamonds (Figure 3c). She then used these operations recursively: she took the *intersection* of this new set ("these four by these five") with the set of three card hands that contained the king of diamonds (Figure 3d), and *eliminated* these three card hands because they contained the king of spades and king of diamonds (Figure 3e).

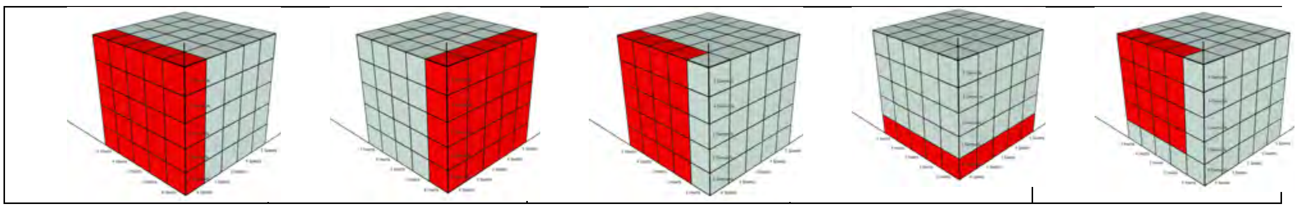

Figure 3a (left), 3b (second from left), 3c (middle), 3d (second from right), 3e (right).

We consider this functional accommodation to entail DeShay using operations external to the ones that produced the set of three card hands represented as a 3-D array. We consider it to be a generalizing action because DeShay's implementation of these operations led to the abstraction of a spatial structuring like the one shown in Figure 1b where she no longer had to implement the operations in order to impose the spatial structuring on future cases of the problem, represented as other cubes (e.g., a seven by seven by seven cube or a one hundred by one hundred by one hundred cube). She could simply assimilate future situations with the result of her operating, a cube that contained eight regions.

Example five reflection generalization involving reflected abstraction. After solving versions of the Card Problem that involved three, four, and five cards, the interviewer asked DeShay to imagine and describe what the cube would look like for the case of six cards, and then asked her if she could write a formula for if there were an unknown number of non-face cards. DeShay first wrote, "$y=1+x$" where y was the total number of cards and x the total number of non-face cards, and then wrote "$(1 + x)^3 = x^3 + (1 \bullet x^2 \bullet 3) + (1^2 \bullet x \bullet 3) + 1^3$." We considered these to be symbols that DeShay could use to call forth the operations outlined in examples two through four without actually having to implement these operations in full. We make this inference based on explanations like the following of how she got "$(1 \cdot x^2 \cdot 3)$":

> D: There would be one option times …Oh. Can't exactly put the multiplication sign. Times x options, also times it by x options again, because with one suit you'd only have one king [points to where the king of spades is represented in the 3-D array] and the other ones you would have
>
> ….

> x different cards to pick from, so x times x. And then you'd also multiply it by three for the three suits. Then you get one times x squared times three.

Her explanation clearly referred back to the Card Problem. Throughout these explanations, she gestured to the five by five by five cube, showing imagined locations of where particular regions would be in the general case. We took this as indication that the mathematical symbols she could use to stand in for operations that could be implemented, but no longer needed to be, and which were closely tied to re-presentations of her activity in prior cases of the problem.

We considered this to be a reflection generalization based on reflected abstraction because it was a final formal statement of generalization that grew from a retroactive thematization of her reasoning. DeShay stated as much when talking about Figure 5b, saying "So, this will be the formula to solve because this is pretty much similar to all these other ones [points to the cases of three, four, five, and six cards in her chart], except we had numbers to plug into them."

Discussion

One central contribution of this work is that our study of generalizing is situated within a framework for learning, which allows for the identification of different processes involved in generalizing. We do not consider our examples to be comprehensive, but rather illustrative. The first example illustrated how a student broadened the assimilatory mechanism of her scheme; the second example illustrated a student that recursively used his scheme, taking the result of a scheme and operating on it with operations that were internal to the scheme; the third example illustrated a student who embedded novel operations into an already extant scheme; and the fourth example illustrated a student who took the result of a prior scheme (a 3-D array) and operated on it with operations that were external to the scheme. Beyond just characterizing these different processes, situating the study in a framework for learning allows for a clear distinction between statements and actions that a student makes that are general, but involve no novel ways of operating (i.e., are based on extant schemes), from those that do involve novel ways of operating (i.e., that involve learning). The final example then shows the connection between the operational structures that students developed during the interview study and their expression in symbolic form. As such the paper provides an exemplar of how symbols function in a way that is compatible with Von Glasersfeld's (1995) model of this process.

Endnotes

[i] Students created an array with snap cubes that were all the *same* color like Figure 1a. We used snap cubes because we could find no good way to physically or virtually represent a 3-D discrete array.

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A PROSPECTIVE SECONDARY MATHEMATICS TEACHER'S DEVELOPMENT OF THE MEANING OF COMPLEX NUMBERS THROUGH QUANTITATIVE REASONING

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This study investigated a prospective secondary mathematics teacher's development of the meaning of the Cartesian form of complex numbers during a teaching experiment. We illustrate that through shrinking/stretching of the distance(s) between the roots and the x-coordinate of the vertex of any quadratic function one might conceptualize complex numbers as a single entity, element of a well-defined set, rather than a prescription of certain operations. Such awareness also yield to answering why quadratic functions have to have conjugate roots once they have a complex root.

Keywords: Teacher Education-Preservice, Algebra and Algebraic Thinking

Introduction

Developing new sets of numbers, such as complex numbers is needed on the part of teachers (Karakok, Soto-Johnson, & Anderson Dyben, 2014) and their students (Nordlander & Nordlander, 2012). However, research has shown that neither (prospective) teachers nor students do have a robust conception of complex numbers such that they have difficulty in thinking of both algebraic and geometric representations of complex numbers representing the same number (Karakok et al. 2014; Panaoura, Elia, Gagatsis & Giatilis, 2006). The primary goal of this research was to investigate how someone might develop the algebraic and geometric representations of the Cartesian form of complex numbers as an extension of real numbers through quantitative reasoning (Thompson, 1994). Quantitative reasoning occurs through quantitative operations. Thompson (1994) defined quantitative operation as "a mental operation by which one conceives a new quantity in relation to one or more already conceived quantities." (p. 184). Dwelling on quantitative reasoning both for the design of the teaching sessions and the analysis, this study particularly investigated the following research questions: How does a prospective secondary mathematics teacher develop the meaning of the Cartesian form of complex numbers? What meanings of the Cartesian form of complex numbers does a prospective secondary mathematics teacher develop during an instructional sequence involving quantitative reasoning?

Method

Participants

The participant of the study was one prospective secondary mathematics teacher, Esra, who was in the fourth year of her five-year undergraduate program. For the selection of the participant, first, a written pre-assessment was given to 21 prospective secondary mathematics teachers in a public university in Turkey where the medium of instruction is English. The participation was voluntary. Then, based on the preliminary analysis of their answers, seven of them were chosen to conduct a 45-minute long clinical interview. Analyzing the transcribed pre-interviews and the written pre-assessment, as opposed to choosing the most mathematically capable ones, we looked for participants who had a range of learned concepts and limited understandings. Esra suited such criteria (See results from the written pre-assessment and the pre-interview).

Data Collection and Analysis of the Study

The teaching experiment methodology was employed in this study. Data collection included three phases: I) a pre-interview after a written pre-assessment (the selection of the participants), II) the teaching sessions, and III) a post-interview after a written post-assessment. Phase I was already explained above. For Phase II, the teaching sessions, we first developed a hypothetical learning-sequence for the participant. The second author implemented the teaching experiments consisting of three 75- to 120-minute sessions and the first author operated a digital video camera and an audio recorder. Then, two weeks after the last teaching session, one-hour long structured, task-based post-interview was conducted with the participant. For the analysis of the pre-and-post interview data, we read the transcripts line-by-line. We focused on Esra's justifications and reasoning behind her answers and also what procedures, representations, and formulas she used. For ongoing analysis for teaching experiments, we reflected on the sessions and interpreted Esra's evolving understandings and constructs of the targeted concepts and also focused on potential understandings to be developed by her. Also, we focused on her weaknesses/difficulties and possible explanations for such weaknesses/difficulties. For the retrospective analysis, we identified interaction sequences in which Esra's actions and utterances provided information about her thinking.

Results

Results from the Pre-Interview and the Written Pre-Assessment

Esra was able to define quadratic functions algebraically and represent them geometrically as a parabola. Also, given the algebraic expressions for the roots she was able to explain the meaning of $-\frac{b}{2a}$ algebraically as the half of the sum of the roots of a quadratic equation. She also stated that it referred to the "abscissa of the vertex" and the midpoint of the roots on the real number line, geometrically. She also was able to explain the meaning of $\frac{\sqrt{\Delta}}{2a}$ as the distance of the roots to $-\frac{b}{2a}$. However, she was not able to reason about what $\Delta = 0$ and $\Delta < 0$ meant geometrically. Also, during the written pre-assessment, Esra had defined complex numbers as ".. the numbers in the form of $a + bi$ where $a, b \in R$ and $i = \sqrt{-1}$". She could refer to the three cases of delta and the roots of the quadratic equations being real and complex numbers and had stated that complex numbers included real numbers. Though, when asked, she stated "I have no idea why real numbers are the subset of complex... That is how I learned". Similarly, Esra had written that x and y algebraically referred to real numbers in the form of $x + iy$. Then, during the interview when asked again, she stated "I still think that both can be real". Also, for the questions why they were real numbers and what they referred to geometrically she said "I don't know".

Results from the Teaching Sessions: Esra's developing the definition of Complex Numbers

To take Esra's attention to the dynamic nature of the distance between the roots and the x-coordinate of the vertex, we asked how many parabolas having the same x-coordinate of the vertex one could draw. She stated that one could draw infinitely many parabolas having the same x-coordinate of vertex as in the figure she drew (See Figure 1). We then asked her "what was changing and what did remain invariant in the parabolas she drew given the algebraic form of $f(x) = ax^2 + bx + c$?".

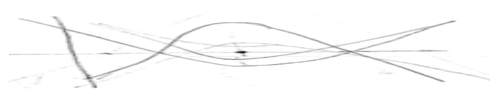

Figure 1. Esra's drawing many parabolas with the same x-coordinate of the vertex.

She stated that the values of a, b, c and also x and y were all changing. She also stated that since these values were changing the roots' distances to the x-coordinate of the vertex was also changing; but, the ratio of $-\frac{b}{2a}$ did not change. Then we asked her to come up with specific examples. She wrote; $y = 2x^2 - 8x + 6$, $y = 4x^2 - 16x + 7$, y= $6x^2 - 24x + 1$. Then we gave her the following examples on GeoGebra such as $x^2 + 2x - 8$, $x^2 + 2x - 4$, $x^2 + 2x - 1$, $x^2 + 2x$, $x^2 + 2x + 1$ to think about: The reason was to allow her to imagine the movability of the distances of the roots to the x-coordinate of the vertex so that she could reason through the geometric meaning of $\Delta = 0$ and

$\Delta < 0$ since she was not able to do so during the pre-assessment. She was able to comment that the x-coordinate of the vertex was the same for all of them albeit the roots' distances to it has changed. She also claimed that the roots' equi-distances from the vertex was invariant in each specific example. Then we asked her to put all the information on the Real number line. She drew (See Figure 2) and explained:

Figure 2. Esra's showing the roots on the real number line she drew.

Esra: Delta is here $[x^2 + 2x + 1]$ with its roots ($x''_{1,2}$,0). None, it is 0. $\frac{\sqrt{\Delta}}{2a}$ is none…It means the overlap of this point with the roots and with the abscissa of the vertex. Eee delta is 0.

R: What was it in the others, here, when there were two roots in here [on the x axis]

E: When $\frac{\sqrt{\Delta}}{2a}$ exists...

The excerpt indicated that acknowledging that the distances of the roots from the x-coordinate of the vertex decreased further and got to a point until there was no distance between the roots and the x-coordinate of the vertex algebraically meant that $\frac{\sqrt{\Delta}}{2a} = 0$. It also meant that there were no distances between the roots for such kind of a quadratic equation geometrically. At that point she stated,

E: ... I can generate parabola from all real numbers... I can add and divide by two. I can draw infinitely many parabolas having the same abscissa of the vertex, and because of that all the roots can be the numbers on the real number line.

The excerpt is important because Esra reversed her thinking in a way that she started from real numbers and given two real numbers she could find the midpoint that would have indicated the x-coordinate of the vertex from which she would have drawn infinitely many parabolas. At that point, we asked what would happen to $-\frac{b}{2a}$ and $\frac{\sqrt{\Delta}}{2a}$ after that point. She stated that the x-coordinate of the vertex would stay on the real number line but she would not be able to put the roots' distances on the real number line anymore because "It $[b^2 - 4ac]$ is smaller than zero". Then we asked her if she could re-write the expression $[b^2 - 4ac]$ in terms of a positive expression. She was able to write: $b^2 - 4ac = -(b^2 - 4ac).(-1)$ and $b^2 - 4ac$=$(4ac - b^2).(-1)$. Then we asked if she could place this expression into the general algebraic expression of the roots: She wrote: (See the first two lines in Figure 3). When we asked if she could write it separately she stated "Normally I cannot take it out". But then she stated that she was not working with real numbers anymore, she said "I run out of them [real numbers]". Then, when we asked if she could state the expression using symbols she wrote (See the last two line in Figure 3) and stated: "Let's say there are infinitely many [quadratic] functions, and the x-coordinate of the vertex of any quadratic function is t...and m is the distance from one of the roots to the x-coordinate of its vertex."

What is interesting is that Esra was able to make sense of the values of "t" and "m" not only algebraically but also geometrically: She knew that "t" stood for $-\frac{b}{2a}$ and "m" stood for $\frac{\sqrt{-\Delta}}{2a}$. She also could relate those values to the quadratic functions such that those values referred to the x-coordinate of the vertex of any quadratic function and the roots' distances to it. Though, it is important to state that Esra's geometrically making sense of "m" was limited because not the value "m" on its own but "$m.\sqrt{-1}$" referred to the roots' distances to the x-coordinate of the vertex. When asked what kind of numbers t and m were, she stated"..ee they are real." and her explanation was "because we have taken a and b as real numbers, $-\frac{b}{2a}$ becomes real and here this number inside $[-\Delta]$ is a real number

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and it becomes real number outside the root. And when we divide it by $2a$, which is real, it $[m]$ is real number again". Then to define complex numbers, she stated "Ee I obtain them from the real roots of quadratic functions. If they are eee.., okay correct, I obtain them from their real roots. Okay, I obtain [complex numbers] from unreal ones [the unreal roots] as well. The numbers obtained from the roots of all quadratic functions are complex numbers. Exactly. They give complex numbers."

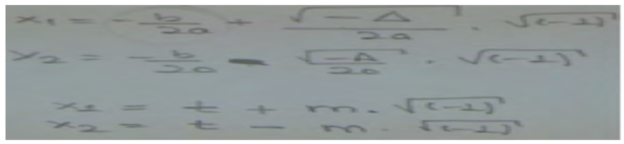

Figure 3. Esra's re-writing the roots of the quadratic equations.

Conclusion

Research on complex numbers has shown that students consider "…the geometric and algebraic representation as two different autonomous mathematical objects and not as two means of representing the same concept" (Panaoura et al., 2006). In this research, through quantitative reasoning focusing on both the algebraic and the geometric meanings of the components of the roots of quadratic equations, i.e. $-\frac{b}{2a}$ and $\frac{\sqrt{\Delta}}{2a}$, Esra was able to develop the meanings of the Cartesian form of complex numbers, $x + iy$, both algebraically and geometrically. In particular, we argue that starting with the examination of any quadratic function and its graph focusing on the quantities (e.g., the roots and the x-coordinate of the vertex), and answering the question of how many parabola(s) someone can construct with the same x-coordinate of the vertex might have the following affordances on reasoning on complex numbers on the part of students: First, thinking of the existence of infinitely many parabolas enabled Esra to focus on (imagine) the 'movability' of the distances of the roots to the x-coordinate of the vertex. That is, as a quantity, $\frac{\sqrt{\Delta}}{2a}$ could *shrink* and/or *stretch* (dilate) and this was imagined by thinking parabolas as shown in Figure 1. Thinking of the movability of the roots' distances to the x-coordinate of the vertex also allowed Esra to think of the placements of them on the real number line too. Once Esra reached such a point in her cognition; that is, once she imagined and thought about the movability of the roots' distances to the x-coordinate of the vertex shrunk to zero it triggered the necessity that real numbers were not sufficient enough to consider the roots of all quadratic functions with real coefficients. Such realization also afforded understanding why complex numbers involved real numbers too. This was important because research has shown that students had difficulty in recognizing that any number is a complex number (Nordlander & Nordlander, 2012). Esra's re-writing the roots as $x_{1,2} = -\frac{b}{2a} \pm \sqrt{-1}\frac{\sqrt{(4ac-b^2)}}{2a}$ allowed her to realize what the components of the complex numbers $z = x \pm iy$ meant algebraically and geometrically. Esra was also able to define complex numbers as the elements of the roots of any quadratic equation with real coefficients (i.e., as members of a well-defined set).

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ACTIVATING A FOURTH LEVEL OF UNITS COORDINATION

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Recent research has highlighted the developmental importance of units coordination for students' continued mathematical learning. In this study, we introduce a task that ostensibly involves four levels of units. We examine student responses to this task and how they relate to student responses from tasks that assess students' coordination of three levels of units. This investigation helps to (1) characterize students' coordination of four levels of units in activity and, (2) explore the potential to use such activity as a valid indicator of students' interiorized coordination of three levels of units. Results of the study provide preliminary evidence for the validity of the task because it positively relates to another measure of units coordination and positively predicts students' construction of an iterative fraction scheme.

Keywords: Number Concepts and Operations, Rational Numbers, Assessment and Evaluation

Purpose

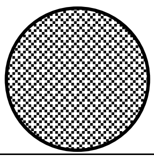
Recent research has highlighted the developmental importance of the ability to interiorize three levels of units coordination for middle grades mathematics concepts (e.g., Hackenberg & Tillema, 2009; Norton & Wilkins, 2012; Ulrich, 2016; Tillema, 2014). In this study, we introduce a task that, from the researchers' perspective, involves the coordination of four levels of units. We examine student responses to this task and how they relate to two other tasks: one measuring students' coordination of three levels of units and one measuring the construction of the iterative fraction scheme (IFS), for which coordinating three levels of units is a prerequisite (Hackenberg, 2007; Norton & Wilkins, 2012). This investigation helps to (1) characterize students' coordination of four levels of units *in activity*, and, (2) examine the potential of such activity as a valid indicator of students' interiorized coordination of three levels of units.

In order for a task to successfully measure the existence of a conceptual construct, such as units coordination, it is important that the task be novel and not be susceptible to solution strategies that could circumvent the need to use the concept in question. For example, Task 3 (see Figure 1), which has been used as an indicator of three levels of units coordination in studies with younger children (e.g., Norton & Wilkins, 2012), might be solved by older children by way of a memorized procedure to find equivalent fractions, thus by-passing the need to mentally coordinate three levels of units (cf. Lovin et al., in press).

In theory, students who can manipulate two levels of units in activity have interiorized the construction of (one level of) units; similarly, if a student can coordinate three levels of units in activity they have interiorized the coordination of two levels of units (Hackenberg & Tillema, 2009; Norton & Boyce, 2015). By extension, if a student were able to coordinate four levels of units in activity, this might suggest that they have interiorized the coordination of three levels of units. The purpose of this exploratory study is to use solutions to Task 1 (see Figure 1) to investigate the potential use of a task that involves four levels of units as an indicator of the interiorization of the coordination of three levels of units, particularly with older students.

(1) Consider the following situation involving chips inside of cups, inside of boxes, inside of crates. There are 3 chips in each cup, 4 cups in each box, and 2 boxes in each crate. If you have 9 chips, how many more cups do you need to make a crate?

(2) Suppose that below is $\frac{6}{5}$ of Sam's pie. Draw Sam's amount of pie.



(3) The pizza shown below is 2/3 of a whole pizza. If each person wants 1/9 of a whole pizza, how many people can share the amount shown below?

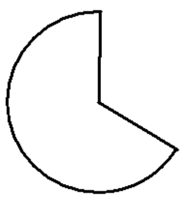


Figure 1. Tasks designed to elicit coordination of units and the iterative fraction scheme.

Theoretical Framework

Units coordination (Steffe, 1992) refers to the ways that students can quantitatively relate different units, and the term is often used to specifically refer to the ability to mentally distribute one composite unit (integer greater than 1) across the elements of another composite unit, yielding what is, to the observer, a multiplication. For example, a student could imagine 4 people who each get 6 brownies, and distribute the 6 across the 4 units of 1 (people) to get the total number of brownies, 24. In this situation, if the student is aware of the 24 as being made up of groups of 6's, then the student can think of 24 as being made up of units of 1 or units of 6. Therefore, the student is working with at least two levels of units in activity.

By reflecting on the activity and results of this kind of units coordination (cf., Simon & Tzur, 2004), students might interiorize the coordination of two levels of units, allowing them to use them when assimilating situations (Norton & Boyce, 2015). For example, such a student may be able to add 24 three times, while keeping track of the number of 6's in the final sum. At that point, 24 itself has become a unit of counting that is coordinated with 6's and 1's, so the student is coordinating three levels of units in activity. Students who have interiorized three levels of units could take the entire structure (3×24 as being made up of 4×3 6's and (4×3)×6 ones) for granted and use it in further operating. This could involve further multiplicative or additive coordinations (Ulrich, 2016), such as realizing that 6 is made up of 5 and 1, so that we can also think of 3×24 as made up of twelve 5's and twelve 1's.

Currently, we hypothesize that students do not need to coordinate more than three levels of units at a time because they can recursively apply their units coordinating schemes to the results of those schemes without mentally keeping track of all the quantities along the way. One might wonder if this implies that only two levels need be coordinated simultaneously with further coordination carried out recursively. However, interiorizing three levels of units seems to be a prerequisite for many mathematical concepts. In particular, Hackenberg (2007) found that understanding fractions as "numbers in their own right" (p. 27) (rather than ratios of two numbers, or a comparison of parts in the fraction to parts in the whole) relies on the interiorization of three levels of units (the fraction itself, the unit fraction, and the whole). The scheme for understanding fractions in this way is called the iterative fraction scheme (Steffe & Olive, 2010).

Method

The study involved 34 pre-service elementary teachers. These teachers were enrolled in two sections of a graduate level elementary mathematics methods course in the fall of 2016 and 2017. Participants completed three tasks as part of their final exam for the methods course (see Figure 1). We designed Task 1 to elicit the coordination of four levels of units *in activity*. The strength of this task is that it does not initially offer known strategies that would allow participants to solve the problem without coordinating units. However, it is possible for participants to solve the task by coordinating two two-level structures simultaneously, and in essence not activate a fourth level of coordination. Task 1 was coded for correctness (0 = incorrect; 1 = correct).

Task 3 has been used in other studies to elicit the coordination of three levels of units (see Norton & Wilkins, 2012) and is used in this study to provide comparative data to evaluate the convergent validity of Task 1. We designed Task 2 to elicit participants' iterative fraction scheme (IFS). Responses to this task are used to evaluate the predictive validity of Task 1. Tasks 2 and 3 were coded for indication of operating with an IFS and three levels of units coordination, respectively (0 = no indication; 1 = indication).

In order to examine the relationships between the tasks we created three contingency tables for pairs of tasks (see Table 1). In order to evaluate the predictive validity of Task 1, we measured the association between responses to Task 1 and Task 2 using the gamma statistic (G) and tested for directionality using the Exact Binomial test. In order to evaluate the convergent validity of Task 1, we examined the relationship between Task 1 and Task 3 based on a comparison of the proportion of correct and incorrect responses to the tasks.

Results

We present the contingency table for Task 1 and Task 2 in Table 1a. Based on these data we documented a relatively strong relationship between responses to Task 1 and Task 2 ($G = .75$, p [one-tailed] $< .05$). By examining the eight off-diagonal entries, we can further ascertain evidence of developmental order, that is, whether three levels of units coordination precedes the construction of an IFS. In this case, we see that of the eight participants who answered one or the other task correctly (but not both), seven answered Task 1 correctly, which is statistically significant, exact binomial $p = .035$. Together, these results provide evidence for the predictive validity of Task 1. That is, three levels of units coordination (measured by Task 1) is associated with and precedes the construction of an iterative fraction scheme (measured by Task 2).

We present the contingency table for responses to Task 1 and Task 3 in Table 1c. Based on these data, 29 out of 34 participants (85%) answered both items correctly providing evidence that success on either item is indicative of success on the other. Furthermore, for the five students that only answered one of the tasks correctly, the prevalence of correctness or incorrectness was not found to be concentrated with either task, exact binomial $p = .813$. Together, these results provide evidence for the convergent validity of Task 1. That is, both tasks tend to be measuring the same construct; in this case, correctness of Task 1 provides evidence for the coordination of three levels of units. However, because no participants answered both items incorrectly, the claim about the association between the tasks may be weakened. Further study involving a sample of students of more varied ability would help to verify this claim.

Table 1: Associations between Responses for Three Tasks

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="3">T2: IFS</td><td rowspan="6"></td><td colspan="2" rowspan="2"></td><td colspan="3">Q2: IFS</td><td rowspan="6"></td><td colspan="2" rowspan="2"></td><td colspan="3">Q3: 3U</td></tr>
<tr><td>0</td><td>1</td><td>Total</td><td>0</td><td>1</td><td>Total</td><td>0</td><td>1</td><td>Total</td></tr>
<tr><td rowspan="3">T1: 4U</td><td>0</td><td>2</td><td>1</td><td>3</td><td rowspan="3">Q3: 3U</td><td>0</td><td>1</td><td>1</td><td>2</td><td rowspan="3">Q1: 4U</td><td>0</td><td>0</td><td>3</td><td>3</td></tr>
<tr><td>1</td><td>7</td><td>24</td><td>31</td><td>1</td><td>8</td><td>24</td><td>32</td><td>1</td><td>2</td><td>29</td><td>31</td></tr>
<tr><td>Total</td><td>9</td><td>25</td><td>34</td><td>Total</td><td>9</td><td>25</td><td>34</td><td>Total</td><td>2</td><td>32</td><td>34</td></tr>
<tr><td colspan="5">(a)</td><td colspan="5">(b)</td><td colspan="5">(c)</td></tr>
</table>

Conclusions

In this study, we introduced a task that ostensibly involves four levels of units. Our goal was to examine the potential of such tasks as indicators of the interiorization of three levels of units coordination. The task was designed to provide a novel situation for which most students do not have a readily available solution strategy, and so, responses are likely to be a true indicator of units coordination. Results of the study provide preliminary evidence for the convergent and predictive validity of the task as it positively relates to an established measure of units coordination and positively predicts students' construction of an iterative fraction scheme. Future research on similar tasks would benefit from a qualitative analysis of participant responses and a wider range of participants, but initial findings show the potential for assessing three levels of units coordination by using tasks that activate a fourth level of units coordination.

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ANALYSIS OF THE RELATIVE DIFFICULTY OF DIFFERENT INTEGER PROBLEM TYPES

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This paper describes an analysis of students' relative abilities to solve different types of integer addition and subtraction problems. Logistic regression and hierarchical methods revealed that, even before instruction, certain problem types proved fairly easy for students while other were extremely difficult. Specifically, certain problems involving subtracting a negative were particularly challenging. These differences have important implications for instructional strategies used to teach integer operations.

Keywords: Number Concepts and Operations

On the surface, addition and subtraction with integers, i.e. positive and negative numbers, are straightforward operations, but middle school teachers consistently find these concepts difficult to teach. When students encounter these types of problems, typically in sixth or seventh grade, they struggle to understand why subtracting a negative actually increases the result. To help their students, teachers often employ a variety of models (e.g., number lines or chips), contexts (e.g. temperature, or money), and/or procedural rules (e.g. "keep-change-change") (Tillema, 2012, Whitacre et al., 2011). These approaches are often most helpful for certain types of integer problems and exhibit weaknesses when applied to other problem types. For example, a student might solve $5 + (-3)$ by thinking of having $5 and owing someone else $3, leaving $2. This approach is much more difficult, however, for a problem like $5-(-3)$, where the idea of removing an owed amount is not as easily understood. Typically, these differences between the effectiveness of models, contexts, or rules for different problem types are not addressed in the literature, and classroom teachers tend to focus on common models and contexts that, while effective for some problems, are not helpful for the most difficult integer subtraction problems. The tacit assumption being that providing students productive ways to think about certain problem will help them to be more successful on all problems. The goal of this analysis is to examine student performance on different integer addition and subtraction problems to highlight the significant differences in difficulty between the problems. Once the most difficult problems are identified, we can examine the weaknesses of common instructional approaches in supporting student thinking around those problems and hopefully move forward in designing better, more effective instructional strategies.

Theoretical Framework

The majority of research on the teaching and learning of integer operations focuses on two areas: how students conceptualize positive and negative numbers, and the models and contexts that teachers use to teach these concepts. Bofferding (2014), Bishop et al. (2016), and Whitacre et al. (2011) for example, examined the ways that young children reasoned through symbolic integer problems, demonstrating that even in early grades students are capable of applying a variety of approaches to integer problems. These authors also found that students often try to apply or extend whole number reasoning strategies to signed number problems with varying degrees of success. While the ways students conceptualize integers and the strategies they use to solve problems are well documented, the effectiveness of those conceptions or strategies for different types of problems is less well understood.

The other major strain of research revolves around different instructional strategies used to teach students to add and subtract integers. Battista (1983) outlined the use of collections of charged particles for modeling integer addition and subtraction, and Hayes and Stacey (1998) described a similar strategy involving integer tiles (yellow and red *chips* representing positive and negative numbers respectively). Stephan and Akyuz (2012) documented the use of *debts* and *credits* (a more intentional application of the context of *money*) in conjunction with a number line model. Tillema (2012) described using contextual stories that embody models such as colored chips or number lines, and Whitacre et al. (2011) pointed out several other models and contexts such as temperature, and directed distance/motion on the number line that appear frequently in textbooks. In most of these cases, the articles discuss in general the merits and disadvantages of each of these approaches; however, these discussions rarely examine the ways these approaches interact with specific types of problems. Very little research has yet attempted to look at the structural properties of different integer problems and connect those properties to students' difficulties or successes with the problems.

Methodology

This study uses both logistic regression and hierarchical linear methods to examine the relative performance of students on different types of integer addition and subtraction problems. The data for this analysis came from a pilot study of 6th grade students at a suburban, midwestern public school.

Research Question

The question addressed with this analysis is "How are structural characteristics of integer addition and subtraction problems related to students' abilities to successfully solve those problems?" Although there are a variety of ways to categorize the structural properties of integer addition and subtraction problems, for this analysis, problems were categorized by the sign of the addends (for addition) or of the minuend and subtrahend (for subtraction), by the sign of the answer, and by the primary operation. Based on this classification, there are 12 types of integer addition and subtraction problems. The problem types and their (arbitrary) identifiers, along with an example for each type, are summarized in Table 1.

Participants and Instruments

The participants in this study were all of the students (n = 159) in the sixth grade class of a suburban, midwestern middle school. The students were spread across three teachers and seven classes, and represent all ability and demographic groups present at the school. In recent years, the school has typically scored slightly below average on state standardized tests, and approximately 40% of students are on free or reduced lunch.

The data in this analysis come from a 15-question free response test involving single digit integer operation problems of different types (see Table 1). "Whole number" problems, i.e. problems that did not require students to engage with negative numbers such as 2 + 3 = 5, were omitted as they were deemed not necessary. The frequency with which each type was tested is also shown in Table 1. Each problem was scored correct (1) or incorrect (0). This test was given prior to instruction as we were interested in students baseline abilities on these problems.

Analysis and Results

The analysis proceeded in two phases. In the first phase, a simple logistic regression was performed to model the probability of success on a specific problem given the problem type and controlling for the student's overall ability as measured by his or her total score on the test. *Figure 1* shows a plot of 95% confidence intervals for the predicted probabilities of success for each problem type after controlling for a students' overall score. Notably, students are fairly successful on the addition problems (Types 2-5) and two of the subtraction problems (Types 11 and 15) even prior to instruction, while Types 12, 13, and 14 showed extremely low success rates.

Table 1: The Twelve Integer Problem Types and Examples

	Identifier	Problem Types	Example	Number of Test Items
Addition	Type0	(+) + (+) = (+)	2 + 3 = 5	0
	Type1	(+) + (-) = (+)	3 + (-2) = 1	0
	Type2	(+) + (-) = (-)	3 + (-5) = -2	1
	Type3	(-) + (+) = (+)	(-2) + 3 = 1	1
	Type4	(-) + (+) = (-)	(-3) + 2 = -1	1
	Type5	(-) + (-) = (-)	(-3) + (-2) = -5	1
Subtraction	Type10	(+) - (+) = (+)	5 – 3 = 2	0
	Type11	(+) - (+) = (-)	5 – 7 = -2	1
	Type12	(+) - (-) = (+)	5 – (-2) = 7	3
	Type13	(-) - (+) = (-)	(-5) – 3 = -8	2
	Type14	(-) - (-) = (+)	(-3) – (-5) = 2	3
	Type15	(-) - (-) = (-)	(-5) – (-2) = -3	2

The logistic regression makes it clear that certain problems are more difficult than others, but it is not able to account for the potential for individual students to respond differently to the more challenging problems. For this reason, a secondary analysis was carried out using a hierarchical repeated measures approach (Raudenbush & Bryk, 2002). In this analysis, the test items were treated as nested within students, and the students' scores on individual problems were aggregated over problem type (as a percent) and were treated as repeated measures of the students' general ability to solve integer problems. Based on the results of the logistic regression analysis, a difficulty score was assigned to each problem type (the log-odds for that problem) and this difficulty score was used as the predictor for the repeated measures. If there were truly no difference between problem types, we would expect the slopes for each student relative to the problem difficulty to be zero. In this case, a two-level model was fitted (problem type scores nested within students), where problem difficulty was used as a level 1 predictor and overall score (grand mean centered) was used as a level 2 predictor. Intercepts and slopes were allowed to vary randomly. This analysis revealed that the slopes were not zero (*difficulty slope* = 0.135, p <0.001), instead implying that between the easiest and hardest problems (a difference in log-odds of about 2.2) we would expect a difference in score of about 30%. Additionally, the slope corresponding to the interaction between problem difficulty and overall score, 0.34, ($p < 0.001$) indicates that higher overall scores actually increase the impact of problem difficulty making the gap between the hard problem and the easier problems even larger.

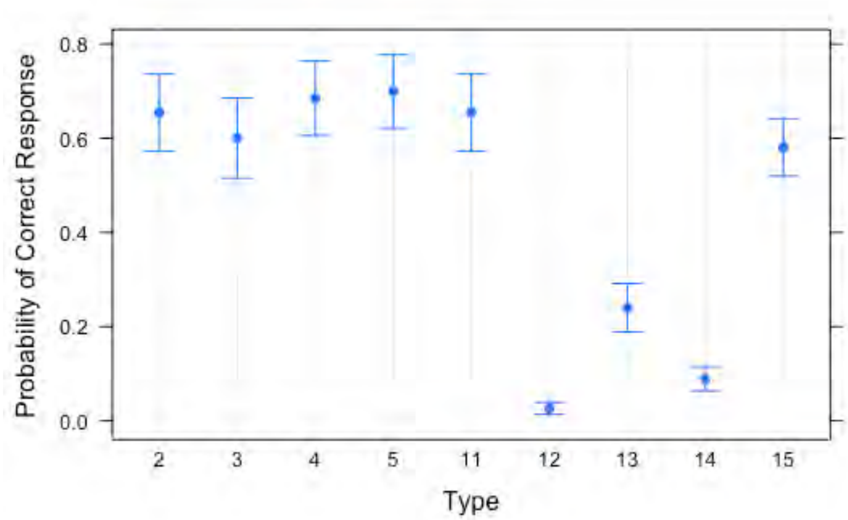

Figure 1. 95% Confidence Intervals for the probability of success for an average student on each problem type after controlling for overall pre-test score.

Conclusions

Based on this analysis, all integer addition and subtraction problems are not alike. Specifically, problem types 12, 13, and 14, appear dramatically different from the rest. Furthermore, the significance of the interaction between the problem difficulty and the students' overall score implies that these problems are, relatively speaking, even more difficult for the stronger students. This indicates that careful thought should be given as to how to help students think about and solve specific problem types, namely those involving subtracting a negative, not just integer operations problems in general. Additionally, because not all models and metaphors work equally well for all types, these results suggest that students would benefit from instruction targeted at those models and metaphors which are most helpful for the most challenging problem types.

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COLLEGE STUDENTS' KNOWLEDGE OF THE EQUAL SIGN AND ITS RELATION TO SOLVING EQUATIONS

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Decades of research have documented young students' misunderstanding of the equal sign. Further, these misunderstandings matter, as young students' knowledge of the equal sign relates to their performance on key algebra problems. However, much less is known about whether knowledge of the equal sign matters beyond elementary and middle school – after students have completed Algebra and have experience with the equal sign in a variety of contexts. In the current study, we assessed 189 college students' knowledge of the equal sign and examined how it related to their performance on formal algebra problems. College students varied in their definitions of the equal sign. Further, students who only provided operational definitions of the equal sign were less likely to solve and interpret key algebra problems correctly. Results suggest that knowledge of the equal sign matters well beyond elementary and middle school.

Keywords: Algebra and Algebraic Thinking, Problem Solving, Standards

Introduction

A large body of research has focused on students' understanding of the equal sign – a concept fundamental to algebra (e.g., Baroody & Ginsburg, 1983; Behr et al., 1980; Matthews et al., 2012; McNeil & Alibali, 2005). The equal sign is ubiquitous at all levels of mathematics, and relational knowledge of the equal sign is included in the Common Core State Standards as early as first grade (National Governors Association Center for Best Practices, 2010).

Unfortunately, decades of research have established elementary and middle school students' difficulties with the equal sign indicating that they often view it operationally, as a signal to do something. Indeed, many students define the equal sign as "the answer" or "the total" (e.g., Alibali et al., 2007; Behr et al., 1980; McNeil & Alibali, 2005), and these definitions matter. For example, in Knuth et al. (2006), approximately one third of middle school students provided a relational definition of the equal sign (e.g., "the same"), and those students were almost twice as likely to solve algebra equations correctly (e.g., $4m + 10 = 70$) as students who did not.

Much less is known about older students' knowledge of the equal sign and whether it matters for their success on algebra problems. Younger students' difficulties with the equal sign are often thought to stem from their overly narrow experience with problems presented in a standard "operations-equals-answer" format (McNeil & Alibali, 2005). College students, however, have already passed Algebra and have experience with the equal sign in a variety of contexts. That said, some limited research suggests that an operational view of equations persists among college students. For example, undergraduates sometimes solve standard equality problems (e.g., $6 + 8 + 4 = 7 + \_\_$) using operational "find the total" strategies, particularly under speeded conditions (e.g., Chesney et al., 2013; McNeil & Alibali, 2005). But no research to date has documented college students' explicit knowledge of the equal sign and whether it relates to their equation-solving success. Moreover, few studies with students of any age have examined the relation between students' knowledge of the equal sign and their conceptual interpretations of variable expressions (e.g., $3a$, $a + 3$). Variable expressions are fundamental in Algebra, yet do not contain the equal sign. We contend that if knowledge of the equal sign is truly foundational for algebraic thinking, it should predict performance on a variety of algebraic tasks – including tasks with expressions that do not include the equal sign (Kieran, 1981).

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Method

Participants

Participants were 189 college students attending a large public university in the western region of the U.S. The student body at the university is 53% female, has an average age of 26, and 92% are in-state residents. Participants (~58% female) were recruited from their college-level developmental math courses and they received course credit for their participation.

Measures

Equal sign definition. Students provided written responses to two questions: "What does this symbol (=) mean?" and "Can it mean anything else?" Responses to each question were coded as relational, operational, or other (see Knuth et al., 2006). A response was coded as *relational* if it expressed the idea that the equal sign means the "same as" or the "same amount" and as *operational* if it expressed the idea that it means "add the numbers" or "the answer." Responses coded as *other* included definitions such as "equal to." Some students provided more than one interpretation, so we also assigned an overall code indicating their "best" interpretation (e.g., if they provided a relational and an operational definition, their best code was relational).

Algebraic equation-solving. Students also solved a set of three basic equations. Approximately half of the students (n = 92) completed Version 1 and the other half of the students (n = 97) completed Version 2. See Table 1 for the items. Responses were coded as correct if the student provided the numerical answer that satisfied the equation.

Table 1: Equation-Solving Items on Each Version of the Assessment

Version 1	Percent Correct	Version 2	Percent Correct
$13 = n + 5$	97%	$10 = z + 6$	98%
$c + c + 4 = 16$	83%	$c + c + 4 = 16$	65%
$z + z + z = z + 8$	78%	$m + m + m = m + 12$	64%

Algebraic expression-interpretation. Students also provided written responses to the following: "Cakes cost c dollars each and brownies cost b dollars each. Suppose I buy 4 cakes and 3 brownies. What does $4c + 3b$ stand for?" On Version 1 of the assessment, the symbols used were mnemonic in that the price of a cake was represented by c and the price of a brownie was represented by b. On Version 2, the symbols used were traditional in that the c and b were replaced by the letters x and y. This contrast has been of interest because use of mnemonic symbols may strengthen students' naïve conceptions that variables in algebraic expressions stand for labels instead of quantities (Küchemann, 1978; McNeil et al, 2010). Responses were coded as correct if it indicated that the letters stood for the cost or price of the cakes and brownies.

Results

Students varied in their definitions of the equal sign. Only 66% of sampled college students provided a relational definition of the equal sign. Nearly half (47%) provided an operational definition, and nearly a quarter (23%) provided a vague, "equal" definition. In terms of students' "best" definitions, 65% were relational, 18% were other/equal, and 17% were operational.

We examined the relation between students' equal sign understanding and their performance solving algebraic equations. We created a total equation-solving score by summing their scores on each of the three equation-solving items. Figure 1 shows their performance.

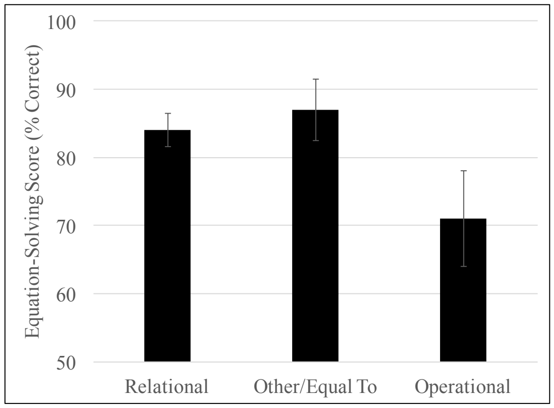

Figure 1. Percent correct on equation solving as a function of equal sign definition.

Providing only an operational definition was predictive of college students' equation-solving scores. Students who only provided an operational definition had lower equation-solving scores (M = 71%, SD = 33%) relative to other students (M = 85%, SD = 24%), $t = 2.30$, $p = .02$. This was primarily driven by performance on the more difficult algebraic equations. For example, students who only provided an operational definition were significantly less likely to solve $z + z + z = z + 8$ correctly relative to students who provided relational or other/equal to definitions (56% vs. 83%), χ^2 (1, N = 92) = 5.35, $p = .02$, and less likely to solve $m + m + m = m + 12$ correctly relative to other students (30% vs. 69%), χ^2 (1, N = 97) = 5.81, $p = .01$.

We also examined the relation between students' equal sign interpretation and their conceptual interpretation of an algebraic expression – a key mathematical object in algebra that does not include the equal sign. Overall, 65% of college students interpreted the expression correctly by indicating that the variables stood for the costs of the cakes and brownies. Of all the incorrect responses, 48% were the common "labels" error (e.g., interpreting the variables as labels for the objects; "four cakes and three brownies"). However, performance varied somewhat by the letter used. Students given the x-and-y version were somewhat more likely to interpret the expression correctly than students given the c-and-b version (72% vs. 60%), χ^2 (1, N = 189) = 2.79, $p = .09$. Further, students given the x-and-y version were significantly less likely to make the common labels error than students given the c-and-b version (9% vs. 24%), χ^2 (1, N = 189) = 7.06, $p = .01$. This suggests that, even among college students, the use of mnemonic letters can interfere with the ability to conceptually interpret the algebraic expression. Importantly, students' equal sign definitions were related to performance. Students who only provided an operational definition were significantly less likely to interpret the expression correctly relative to other students (47% vs. 71%), χ^2 (1, N = 189) = 6.48, $p = .01$, and somewhat more likely to make the labels error (27% vs. 14%), χ^2 (1, N = 189) = 2.97, $p = .09$. These relations were similar for both expressions, but somewhat stronger for the expression containing the mnemonic c-and-b letters.

Conclusion

We examined college students' knowledge of the equal sign and its relation to their performance on key algebraic problems. We found that college students varied in their definitions of the equal sign. Although many students endorsed a relational understanding, many also endorsed an operational understanding, and a small minority *only* provided an operational definition. Importantly, knowledge of the equal sign mattered. Students who only provided an operational definition were less likely than their peers to solve key algebraic equations correctly and to interpret an algebraic expression in a conceptually correct way. These results extend previous research on the importance of the equal sign with elementary- and middle-school students (e.g., Knuth et al., 2006; Matthews et al., 2012), and suggest that knowledge of the equal sign is important beyond these early grades, even after students have completed Algebra.

These findings provide support for the theoretical relation between understanding of equality and formal, algebraic reasoning (e.g., Linchevski & Herscovics, 1996). As in prior work with younger students (e.g., Alibali et al., 2007; Knuth et al., 2006), we found that knowledge of the equal sign was related to equation-solving success among college students. However, we also found that knowledge of the equal sign was related to students' conceptual interpretation of an expression with variables – a key algebraic problem that does *not* include the equal sign. Thus, this research establishes a more general connection between knowledge of equality and algebra – students who view the equal sign operationally also tend to view equations and expressions in terms of processes and operations rather than as objects or structures that can be manipulated (Kieran, 1981; Linchevski & Herscovics, 1996). In general, this research highlights a need to focus on the importance of the equal sign beyond middle school.

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DEVELOPING FUNCTION UNDERSTANDING THROUGH DEPENDENCY RELATIONS OF CHANGE

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Understanding function is a critical aspect of algebraic reasoning, and building up functional relationships is an activity increasingly encouraged at the elementary and middle school levels. This study identifies how one group of middle-school students leveraged their rate of change thinking to inform the development and understanding of correspondence rules. Drawing on an analysis of a 15-day teaching experiment with 6 eighth-grade students, we introduce three dependency relations of change concepts – recognition, identification, and translation – and discuss how these concepts support students' transitions to more formal algebraic expressions.

Keywords: Algebra and Algebraic Thinking, Middle School Education

Research Issues and Purpose

Recommendations for fostering students' algebraic understanding include introducing functional relationships in upper elementary and middle school (e.g., Blanton & Kaput, 2011), with researchers arguing that functions can unite a wide range of otherwise isolated topics, encourage student inquiry, and provide a rich context for generalization and justification. However, students' difficulty in developing function understanding is well documented (Stephens et al., in press), highlighting the need to better support an emerging function concept that is flexible and mathematically productive. Approaches to function often distinguish between correspondence views (e.g., Smith, 2003) and covariation views (e.g., Confrey & Smith, 1994; Thompson & Carlson, in press). Separately these approaches emphasize either a direct mapping relationship between x and y, or attention to a coordinated change in one quantity with associated change in the other. These two ways of thinking about function are often presented as separate approaches, with researchers advocating for flexibility across these modes of reasoning in order to inform a more robust understanding of function (Stephens et al., in press). It remains an open question, however, how one can leverage a covariation perspective to inform the correspondence view and vice versa. In this paper, we address the following research question: How might students capitalize on their understanding of function from one mode of reasoning in order to inform their thinking in the other? In particular, we characterize the transition made by middle-school students who leveraged their coordinated change thinking to inform the development of correspondence rules in the context of instruction about quadratic functions. We identify three student concepts, which we term *dependency relations of change* concepts, that address how students recognized and developed links between these modes of reasoning.

Theoretical Framework and Background

Two perspectives typically drive a function-based approach to algebraic reasoning: the correspondence perspective and the coordination/covariation perspective. The correspondence view emphasizes a function as a relationship between members of two sets, in which $y = f(x)$ represents a mapping, with each value of x is associated with a unique value of y (Smith, 2003). This view underlies traditional instruction on functions (Yerushalmy, 2000), and offers the affordance of emphasizing the development of closed-form rules that can be used to analyze and predict function behaviors. In contrast, Thompson and colleagues (e.g., Thompson & Carlson, in press) and Confrey and Smith (e.g., Confrey & Smith, 1994) offer a covariation model, although they address ideas of covariation differently. Confrey and Smith describe an approach relying on coordinated changes of x- and y-values, in which one can coordinate a shift from y_m to y_{m+1} with a corresponding shift from x_m to x_{m+1}. Such an emphasis on the coordination of sequences is a natural fit for tasks relying on images or tables that present successive states of variation.

Thompson and Colleagues (e.g., Saldanha & Thompson, 1998) also discuss ideas of covariation, but from the perspective of identifying students' reasoning about quantities that vary, together, either simultaneously or interdependently. This form of reasoning involves mentally coordinating two varying quantities while attending to how they change in relation to one another, and thus represents a dynamic perspective of covariation. This work also leverages the notion of quantity as a conceptual entity composed of a person's conception of an object, a quality of the object, such as length or height, an appropriate unit for measurement, and a process of assigning a value to the quality (Thompson & Carlson, in press). For the purposes of this paper we use the term *covariation* to refer to students' dynamic coordinated images of change, and *coordination* to refer to students' static coordinated images of change. In both cases, students can begin to develop an understanding of classes of functions in terms of their characteristic actions (e.g., recognizing that linear functions represent a constant rate of change while quadratic functions represent a rate of change that is constantly changing).

Methods

The data presented are from a 15-day videotaped teaching experiment conducted with 6 8th-grade students. The participants included 3 students in general 8th-grade mathematics, 2 students in pre-algebra, and 1 student in 8th-grade algebra. None of the participants had been introduced to quadratic function in their mathematics courses at the time of the study. Each session lasted 1 hour, and all sessions were transcribed. Our aim was to build students' understanding of quadratic function from a rate-of-change perspective, in which quadratic growth is conceived as a constantly changing rate of change. We therefore grounded students' exploration in one context, that of a rectangle that grew while maintaining its length/height ratio. Students worked with computer simulations of different growing rectangles and made area predictions, created algebraic rules for height and area, and made graphs representing those relationships.

Data sources included video and transcripts from the teaching sessions and copies of the students' written work. Following the constant comparative method (Glaser & Strauss, 1967), we developed emergent categories of student concepts about quadratic growth and function based on evidence from students' written work, descriptions of their ideas, drawings, and gestures. Each co-author independently analyzed each of the 15 sessions before coming together to discuss and reconcile code decisions. This led to a series of iterative revisions to the coding scheme on a day-by-day basis until no new codes emerged.

Results: Identification of Dependency Relations of Change

We present three concepts within a family of related ideas we call *dependency relations of change*. These concepts address how students can leverage coordination thinking into the development of correspondence relations by attending to how the magnitude of change in one quantity determines the magnitude of change in an associated quantity. Students evidenced three stages of thinking as they began to shift from coordinated changes to relating those changes to direct correspondence rules: *recognition*, *identification*, and *translation* (Table 1).

Below we provide excerpts characterizing students' thinking across the three concepts. The students invented a number of terms to capture the change in quantities. They described the change in height and length values as "DiH" and "DiL", respectively, referring to the difference in successive height or length values when organized in a table or a series of figures. The students used the term

"RoG" to refer to the rate of growth of the rectangle's area for successive increases in height/length, and they used the term "DiRoG" to refer to the difference in the rate of growth of the area, $\Delta(\Delta A)$.

Table 1: Recognition, Identification, and Translation

Concept Definition	Data Example
Recognition: Understanding that the magnitude of the change in one quantity determines the amount of change in another quantity. Student understands that there is a dependency relation without determining what that relation is.	*Jim*: So, couldn't your rate of growth be, like, really high instead of just…if you're going, like, every other one, or you can go like every 5 or whatever (referring to increases in height values). So your rate of growth (for the area) can change no matter what.
Identification: Quantifying the relation between changes in associated quantities. Student can identify a quantity (or its change) as dependent on and determined by the change in the associated quantity and can characterize this dependency relation.	*Tai*: (Referring to a table of values coordinating successive height (H), length (L), and area (A) values for a growing rectangle) DiRoG divided by DiL equals…and, divided by DiH equals 2. [Here he verbally expressed the relationship $\left[\frac{\frac{\Delta\Delta A}{\Delta L}}{\Delta H}\right] = 2$.]
Translation: Expressing the relation between changes in two quantities in an algebraic rule, such as $y = ax^2$. Student understands the rule in terms of a relation of change.	*Daeshim*: [Wrote the equation $A = \frac{5}{3}h^2 + 4$] *Bianca*: Where did you get that? (Referring to 5/3). *Daeshim*: DiL over DiH. [Here $a = \frac{\Delta L}{\Delta H}$]

As shown in Table 1 with Jim's comment, the students first understood that the change in height, for instance, would determine the rate of growth of the area and $\Delta(\Delta A)$, but they could not yet determine how that change would affect the other changes. Tasks providing different tables of values for the same rectangle induced the students to begin to account for how multiple quantities changed together. For instance, consider a task that presented four tables of height/area pairs for the same growing rectangle. Each table grew by uniform increments in height, which were 1 cm, 2cm, 5 cm, and 10 cm respectively. Prior to this task the students had characterized the DiRoG as the rate at which a rectangle added units as it grew in height and length, so they knew that it would be greater for larger increments (*recognition*), but they did not know how much larger. For the Four Tables task, however, Tai found a way to relate the DiRoG to the changes in both the height and the length (*identification*): "Take the DiRoG of the area, and you divide by the difference in the length…and also divided by, the difference in height…and, always equals 2." Tai identified a dependency relation between ΔH, ΔL, and $\Delta(\Delta A)$, but he did not connect that relation to a correspondence rule. Similarly, Bianca noticed that $\Delta(\Delta A)$ could be expressed solely in terms of the change in height values: "So it's basically like 3 DiH squared times 2", which Tai then expressed as "$3 \times [\text{DiH}]^2 \times 2$ = DiRoG of Area". The students correctly wrote the rule $A = 3h^2$ to express the area in terms of height, but they struggled to understand how the parameter 3 related to the quantities length, height, or area.

The *translation* concept emerges when students leverage their coordination thinking to create and make sense of correspondence rules. For instance, two days later the students worked with a rectangle with the following height and area values: (2, 1), (5, 6.25), (7, 12.25), (8, 16), and (10, 25). Bianca realized that the length was ¼ of the corresponding height for each pair and wrote "$h(.25h)$" to express the area. Tai then reacted to Bianca's rule by exclaiming, "The DiL divided by DiH equals that number (0.25)." Tai had divided the *change* in length by the *change* in height for each successive pair of values, and found the ratio to be 0.25. Subsequently, Tai and the other students began to determine ratios of change in order to develop correspondence rules (*translation*). For instance, given a new task with three pictures of a growing rectangle with an unchanging "tail" of 4 square cm, the students determined that for each 3-cm increase in height, there was a 5-cm increase in length. They were then able to form the rule A = $5/3h^2 + 4$. Daeshim explained the parameter 5/3 as the ratio of ΔL to ΔH, and Bianca noted that one could also think of this as the ratio of sides for any given rectangle. Ultimately the students also related the parameter "a" to Δ(ΔA), but learned that they had to account for Δ*h*, expressing this relationship as "a = DiRoG/2" when Δ*h* was 1.

Discussion

The teaching experiment students began attending to functional relationships through coordinated changes, an approach common with middle-school students (Stephens et al., in press). Ultimately they were able to leverage coordinated changes in the associated quantities height, length, and area in order to make sense of the correspondence rules they wrote. The students understood those correspondence rules – particularly the parameter "a" – in a number of ways, with the more powerful meanings being those that emerged from their activities of coordinating growth in the relevant quantities. These findings suggest that coordination and correspondence views do not have to arise separately or in parallel, with robust understanding being limited to translating across forms of reasoning. Rather, correspondence rules can emerge from a coordination view and can also represent a formalization of the activities of coordination and covariation. This suggests an avenue for future research in investigating the ways in which covariation and coordination can support function understanding across the function families.

Acknowledgments

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FOSTERING GENERALIZATIONS: A CLASSROOM DISCOURSE ANALYSIS

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This study explores the conditions that support students in forming verbal generalizations in one Grade 4 classroom's discussions. By analyzing the student-teacher discourse in ten teaching segments, we coded for purpose of statement, as well as technique for fulfilling the purpose. We identified sixteen statements of generalization. The data reveal that students' generalizations are linked to discursive moves associated with the purpose of extending and the techniques of requesting for justification and justifying. We argue the findings have implications for instruction that fosters generalizing.

Keywords: Algebra and Algebraic Thinking, Elementary School Education

Generalizing is "intrinsic to mathematical activity and thinking" (Kaput, 1999, p. 137), yet students struggle to generalize, often make weak generalizations, and rarely justify their generalizations (Breiteig & Grevholm, 2006). Supporting generalizing in the mathematics classroom requires a better understanding of the conditions that encourage students' generalizing. This issue is central to our research, which seeks to understand the nature of the discourse that supports students' generalizing. In particular, the study reported here explores the conditions that support students in constructing generalizations in Grade 4 classroom discussions. Working from the premise that discourse represents the interrelationships between thought and speech (Truxaw & DeFranco, 2008), we examine teacher-student discourse to gain insight into students' generalizing.

Generalization and Discourse

We define generalization as the situated activity of "lifting" and communicating reasoning to a level where the focus is no longer on a particular instance, but rather on patterns and relationships of those particular instances (Kaput, 1999, p. 137). To explore how discourse supports generalizing we use the concepts of linguistic moves. Moves are utterances that can be categorized based on the speaker's purpose. For example, initiating moves occur when the teacher introduces an idea, knowing that it will likely prompt centering or extending moves (Lobato, Clarke, & Ellis, 2005). Re-centering moves are responses to the conversation changing track or needing redirection (Love & Suherdi, 1996) and provide conceptual anchoring. Extending moves are used to obtain or offer additional information through elaborating or enhancing ideas (Gonzalez & DeJarnette, 2012). And centering moves continue the conversation without changing the track. Recentering, centering, and extending moves are responsive in nature, meaning they are usually a part of an exchange (Truxaw & DeFranco, 2008). Each move may also be assigned to a technique. The technique is the way in which the speaker fulfills their purpose. For example, if the purpose of a teacher's move is to extend, she might use the discursive technique of requesting justification.

Methods

Participants (n=21) were from one Grade 4 classroom participating in an early algebra intervention that was taught by a member of our research team. All lessons in the intervention were video recorded and transcribed. Each video recording was segmented into clips of classroom discussion, of which the first ten were selected for analysis.

This research is a part of a larger project (viz., Blanton et al., 2015), aimed at designing and evaluating the effectiveness of a longitudinal early algebra intervention in Grades 3-5. Clips were selected by the larger research team based on their representation of exemplary teaching to be used in professional development supporting the implementation of an early algebra intervention. Analyzing these clips allowed us to focus on the discourse-rich moments of the lesson. Clips were de-identified using gender appropriate pseudonyms and numbered randomly.

We conducted a multi-level analysis of the classroom discourse. That is, the transcribed videos were organized by each speaker's (students or teacher) statement. Then, based on the inferred goal of the speaker, each move (or utterance) was coded for purpose and technique for fulfilling that purpose (Gonzalez & DeJarnette, 2012). Additionally, we recorded generalizations (Ellis, 2011), and situated these generalizations in the discourse in order to describe the discursive moves and techniques that contributed to and resulted from the generalizations. We identified statements within discussions that could be characterized as generalizations according to Kaput's (1999) definition. Namely, we coded statements made by students that indicated their reasoning was no longer on a particular instance, but rather on general patterns and relationships of those particular instances (Kaput, 1999). At least two members of our research team coded each transcript. Codes were discussed until agreement was reached.

Findings

In the ten clips that were analyzed, we identified 14 techniques. Here we report on two interrelated techniques. The first technique, requesting justification, occurred when the speaker, usually the teacher, responded to an idea by encouraging the original speaker to support their idea by providing an explanation. The second technique, justifying, occurred when the speaker (formally or informally) described why they believed his or her idea is true.

We found 16 student utterances that were characteristic of Kaput's definition of generalization. Clips were ranked based on the frequency of generalizations with respect to the length (in number of turns and time). Clips 3, 5, and 6 were the most generalization-dense based on the total number of generalizations and the total turns in each segment. Whereas clips 2, 7, and 8 were the most generalization-sparse. Important differences between these clips in terms of purpose, techniques of the teacher, and techniques of the students are captured in Table 1. The percentages refer to the frequency of a particular purpose or technique within the discussion.

Of the 16 utterances coded as generalizations, 11 were coded as Student Response as Justifying Technique. This finding supports prior findings (e.g., Ellis, 2011) regarding the relationship between generalizing and justifying; specifically that generalization and justification are interrelated, inseparable mental activities.

We identified a variety of techniques, but here we focus on two techniques—Teacher Request for Justification Technique (rejust) and Student Response as Justifying (just)—because we noticed a relationship between these techniques and generalizations. In particular, we observed that in comparison to the generalization-sparse clips the generalization-dense clips had a higher percentage of the techniques, teacher requests for justification and student responses as justifying. Conversely, in the generalization-sparse clips, we found an overwhelming percentage of student responses that do not map to a technique. This finding indicates that students frequently reply with an answer only (no explanation) in the generalization-sparse clips.

Table 1: Generalization-Dense Clips Vs. Generalization-Sparse Clips

	Generalization-dense Clips			Generalization-sparse Clips			Overall
	Clip 3	Clip 5	Clip 6	Clip 2	Clip 7	Clip 8	
Extending Purpose Moves	26.3%	13.7%	12.5%	6.4%	2.6%	8.3%	11%
Teacher Request for Justification Technique	6.7%	15.4%	0%	0%	3.5%	0%	8.4%
Student Response as Justifying Technique	50%	27.3%	50%	0%	10%	0%	12.3%
Student Response with No Technique	25%	48.5%	0%	74%	80%	83%	15.6%

Furthermore, we observed that the generalization-dense clips have a higher percentage of moves with the purpose of extending. For instance, extending occurred when a teacher used domain-specific vocabulary to rephrase a student's idea, or when a question is posed to encourage a student to provide additional explanation.

Drawing from the data presented in Table 1, we conclude that generalizations occur in an environment where: (1) The teacher pushes for more extending purpose moves; (2) The teacher requests justification from student answers; and (3) The students provide justification for previous answers that do not contain reasoning. Because moves and techniques occur on different levels, finding (2) and (3) are subsets of finding (1). Additionally, finding (2) and (3) often occur sequentially because one prompts the other.

Discussion

In the generalization-dense clips, we observed a higher percentage of extending purpose moves. Whereas, we noticed the generalization-sparse clips lack a high frequency of extending moves. Instead, the teacher spends a majority of class time recentering or centering the class discussion, which implies that more time is needed to help students understand the basis of the problem being addressed. Such situations minimize actions that push students into deeper problem solving. We claim teaching segments that utilize extending moves are more likely to result in student generalizations because those instances push the students to provide more information within their responses.

In the generalization dense clips, we observed a higher percentage of teacher requests for justification. These requests for justification often take the form of a follow-up question to a student response. For example, the teacher may ask, "How do you know?" or "Why?," encouraging students to explain their reasoning. This was the most common catalyst for student justification as well as generalization. Since the teacher's requests for justification were linked to students' generalizations, we saw a higher percentage of student justifying techniques in generalization-dense clips. Justification techniques occurred when students explained why their answer was correct. For instance, we observed one student (Steven) respond to the question "$m = 1 \times$ ___" as "m." Following the teacher's request for justification ("How do you know?"), Steven justified his answer as shown in Table 2. Due to our multi-level analysis of purpose, technique, and generalizations, this example represents a statement that could be characterized as both a generalization and a Student Response as Justifying (just) technique. The teacher's response technique is confirming (cf). It is also important to note that students often begin justifying statements by using the word "because."

Table 2: Excerpt from Clip 5 on Student Generalizing

Speaker	Move/Utterance	Purpose	Technique
Teacher	How do you know?	Extend	rejust
Steven	Because *m* can equal any number and 1 times anything would equal that number.	Recentering	just
Teacher	Very, very, good. Excellent	Centering	cf

We argue that the results of this study have a high-impact potential to inform instruction, because educators are able to incorporate the techniques that encourage generalization into curriculum. We suggest these findings be viewed as a starting point for research on supporting mathematically productive generalizing in the elementary classroom. Moving forward, we aim to build on and refine our claims by analyzing a broader sample of classroom discussion segments, so they have direct implications for educators. We also recognize that there are many ways to support generalizing, and that this is only one approach. We hope that future research explores this avenue and aims to uncover other ways to support generalizing, because research in this area is critical to advancements in mathematics education.

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GRADE 5 CHILDREN'S NUMBER LINE DRAWINGS FOR INTEGERS

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Three Grade 5 children participated in twelve weeks of a teaching experiment on integer addition and subtraction, and participated in four individual interviews across the 12-weeks of the teaching experiments. Drawing on the theoretical perspective of learner-generated drawings, the types of number line drawings produced by the children are qualitatively described. The children often unconventionally drew different types of number lines drawings (Number Sequence, Empty Number Line, Number Line). The results point to the importance of making sense of children's constructions before imposing integer instructional models.

Keywords: Number Concepts and Operations, Elementary School Education, Cognition

We know that children often use strategies that incorporate drawings for solving addition and subtraction problems with positive integers (Carpenter, Fennema, Franke, Levi, & Empson, 2015). Children often use drawings paired with direct modeling or counting strategies as they invent strategies for solving whole number addition and subtractions problems. Although there is significant research into the different instructional models for the teaching and learning of integers (e.g., Javier, 1985), we know little about the drawings that children employ as they transition from using positive integers to negative integers. Bofferding (2010) demonstrated that children often use a number path, or a sequence of numbers written in boxes, when solving integer problems. Other researchers have shown that children will use a variety of ways to reason about the integers which include order-based reasoning (e.g., Bishop et al., 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). Order-based reasoning is when a child draws upon the order of integers to make decisions about integer addition and subtraction (e.g., -2 + 3 may be determined by counting up 3 from -2). Despite the vast amount of research on the ways that student think about integers (e.g., Bofferding, 2014; Bishop et al., 2014), we need to know more about the ways that children reason about integers in relationship to the types of drawing they produce, such as number lines. Understanding children's invented drawings has potential to provide insight into ways we design instruction for negative number.

Theoretical Framework

This study draws on learner-generated drawings (Van Meter & Garner, 2005). Learn-generated drawings provide insight into the cognitive processes. Learner-generated drawings are: (a) "intentionally constructed to meet a learning goal"; (b) "meant to depict represent objects accurately"; and (c) "for which the learner is primarily responsible for construction" (p. 290). This research brief addresses these learner-generated drawings by looking at: What types of number line drawings did the Grade 5 children construct as they solved integer addition and subtraction open number sentences?

Methods

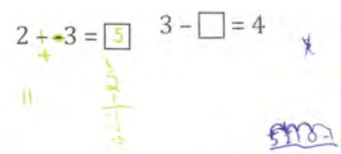
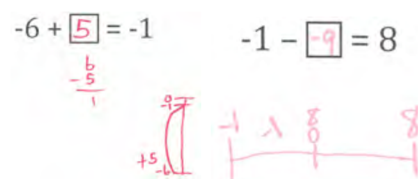
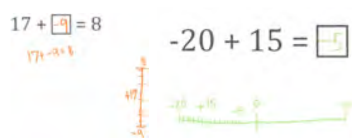
Three Grade 5 students from a rural Midwest school participated in a 12-week teaching experiment (Steffe & Thompson, 2000) centered on integer addition and subtraction, using both contextual problems and open number sentences. I used several structured task-based interviews (Goldin, 2000) to evaluate the children's' understandings of solving open number sentences in four individual sessions across the teaching experiment. During these sessions the open number sentences were provided on paper, with no manipulatives and only a box of markers available. The students were asked to explain their reasoning for solving the open number sentences. In these four sessions, the students solved 20, 23, 25, and 25 integer addition and subtraction open number sentences, respectively.

Each drawing produced by the children constituted the unit of data— there were 93 units for each child and 279 total units of data for this investigation. A constant comparative method (Merriam, 1998) was utilized for each of these units of data, beginning with an expectation that children may draw objects as they do for whole numbers (Carpenter et al., 2015), number lines (Saxe et al., 2013), empty number lines (Verschaffel, Greer, & De Corte, 2007), or number paths (Bofferding, 2010).

Types of Number Line Drawings Produced

The different types of Number Line visual mediators that emerged are highlighted and defined in Table 1. I present the frequency in which each mediator was used in Table 2. Then, I describe examples of Number Sequence, Empty Number Line, and Number Lines produced by the children.

Table 1: Different Number Line Visual Mediators Produced by Grade 5 Students

Type of Visual Mediator	Example of Visual Mediator
Number Sequence: Numbers are used in an ordered manner or list. Lists of numbers include negative or positive integers in a sequence.	
Empty Number Line: There is a segment of a number line that does not use equipartitioning, but numbers are listed on the number line. The distances on the empty number line may be highlighted. Negatives may be on the right or left on a horizontal line. Negative may be on the top or bottom of a vertical line.	
Number Line: There is a segment of a number line that attempts equipartitioning. Negatives may be on the right or left on a horizontal line. Negative may be on the top or bottom of a vertical line.	

Number Sequence

Alice produced Number Sequences paired with objects to help her describe and solve the number sentence -2 – 8 = □ (see Figure 1). For example, Alice first produced objects, using tallies, determined a solution of 6 for the number sentence -2 – 8 = □ and then she connected her Number Sequence to her objects. In contrast, when Jace solved □ – -2 = 1 (see, e.g., Figure 1) he created a Number Sequence by drawing numbers vertically and ordered (1, 0, -1). He used this ordered list of integers to count 2 backwards from -1 to 1, determining a solution of -1.

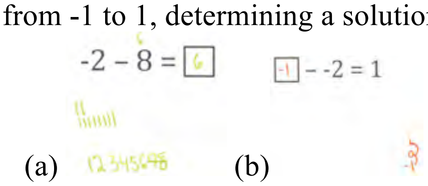


Figure 1. Number sequences produced by Alice (a) and Jace (b).

Empty Number Line

When Jace produced Empty Numbers lines he demonstrated flexibility with the positioning of the Empty Number Line—whether horizontal or vertical. He, for instance, also placed the negative integers flexibly on the right or left the Empty Number Line (see, Figure 2).

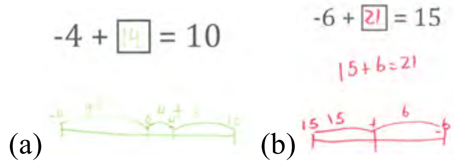


Figure 2. Empty number lines produced by Jace with negatives on left (a) and right (b).

As illustrated in Figure 2, Jace also used the Empty Number Line for determining the distance in between two integers as he solved addition or subtraction problems. In contrast, Kim focused more on moving from one point to another, using the Empty Number Lines.

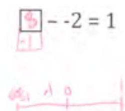


Figure 3. Empty number line produced by Kim, using distance as a directed movement.

For example, As Kim solved □ – -2 = 1 she first thought the answer was 3 and then decided to change her solution and drew a number line.

> The answer was one and here was a negative two (points at -2). So I sort of knew the only way I could get to a positive, which was the one (points at 1), which was to like have a smaller negative number (points at -2) besides 0 and then negative two. And, the only number was negative one (points at box with -1)

She started at -1 and moved to the right to 0, "losing -1," and then moved right to losing another -1 to 1.

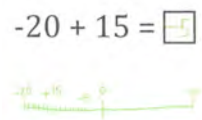
> When you subtract the two off of it, it would go, but when it hit zero it's lost one (marks number line). So, it has zero. It has one remaining over, so you could just add onto and go into the positive area. And it, when you got done using your remainders it'd be one...

Number Line

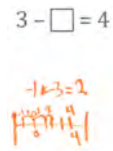
Jace and Kim drew Number Lines. Jace, for example, created a line segment from -20 to 20 when he first solved -20 + 15 = □ (see Figure 4). This was the first problem he solved in the first session and he began with equipartitioned units from -20 to -5 by starting with -20 and moving to the right 15 units drawing a tick mark each time. Similarly, Kim drew a line segment and partitioned her segment, from -2 to 5, as she attempted to solve 3 – □ = 4 (see Figure 4). Kim also used this Number Line drawing as she addressed a problem in the second session, which she was not able to solve.

The different types of drawings created are a significant component to understanding children's thinking because they illustrate the students' constructed meanings. As this study was conducted with these children prior to formal school instruction with negatives and they were not provided manipulatives or explicit instructional models, their drawings highlight the sophisticated nature of their mathematical inventions of the number line. Although we have insight into the types of drawings that children may produce for whole numbers (e.g., Carpenter et al., 2015), these descriptions of number line drawings illustrate what this may look like for children that work with integers.

(a)



(b)



l4. Number lines produced by Jace (a) and Kim (b).

Discussion

Top-down approaches where students are required to utilize a particular instructional model (e.g., number line models that specify particular movements to solve integer subtraction) may not be the best way to begin integer instruction. Rather, children should be allowed to produce and create their own drawings, which could then serve as the instructional models.

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INVESTIGATING ELEMENTARY PRE-SERVICE TEACHERS' DISTRIBUTIVE REASONING AND PROPORTIONAL REASONING

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To investigate the relationship between PSTs' quantitative understandings of distributive reasoning and proportional reasoning, two 12-episode teaching experiment were conducted with two pairs of PSTs. During the experiments, PSTs were asked to work on particular mathematical tasks tailoring their ways of thinking regarding to proportional reasoning. Preliminary findings indicate that one of the participants was mostly illustrating the results of her way of thinking as opposed to engaging in distributive reasoning. She also had some powerful ways of thinking and did not need to show a distributive partitioning operation (DPO). On the other hand, another participant showed evidence of distributive partitioning schemes and operations throughout the experiment, which enabled her to find unit ratios and create equivalent ratios.

Keywords: Teacher Education-Preservice, Cognition, Algebra and Algebraic Thinking

Distributive reasoning plays an important role in the development of ratio reasoning (Steffe, Liss II, & Lee, 2014). An illustration of this statement would be that one way of thinking about ratios is measuring one quantity in terms of the other. If there are three tbsp. of powder and five oz. of water, the three tbsp. must be distributed across those five oz. to determine the quantity of powder that goes with one ounce. Alternately, the five oz. must be distributed across those three tbsp. to know how much water goes with one tbsp. of powder. Establishing a unit ratio, say 3/5 tbsp. of powder for each ounce of water, involves distributive reasoning. Making these unit ratios and being able to distribute them is a powerful way of establishing equivalent ratios. So, construing the distribution as measuring one quantity in terms of another quantity demonstrates that distributive reasoning is involved in proportional reasoning and reasoning with ratios.

The main purposes of this research, which is part of my dissertation project, are to understand how pre-service teachers (PSTs) reason distributively and proportionally, and to examine relationships between PST's distributive reasoning and proportional reasoning. Specifically, the research questions for this study are: (1) How do PSTs solve problems that involve proportional relationships? (2) How do PSTs reason distributively? (3) What schemes, operations, and concepts are involved in PSTs' construction of proportional reasoning? (4) What are relationships between PSTs' distributive and proportional reasoning?

Conceptual Framework

A Quantitative Approach

Thompson (2010) defines a quantity as a scheme consisting of an object, a quality of the object, an attribute of this object that has a unit of measure, and a process that the attribute's measure entails a proportional relationship with its unit. Throughout this paper, following Thompson, I interpret quantitative operations as mental operations and regard these operations as essential aspects of quantitative schemes because this approach allows me to account for PSTs' schemes and operations when they reasoned quantitatively with ratios.

Operations and Schemes

The concept of schemes, or goal-directed ways of operating that involve an assimilatory mechanism, activity, and result, is a substantial part of Piaget's theory of knowledge (von

Glasersfeld, 1995). My study is based on the mathematical thinking in PSTs' operations, which are components of schemes. Based on Steffe and Olive (2010), I define a distributive partitioning operation (DPO) as partitioning n items among m shares by partitioning each of the n items into m parts and distributing one part from each of the n items to the m shares. For example, when asked to equally share three candy bars with five people, a student with a DPO would partition each of the three candy bars into five equal parts and take one part from each bar to determine a share. Steffe and colleagues (2014) also originated the ideas of a distributive partitioning scheme (DPS) and a reversible distributive partitioning scheme (RPDS). A child who has constructed a DPS could interpret one share as n/m of one unit by taking 1/m of each n item, while a child with an RDPS could interpret one share as 1/m of all the units and see that n/m of one unit is equal to 1/m of all the units, and could also justify that iterating n/m m times does indeed produce n items (Liss, 2014).

Methods

To launch each of the two small scale intensive 12-episode teaching experiments, I conducted 60-minute task-based selection interviews with 8 PSTs to assess their multiplicative reasoning, distributive operations, and initial ways of reasoning with ratios. Then, I invited four PSTs to participate in my dissertation project. Based upon the similarities in their reasoning, I matched them in two pairs. In this paper, I will present some data from one of the pairs: Maggie and Rose. The teaching experiment consisted of 12 45-minute sessions (two episodes in each week for each group) for a total of 6 weeks. They worked in pairs on particular mathematical tasks related to distributive and proportional reasoning. All teaching experiment sessions were video-recorded and all written work were collected. During episodes, PSTs often used a software program called JavaBars (Biddlecomb & Olive, 2000). After each teaching session, I processed data, kept a digital research journal, watched video, took notes, and discussed conjectures with my advisor to organize my plans and develop the tasks for the next episode. Following teaching experiments, each PST participated in a 60-minute follow up interview to elicit their reasoning and experiences after the teaching experiment.

The data were analyzed retrospectively (Steffe & Thompson, 2000). Through retrospective analysis, I formulated a second-order model of PSTs' reasoning with ratios. A second order model is a researcher's constellation of constructs to describe and account for another person's ways and means of operating (Steffe, von Glasersfeld, Richards, & Cobb, 1983). For this purpose, I repeatedly viewed the relevant video files, transcribed major portions, and took detailed analytic notes. I also wrote memos and conjectures about changes and persistent constraints in PSTs' ways of thinking and operating, and about interactions that may have supported these changes and constraints. Finally, I wrote a document comparing these models and synthesizing the interactions that contributed to changes.

Preliminary Analysis

So far, I found qualitative differences in how PSTs represented and thought about proportional reasoning. I support this claim by demonstrating how they worked on the Lemonade Mixture Problem in the teaching experiment. Because I am still analyzing my data, I will present a very small but important portion of what I have right now.

Near the end of episode 7, I had already created a sketch showing 1 tbsp. and 1 ounce bars in JavaBars (Figure 1, upper left). The main question was "How much tbsp. of lemonade powder would go with 1 ounce of water if 3 tbsp. of lemonade powder with 16 ounces of water are the ingredients of the "best" lemonade?" To solve that problem, Rose worked on JavaBars and Maggie worked on paper. Rose first created her 3 tbsp. and 16 oz. bars (Figure 1, upper right). She then copied the 3 tbsp. bar, divided each part into 16 mini-parts, and filled them in blue, peach and red, respectively

(Figure 2, bottom right). She pulled out one mini-part from each tbsp. bar and joined these 3 mini-parts under 1 ounce of water bar (Figure 2, bottom left). Looking at 3 mini-parts she pulled out, Rose stated that 3/48 tbsp. of powder, which can be simplified to 1/16, would go with 1 ounce of water. Rose has demonstrated a DPO when pulling out one mini-part from each bar to get 3 mini-parts for 1 ounce of water after dividing each of the tbsp. bars into 16 mini-parts. However, her answer was 1/16 tbsp. not 3/16 tbsp. since she did not relate her answer to the unit bar, 1 tbsp.

In Episode 8, after probing with me, Rose realized that since 3 mini-parts together are 1/16 of 3 tbsp., if she repeats 3 mini-parts 16 times it would give her 3 tbsp. bar. So she corrected her answer as 3/16 tbsp. of powder. Then she repeated 3 mini-parts 16 times in her JavaBars picture. I infer that repeating 3 mini-parts 16 times might have helped her see her reasoning with the picture. During that justification, she has also constructed an RDPS when she repeated 3 mini-parts 16 times to see that would give her whole 3 tbsp. bar.

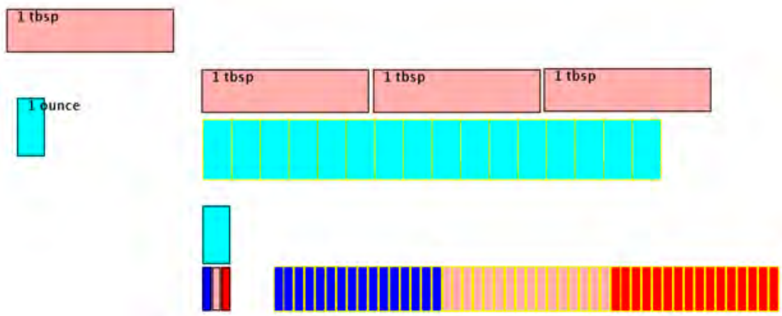

Figure 1. Rose's work on the Lemonade Mixture Problem.

Maggie first drew out 3 tbsp. of powder and 16 oz. of water. She said she wanted to start with how many tbsp. of powder goes with how much ounces of water. After figuring out 5 1/3 oz. goes with 1T of powder, she drew 5 1/3 oz. of water in three groups and she wanted to find out how much tbsp. of powder goes with 1 ounce by highlighting the first 3 mini-parts of her first 5 1/3 drawing (Figure 2). She stated that she needed to be her pieces even that is why she divided every 1 ounce in 3 mini-parts to find the proper fraction. Then she realized every three mini-parts would give her how many tbsp. of powder in 1 ounce and said she needed to look at the whole picture—including other two 5 1/3 drawings, which has totally 48 mini-parts. She said the answer is not 3/16 because she needed to think all three 5 1/3 oz. groups. So she said the answer is 3/48 (she counted all mini-parts) and she simplified it to 1/16. Maggie has not demonstrated a DPO in solving this problem. Her way of thinking is pretty similar to what she did before with other lemonade mixture problems. Representing the powder and water together in the same bar might have masked showing the distribution idea and focusing on the unit bar (1 tbsp.) as well.

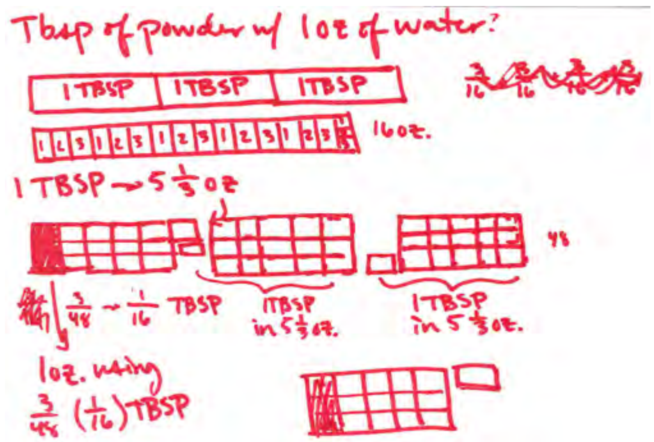

Figure 2: Maggie’s work on the Lemonade Mixture Problem.

In Episode 8, although Maggie might have interpreted Rose’s idea/justification, she did not make it her own. When I asked about her justification, she realized that she needs to focus on the 1T bar not all 3T bars saying "But then I guess it is 3/16 [she meant 3/16 tbsp. of powder would go with 1 ounce of water] because I counted up all of the 48s. But if you are counting up for 1T it is just three-sixteenths [pointing out her first drawing]. Cause I counted up all 48. Well, I know that it is 1/16 of 3T then I was just trying to think in my head how you can like mathematically prove it."

Discussion

The teaching experiment data presented above provides several important results. First, Rose mostly showed evidence of demonstrating a DPO and even an RDPS when she was reasoning with ratios. This helped her create unit ratios and use these unit ratios to find any other equivalent ratios. Second, Maggie had powerful ways of thinking and a desire to understand other people’s reasoning. She did not spontaneously produce these ideas when she was working on problems even though she could understand and justify them in interactions with me and Rose. She did not necessarily do so, perhaps because she did not see them as more powerful than her own. Therefore, having her own powerful ways and not adopting others’ ideas as her own could also help us understand why Maggie did not need to show use of a DPO in reasoning with ratios.

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EXAMINING THE FIDELITY OF IMPLEMENTATION OF EARLY ALGEBRA INTERVENTION AND STUDENT LEARNING

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Through a theoretical framework emphasizing the importance of fidelity of implementation (FOI), this paper explores how 3rd grade teachers implemented an early algebra intervention, and the extent to which the FOI related to student learning. The data for this report are taken from the first year of an experimental research project. Videotaped classroom observations, our primary measure of FOI, were coded by adding to and adapting the Mathematical Quality of Instruction (MQI) instrument, and student performance was measured by overall score (correctness) on an algebra assessment. Results revealed a significant positive relationship between teachers' implementation and their students' performance.

Keywords: Algebra and Algebraic Thinking, Elementary School Education, Assessment and Evaluation

This paper reports on the fidelity of implementation (FOI) of 3rd grade teachers as they implemented an early algebra intervention, and the relationship between FOI and student learning. The data from this study are taken from the first year of an experimental research project (Project LEAP: Learning through an Early Algebra Progression) that tests the hypothesis that children who receive comprehensive, longitudinal early algebra instruction during the elementary grades are better prepared for algebra in middle school than children who have only arithmetic-based experiences during elementary grades.

Theoretical Framework

The treatment of algebra in school mathematics has changed dramatically over the past two decades. The *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center & CCSSO], 2010) calls for algebraic reasoning to start in Grade K and span across the grades. In response to this challenge, we initiated a study to examine the effectiveness of an early algebra intervention in Grades 3-5. The intervention consisted of 18 lessons and teachers also attended ongoing professional development to support their implementation of the intervention.

The focus in this paper is on the relationship between student performance outcomes and teachers' FOI of the early algebra intervention. Our goal was to measure the fidelity with which teachers in diverse demographic settings implemented the intervention and how this intervention affected student learning outcomes. Measuring FOI revealed differential patterns of implementation and their relationship to the intervention (Mowbray et al., 2003), a critical factor in evaluating an intervention's effectiveness (NRC, 2004; Summerfelt & Meltzer, 1998) and promoting external validity (O'Donnell, 2008).

Methodology

Three school districts (including approximately 240 classrooms and 3,400 children) participated in the cluster randomized trial, with entire schools being assigned to either the experimental or control condition. During the first year of implementation, we focused on third grade classrooms, and subsequently followed these children into fourth grade.

Data sources for the study reported here are classroom observations and student assessment data. Classroom observations (videotaped LEAP lessons) were conducted with a subsample of experimental teachers (n=50). All teachers were observed twice except for one, for a total of 99 observations.

Student participants were given a one-hour, written algebra assessment as a pre/post measure. The assessment was designed by the project team (Blanton et al, 2015) and measured students' understanding of early algebra. We have assessment data from approximately 800 students in the observed classrooms. For the purposes of the analyses reported here, student assessment data were coded according to item correctness.

We coded classroom observation data with a specific focus on the degree to which teachers implemented the early algebra materials with fidelity as well as the quality of mathematics instruction. Teachers were rated on 5-point Likert scales on each of the three cognitive demand variables created by the project team: *justify an answer*, *generalize a mathematical relationship*, and *represent with variables*. Separate codes were given for whole class work and individual/group work. Observations were also coded for six items adapted from the Mathematical Quality of Instruction (MQI) instrument (Hill et al, 2008): efficient use of class time, clear presentation of mathematics, student engagement, teacher attention to student difficulty, teacher use of student ideas, and precise use of mathematical language and notation.

Approximately 15% of videos were double coded in order to assess inter-rater reliability. Factor analysis was then employed in order to create composite variables that could be used as teacher-level predictors of student outcomes (i.e., student performance on the early algebra assessments) in a multilevel analysis.

Results

As a first step, we looked at the implementation of the intervention. Lessons in this early algebra intervention consist of two parts – the Jumpstart (a review and warm up activity related to the objectives of the lesson) and the main early algebra activity. The jumpstart activity was completed in 95% of observed classes. Of those that did, 100% of the observed classes included whole class discussion led by the teacher, 67% included individual student work, 32% included group work, and 35% included student-led presentations. Jumpstarts activities lasted, on average, 15 minutes and 32 seconds (*SD* = 07:54).

Once the Jumpstart was completed, lessons moved on to the early algebra task. Overall, in 90% of observed classrooms teachers read the problem aloud or had a student read the problem. In 52% of classrooms, teachers ensured students understood any terms or concepts that might be unfamiliar to them. Teachers demonstrated methods of presenting numerical information that might be unfamiliar to students in 42% of classrooms.

After the lesson was introduced, in 71% of the observed classrooms students worked on their own (either independently or in groups) on the bulk of the remainder of the activity. During individual/group work, teachers were rated on whether they were active or passive. An active teacher would actively visit individuals/groups to help students with questions, but also to challenge their thinking. A passive teacher would go around to groups only when asked, and would be largely reactive to students' needs rather than being proactive and challenging mathematical thinking. In 69% of observed classrooms, the teacher was coded as "active."

LEAP Cognitive Demand Variables: Justify, Generalize, Represent

A rating of 1 indicates that a teacher did not ask students to justify, generalize, or represent at all, while a rate of 5 indicates that the teacher asked students to justify, generalize, or represent in a way that went beyond the lesson expectations (see Table 1).

Table 1: Mean (SD) Justify, Generalize, and Represent Ratings

Whole Class			Individual Group		
Justify	Generalize	Represent	Justify	Generalize	Represent
3.75 (1.02)	3.51 (1.08)	3.00 (1.25)	2.91 (1.39)	2.55 (1.33)	2.10 (1.34)

Adapted MQI Codes

With the exception of *imprecision*, which was coded on a 4-point scale, all MQI items were coded using a 5-point Likert scale. For all items, 1 is the most "negative" rating, indicating inefficient use of class time, severely distorted mathematics, total lack of student engagement with the lesson, student difficulty without any teacher remediation, no substantive use of student ideas, or imprecision that obscured the mathematics of the lesson (see Table 2).

Table 2: Mean (SD) Adapted MQI Ratings

Efficiency	Math is Clear	Engaged	Attends to Student Difficulty	Uses Student Ideas	Precision in Language
3.33 (1.11)	4.24 (0.96)	3.80 (1.00)	3.56 (0.99)	3.85 (0.93)	3.31 (0.83)

Inter-rater reliability was assessed for the cognitive demand and MQI data using weighted Cohen's kappa. The analysis suggested that raters had acceptable levels of agreement (κ_w>.60) for all MQI variables, for the three individual/group cognitive demand variables, and for the whole class represent variable. However, there was only moderate agreement for two of the whole class cognitive demand variables: whole class generalize (κ_w =.54) and whole class justify (κ_w =.53). For this reason, subsequent analyses do not include the whole class variables.

Relationship Between FOI and Student Performance

We hypothesized that several of our observed variables would be correlated due to their association to latent (unobserved) variables. In order to identify these underlying latent variables, factor analysis using principal components analysis was utilized.

The six MQI variables and the three individual/group cognitive demand variables were entered. A two-factor solution, which explained 71% of the variance, emerged. The three individual/group cognitive demand codes (justify, generalize, represent) were added together to create a composite variable, "cognitive demand," M = 7.55, SD = 2.82, and the six MQI variables were added together to create a composite variable, "MQI," M = 22.25, SD = 3.79.

Using these composite variables as level 2 (teacher-level) predictors, we conducted a multilevel regression analysis to explore the relationship between the teacher-level FOI variables and student performance. The baseline measure of performance, grade 3 pre-test, was included as a student-level (level 1) predictor, and a measure of SES (percentage of students with free or reduced lunch) was included at the school-level (level 3).

After controlling for baseline performance and school-level SES, the cognitive demand composite variable was found to be a significant predictor of students' score on the LEAP post-assessment, $\gamma = .015$, $t(30) = 3.093$, $p < .001$. A one-unit increase in cognitive demand score was associated with a 1.5% increase (.015 points) in post-test score. The MQI composite variable was not a significant predictor of student performance.

Discussion

Understanding the ways in which teachers implemented the early algebra intervention and how it impacted student learning have important implications for this particular study and, more broadly, how we as a community understand the complexity of finding ways for educational research to influence actual instruction and have an impact on the mathematics learning of large numbers of students. This is critical if we are to take educational innovations to scale.

Aspects of teachers' implementation were significantly positively related to their students' performance on the early algebra assessments. Given the range of the cognitive demand composite variable (3 to 14), students in classrooms where the teacher received the highest rating outperformed their peers in the classroom of the lowest rated teachers by an average of 16.5%. Therefore, students of teachers who implemented the intervention with higher fidelity had higher mean scores on the early algebra assessment, suggesting that these students are better prepared for algebra in middle school than their peers whose teachers implement with lower fidelity.

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REASONING WITH CHANGE AS IT RELATES TO PARTITIONING ACTIVITY

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This paper discusses a pre-service secondary teacher's mental actions in the context of her partitioning activity and reasoning about amounts of change (i.e., coordinating the amounts of change of one quantity for changes of another) to structure dynamic situations. We illustrate ways of thinking that necessitated that she carried out partitioning activity, and thus had difficulty re-presenting actions when reasoning with changes on graphs and situations.

Keywords: Cognition, Advanced Mathematical Thinking, Algebra and Algebraic Thinking

Background

Researchers have shown that quantitative reasoning—the mental actions involved in conceptualizing a situation in terms of a structure of measurable attributes (Thompson, 2011)—and covariational reasoning—the mental actions involved in coordinating quantities changing in tandem (Carlson et al., 2002)—are critical for students' rate of change understandings (Johnson, 2015; Thompson, 1994). Highlighting the complexities in the ways students conceive of relationship between covarying quantities, these researchers have called for a closer look into students' reasoning about *amounts of change* in order to explain nuances in students' rate of change understandings. Inspired by the aforementioned research, this study aims to gain a better understanding of students' reasoning with amounts of change in relation to their partitioning activity. We draw on both Carlson et al. (2002) and Saldanha and Thompson's (1998) theories of covariational reasoning, with particular attention to Level 3 (i.e., students' coordination of amounts of change of one variable with respect to changes in another) of Carlson et al.'s (2002) covariation framework. For example, given a graph or a situation, a student can partition one quantity into equal increments and visualize the amounts of change of the other quantity corresponding to the endpoint of each increment. We refer to this particular kind of activity including the mental operations involved when we use *partitioning activity* throughout the paper.

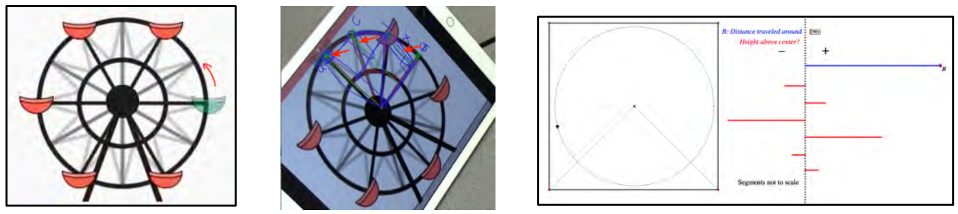
Methods

This paper reports results of a semester-long teaching experiment (Steffe & Thompson, 2000) with Lydia, who was in her first semester of a four-semester secondary math education program. We conducted 12 teaching sessions with her and videotaped and digitized them for analysis. In both ongoing and retrospective analyses efforts, we conducted a conceptual analysis (Thompson, 2008) in combination with open and axial techniques (Corbin & Strauss, 2008) in order to develop models of her mathematical thinking. Specifically, our iterative analyses efforts involved constructing hypothetical mental actions that viably explained Lydia's observable and audible behaviors. We also continually searched for instances that the models could not account for, and modified our models or attempted to explain developmental shifts in her meanings.

Results

The Taking a Ride and Which One Task

In our first teaching sessions, we designed the Taking a Ride task (Figure 1a) to support students in reasoning about the relationship between the height of the green rider above the horizontal diameter of the wheel and the arc length the rider has traveled. Lydia initially claimed, "the arc length has increased to this [*drawing an arc on the first quarter of the circumference of the wheel*] while the distance from the center has increased to that [*drawing a vertical segment from the top position to the center of the wheel*].” Eventually, Lydia constructed successive amounts of change of height (see the blue segments in Figure 1b) for successive, equal changes in arc length. Noticing that these blue segments decreased in magnitude, Lydia concluded that as the arc length is increasing by equal amounts, the amount of increase in height is decreasing.



(a) (b) (c)

Figure 1: (a) Animation for the Taking a Ride task; (b) Lydia identified the amounts of change of height on the animation; (c) The Which One task.

Immediately following this task, we presented the Which One task (Figure 1c), where we presented a simplified version of the Ferris Wheel with the position of a rider indicated by a dynamic point on a circle. We informed Lydia that the topmost blue segment represents the arc length the rider has traveled counterclockwise from the initial three o’clock position. We then asked Lydia to determine which of the six red segments, if any, accurately represents the rider’s height above the horizontal diameter as the rider’s arc length traveled varies. Lydia chose a red segment (which represented a normatively correct solution), oriented it vertically, and put it inside the circle (Figure 2a). She then confirmed that the length of the segment matched the height of the dynamic point for different states (Figure 2b). We then asked if the segment entailed the amounts of change relationship constructed in Figure 1b, to which she responded:

Lydia: Not really. […] Um, don’t know. [*laughs]* Because that was just like something that I had seen for the first time, so I don’t know if that will like show in every other case […] for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion. […] I saw what I saw, and I saw that difference in the Ferris wheel, but I don’t see it here, and so –

I: And by you don’t see it here, you mean you don’t see it in that red segment?

Lydia: Yes.

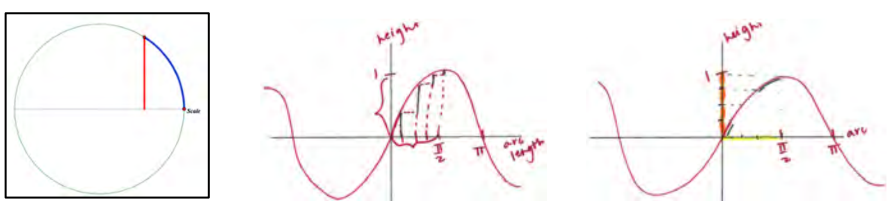
(a) (b) (c)

Figure 2: (a) Lydia was working on the Which One task; (b) Lydia was checking the red segment point-wisely; (c) We were assisting Lydia to identify amounts of change of height.

We draw attention to Lydia referencing height increasing by decreasing amounts as a “theory” that needed to be tested despite her having already identified that the red segments worked point-wisely with respect to traversed arc length; her knowing that the red segment worked for each state did not imply by necessity that the red and blue segments existed in a covariational relationship consistent with the relationship between height and arc she identified in the Taking a Ride task. Furthermore, after we assisted her in denoting amounts of change of the red segment (i.e., height) for successive cases (Figure 2c), she responded in surprise that her "theory" held true in the current situation. Collectively, her activity was a contraindication that she had constructed a re-presentable quantitative relationship between co-varying quantities; rather, her understandings were rooted in carrying out particular activity of partitioning in order to make amounts of change perceptually available for comparison on the specific representation (Figure 1b). That is, when moved away from the original representation to a novel context with magnitudes changing continuously, she did not envision the same quantitative relationship of those magnitudes because from her perspective, she was carrying out a different activity (i.e., point-wise checking of the red segment) than the one in the previous situation.

The Circle Task

In the ninth teaching session, we asked Lydia to recall her graph related to the Circle Task in the previous session. The Circle Task included an animation of a circle with a point moving along the circle, with the red segment representing the height above the horizontal diameter and the blue segment representing the arc length traced out as the point traveled along the circumference of the circle (Figure 3a). Lydia drew from memory the graph shown in Figure 3b, created equal partitions along the horizontal axis, and denoted corresponding height magnitudes and amounts of change of height (Figure 3b). She claimed, "arc length is increasing at a constant rate, and height is increasing at a decreasing rate." We note that Lydia was only talking about *one* quantity (either height or arc length) when she used the word "rate" as opposed to using "rate" to describe the multiplicative comparison between *two* co-varying quantities. In response to Lydia discussing each quantity separately with respect to "rate," the teacher-researcher drew a copy of the graph with equal partitions along the vertical axis (Figure 3c). She described, "you're increasing in height at a constant rate, and then our arc length is increasing at an increasing rate."



(a) (b) (c)

Figure 3: (a) The Circle Task; (b) Lydia's graph; (c) The teacher-researcher's graph.

Noticing that Lydia described height as both "increasing at a decreasing rate" and "increasing at a constant rate," the teacher-researcher questioned her in ways that would draw her attention to comparing the two claims. She explained:

> *Lydia:* […] Because [height] increases at a decreasing rate [*pointing to Figure 3b*], and I'm assuming, I am assuming that the pattern would probably hold [on Figure 3c] if we did this [*pointing to the partitions in Figure 3b*], um, kind of test […] so I'm kind of seeing that maybe that pattern would hold. I haven't tested it, so I can't say that –

This demonstrated Lydia's difficulty with imagining or anticipating the partitions of Figure 3b on Figure 3c, and her continued work on the task indicated uncertainty with conceiving "height is increasing at a decreasing rate" on Figure 3c due to the perceptually available equal partitions of height. She was constrained by the partitions she perceived on the graph to the extent that without physically carrying out subsequent partitioning activity, she could not take those partitions represented on Figure 3b as relevant to Figure 3c.

Discussions

Through our analysis, we found that although the partitioning activity assisted Lydia's reasoning with amounts of change, it did not necessarily support her abstracting an operative covariational relationship between quantities. That is, when confronted with a series of similar situations, and without carrying out particular sensorimotor actions (e.g., drawing lines to partition one quantity and highlights the amounts of change of another) to produce perceptually available results, she had difficulty *re-presenting* the actions and its results mentally. By "re-presenting" an activity, we employ Piaget's notion of *re-presentation* to refer to a student's ability to mentally run through an activity and coordinate the results of the activity when the perceptual situation that originally led to the coordination is not actually present (von Glasersfeld, 1995). Lydia's difficulty with mentally re-presenting partitioning activity particularly constrained her thinking in situations that differed in segment orientation and placement (e.g., the Which One task) or situations where perceptual features of previous results were absent (e.g., Figure 3c). We highlight that when carrying out an activity dominates a student's thinking of graphs and situations, that student might have difficulty with reasoning with novel representations because of not having the coordination of such activity (e.g., the underlying quantitative relationships and operations) mentally available in those representations. Moving forward, we call for continued explorations into how students reflect upon their partitioning activity and abstract quantitative relationships and structures (e.g., rate of change).

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STUDENTS' APPROPRIATION OF MATHEMATICAL DISCOURSE IN A DISCOURSE-DRIVEN CLASSROOM

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This study explored how students appropriated mathematical and discursive practices in a class whose teacher took a discursive approach to instruction while supported by professional development and a curriculum designed with equity and discourse in mind. The study deployed a theoretical framework that used Academic Literacy in Mathematics (Moschkovich, 2015) as a framework for analysis of visual representations and Talk Moves (Anderson, Chapin, & O'Connor, 2011). The study examined a 4th-grade classroom in the San Francisco Bay Area where roughly three out of five students were English Language Learners and 19 out of 20 were low-income. Results showed that students appropriated the Talk Move, "Do you have any ideas?" for multiple purposes.

Keywords: Classroom Discourse, Elementary School Education

The National Council of Teachers of Mathematics (NCTM) and California Common Core State Standards (CCCSS) call for students to participate in mathematical discussions (CDE, 2014). However, few students enter classrooms already knowing how to participate in these kinds of discussions. The purpose of this qualitative study was to examine how students appropriated mathematical and discursive. The study is part of a larger research project, led by the Early Mathematics Education Research Group (EMERGe) at Stanford University. The following questions, in part, guided the exploration of mathematical communication:

1. In a setting where a teacher attempts to provide mathematics instruction that supports a discursive approach, do students appropriate discourse practices provided and/or promoted by the teacher or curriculum?
    a. If so, which ones?
2. How do students make use of multiple representations when engaged in mathematical discussions during group work?

Theoretical Framework

Academic literacy in mathematics has three integrated components: mathematical proficiency, mathematical practices, and mathematical discourse. Mathematical proficiency refers to the "expertise, competence, knowledge, and facility" that is required to successfully learn mathematics and is a "cognitive account" of mathematical activity vis-à-vis the five strands of proficiency (NRC, 2001); Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, and a Productive Disposition. Defining characteristics of mathematical practices come from the NCTM standards and the Common Core State Standards include problem solving, sense-making, reasoning, modeling, and looking for patterns, structure or regularity (California Department of Education, 2014). Finally, Moschkovich describes mathematical discourse as the "communicative competence (Hymes, 1972) necessary and sufficient for competent participation in mathematical practices" (Moschkovich, 2015b, p.47)

A sociocultural perspective on mathematics learning views mathematics learning as a "discursive activity that involves participating in a community of practice" and mathematical activity is framed by mathematical knowledge, practices, and discourse (Moschkovich 2002, 2015b). Within a

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sociocultural approach, the assumed mechanism for learning is appropriation. Appropriation refers to the process by which learners come to use "cultural tools" for themselves and transform them for their own purposes (Moschkovich, 2004 and Rogoff, 1990).

Setting and Curriculum

The professional development and curriculum supported the teacher's efforts to facilitate mathematical discussions through the promotion of Talk Moves (TMs). TMs are described as the tools that enable productive classroom discussions in mathematics through four steps:

Step 1. Helping individual students clarify and share their own thoughts,

Step 2. Helping students orient to the thinking of other students,

Step 3. Helping students deepen their reasoning,

Step 4. Helping students to engage with the reasoning of others (Anderson, Chapin, & O'Connor, 2011, p. 13).

The unit of instruction, *The T-shirt Factory (2007),* was designed to provide students with a rigorous curriculum and invite their active participation. It is written as a simulation where groups of students establish a T-Shirt factory and keep their warehouse organized while focusing on unitizing, place value, and equivalence (Fosnot, 2007).

Methods

Field observations, student work, and video recordings (of group work and whole-class discussions) were collected during the instructional unit (17 instructional days, total). Field observations were taken primarily as an observer. Interactions with students were minimized. Researchers gathered samples of student work on two occasions (Days 3 and 6 of the instructional unit). Video data for this study were selected from a subsample of six lessons (two each from the beginning, middle, and end of the unit) to see if there were any changes in student discussions. Each lesson typically included time for Small Groups, during which students worked on assigned tasks in groups of 1 – 4. For each of the six lessons, 1 – 2 small group work sessions (which were student-centered activities) were transcribed and coded. The analysis started by locating episodes in the video recordings of small groups where students were engaged in the focal mathematical practice, Model with mathematics. Within these episodes, I looked for evidence of mathematical proficiency (either Conceptual Understanding or Procedural Fluency) and cataloged which discursive practices students appeared to appropriate. Then, I looked for evidence of student coordination of multiple representations, including utterances, when engaging in mathematical discussions. In the interactions I looked for situated meanings of words and phrases.

Results

The teacher often used the "Do you have an idea?" TM as an invitation to explore students' prior knowledge and followed up with a "Say More" TM for elaboration or clarification. There was one student that demonstrated the variety of situated meanings that a single TM could engender. During Day 2 of the instructional unit (see Figure 1 for a copy of the problem on which students are working), Jefferson used the "Do you have an idea?" TM five times (pseudonyms are used for all participants in the study). However, two distinct purposes emerged: 1) use of the TM as a request for answers, and 2) use of the TM as a deflection. In all cases, Jefferson initiated the conversational exchange using the TM and meaning was determined by the characterization of the outcome initiated by the TM when Chloe responded.

Number of Rolls	Number of Loose T-Shirts
	13 33

Figure 1. Student Problem. Jefferson and Chloe's instructions: *Bundle the shirts. How many ways can you find?*

For example, in the excerpt below, Chloe answered Jefferson's question in a manner that suggests she interpreted the question as a request for an answer; not an idea or a solution process.

1 *Jefferson:* Ok. Do you have any idea?
2 *Chloe:* Yeah. Yeah. Yeah. Yeah. What's the number?
3 *Jefferson:* It is 13.
4 *Chloe:* We could make one pile… One row of ten and three left over.
5 *Jefferson:* Oh yeah! So, one pile of ten. Three loose ones. ((writes down response and raises hands victoriously)) So, ok, we need another two.

When Chloe responded with an answer, "One row of ten and three left over" (Turn 4), Jefferson didn't hesitate to acknowledge the response with hands in the air and stated the need for 'another two'. When the students attempt the second problem (see Figure 1 for the problem and the excerpt below), the students similarly construct the purpose of the TM as a request for answers.

60 *Jefferson* …Now we're on to thirty-three. Do you have any ideas?
61 *Chloe* Ok. We could make.. three rows of ten and another three rows of tens.

Chloe's suggestion is incorrect; however, these instances suggest that Chloe interpreted the question as a request for answers. Other uses of the TM seem to have been used for the purpose of deflection. In Turn 36, below, Chloe indicated that she "already gave [him] two ideas" and, in Turns 38 and 40, asserted that she has no more 'ideas'. Jefferson's persistence and the pair's ultimate abandonment of the task suggest that he was using the TM to deflect having to contribute a novel solution or process (see excerpt, below).

35 *Jefferson*: Ok. Let's get to work. I said do you have any ideas?
36 *Chloe:* I already give you two ideas.
37 *Jefferso*n: I know you have more.
38 *Chloe:* No, I don't.
39 *Jefferson:* Why are you laughing, huh? Dude, I know that you have ideas.
40 *Chloe:* No, I don't.
41 *Jefferson:* I think we're already done with this. So,…
42 *Chloe:* Go get a new one.

In the excerpt below, Jefferson appears to use the TM to deflect responding, again.

76 *Chloe* Write three T-shirts and three left-overs.
77 *Jefferson* **I was gonna do that. Your turn to do ideas.**
78 *Chloe* I already told you one.
79 *Jefferson* No! Those were me. I did do it.
80 *Chloe* No. You just do that one.
81 *Jefferson* We need to hurry up. … I don't think we need these [unifix] cubes.
[Turns 82 – 83 omitted]

84 *Chloe* Ten. Twenty. Thirty. Thirty-one. Thirty-two. [modeling with Base-10 blocks while speaking] You could make three piles of ten…
85 *Jefferson* Ok.

Chloe, reluctant to give additional answers, stated, "I already told you one" (Turn 78). Jefferson claimed credit saying, "Those were me" (Turn 79) and pressuring Chloe by saying "We need to hurry up" (Turn 81). Chloe conceded and offers another possible answer, "You could make three piles of ten…" (Turn 84). Jefferson successfully deflected his obligation. There was scant evidence of the appropriation of TMs by the students at large. The interactions above illustrate how the TM "Do you have any ideas?" was appropriated and transformed by a student to suit his own purposes.

Discussion/ Conclusion

In this study, students appeared to appropriate a TM when engaged in mathematical activity. The purposes seemed to reflect a request for answers to a problem or a deflection of responsibility to answer. This study reflects how students can appropriate and transform practices in unpredictable ways. Further research is warranted to determine how 'purpose' can be addressed and incorporated into mathematical discussions so that students can reap their full benefit.

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STUDENTS' MEANINGS FOR EXTENSIVE QUANTITATIVE UNKNOWNS

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A series of three design experiments was conducted with middle school students to investigate relationships between students' rational number knowledge and algebraic reasoning. After the first experiment a change was made in the investigation of students' construction of extensive quantitative unknowns. Students were asked to represent in pictures and equations the values for an unknown height measured in two different, multiplicatively-related measurement units. The work of 13 students operating at two levels of multiplicative reasoning was analyzed to identify differences and similarities. Students operating at the lower level of reasoning required substantial support to construct unknowns with implicit quantitative relationships, while students operating at the higher level of reasoning constructed unknowns with explicitly embedded units.

Keywords: Algebra and Algebraic Thinking, Middle School Education

It is well known that secondary students develop concepts for unknowns and variables that are often quite different from what teachers and curriculum developers intend. Common and persistent issues include that students see letters as standing for labels of objects rather than quantities and for known rather than indeterminate values (e.g., Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Küchemann, 1981). In addition, an increasing number of secondary students take algebra courses (Stein, Kaufman, Sherman, & Hillen, 2011), and so algebra teachers are tasked with working with a greater diversity of students. This phenomenon has been managed in several ways, from tracking lower-skilled students into double periods of algebra (Nomi & Allensworth, 2013) to teaching all students in heterogeneous groups with supports such as a 2-year algebra course (Boaler & Staples, 2008). In whatever ways diversity is managed, more needs to be known about the algebraic thinking and learning of a wide range of secondary students in order to inform the kind of supports that both students and teachers need.

To address these issues, we conducted three iterative design experiments with small groups of cognitively diverse middle school students in which we studied relationships between their rational number knowledge and algebraic reasoning. Analysis of the first experiment led us to revise our approach to developing concepts of unknowns. The purpose of this paper is to describe and account for how the students in the second and third experiments conceived of what we call *extensive quantitative unknowns* (EQUs). For example, consider the relationship between two values of the unknown height of a school measured in feet and inches: The number of inches that can fit into the height is 12 times the number of feet that can fit into the height because each foot in the height is equivalent to 12 inches. We call these problems Single Unknown Problems, and we posed them with non-standard measurement units to promote rethinking of relationships that students might take for granted when working with standard ones. Our research questions are: (1) How did the students conceive of EQUs? (2) How can we account for their conceptions?

Approach to Algebraic Reasoning and an Analytical Tool

A Quantitative Approach

In our experiments we took a quantitative approach to algebraic reasoning, in which a *quantity* is a property of one's concept of an object or phenomenon "that can be subjected to comparison" (Steffe & Olive, 2010, p. 49). Quantities can be extensive or intensive (Schwartz, 1988): Extensive quantities can be directly counted or measured (e.g., a distance), while intensive quantities are often created out of multiplicative comparisons of extensive quantities (e.g., amount of distance covered per unit of time). In our experiments we focused first on extensive quantities because, generally speaking, they are more basic than intensive quantities.

To conceive of an extensive quantity requires conceiving of a measurement unit, of the property as subdivided into these units, and of a way to enumerate these units to find a value (Thompson, 2011). However, extensive quantities can be thought about absent values, which makes them useful algebraically (Smith & Thompson, 2008). We conceptualized an EQU as an extensive quantity for which a value is not known, but for which a value could be determined. In our experiment we focused primarily on distances. So, an EQU could be thought of as a distance for which we have a measurement unit; we can imagine subdividing the distance into those units; but we don't know how many of those units will be needed to span the distance.

Students' Multiplicative Concepts

In our work we use students' multiplicative concepts as a key analytical tool. We conceive of students' multiplicative concepts as the interiorized results of students' units-coordinating schemes (Steffe, 1994). *Interiorization* refers to re-processing the result of a scheme so that students can anticipate it prior to activity. A *units coordination* involves two composite units (units of units), and it means to distribute the units of one composite unit across the units of another composite unit. For example, a units coordination of 5 and 7 involves distributing 7 units of 1 across each of the units of the 5 to get a unit of 35 that students structure in various ways.

Students who have interiorized two levels of units (MC2 students) can treat a length as a unit of units, or composite unit, prior to activity (Hackenberg & Tillema, 2009; Steffe & Olive, 2010). For example, MC2 students can imagine taking a 1-meter length and partitioning it into 5 equal parts without having to actually make the partitions. In other words, they can treat a length that represents 1 meter as a unit containing 5 units, a two-levels-of-units structure.

Furthermore, MC2 students can make three levels of units in activity: They can insert units into each unit in solving a problem. For example, they can insert 7 parts into each of the 5 parts in the 5/5-meter and determine that they have made 35 parts in all. However, in further activity the 35/35-meter becomes only a unit of 35 units; these students do not continue to view the 35/35-meter as a unit of 5 units each containing 7 units. So, MC2 students can consistently take two levels of units as given and create three-levels-of-units structures, but they don't maintain these structures in further operating. In contrast, students who have interiorized three levels of units (MC3 students) can maintain three-levels-of-units structures in further operating.

MC2 and MC3 students are the focus of this paper, so we don't discuss students who have interiorized only one level of units (MC1 students) here. We note that operating with a multiplicative concept is relatively stable: Progressing from one concept to another requires a significant reorganization of schemes that can take two years (Steffe & Cobb, 1988). Current estimates are that about one-third of incoming sixth-grade students are MC1 students, with MC2 and MC3 students making up the rest of the population (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015).

Method and Data Analysis

To launch each of the three 18-episode design experiments, we implemented a selection process with 21-24 7th- and 8th-grade students: a 30-minute interview and a 12-item worksheet. The interview questions and worksheet were designed to assess students' multiplicative concepts and fractions knowledge. Our aim was to use this process to invite three MC1, three MC2, and three MC3 students to participate in each experiment; however, MC1 students declined to participate. So, we invited six MC2 and three MC3 students to participate in each experiment. In the second and third experiments there was some attrition, resulting in 7 students in the second experiment (4 MC2, 3 MC3) and 6 students in the third experiment (3 MC2, 3 MC3).

In each experiment the 18 1-hour episodes ran twice per week and were video-recorded with one stationary and two roaming cameras. During episodes students often worked in groups of two or three using a software program called JavaBars (Biddlecomb & Olive, 2000), and student work was recorded with Screenflow (Telestream LLC, 2013). Sometimes students worked in groups that were cognitively more homogeneous (e.g., all MC2 students) and sometimes more heterogeneous (e.g., both MC2 and MC3 students).

One researcher (the first author) served as the teacher. Other team members operated roaming cameras, took notes, and interacted with students. Between episodes the team processed data, kept an Episode Index, watched video, took notes, and discussed conjectures to prepare for the next episode. Following each experiment each student participated in a 45-minute interview to assess the student's understanding of topics from the experiment and experience of the class.

For this paper we engaged in two phases of analysis. First we formulated a second-order model (Steffe, von Glasersfeld, Richards, & Cobb, 1983) of each of the 13 students' rational number and algebraic reasoning as it was addressed in the experiments, including students' concepts of unknowns. Second-order models are generated out of researchers' theoretical constructs, models from prior research, and a commitment to use constructs in an orienting but not deterministic way (Clement, 2000). To accomplish this analysis, we repeatedly viewed video files, transcribed major portions of video, and took detailed notes (Cobb & Gravemeijer, 2008). We also wrote and discussed memos (Corbin & Strauss, 2008): interpretations of and conjectures about students' ways of operating and about interactions in which those ways of operating occurred. Then, in the second phase we looked across the students to articulate differences in how students operated with unknowns and to account for these differences based on our models.

Analysis

We found qualitative differences in how MC2 and MC3 students represented and thought about EQUs: MC2 students constructed *implicit* quantitative unknowns, while the MC3 students constructed unknowns with explicitly embedded units. We support these claims by demonstrating how students worked on Single Unknown Problems. Often MC2 students did not represent the two values for the unknown height accurately (from our perspective) in pictures or equations; those who did represent the two values accurately in pictures and equations still demonstrated conflations that to us indicate their equations did not reflect embedded measurement units. Sometimes MC3 students also developed incorrect equations (from our perspective) but revised their work upon questioning; furthermore, their comments about their revised equations demonstrated that the equations did reflect embedded units. So, while both kinds of thinkers showed evidence of the construction of EQUs, their concepts of them differed significantly.

Due to space limitations, we cannot demonstrate student work here; we will do so in the presentation of this paper and in a manuscript that we will develop out of this work.

Discussion and Conclusions

This study suggests that MC2 students require substantial teacher support to create equations representing quantitative unknowns, and they may develop only an implicit understanding of the quantitative structure that we see. In contrast, the construction of EQUs, complete with an awareness of embedded units, is in the province of MC3 students. However, in our study, both MC2 and MC3 students benefitted from explicit discussion about the meanings they attributed to letters. We see developing both common and different supports for algebraic learners like these students to be at the heart of what is needed for access to algebra.

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VIDEOS OF PRESCHOOL MATHEMATICAL THINKING FOR TEACHER LEARNING

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Recent reports on the state of mathematics education call for a deeper understanding of mathematics developmental progressions and recommend improved teacher training for early childhood educators. However, few studies have explored the nature of preschool teachers' existing knowledge, beliefs, and orientations toward children's mathematical thinking. This report describes the initial phase of a project aimed at deepening teacher knowledge of how young children construct mathematical understanding. The video interviews described in this report illustrate the counting stages through which preschool children progress and show potential for use in professional learning settings as teachers reflect on their existing knowledge and work to develop more sophisticated interpretations of children's mathematical thinking.

Keywords: Early Childhood Education, Learning Trajectories, Teacher Knowledge

Introduction and Purpose

Early childhood mathematics education has been a topic of increased interest and numerous recommendations (NAEYC and NCTM, 2010; Duncan et al., 2007). Early mathematics teaching requires a particular knowledge base that includes an understanding of children's developmental and learning processes (Sarama, Clements, Wolfe, & Spitler, 2016). While enhanced knowledge and skills have been shown to produce positive influence on change in teaching practices (Garet et al., 2001), less is known about the specific knowledge base needed for early childhood mathematics teaching (Parks & Wager, 2015).

We are engaged in a research and development project to explore ways to provide early childhood educators with increased opportunities to identify children's existing mathematical knowledge and maximize learning opportunities in mathematical play. In the initial phase of this project–the focus of this research report–we collected video examples of children's mathematical thinking in one-on-one play-oriented interviews with preschoolers and explored what might be noticed about children's knowledge and ways of thinking. In a future phase, we will use these videos with early childhood mathematics educators as potential routes for developing teacher knowledge, addressing beliefs, and strengthening child-oriented pedagogical practices. In this study, we addressed the following research question: What aspects of children's counting schemes are evidenced through interactions with play-oriented tasks in video-recorded interviews?

Theoretical Framework

This project's goals and activities are situated within a constructivist epistemic framework. Consistent with the theories of Piaget, we assume that children construct understanding and develop knowledge as a function of their natural ability to think (Kamii & DeClark, 1985). This epistemology points to the role a child's existing knowledge plays in the construction of new knowledge. We draw on work by Steffe, Richards, and Cobb to support the identification of the stages through which children progress as they develop a counting scheme (1983). Of particular interest to us is the "child's progress in counting…marked by decreasing dependence on perceptual material. The first step in that direction is the ability to count figural representations of perceptual items (i.e., visualized images), which, though presented in the context of the task, are not perceptually available at the moment" (Steffe, Richards, & Cobb, 1983, p. 36-37).

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For teachers, we acknowledge that enhanced pedagogical skills are needed to recognize and make use of children's knowledge during play (Wager & Parks, 2016). To structure our future work with early childhood educators, we will rely on the construct of teacher noticing (Jacobs, Lamb, & Philipp, 2010) and the use of videos of children's thinking in a professional learning setting to explore potential pedagogical strategies with teachers. Studies by Sherin and Van Es suggest that video use can support teacher noticing and may influence the focus of teachers' attention (2005). In fact, video from teachers' own classrooms has been shown to be a useful tool in helping teachers consider their current practices and explore new strategies (Borko, Jacobs, Eiteljorg, & Pittman, 2008). We also consider the role of teachers' beliefs as they develop and enact new teaching practices (Borko & Putnam, 1996). Consistent with findings that written vignettes of classroom activity have the potential to elicit preschool teachers' beliefs (Lee & Ginsburg, 2007), we suggest that the use of these videos will show similar potential.

Method

The data for this phase of the project were collected through video-recorded interviews with preschool children from two Head Start preschools in the Western United States. Research associates interviewed all preschool children whose parents provided consent. Twenty children were selected for ongoing interviews based on the goal of obtaining a sample of children representative of gender, initially observed counting level, and primary language. These 20 children were interviewed biweekly over the course of 12 weeks.

Interviews were conducted consistent with the constructivist teaching experiment goals of "formulating and testing hypotheses about various aspects of the child's goal-directed mathematical activity in order to learn what the child's mathematical knowledge might be like" (Steffe, 2002, p. 177). Within each 10-15 minute interview, children were presented with several short play-oriented tasks in which they were asked to count items, say a number word sequence, produce collections of a given number, tell the number of items in small sets without counting, sort plastic animals into groups by color and count them, count claps or marbles dropped into a cup, count hidden items, or build block towers and count the number of blocks used. Following each week of interviews, video episodes were analyzed for evidence of children's mathematical thinking and anecdotal notes were taken. During this analysis interviewers discussed observations and planned tasks and questions they would consider in the following interview.

Preliminary Findings

All interviews were video-recorded and analyzed by the team of research associates. Initial findings revealed that children brought a range of counting concepts and skills to the tasks, evidenced in the ways they counted items and responded to questions. Table 1 presents a sampling of our observations consistent with established research drawn from the analysis of the video interviews. We then offer one vignette from video collected to illustrate a specific example noted in the table.

In one interview [noted in bold in the table], the interviewer asked Mario (pseudonym) to count the number of marbles that were dropped into a cup. The marbles remained hidden from Mario's view while being dropped and while in the cup. The following portion of the interview took place after Mario had correctly produced counts of three and two marbles.

Mario: (Summarizes his previous two counts). I got three and two. (Shows finger pattern of three on left hand with thumb, forefinger, and middle finger. Shows finger pattern of two on right hand with forefinger and middle finger.)

Interviewer: Good job, you made them with your fingers. OK, let's do it again. (Slowly drops four marbles into the cup while Mario listens.)

Mario: Four. (Quickly shows finger pattern of three on his right hand with thumb, forefinger, and middle finger).
Interviewer: (Shows Mario the cup).
Mario: (Touches each marble inside the cup as he counts.) One, two, three, four. I got four!

Table 1: Observations Based on Counting Types (Steffe et al., 1983; Wright et al., 2006)

Emergent	Perceptual	Figurative
may use number word sequence, but may not coordinate words with items	counts to find out how many items are present	counts items in a screened collection or may count items in more than one screened collection
may count items of a collection multiple times and arrive at different number words	**counts perceptual unit items, coordinates correct number word sequence with items**	counts substitutes for perceptual items (visualized spatial patterns, sequentially raised fingers or other movements, verbal items)
may use number word sequence from "one" in response to "How many are there?"	may count two addends and not count joined collection when asked how many	
may make finger patterns for numbers one to five	may reorganize items to facilitate counting	

Mario recognized the sound of the marbles hitting the bottom of the cup as something to be counted, and progressively produced more accurate counts of the marbles. While Mario has some finger patterns for numbers to five, he may lack a consistent pattern for four. In further interactions with Mario, asking him about the finger patterns he shows for his counts may help him to reconcile this inconsistency.

Discussion

The videos collected during this phase of the project provide specific examples of preschool children's thinking in counting activities and demonstrate the range of counting types that may exist in a preschool classroom. Furthermore, our analyses suggest that early learning environments are comprised of children who are operating at varied levels of understanding. Our goal in the subsequent phase of this project is to use these videos as models of early mathematical thinking to encourage discussion and reflection among teachers in professional learning settings. Using the construct of noticing, we plan to support teachers in learning to identify ways in which children may be operating and encourage the implementation of child-oriented pedagogical practices that acknowledge these existing understandings. We believe these video artifacts have the potential to surface the implicit theories teachers hold about how young children come to know and understand mathematics and could prove to be helpful tools in developing early educators' knowledge, beliefs, and practices.

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WHOLE NUMBER AND INTEGER ANALOGIES

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This research brief highlights analogy use of a Grade 5 student, Kim. Kim participated in four individual interviews across 12-weeks, embedded within a teaching experiment. Drawing on the theoretical perspective of in-between and within-domain analogies, I analyzed how Kim drew upon analogies as she solved the integer open number sentences during the individual interviews. These analogies are distinguished between whole number and integer analogy use. I connect existing research with descriptions of whole number and integer analogy.

Keywords: Number Concepts and Operations, Elementary School Education, Cognition, Middle School Education

Children are capable of reasoning about negative integers in productive and sophisticated ways prior to school instruction (e.g., Bofferding, 2014; Bishop et al., 2014; Featherstone, 2000). Although there are different ways to describe children's thinking about integers—mental models (Bofferding, 2014), ways of reasoning (Bishop et al., 2014), and conceptual models (Wessman-Enzinger, 2015)—there is much agreement that children frequently utilize analogies as they make sense of integer addition and subtraction (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016; Bofferding, 2011; Bofferding, 2014; Wessman-Enzinger, 2015). In fact, analogical reasoning is likely a component of all learning (Vosniadou, 1989).

Bofferding (2011) described the different types of analogies that children make, ranging from comparisons that involve surface features to deep structural comparisons, and comparisons that involve both. A surface feature comparison, for example, includes comparisons like $-5 - 9 = \square$ to $9 - 5 = \square$, because both number sentences involve the same digits. A deep structure comparison includes recognizing that $3 - 7 = \square$ is different from $7 - 3 = \square$. Bishop et al. (2016) described different comparisons, or analogies, that are logical necessities: variations in signs, variations in operation, and variations in addends. A variation in sign when number sentences like $3 - 7 = \square$ and $3 - -7 = \square$ are compared. Variation in operations includes comparisons like $-5 + 2 = \square$ and $-5 - 2 = \square$. Variations in addends includes comparing $-1 + 4 = \square$ and $4 + -1 = \square$.

Although differentiating these types of comparisons is important, researchers do it differently (Bishop et al., 2016; Bofferding, 2011; Wessman-Enzinger, 2015). This work seeks to understand the structural similarities of these different types of analogies and potentially connect this work by drawing on Vosniadou's (1989) definition of analogical reasoning.

Theoretical Perspective

Vosniadou (1989) characterized the mechanism behind analogical reasoning as the following: (a) "retrieving a source system (Y), which is similar to X in some way"; (b) "mapping a relational structure from Y to X"; (c) "evaluating the applicability of this relationship structure for X" (p. 422). She also highlighted two different types of analogies: between-domain and within-domain analogies. *Between-domain analogies* are when two comparisons are made (i.e., comparing Y to X), but these comparisons have similarity in structure, but are in two different systems (i.e., $X \ni \{A\}$, $Y \ni \{B\}$). *Within-domain analogies* are comparisons that belong to the same, or very close, conceptual domains. For example, if I compared wavelengths of air to wavelengths of water this would be a between-domain analogy since water and air are different structures. But, if I compared amplitude and frequency of sounds waves these would be within-domain analogies since amplitude and frequencies of sounds waves belong to the same conceptual structure. Vosniadou stated, “The distinction between within-domain and between-domain analogical reasoning is not a dichotomous one. Rather, it represents a continuum from comparisons involving items that are clear examples of the same concept to items that belong to different and remote domains” (p. 415). Specific to this brief report, I address the distinction between whole number and integer analogy (between-domain and within-domain analogies), and use one student, Kim, to illustrate this difference.

Method

Context of Study

Three 10-year old Grade 5 children from a rural Midwest school in the United States participated in a 12-week teaching experiment (Steffe & Thompson, 2000). These Grade 5 children made ideal participants because they did not have prior instructional experiences with operations with negative integers. As part of this study, the children engaged in four individual, structured task-based interviews (Goldin, 2000). These structured task-based interviews provided space to evaluate the children’s understandings of open number sentences individually, while they solved the same integer number sentences types (Murray, 1985) across the 12-weeks. Group sessions of the teaching experiment were about solving integer problem in contexts, not intended to support analogy use, and are described elsewhere (Wessman-Enzinger, 2015). The data generated for this study comes from when the children solved open number sentences during the individual sessions.

Individual Interviews

Across the individual sessions, the student solved open number sentences of the same type. For instance, -20 + 15 = ☐ is considered the same problem type as -16 + 4 = ☐, aligning with the problem type -a + b = ☐, a > b (Murray, 1985). The transcript for solving each of these open number sentences constituted the unit of data— there were 93 units for each child and 279 total units of data for this investigation. For this report, Kim is selected as case study because she had the most units of data coded with analogical reasoning.

Data Analysis

A second researcher and I examined the data utilizing a constant comparative method (Merriam, 1998), looking for the use of analogy as defined by Vosniadou (1989). We compared and negotiated all differences on this first coding of analogy. During the secondary pass of data, I developed the definition of analogy further with the distinction between whole number analogies and integer analogies, relating to Vosniadou’s within-domain and between-domain analogies.

Descriptions of Analogical Reasoning with Integer Addition and Subtraction

Children utilized *analogy* when they compared integers in a way that connects an integer addition and subtraction number sentence to different addition and subtraction number sentences. The two subcategories of analogy, whole number analogy and integer analogy, constitute types of analogy use. *Whole number analogies* are comparisons of integer addition or subtraction open number sentences to other addition or subtraction problems with only whole numbers, or positive integers. For example, Kim compared -15 – -4 = ☐ to 15 – 4 = 11 (correctly determining the solutions to -15 – -4 = ☐ as -11). Her comparison, 15 – 4 = 11 is a number sentence with only whole numbers. *Integers analogies* are comparisons of integer addition or subtraction problems to different, and not necessarily equivalent, integer addition or subtraction problems. Integer analogies include the comparison of an integer addition and subtraction problem to a number sentence with at least one negative integer. For example, Kim compared ☐ – -3 = 2 to 1 – -3 = 4 (correctly determining the solution to ☐ – -3 = 2 as -1). Her comparison, 1 – -3 = 4, involved negative integers in the number sentence this time. For both of these examples, the comparisons are not mathematically equivalent: $-15 - -4 \neq 15 - 4$ and $-1 - -3 \neq 1 - -3$. This aligns to Vosniadou's (1989) definition of analogy that two comparisons are related, but not necessarily structurally equivalent. What distinguishes analogical reasoning is the relation is not the equivalence, but the comparison of addition and subtraction problems to other addition and subtraction problems. Thus, both whole number analogies and integer analogies are examples of analogical reasoning (Vosniadou, 1989). This distinction between whole number and integer analogy is described in more depth next.

Whole Number Analogy

Kim utilized whole number analogy when she solved $-12 + -5 = \square$ in the following excerpt:

> (Writes -17 in the box.) Both of the numbers have the negative sign in front of them. That means that they are both negatives. And, that's pretty much the same thing as twelve plus five when you add a negative twelve plus five. It's just negatives this time. So you just (shrugs) ... the twelve plus five and got seventeen. And I just added the negative onto it. … It comes out as the exact same answer, but the only difference is that they add a negative sign to it.

In this excerpt, Kim stated that $-12 + -5 = \square$ is "pretty much the same" as $12 + 5 = \square$. Kim's direct comparison of $-12 + -5 = \square$ to $12 + 5 = \square$, only includes whole numbers is a typical example of whole number analogy.

In the excerpt above Kim drew upon whole number analogy and determined a correct solution. However, sometimes as Kim drew upon whole number analogy she obtained incorrect solutions. For example, she compared $-5 - 4 = \square$ to $5 - 4 = 1$. As Kim solved $-5 - 4 = = \square$, she concluded the answer was -1, "Five minus four equals one. Pretty much the same thing. You don't really need the negatives."

Integer Analogy

Kim constructed integer analogies included when she made comparisons to number sentences with negative integers. For instance, Kim compared $-4 + \square = -19$ to $-4 + 15 = 11$ (correctly determining the solution of $-4 + \square = -19$ as -15). She made her comparison, $-4 + 15 = 11$, to help her determine that the solution to $-4 + \square = -19$ needed to be negative. She, then made a second comparison, a whole number analogy, comparing $-4 + \square = -19$ to $4 + 15 = 19$. In contrast to whole number analogy, comparing $-4 + \square = -19$ to $-4 + 15 = 11$ constitutes an integer analogy because it involves negative integers.

Kim also constructed integer analogies that did not lead to a correct answer or made comparisons to number sentences that are not true. For example, Kim drew upon integer analogy as she compared $3 - \square = 4$ to $3 - -1 = 2$. But, $3 - -1$ does not equal 2. Consequently, Kim's use of integer analogy could not help her produce a correct solution since $3 - -1 \neq 2$.

Discussion & Final Remarks

The distinction between whole number and integer analogies reported here relates to Vosniadou's (1989) within-domain and between-domain types of analogies. Whole number analogy related to between-domain analogy in the sense that these types of analogies connect the whole numbers to integers. Integer analogy is a type of between-domain analogy in that, although the two integer number sentences are not necessarily mathematically equivalent, the children relate these two different types of integer number sentences in their own unique ways.

Crossroads are a place where a road crosses a main road or connects two main roads, one of the conference themes this year. This theme resonates in this research. First, this research represents a crossroad as it connects well-traversed work in analogy (Vosniadou, 1989) and current research with integers (e.g., Bishop et al., 2016). In alignment with prior work on children's thinking about integers that illustrates that different types of comparison are made (Bishop et al., 2016; Bofferding, 2011), Kim used whole number and integer analogy differently, developing use of integer analogy and using analogies to obtain correct solutions. These results extend this work of Bofferding (2011) and Bishop et al. (2016), but also complement and connect this work by illustrating that these types of comparisons (e.g., surface features, deep structure, variation in operations, varying order of addends) are capable of being grouped as whole number analogy and integer analogy. A surface feature comparison (Bofferding, 2011) or variations in operations (Bishop et al., 2016), for example, could be a whole number or integer analogy. Analogy use represents a crossroad where integer research is connected.

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ALGEBRAIC TASK DIMENSIONS: A TOOL FOR INTERPRETING A CURRICULUM-BASED HLT

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Introduction

This analysis is situated within the Improving Formative Assessment to Support Teaching Algebra (iFAST Algebra) Project, an NSF-DRK12 grant focused on enhancing middle-school mathematics teachers' formative assessment practices in algebraic function. The project involves design and study of a professional development program that aims to develop teachers' understanding of the hypothetical learning trajectories (HLTs) embedded in their classroom curriculum. The integration of HLTs into teaching requires framing them in ways that are interpretable and useful for teachers. One such framing establishes a set of dimensions for linear-function tasks to highlight key conceptual obstacles and landmarks along an HLT. We document how dimensions and representations play out in a sequence of tasks across a middle-grades curriculum (CMP3). We show how key task features signal conceptual concerns along HLTs in algebra.

Analysis of Dimensions and Representations of Linear-Function Tasks

Our framing of *task dimensions* draws from an approach to algebraic task analysis (Leinhardt, Zaslavsky, & Stein, 1990) in which functions-related graphing tasks were investigated using four constructs: action called for, contextualizing situation, type of variables, and location of mathematical attention. In addition, the importance of translating among *representations* of algebraic functions— tables, graphs, equations, pictures, and verbal situations—is well documented (see Brenner, et al., 1999). We first analyzed several pilot tasks to further specify dimensions and levels for each, and refine representation codes (Figure 1). We then applied the refined tools to all tasks.

Results

Preliminary analysis identified key dimensions levels with a hypothesized direction of increasing complexity. When analyzed against these dimensions, the sample of tasks in CMP3 reflected a robust distribution of task features, particularly with respect to translations among representations, mathematical actions, and explicitness of parameters in the function representation. Analysis also identified key conceptual transitions in the curriculum (e.g., introduction of non-proportional linear functions, non-positive slopes, tables with non-unitary input increments).

DIMENSION	LEVELS		
Situation	Contextual Abstract		
Data fit	Exact Approximate		
Proportionality	Proportional	Non-proportional	
Slope value	Positive only	Negative, zero, or undefined	
Inputs/outputs	Whole-number values only	Include negative values and/or rational values	
Explicitness	m and b explicit	m and b partially explicit	m and b implicit
Actions	Interpretation Value calculation Construction		

Hypothesized direction of increasing complexity

Figure 1. Task dimensions.

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ASSESSING EARLY CARDINAL-NUMBER CONCEPTS

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The give-me-*n* task (e.g., asking a child for 3 or 5 chips from a pile of 10 chips) is used widely in developmental research as the operational definition of a child's understanding of small numbers (*n*-knower level) and the cardinality principle (CP) with larger *n*s (CP-knower level). The CP entails recognizing that the last number word used in the counting process represents the total. However, the give-me-*n* task, which entails mapping a number word to a collection, may be cognitively more challenging than the how-many task, which involves mapping a collection to a number word and, thus, provides a direct measure of these constructs. Two studies were undertaken to evaluate whether the two tasks produce equivalent results for subitizing small numbers (n-knower level) and an understanding of the CP (CP-knower level).

Study 1 entailed comparing 60 3-year-olds' (M = 43.2 months, SD = 3.1 months) performance on the give-me-*n* and the how-many tasks with sets of 1 to 3 at three time points 3 weeks apart. A 2 (Task: how many v. give-me-*n*) x 3 (Size of $n = 1, 2$, or 3) x 3 (Time: T1, T2, or T3) repeated-measure ANOVA was conducted. This analysis yielded main effects for Task ($F[1, 59] = 17.078$, $p < .001$) and Size ($F[2, 118] = 39.069$, $p < .001$) but not Time ($F[2, 118] = .448$, $p = .64$). In addition, a Task x Size interaction was significant. Specifically, when n=3, children performed significantly better on the how-many task (mean for all three sessions = 1.494) than on the give-me-*n* task (M = 1.022), $F(1, 59) = 34.061$, $p < .001$, Cohen's $d = 0.635$.

Study 2 was undertaken to directly compare 3- and 4-year-olds' (M = 49.4 months, SD = 3.95 months) performance on the how-many and give-me-*n* tasks with larger numbers (6 & 8). In addition to a how-many task with linear arrays, a more difficult how-many task was also administered. Unlike the linear version and similar to the give-me-*n* task, the challenging how-many task entailed counting non-linear arrays and, thus, entailed more than minimal effort to keep track of which items have been counted and which need to be counted. As children may merely memorize a state-the-last-*n* rule for responding to how-many questions and may not truly understand the CP, two "how-many application tasks" (identity conservation and cardinality equivalence tasks) were also administered. These tasks involved determining a cardinal value of a collection *and* then applying this information in a meaningful manner.

Performance on the give-me-*n* task was significantly and substantially (as measured by effect size) lower than that on the how many—easy version ($p = .016$, Cohen's $d = 0.434$) and the how many—hard version ($p = .001$, Cohen's $d = 0.609$). The difference was marginally significant but still substantial for the identity conservation task ($p = .052$, Cohen's $d = 0.356$) and cardinality equivalence task, ($p = .083$, Cohen's $d = 0.352$).

The results of the present research confirm that the give-me-*n* task is but an indirect or proximate measure of the 3-knower level and the CP-knower level. Importantly, using the give-me-*n* task as the sole measure of these key aspects of early numeracy may seriously underestimate children's early number development. Performance on the give-me-*n* task more accurately reflects "*n-producer* levels": verbal subitizing-based set production and counting-based set production, which requires a cardinality concept more advanced than the CP.

ASSESSING SYMBOL SENSE BY IDENTIFYING STRATEGIC SOLUTIONS

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When solving math problems, we can often reach the correct answer using different approaches. Some approaches are more efficient than others—for instance, you might notice a shortcut, or use a trick to quickly simplify a complex equation. The ability to identify clever solutions is part of what's known as 'symbol sense.' Researchers suggest that people who are stronger at math should be better able to recognize strategic solutions and avoid less-strategic approaches (Bokhove & Drijvers, 2010), but this notion has not been tested systematically.

The current study explored whether people can differentiate between incorrect solutions, correct but less strategic solutions, and correct solutions that are also highly strategic. In order to do this, we showed each participant a series of algebra problems. For each problem, they also saw the first step of four different approaches to solving it: an *invalid* manipulation; a valid but ultimately *useless* manipulation; a valid, helpful, but *less strategic* manipulation; and a valid manipulation that would lead to a *clever*, strategic solution (Table 1). On the basis of these four first-steps, participants rated each approach to solving the problem, using a scale from 0 (a good student should never use this strategy) to 100 (a good student should always use this strategy).

We found that participants exhibited two distinct patterns of results. Many participants gave higher ratings to clever answers, thus distinguishing these strategic approaches from formally correct manipulations that were ultimately useless or inefficient. Others did not distinguish between formally valid approaches, giving equally good scores to any approach that did not include an error (i.e., clever, less strategic, and useless approaches). Critically, the ability to distinguish strategic, clever approaches from less strategic or useless approaches was related to both algebraic skill (as measured by accuracy) and mathematical experience (as measured by the number of math courses completed). These findings suggest that the ability to recognize clever or strategic approaches—'symbol sense'—is a critical component of complex mathematical skill.

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CONCEPTUAL MODEL-BASED PROBLEM SOLVING: A RESPONSE-TO-INTERVENTION PROGRAM FOR STUDENTS WITH LEARNING DIFFICULTIES IN MATHEMATICS

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About 5-10% of school age children are identified as having learning disabilities in mathematics (LDM), many of whom become at significant risk of failing the secondary mathematics curriculum. Given the increases in sophisticated and affordable technology for in and out of school, parents and teachers often turn to the many games, ‘apps,’ and web-based teaching/learning programs for help. However, there is a lack of Resources/Materials/Tools (RMTs) that focus on building *conceptual understanding* of fundamental mathematical ideas--concepts that are essential to enabling these students to understand and solve word problems and catch up with their peers. Without such RMTs, students with LDM are left further behind and virtually denied access to mathematics learning opportunities both inside and outside the elementary classroom. The purpose of this National Science Foundation funded project (Xin, Kastberg, & Chen, 2015) is to create a web-based intervention program to address the skill deficit and immediate needs of second- and third-grade students with LDM in meeting the Common Core State Standards for Mathematics. The objectives of this project include:

1. Create the curriculum content, screen design, and teacher manual for four modules of this computer-assisted tutoring program in the area of additive word problem solving;
2. Design and develop the cross-platform computer application that can be ported as web-based, iPad, Android, or windows app, which can run on different kind of computers and be accessible to students with various social economics status in a range of environment;
3. Conduct a small-scale single subject design study as well as a randomized controlled trial (RCT) study to evaluate the potential of this program in enhancing elementary students’ word problem-solving performance.

Multiple innovative features of this computer assisted tutoring program include: (a) an instructional program designed to emphasize mathematical model-based approach to promote students’ generalized problem-solving skills, (b) an emphasis on computer science technology to facilitate instruction that is computer-generated and specifically tailored to each individual student’s learning profile in real time, and (c) an emphasis on making the reasoning behind mathematics explicit to students through preparing fundamental mathematical ideas (e.g., a number as an abstract *composite unit*). This computer-assisted tutoring program represents a shift from traditional problem-solving instruction, which focuses on the *choice of operation* for solution, to a *mathematical model-based problem-solving* approach that emphasizes an understanding and representation of mathematical relations in algebraic equations and therefore promotes generalized problem-solving skills. This project is expected to make a broader impact due to its *cross-platform* (web-based, iPad, Android, or windows app) tool’s flexibility –with group or one-on-one instruction within the regular classroom settings or in pull-out settings during or after the school day, and with individuals at home.

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DEVELOPING FRACTION ADDITION WITH CONCEPTUAL UNDERSTANDING AND PROCEDURAL FLUENCY USING DELIBERATE STRIP DIAGRAMS

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The Problem

Development of conceptual understanding of the algorithm for adding fractions is built “using visual models, estimation, unit-fractions understanding, equivalence, and properties of operations” (Petit et al., 2016, p. 1420). These relevant concepts are investigated across grades 3-5 progressively, and by the end of grade 5, students should develop procedural fluency using the algorithm building on conceptual understanding (CCSSI, 2010). Research has suggested that addition of fractions should be developed through solving contextual problems using area models (Carmer, Wyberg, & Leavitt, 2008) and/or linear models (Izsak, Tillema, & Tunc-Pekkam, 2008). However, putting these research ideas into daily classrooms has been a challenging task. We report on a research-based lesson that promotes students’ understanding and fluency through Chinese lesson study (LS).

Method and Results

The LS occurred in a school system in a mid-size city in the southeastern USA. Three K-5 teachers implemented two cycles of a research-based lesson of addition of fractions in 5th grade classrooms. The LS was facilitated by three professors in math education from a large, public university in the city, and one math specialist from the school system. Data included lesson plans, videotaped research lessons, and post-lesson debriefings and reflection reports. Data analysis focused on capturing major changes between the two lessons and major features of lesson two. Reflection essays were analyzed through constant comparison to identify what teachers learned by the LS.

The lesson improved significantly toward developing the algorithm both conceptually and fluently. This improvement was facilitated by the development and inclusion in the second lesson of a graphic organizer involving three strip diagrams, bringing an area model and a number line together. Students represented the two fractions to be added in two of the strips of the diagram, and subdivided these into equal-sized pieces, which modeled finding fractions equivalent to the original addends only with a common denominator. This organizer aided teachers and students in more deliberately connecting the symbolic steps of the algorithm with the models. In lesson one, the students used area models to add fractions and then performed the algorithm, but the connection between the two was weak. The teachers appreciated the process of the LS and were proud of the product. They perceived their growth in knowledge about content, pedagogy and student learning, the specifics of which will be specified in the poster.

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INTERSECTIONS OF QUALITATIVE AND QUANTITATIVE RESEACH: RELATIVE SIZE OF NUMBERS

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Keywords: Number Concepts and Operations, Teacher Education Pre-Service, Research Methods

Research studies about both children's and adults' conceptions of the relative size of large numbers have employed number line tasks and quantitative measures of analysis. While these studies have revealed important findings, missing from these analyses are the participants' reasons for their placements of large numbers on a number line. Because pre-service teachers (PSTs) are tasked with planning and implementing learning experiences, their conceptions are important for mathematics teacher educators to understand. Brass and Harkness (2017) used a socio-cultural qualitative research design and asked the question: How do PSTs think about large numbers and the relationships between large numbers? Pre-service elementary, middle, and secondary teachers' (n=128) were asked to place one billion on a number line segment with endpoints at zero and one trillion and write explanations for their placements. Nine distinct explanations for these placements were identified (Brass & Harkness, 2017).

For this poster session we will show examples of these nine different explanations (Brass and Harkness, 2017) and highlight the intersections between them and the findings of quantitative studies involving the relative size of numbers. Quantitative studies have largely been cognitive or psychological in nature and have focused on: logarithmic-to-linear shifts (Moeller, Pixner, Kaufmann, & Nuerk, 2009; Slusser, Santiago, & Barth, 2013); proportionality (Barth & Paladino, 2011; Landy, Silbert, & Gilbert, 2013; Siegler & Opfer, 2003); familiar numbers (Rips, 2013); number nomenclature (Landy et al., 2013); and, place value (Moeller et al., 2009). Most studies have treated these topics as parallel to each other; however, our aim is to show how they overlap within mathematics education. For example, concepts of place value are contingent upon both understanding of proportionality and familiarity with number names.

Our aim is to provide a deeper understanding of PSTs' conceptions of the relative size of numbers. This work will inform mathematics teacher educators so that they plan activities that help PSTs move toward proportional, spatial, and place value number sense related to the relative size of large numbers.

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OPEN NUMBER SENTENCES: A BETTER WAY TO ASSESS PRECISELY THE CONCEPTION OF EQUIVALENCE

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Key Words: Elementary School Education, Number Concepts and Operations

It is generally accepted that a relational conception of equivalence (sameness or interchangeability of expressions), is critical for elementary students' success in latter mathematics (Blanton & Kaput, 2005). By contrast, an operational view of the equals sign (i.e. producing the answer) hinders their success. Researchers often use different types of items to assess students' conception of equivalence (Stephens et al., 2013). These items mainly fall into two categories: true/false items with closed number sentences (i.e. $a + b = c + d$) and open number sentences (i.e. $a + b = c + \square$). However, research tends not to focus on the specific advantages each type of item has in assessing conceptions of equivalence. The purpose of this study is to examine how each open number sentences and true/false closed number sentences elicit evidence of students' relational conception of equivalence.

The data used in this study includes observations of two second-grade students' in two sessions from a year-long teaching experiment. We found that both students' responses on true/false closed number sentences indicate that they have relational conception of equals sign (see figure 1). In true/false closed number sentences both students compute the answer for each side (i.e. expression) and then compares to see if both sides equal to the same answer. However, in the very next session when they got open number sentence ($8 + 5 = \square + 10$) both students represented operational conception of equal sign and put 13 in the box.

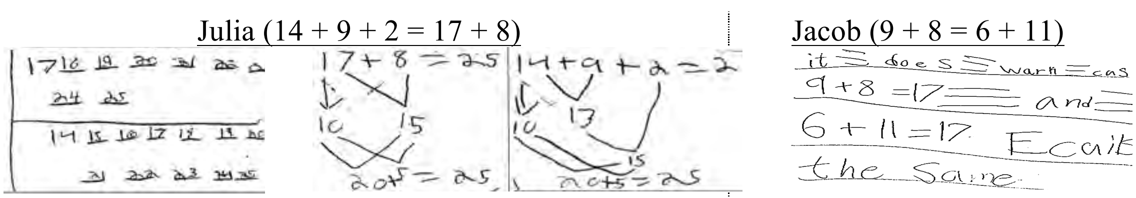

Figure 1. Participants' responses in session 13 for true/false closed number sentences.

The possible reason behind students' success in closed number sentences is that the structure of such problems allows students to treat each expression separately without seeing the equation as a math object on its own. On the other hand, in open number sentences counting and comparing scheme does not work due to the presence of unknown. Thus, open number sentences make it more obvious if students are not seeing the whole equation as an entity. It is therefore, argued that open number sentences provide better assessment items for understanding of equivalence as compared to closed number sentences.

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USING MAGIC ACTIVITES TO ENGAGE STUDENTS INTELLECTUALLY

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Most students are generally uninterested in mathematics because of its abstractness. Magic can captivate students because of their innate capacity to be intrigued and their proclivity to resolve curiosity. Educators have advocated the use of math magic to teach algebra (Benjamin & Brown, 2014; Matthews, 2008). In her study involving 23 ninth graders, Koirala (2005) found that math magic motivated students to learn basic algebraic concepts, and resulted in a pre-to-post improvement in solving basic algebraic problems. Recognizing the power of magic in reinforcing foundational concepts like variables, expressions, equations, and inverse functions, we implemented two magic activities in a capstone course for preservice 4-8 math teachers. In the *5-4-3-2-1-½ Magic*, students think of a secret number and perform arithmetic operations (add 5, times 4, minus 3, divide 2, add 1, add ½). Upon hearing a student's final number, the teacher-magician can spontaneously say out the corresponding secret number. In the *I-Know-Your-Final-Answer Magic*, the magician made every participants end up with the same final number.

The research questions are: (a) How do students respond to the magic activities? (b) Which type of magic activities do students enjoy more? (c) In what ways do students feel that the activities have increased their understanding and/or appreciation of algebra? Students' written work on their thoughts about how each magic worked were collected and analyzed. The other two research questions were answered using students' responses in an online survey and a focus group. There were 25 participants in this study; a few students being absent on certain days.

For the 5-4-3-2-1-½ magic, 13 out of 22 students figured out the trick using strategies like arithmetic approaches, reasoning with number patterns, and working backwards. Only two students thought of using algebra but could not figure out the trick algebraically. For the second magic, 39% successfully used algebra to show how the magic worked, 43% had errors in their use of algebra, and 18% did not even attempt to use algebra. Based on the survey and the focus group data, students overwhelmingly liked the magic activities especially the 5-4-3-2-1-½ magic, found them to be engaging and fun, and enjoyed the challenge to figure things out on their own. They appreciated its connections to algebra concepts and thereby gained a better understanding of the usefulness of algebra (a variable to represent the secret number, an expression to represent the steps, and solving for x to find the inverse function that corresponds to the trick). They found that the magic activities had reinforced what they had learned, but forgotten, about algebra. They also expressed their intent to use the magic activities in their own classroom.

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REVISITING AREA MODELS FOR FRACTION MULTIPLICATION

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The Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2011) highlight the importance of using "visual fraction models" for teaching and learning fractions to help students explore the underlying ideas of fraction multiplication and deepen their understanding of multiplying fractions. However, there are various misrepresentations and misinterpretations of fraction multiplication models made by teachers and students (Webel, Krupa, & McManus, 2016). Although teachers and students may assume the typical area model for the multiplication of fractions is a conceptual extension of the area model for whole number multiplication, the area model — prevalently used in current U.S. curricula for fraction multiplication — seems not a direct conceptual extension of the area model for whole number multiplication (Son, 2012). Predinger (2008) also shows (dis)continuity of the mental models of multiplication when students having transitions from whole number multiplication to fraction multiplication. It thus requires a careful investigation of the effective use of area models to help students deepen their understanding of multiplying fractions.

The purpose of this study is to consider how various area models in mathematics curricula support student understanding of fraction multiplication. Three U.S. textbooks and one Korean textbook are examined for how they offer opportunities with fraction representations, corresponding area models, and related word problems to foster understanding of fraction multiplication.

The results point out that many models are just reflecting the appearance or different characters of models themselves, not carefully reflecting students' action, thinking, and thought process in relation to models and the nature of multiplication. They also indicate that two different types of area models are used in various contexts of fraction multiplication: *the area-to-area model* and *the lengths-to-area model*. Each model reflects different aspects of a fraction and different thinking processes involved with fraction multiplication. Korean textbook uses both two models in teaching fraction multiplication, whereas U.S. curricula rarely introduce the lengths-to-area model for fraction multiplication.

These findings suggest that without a careful conceptualization of the two area models, teachers and students may overlook the conceptual ideas of fraction multiplication and connection to the models for whole number multiplication. We thus address the two types of area models representing the fraction multiplication processes in detail and discuss how to carefully introduce those models to help students attain a better understanding of the multiplication of fractions.

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RESPONSE TIME CRITERION FOR DEFINING FLUENCY

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Introduction and Method

Being *fluent* with basic combinations is crucial for achieving mathematical proficiency and entails the ability to find the sums and differences of basic number combinations *efficiently* (generate the answer accurately and quickly), appropriately and adaptively (Baroody,1985; NMAP, 2008). Researchers agree that children typically progress through three overlapping phases to deduce answers to unknown facts— a) counting, b) deliberate reasoning with known facts and, c) automatic retrieval—*before* achieving fluency with basic facts or family of facts (Geary, Hoard, Byrd-Craven, & Desoto, 2004). Although words such as *quick* and *fast* are often used to define basic fact fluency, what exactly counts as *fast* is often unclear. The current study evaluated the response time criterion for defining first graders' fluency (retrieval) with basic addition and subtraction combinations. Part of a larger experimental study with a diverse group of first graders from the Midwestern U.S., the data considered for this study was drawn from 75 students who completed the larger study. The data consisted of students' answers to basic addition and subtraction problems, response-time to generate each answer and, a description on how each answer was generated.

Findings and Conclusion

All correct responses were coded for strategy type and use: *overt* (deliberate or apparent use of the strategy) and *covert* (reporting a counting/reasoning strategy when asked during follow up questioning). The proportion of overt reasoning and counting responses for addition and subtraction problems were calculated with various response time (RT) cut-offs —from 2 to 5 seconds. The 3-second response time criterion successfully differentiated between fluent and non-fluent counting, with zero responses created under 3 seconds for all problems. However, this was not the case for responses generated under 3 seconds using a reasoning strategy, with approximately 3.5% of the addition and 29% of the subtraction responses attributed to reasoning. These findings indicate that different RT cut-offs may be appropriate to define fluency in addition and subtraction problems as opposed to using one RT cut-off as a proxy for defining retrieval (or fluency).

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RULE FOR PATTERNS AND RULE FROM PATTERNS: THE TENSION BETWEEN ALGEBRAIC MANIPULATION AND ALGEBRAIC UNDERSTANDING

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Pattern generalization is acknowledged in its importance as an introduction to algebra as well as in the development of algebraic reasoning (e.g., Jurdak & El Mouhayar, 2014). Figural patterns, in particular, provide opportunities to reason about diagrammatic and algebraic structure (Rivera & Becker, 2008). This study investigated the ways in which a group of prospective teachers (PSTs) attempted to provide generalizations for figural patterns. More specifically, we analyzed PSTs' approaches toward expressing generality and the relation of such approaches to PSTs' algebraic understanding of figural patterns. The research question was: how do PSTs' approaches to algebraic generalization relate to their understanding of figural patterns?

Eight PSTs at a Midwestern university in the United States participated in our study. They were in their third year of a secondary mathematics teacher preparation program. During the study, the PSTs were asked to provide generalizations for four different figural patterns. The data sources included both video-recorded clinical interviews with the PSTs and their written work on the tasks. Based on the research goal, we coded the data qualitatively and categorized the ways in which the PSTs produced their generalizations. The findings show that the PSTs employed two approaches to generalization: *rule for patterns* (attempting through trial-and-error to fit a symbolic rule onto the numeric pattern that was extracted from the figural pattern) and *rule from patterns* (drawing on numeric patterns in conjunction with the diagrammatic structure of figures to generate a symbolic rule). One PST, Amy, employed a *rule-for-patterns* approach on all of the tasks. In solving them, she attempted to fit exponential expressions onto both linear and quadratic number patterns without considering how such expressions related to the diagrammatic structure of the figures. Another PST, Jim, employed a *rule-from-patterns* approach. Across each of the tasks, Jim consistently noticed and utilized the relationship between the figure number and the figural structure to produce generalizations. A third PST, Zoe, employed both approaches. She attempted a *rule-for-patterns* approach in the first task, but later shifted her approach to rule from patterns upon recognizing an algebraic relation within the figural structures of the pattern. Our results suggest that only the *rule-from-patterns* approach is associated with drawing connections between algebraic expressions and the figures.

The algebraic understanding of students in K–12 schools has often lacked depth due to the traditional focus on symbol manipulation without meaningful connections (Lannin et al., 2006). If we wish to support students (and PSTs) in making connections and deepening their algebraic thinking as it relates to generalization and functional relationships, then there may be benefits in emphasizing the importance of sense-making in the generalization process.

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THE IMPACT OF A TEACHER-LED EARLY ALGEBRA INTERVENTION

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The view of algebra as a purely secondary school course has been challenged in recent decades, with algebra now being seen as an important strand of study for earlier grades (NCTM, 2006). Our work addresses the premise that a comprehensive early algebra education will better prepare students for a more formal treatment of algebra in secondary schools than an arithmetic-focused curriculum (Blanton et al., 2015). We share preliminary results from a randomized study (with 46 schools) in which we compare the performance of students who experience a teacher-led early algebra intervention in Grades 3-5 to the performance of students who experience a "business as usual" arithmetic focused curriculum. Intervention students significantly outperformed the control students in Grade 3, increasing their overall assessment scores 13% more than control students. Intervention students also demonstrated more sophisticated algebraic strategies in comparison to control.

While previous research has produced "pockets" of success developing students' algebraic reasoning, often with researcher-intensive involvement and a narrower algebraic content focus, our work applies a comprehensive and sustained model for early algebra by engaging students in a greater scale and breadth of early algebra concepts under the direction of regular classroom teachers. Preliminary results suggest that early algebra can make a difference and motivate future research in early algebra instruction.

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THE LOW-ACHIEVING LABEL: WHAT DOES IT TELL US ABOUT STUDENTS' CONCEPTUAL UNDERSTANDING OF VARIABLES?

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Keywords: Algebra and Algebraic Thinking, Equity and Diversity

Algebra is often identified as requisite for the success of individuals by providing them with more college and career options. Kaput and Blanton (2000) refer to aspects of teaching practices and curricula that are focused on procedural skills instead of conceptual understanding as problematic. Students' understanding and use of variable may also be problematic as a significant aspect of algebra. Research has shown mathematics difficulties (MD) may differ from their typically-achieving peers in mathematics-specific characteristics (e.g., Geary et al., 2012). Lewis (2014) warned a limited focus of research on MD students has resulted in low procedural fluency as a de facto defining characteristic of students with MDs. This neglects the more complex and conceptually-based aspects of mathematics. This study investigated the differences between conception and use of variable of low- and typically-achieving students.

Theoretical Framework and Methodology

The framework is based on Blanton, et. al's (2015) description of a learning trajectory (LT) that characterized increasingly sophisticated levels in students' thinking about variable. Students' responses to items on a conceptual algebra progress monitoring measure (created by Foegen & Dougherty, 2010) were coded with respect to TL levels and then compared across the MD and Typical Mathematics Achievement (TMA) groups with descriptive and inferential statistics.

Results and Implications

This investigation revealed that the differences between MD and TMA students are limited and related to item complexity. More than these differences, the similarities that were illuminated by this study impact how we perceive what successful and struggling students look like. The overall low proportion of students who have a sophisticated conception of variable suggests that all students need access to better conceptual instruction and experiences in algebra. Students identified as MD may be disproportionally disadvantaged by traditional instructional practices and data-based decision making based on procedurally focused assessments and instruction. The limitations of the current ways in which students are labeled provide an opportunity to reexamine how students are identified as "successful".

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USING NUMERICAL EXPRESSIONS TO SUPPORT STUDENTS TO DEVELOP A QUANTITATIVE UNDERSTANDING OF ALGEBRAIC NOTATION

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The abstract nature of algebraic notation serves as a barrier to higher mathematics for many students, particularly those from historically marginalized populations (Kaput, 1999). Without access to a meaningful understanding of algebraic symbols, students become disenfranchised with mathematics, a situation exasperated by the predominant focus on symbol manipulation in. In response to this challenge, many scholars have advocated for the ability for students to not only manipulate symbols, but interpret the contextual quantities that expressions represent. The Common Core State Standards (2010) emphasizes this understanding as a core component of algebraic thinking, including it as one of the eight practice standards (SMP 2: Reason abstractly and quantitatively) as well as a high school algebra content standard (HSA.SSE.A.1). In addition, many researchers have characterized this understanding and documented difficulties that result when students reason and manipulate symbols void of a coherent system of referents (e.g. Harel, 2007; Kaput, Blanton, & Moreno-Armella, 2008). While the field has underscored the need for students to develop a contextual understanding of symbolic representations, limited work has explored ways to develop this type of thinking in practice.

As part of a study in which I examined the instructional practices of four experienced algebra teachers in high needs schools, a concrete method for instilling this type of thinking emerged in one particular classroom, a finding that makes a valuable connection between research and practice. Engaging students in a series of inquiry based lessons involving figural pattern generalization, one of the teachers repeatedly asked students to express their understanding of the pattern using a numerical expression. Each time he followed up by asking the class to articulate the exact quantity each symbol represented. Results indicate that this instructional move served as a *pedagogical content tool* (Rasmussen & Marrongelle, 2006). Rather than telling students or asking them to produce a new way of thinking on their own, this instructional move gently pushed them to flexibly build on their own informal thinking and approach the problem differently. Specifically, it encouraged students to make the quantities within the pattern decomposition as well as their relationships more explicit. It initiated a shift for many students from recursive thinking to an explicit view of the pattern and finally embedded their understanding of the pattern into an abstract representation. In the end, students began to automatically use numerical expressions to describe their thinking about the pattern, indicating that this act had transformed the number sentence from a calculational tool to a descriptive representation that captured the quantitative structure of the figure.

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Chapter 4

Geometry and Measurement

Research Reports

Brief Research Reports

Posters

DYNAMIC MEASUREMENT: THE CROSSROAD OF AREA AND MULTIPLICATION

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In this exploratory study, our goal was to engage students in dynamic experiences of area as a continuous quantity that can be measured by multiplicatively composing two linear measures (lengths), an approach we refer to as 'dynamic measurement,' or DYME. In this paper, we present the learning trajectory constructed from two cycles of teaching experiments with sixteen third-grade students. We discuss the types of tasks used for developing students' DYME reasoning as well as the forms of DYME reasoning students developed as a result of their engagement with these tasks.

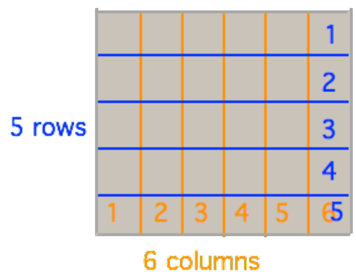
Keywords: Measurement, Technology, Learning Trajectories

Background and Aims

Measurement is defined as assigning a number to a continuous quantity (Clements & Stephan, 2004). In terms of area, several studies focused on using square units to cover surfaces and quantify that covering based on the number of square units needed to cover the surface (e.g. Barrett & Clements, 2003; Battista, Clements, Arnoff, Battista & Borrow, 1998; Clements & Sarama, 2009; Izsak, 2005; Kamii & Kysh, 2006). In these studies, structuring area first involves students counting the individual square units used to cover the surface, then counting unit composites of a row and using repeated addition to find all units, and lastly recognizing that they can count composite units in a row and a column and multiply *rows x columns*. A key structure of this approach is the construction of the grid/array that results when a rectangle is covered with square units (Figure 1a), a difficulty that students face even after extensive covering and tiling activities (Outhred & Mitchelmore, 2000). Ultimately, children must "switch" from the generalization *rows* x *columns* to the area formula of *length* x *width* (Outhred & Mitchelmore, 2000) but this "switch" is not always intuitive and results in students connecting multiplication to area by reciting, not by understanding the formula (Izsak, 2005; Simon & Blume, 1994).

Indeed, to understand how area is generated by multiplying lengths is a very different notion conceptually from the construction of a matrix like shown in Figure 1a. As Piaget argues, "the difference between the two operational mechanisms is the difference between a matrix which is made up of a limited number of elements and one which is thought of as a continuous structure with an infinite number of elements" (p. 350). Thus, "switching" from the notion of counting discrete (discontinuous) squares in rows and columns to the multiplicative relationship of combining two linear continuous measures (lengths) in an area formula can be extremely difficult for students (Baturo & Nason, 1996; Kamii & Kysh, 2006). Piaget et al. (1960) suggested that these difficulties arise because "the child thinks of the area as a space bounded by a line, that is why he cannot understand how lines produce areas" (p. 350). Area measurement involves the coordination of two dimensions (length and width) and is a multiplicative process while covering a surface with unit squares is a one-dimensional process and additive in nature (e.g. Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996; Simon & Blume, 1994).

Consequently, our plan was to test the conjecture that that it was possible for children to experience the two-dimensional multiplicative relationship of area without relying on the switch from the rows by columns structuring; in effect, to provide evidence that students can visualize area as a dynamic continuous structure that can be measured by coordinating two linear measures (lengths). We drew on research on visualizing area as 'sweeping' through the power of motion (Confrey et al. 2012; Lehrer, Slovin, Dougherty, & Zbiek, 2014; Thompson, 2000) for designing experiences for students to visualize area as a continuous structure dynamically. For instance, as suggested by Confrey et al. (2012), we considered engaging students in dynamic experiences of generating surfaces and visualizing a meaning for area as a ‘sweep’ of a line segment of length *a* over a distance of *b* to produce a rectangle of area *ab* (Figures 1b & c).



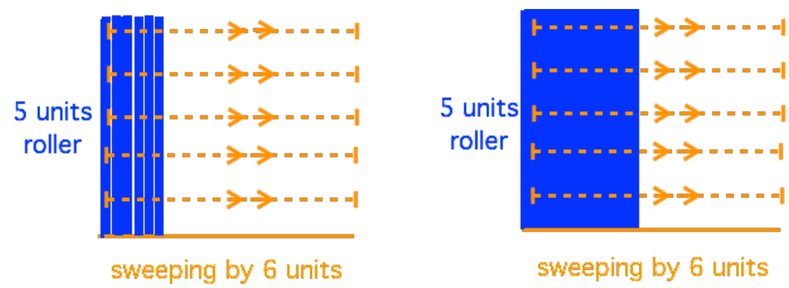


Figure 1a. Area as a discrete structure. **Figures 1b & 1c.** Area as continuous structure.

We distinguish this dynamic continuous approach from other approaches to measurement (e.g. *rows* x *columns* structuring) by referring to it as *Dynamic Measurement* (DYME) (Panorkou & Vishnubhotla, 2017). By looking at the example in Figures 1b & 1c, students visualize area as a continuous dynamic quantity which depends on both the length of the roller and the length of the swipe. This dynamic approach emphasizes the relationship between the boundaries of a shape and the amount of surface that it encloses, so that as the boundaries converge the area approaches zero (Baturo & Nason, 1996).

Although prior work (e.g. Confrey et al., 2012; Lehrer et al., 2014) identified the significance of teaching this dynamic approach, little information exists about how students’ DYME reasoning can be developed. Therefore, our goal was to examine: (1) What kinds of reasoning do students exhibit as they encounter DYME tasks? (2) What are critical aspects of students’ DYME reasoning that constitute increasingly sophisticated ways of understanding area measurement? (3) What kinds of measurement tasks, questioning, and scaffolding help students generalize concepts related to the spatial structuring of DYME?

Methods

Aiming to explore how students’ DYME thinking might be developed and progressed, our attention was drawn to research on learning trajectories (LTs), which have been widely used as an organizing framework for student conceptual growth (e.g. Clements & Sarama, 2009; Barrett et al., 2012; Simon, 1995). The design of the DYME LT followed Simon’s (1995) three components of an LT: a learning goal, a set of learning activities, and a hypothetical learning process. These components were constructed simultaneously during our LT design process; in other words, we defined our learning goals by having some instructional tasks in mind for promoting these goals and also postulated how students’ thinking of DYME may develop when they engage with our specific tasks and goals.

When formulating our initial conjectures about students’ reasoning of DYME, we synthesized existing LTs on length and area measurement (e.g. Clements & Sarama, 2009; Barrett et al., 2012; Confrey et al. 2012), and for each measurement construct we began by asking ourselves, “How can this construct be interpreted/modified/used in terms of DYME?” In contrast to other measurement LTs, the spatial structuring of DYME focuses on visualizing composites of 1-inch paint rollers iteratively dragged over a specific distance to cover a surface. In other words, a surface is described in terms of the number of 1-inch swipes (length) and the distance of each swipe (width). For example, to cover a surface of length 4 cm and width 7 cm, we need 4 swipes of 7 cm. Our design promotes students' thinking about commutativity (e.g. 4 swipes of 7 cm is the same as 7 swipes of 4 cm) and also reversibility (e.g. constructing surfaces by iteratively dragging rollers and deconstructing surfaces of length *a* and width *b* by equally splitting the surface into *a* sections of length 1 and width *b* to find area.) The target understanding of DYME involves a dimensional deconstruction (Duval, 2005), in other words analytically breaking down a 2D shape (its area) into its constituent 1D elements (length and width measures) based on relationships. Thus, two quantities (length and width) are coordinated simultaneously when making judgements about size.

In terms of task design, we used dynamic motion and paint rollers to enable students to both visualize area as a continuous quantity and coordinate the two linear measures. Our conjecture was that the paint rollers would act as bridge between the shape, and the number used to describe the size of the shape (Hiebert, 1981). To illustrate this dynamic motion, we used dynamic geometry environments (DGEs) to design our tasks. In addition to the dragging tool, most DGEs have a trace tool, which gives a trace of all the points on line segment (paint roller) following a locus as they move on the screen (see Figures 1b & c). Our conjecture was that the user would associate this discrete trace with the continuous surface formed.

We used a design-based research methodology (Barab & Squire, 2004; Brown, 1992) to develop and refine the LT and the tasks and tools focusing on two cycles of design, enactment, analysis, and redesign. The goal of Cycle 1 was to test our initial conjectures by experiencing students' first hand mathematical learning and how they construct DYME reasoning. We conducted a series of design experiments (DEs) (Cobb et al., 2003) with six pairs of third grade students. For the design of the tasks we used Geometer's Sketchpad (Jackiw, 1995). We had 6-10 sessions of 45-90 minutes with each pair of students. The outcome of Cycle 1 was a significant revision of the LT based on how students interacted with our tasks and tools.

The aim of Cycle 2 was to further demonstrate the feasibility of learning the DYME concepts and evaluate the effectiveness of the revised LT and also to test our tasks and tools with a group of students (Cycle 1 DEs were in pairs). We conducted a DE with a group of four students who participated in a STEM summer camp. Similar to Cycle 1, we had 9 sessions of 45-90 minutes each. For the design of the tasks we switched to the online version of Geogebra because of limitations of Chromebooks for downloading software.

Two stages of analysis occurred with the DE data, on-going analysis following each episode of the experiment (Cobb & Gravemeijer, 2008) and retrospective analysis at the conclusion of all DEs (Cobb et al., 2003). During the ongoing analysis, our initial conjectures evolved and we modified the tasks in light of iterative examinations of changes in students' thinking about area when interacting with the DYME tasks. During the retrospective analysis, we viewed the session videos and other data to create chronological accounts that tracked the forms of reasoning that emerged in the DE, the ways in which they emerged as reorganizations of prior ways of reasoning, and the aspects of tasks and tools that seemed to mediate those changes in reasoning. We drew on these analyses to refine the LT which included both our inferences about students' reasoning and the relationship between students' reasoning and DYME tasks.

A Learning Trajectory for DYME Reasoning

In this paper, we describe the most recent version of the DYME LT which resulted after conducting two cycles of DEs. We present each level of the LT by referring to Simon's three components: learning goals, sample tasks and examples from the student generalizations.

1. Exploring Dimensions and Area as Continuous Quantities

The goal of the first level is for students to build the idea of 2D space by using two linear dimensions (Clements & Stephan, 2004); in other words, visualizing area as a continuous structure that can change dynamically. The tasks engage students in coloring surfaces by dragging a given roller in multiple distances (Figure 2) and also matching different-sized rollers with shapes to color them and reason about the paint distance as well as the length of the roller. Students' generalizations include recognizing that the dimensions, length of the roller and the distance dragged, define how big or small a shape is, such as *"the further we drag the roller, the bigger the shape we create"* and *"the bigger the roller, the bigger the shape we create.* "At the same time, they begin to form relationships between the dynamic action of painting and the dimensions of the shape being painted by recognizing that *"the length of the roller needs to be the same as the height of the rectangle"* and *"the distance of paint is same as the base of the rectangle."*

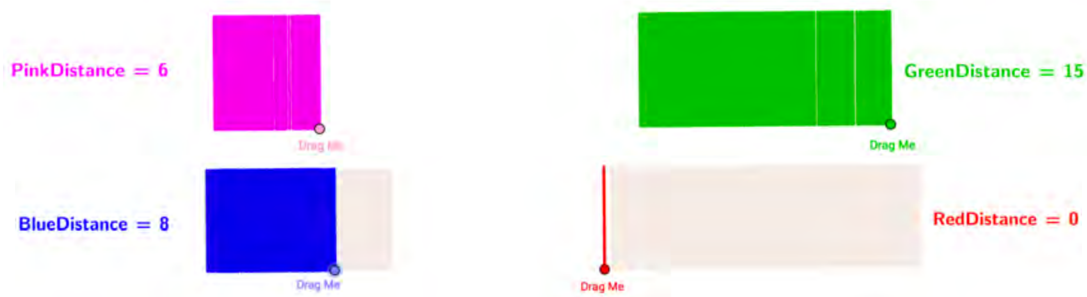


Figure 2. How far did you drag the paint roller to paint each shape?

2. Coordinating Two Dimensions to Compare Area

While in Level 1 students explored each dimension (length and width) independently, the goal of Level 2 is for the students to recognize that the measurement of a surface requires the coordination of two dimensions (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996; Simon & Blume, 1994). The tasks involve asking the students to fit a card into an envelope by modifying the dimensions of the envelope (Figure 3). Students are asked "What do we need to change? What stays the same?" They are asked to first write their predictions and then try it on the screen by modifying the envelope using the dragging tool. The card was movable so students could actually check if it fits in the envelope they created. Their generalizations include recognizing that to compare two shapes, they need to compare both dimensions. They also recognize that if one dimension is same they just have to compare the other dimension, e.g. *"it's bigger because it has the same base but it doesn't have the same height."*

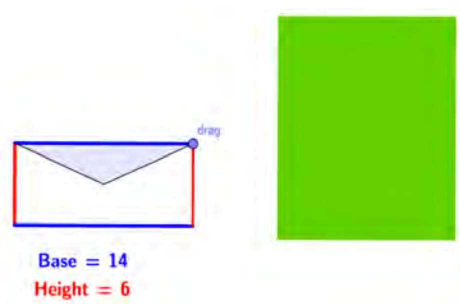


Figure 3. Modify the envelope to fit the size of the card! How big is the envelope you created?

3. Multiplicative Relationship of Length, Width and Area

The goal of this level is for the students to recognize the multiplicative relationship between the two dimensions of a rectangle and its area (Izsak, 2005; Simon & Blume, 1994). To help students identify this relationship, the tasks first involve the use of a 1-inch roller to paint shapes of different lengths and widths and constructing a repeating pattern for covering the shape (Outhred &

Mitchelmore, 2000; Reynolds & Wheatley, 1996), by considering the distance covered in one swipe with the number of swipes (Figure 4). This experience is critical for generating the use of the multiplicative 'times' language to find the space covered, such as *"this is 30 because the base is 10 and we are going to swipe three times."*

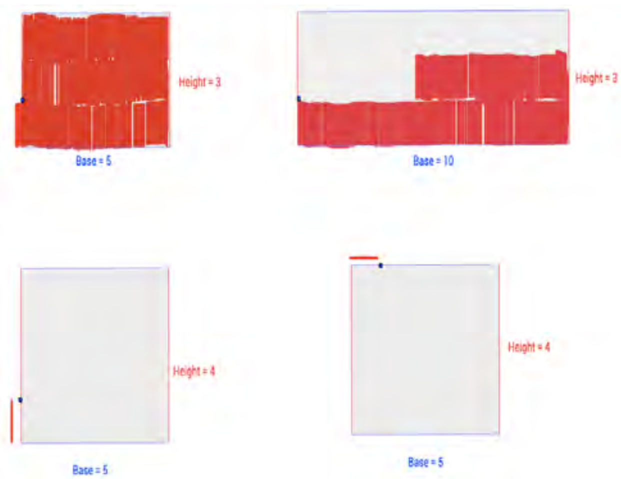


Figure 4. How far did you drag the roller? How many swipes did you need to cover the wall? How much space did you cover?

Central to the construction of area is understanding that the length measure indicates the number of unit lengths that fit along that length (Clements & Stephan, 2004). Consequently, our next goal is for the students to recognize that a roller of size *l* covers the same area as *n* rollers of size *l/n* dragged for the same distance. Our tasks include asking the students to paint the same rectangle first using 1-inch rollers and second by using rollers that are of different sizes and reason about the space covered in both occasions. At this stage, students begin splitting rollers to find the space covered, such as: "*This is a 3-inch roller. But if we cut it into 3 parts and you go across one time it is 4 and then if you go across another time it will be 8 and if you go one more time it will equal 12.*" Gradually, they begin to recognize that the space covered can be found by the length of the roller *times* the base of the rectangle or the distance of swipe. As students recognize that the length of the roller is the same as the height, they also begin using "length of roller" and "height" interchangeably, and this intuitively leads to *height times base*.

4. Multiplicative Coordination of Length and Area

The goal of this level is for students to recognize the effects on the dimensions when the area of a shape is scaled. To explore these effects, our tasks engage students in doubling and tripling areas (Figure 5) and identifying that they can double/triple areas by multiplying only one of the dimensions by the same factor, generalizing that *"to change area we need to change the base or height" or "to double the area we double just the length or just the width."* As a reverse process, we also designed tasks which engage students in doubling, tripling and halving lengths and widths of rectangles and reasoning about how area changes, such as *"since the length is going two times bigger, then the area should go two times bigger."*

As students' multiplicative thinking of area develops further, our next goal is for students to recognize that in order to split area (fractional thinking), they need to split the length or the width. The tasks engage students in creating shapes that have a fraction of an area of another shape. For example, students create a cafeteria which is 1/4 of an 8 by 5 inches garden and argue "*If we split this into four parts, then one of the parts will be the cafeteria. It would be 2 inches [the height of the* *cafeteria] because the if we use only 1-inch roller it would go 8 times across but if you use 2-inch roller then it would go 1,2,3, and that would go 4 parts."*

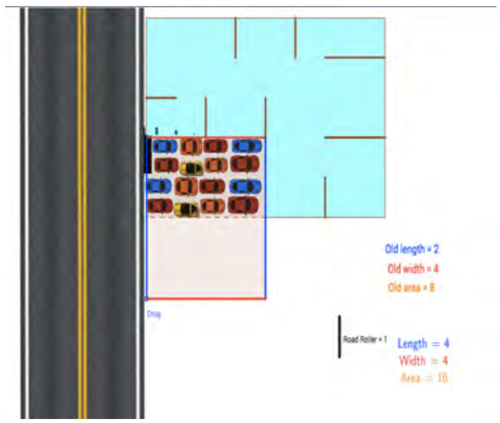


Figure 5. How can you make the parking place twice as big as it is now?

5. Identifying Area as a Multiple of its Dimensions

The goal of this level is for students to recognize area as a multiple of its dimensions and identify factors that give the same area. Our tasks involve asking students to create different rectangles of the same area (e.g. 12 sq. units) (Figure 6). This connects area measurement to geometry and the concept of congruence by recognizing congruent shapes in different orientations (e.g. 2 x 6 or 6 x 2) and describing congruence by using geometric motions such as rotation (Huang & Witz, 2011). It also directly relates to the properties of multiplication (e.g. commutative property) as well as factors and multiples. Students' generalizations include, "*length 4 and width 3 is doing 4 swipes of 3. This is same as two swipes of 6, so length 2 and width 6."* After students create all the rectangles, we ask them which rectangle has more space:

Researcher: Which one has more space?
Student 1: Everything is equal. Everything.
Student 2: The space is the same. The lengths and widths are different.

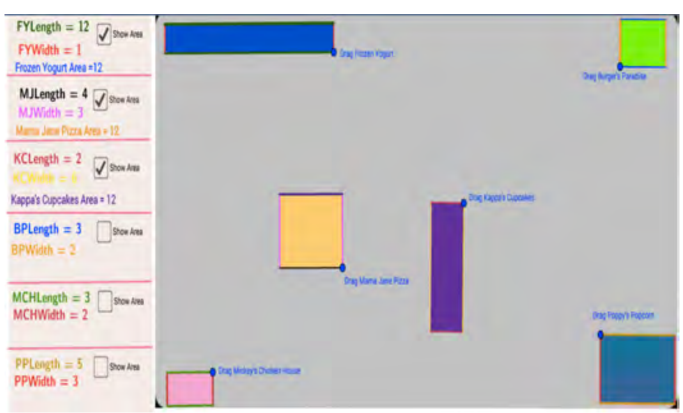


Figure 6. Each store should have an area of 12 sq. meters and different length and width from the other stores.

6. Coordination of Relative Areas

The goal of the final level is for students to coordinate relative areas. The task engages students in creating a robot with a fixed area (e.g. no bigger than 190 cm^2) by composing and decomposing shapes of fixed lengths and widths (Figure 7). As part of the task, students need to find the area of each leg (right triangle), by recognizing that if the leg is half the rectangle, then its area is half of the area of the rectangle. For instance, for calculating the area of each blue leg, students argued:

Student 1: You have to do a half of 7 and 2. We do 7 times 2 and a half of that.
Student 2: Both of them are 7 and both of them together are 14.

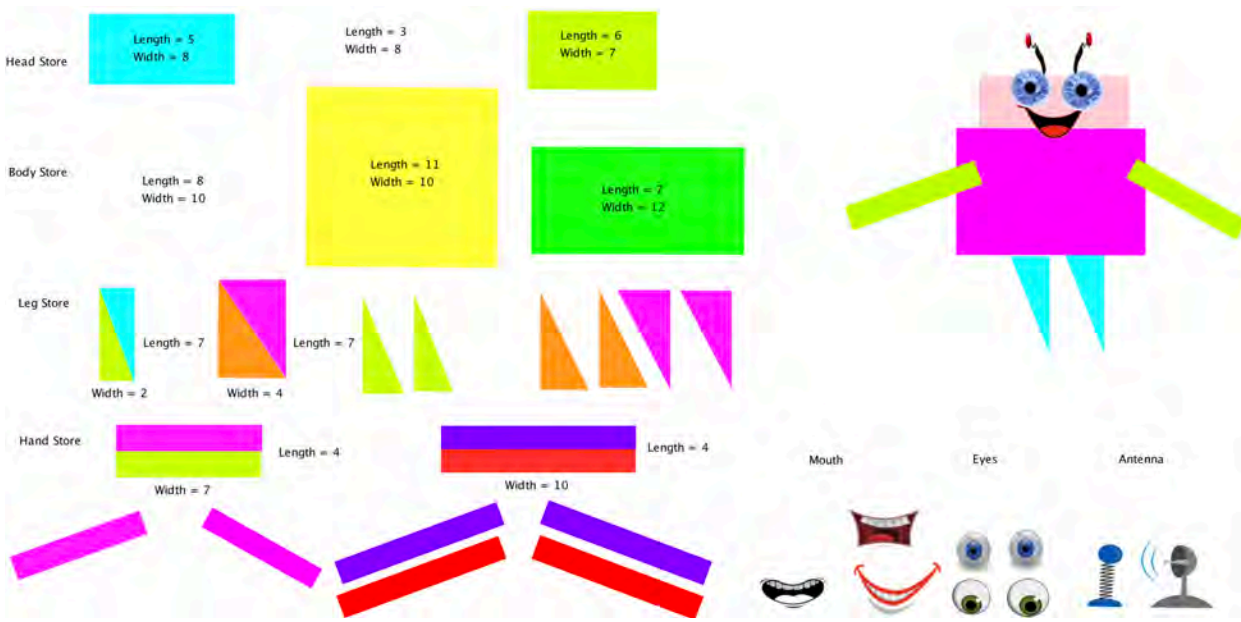


Figure 7. Make the robot as fancy as you like but its area should be no more than 190 sq. inches.

Concluding Remarks

This is an exploratory study examining a dynamic way of learning and teaching measurement. The DE findings show DYME's potential as a route to area measurement that would make the multiplicative relationship of the area formula more intuitive and accessible. The DEs helped in the design of the DYME LT, which is a conjecture of how students' DYME reasoning may evolve in the context of the specific learning activities. The LT shows that DYME lies at the crossroad of multiple mathematical ideas such as multiplication, division, fractions, (shape and unit) transformations, and covariation. Among our future goals is to explore these connections further as well as to examine how the DYME approach could complement the existing *rows* x *columns* structuring approach that is emphasized in research and schools.

Additionally, we are currently preparing a DE for a whole class to evaluate the effectiveness of learning the DYME concepts in a real classroom environment, and designing pre- and post-assessment items to evaluate students' thinking of area (as it develops) and validate the LT. We consider our findings to be very important for initiating a discussion around dynamic measurement and how it can be used for developing a conceptual understanding of area.

Acknowledgments

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ELEMENTARY STUDENTS' REASONING ABOUT ANGLE CONSTRUCTIONS

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In this report, we discuss the findings from 2 pilot studies investigating the effects of interventions designed to provide students in Grades 3–5 with opportunities to work with dynamic and static models of angles in a dynamic geometry environment. We discuss the effects of the interventions on the children's development of quantitative reasoning about angle measure.

Keywords: Elementary School Education, Geometry and Geometrical and Spatial Thinking, Measurement, Technology

Geometric measurement is a branch of mathematics that integrates number and space and includes length, area, volume, and angle measurement. Based on Barrett and Smith's review of the literature in the measurement chapter of the research compendium, Smith (2016) asserted that they found markedly less research on angle measurement in comparison to the other three forms of geometric measurement (length, area, and volume). The present research on angle measurement focuses on student difficulties with several measurement concepts, including identifying and attending to the correct attribute of angle as well as angle unit, unit iteration, and origin. Specifically, students struggle to identify what is being measured when referring to size of angles and what is one degree (Keiser, 2004). Often elementary and middle school students attend to ray lengths (e.g., Clements, 2003; Keiser, 2004), and elementary, middle, and high school students are distracted by angle orientation (Mitchelmore, 1998; Noss, 1987; Fyhn, 2008). Mitchelmore (1998) argued that students need opportunities to work with both dynamic (the motion of an angle opening) and static (the resultant figure after the opening) angle models to confront their misconceptions.

Previous research has incorporated both models. Several previous studies utilized the LOGO environment (e.g., Clements & Burns, 2000; Noss, 1987; Simmons & Cope, 1993) or used a sequence of static models to indicate motion to varying degrees of success.[i] In one study, Clements, Battista, Sarama, and Swaminathan (1996) explored third grade students' understanding of angle measurement in a modified LOGO environment. These researchers found that immediate feedback helped students reflect on their turn commands and thus angle measurement (cf. Simmons & Cope, 1993) and that using benchmarks helped students assign numbers to turns. However, because there was no record of the turn during the turn in Clements et al.'s modified LOGO environment, turn commands were less salient to their students than forward or backward commands, which could be because the forward and backward commands leave a line segment as a trace, whereas there is not a similar record with turns.

Mathematics education has yet to fully determine how to address the misconceptions present in the literature since the 1980s. The authors of the Common Core State Standards in Mathematics (CCSSM, National Governor's Association for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) have renewed mathematics educators' interest in elementary students' conceptions of angle and angle measurement through their definition of angle and mandate for how angle should be understood by fourth grade:

> An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles. (p. 31)

To promote this interpretation of angle and extend the literature on angle, we designed interventions. Both were enacted in a dynamic geometry environment utilizing the computer software, Geometer's Sketchpad, to provide children with opportunities to work with movable angle situations as well as reflect on dynamic (the motion of an angle sweeping open) and static (the resulting image of an angle after sweeping open) angle models. In this report, we discuss the results from our testing of these interventions. We posed the research question: *In what ways does interacting with dynamic and static angle models affect students' reasoning about angle constructions in Grades 3, 4, and 5?*

Theoretical Perspective

We approached this study from a quantitative reasoning approach. According to Thompson (1990), "a quantity is a quality of something that one has conceived as admitting some measurement process. Part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit" (p. 5). In the case of angle measure, quantifying involves reasoning about the unit (e.g., a degree) in terms of a quantitative relationship (i.e., a multiplicative relationship) between a fraction of the circular arc of a circle and the circle's circumference, which is consistent with the CCSSM authors' (NGA & CCSSO, 2010) recommendations for how fourth grade students should understand angle measure.

There are three types of quantity—gross, extensive, and intensive (Piaget, 1965). Gross quantity depends on perception—one object is bigger, smaller, more, less, or the same as another in terms of some attribute. Extensive quantity is additive (Piaget, 1965) and the result of unitizing activity (Steffe, 1991), whereas intensive quantity is not additive (Piaget, 1965). Instead, it requires proportional reasoning. For example, if Person A is traveling 20 mph and Person B is traveling 40 mph, we are not traveling 60 mph. We used a quantitative reasoning approach in our design of the interventions as well as in our interpretation of the findings.

Method

To investigate dynamic and static angle models affected students' reasoning about angle constructions, we wanted to observe and document changes. Thus, we utilized the microgenetic method (Siegler & Svetina, 2006). The microgenetic method has three main tenets:

> (1) observations span the whole period of rapidly changing competence; (2) the density of observation within this period is high, relative to the rate of change; and (3) observations of changing performance are analyzed intensively to indicate the processes that give rise to them. (Siegler & Svetina, 2006, p. 1000)

The data presented in this report were collected during the 2014–2015 school year at a suburban public school in the Midwestern region of the United States. In the first pilot study, we interviewed 18 students in Grades 3–5 (ages 8–12), six students per grade. For the second pilot study, we interviewed 19 students in Grade 3 (ages 8–9). Each student participated in three 4 to 18 minute individual interviews with one of the three authors of this report during the normal school day. We used a structured interview protocol and recorded the interviews using screen-capturing software, Screencast-o-matic, which also records audio. Consistent with the microgenetic method, observations were dense. The three interviews occurred on three separate days, and the mean elapsed time between first and third session was 2.9 school days (max of 5 school days). Prior to the first interview and after the third, children took a written survey. On this survey, children were asked to give a definition of angle, estimate the measure of a given angle, and construct an angle. On six items, children were asked to select one out of three angles that (a) had a specified measure, (b) had the largest measure, or (c) had the smallest measure.

Pilot 1

During the first interview, we guided each student through a tutorial on how to use four sliders in a dynamic geometry environment (i.e., Geometer's Sketchpad). Each of the sliders had a different effect. One slider opened and closed the angle, one translated the image left and right, one rotated the image of the angle, and one lengthened and shorted the rays.

During the second interview, the student went through eight trials, which we define to be a task-intervention pair (Siegler & Crowley, 1991). Specifically, the student was asked to construct an angle of a specified measure and to let the interviewer know when he or she was ready to check. During the check, the researcher clicked a check button and a ray swept from the initial ray to the angle of the terminal ray of the desired angle. In addition, benchmark rays appeared at each 30-degree interval until the ray stopped at the terminal side of the angle (see Figure 1). Note that the ray left a fading trace as it swept across the screen to provide a record of the turn (cf. Clements et al., 1996). The measure of the angle the students constructed was also briefly displayed, providing them with feedback. This feedback allowed them to compare the size of two angles (the created and desired) to their associated measure as well as to compare the difference in the created and desired angle. For example, in the sequence below a student would have had the opportunity to see a 120-degree angle, a 128-degree angle, and to see that small difference between the two was about eight degrees. During the third interview, the student went through nine trials with the same design principles.

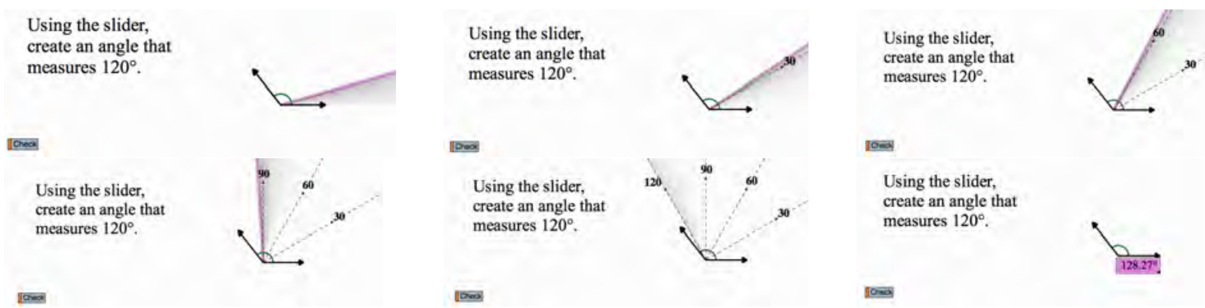

Figure 1. A sequence of screen shots displaying the labeled benchmark rays (e.g., 30, 60, 90) appearing as the student checked her attempt at a 120-degree angle.

Pilot 2

During the first interview, we asked the student to use one slider to create an angle of a specified measure on eight trials. This process was repeated for nine trials during the second interview and nine trials during the third interview. For the construction of all 26 angles, the student started with one initial horizontal ray and one slider that opened and closed the angle. For each angle, the interviewer told the student to use the slider to create an angle of a specified measure.

The 19 participants were divided between the two intervention groups: 10 were in Intervention Group 1 (IG1) and nine were in Intervention Group 2 (IG2). For children in IG1, during alternating trials in Interviews 1 and 2, unlabeled benchmark rays of 30 degrees would appear as the child used the slider to open the angle (see Figure 2). During the trials in Interview 3, the benchmark rays did not appear during any of the trials. When the child indicated that he or she was ready to check, the check button was clicked, and a ray swept from the initial ray of the angle to the terminal ray of the angle the child had been instructed to create. The main components of Pilot 2 IG1 were a subset of those in Pilot 1: Benchmark rays appeared at each 30-degree interval until the ray stopped at the terminal side of the desired angle. The measure of the angle the child constructed was briefly displayed for the interviewer to record (see Figure 1). In contrast, for children in IG2, benchmark rays appeared neither during construction nor during the check. Instead, when the check button was clicked, only the measure of the angle the child constructed was displayed. Figure 3 summaries the key components of the two pilot studies.

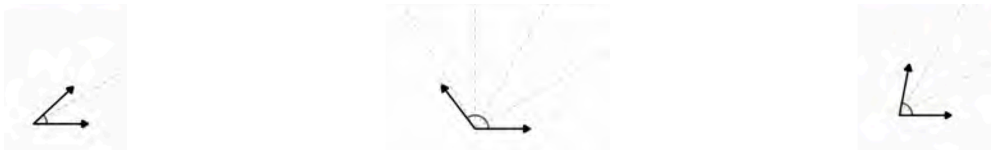

Figure 2. A sequence of screen shots displaying the 30-degree benchmark rays appearing for students in Pilot 1 and Pilot 2 IG1 as the terminal ray of the angle swept open.

Pilot 1	Pilot 2, Group 1	Pilot 2, Group 2
Grades 3–5	Grade 3	Grade 3
18 participants (6 per grade)	10 participants	9 participants
3 interviews	3 interviews	3 interviews
17 trials	26 trials	26 trials
4 sliders	1 slider	1 slider
Alternating trials, unlabeled benchmark rays of 30 appeared as created angle (*)	Alternating trials, unlabeled benchmark rays of 30 appeared as created angle (*)	NA
Trial 9, 90 Trial 10, 80* Trial 11, 30 Trial 12, 40* Trial 13, 60 Trial 14, 70* Trial 15, 120 Trial 16, 110*	Trial 9, 90 Trial 10, 80* Trial 11, 30 Trial 12, 40* Trial 13, 60 Trial 14, 70* Trial 15, 120 Trial 16, 110*	Trial 9, 90 Trial 10, 80 Trial 11, 30 Trial 12, 40 Trial 13, 60 Trial 14, 70 Trial 15, 120 Trial 16, 110
After check button clicked, a ray swept from the initial ray of the angle to the terminal ray of the desired angle, and that ray left a trace.	After check button clicked, a ray swept from the initial ray of the angle to the terminal ray of the desired angle, and that ray left a trace.	NA
After ray swept from initial ray to the terminal ray of the desired angle, the measure of constructed angle appeared.	After ray swept from initial ray to the terminal ray of the desired angle, the measure of constructed angle appeared.	After check button clicked, the measure of constructed angle appeared.

Figure 3. Comparison of the key components of the two pilot studies.

In the design of both pilots (including Pilot 2 IG2), we privileged approximations of 10 degrees. During Interview 2 (Trials 9–16), we sequenced trials to pair benchmark angles with near benchmark angles (i.e., angles measuring 10 degrees more or less than one of the bench mark angles). There were four sets of paired trials (see Figure 4) to provide children with experiences that would support their development of a sense of 10 degrees. (It was our conjecture that developing a sense of 1 degree would be more difficult than developing a sense of 10 degrees.)

Trial 9, 90 degrees	Trial 10, 80 degrees
Trial 11, 30 degrees	Trial 12, 40 degrees
Trial 13, 60 degrees	Trial 14, 70 degrees
Trial 15, 120 degrees	Trial 16, 110 degrees

Figure 4. Four sets of paired trials.

We also privileged the integration of number and space. At the end of each trial, we displayed the measure of the constructed angle, providing students with the opportunity to pair the image of the constructed angle with its measure. For students in Pilot 1 and Pilot 2 IG1, this was taken further because of how the terminal ray of the angle swept open, providing students with the opportunity to pair the image of the intended angle with its measure.

In the design of Pilot 1 and Pilot 2 IG1, we privileged 30-degree benchmarks by displaying unlabeled benchmark rays at 30-degree intervals during construction on alternating trials. Additionally, the check on every trial displayed 30-degree benchmark rays with labels (e.g., 30, 60, 90). Our purpose for including these supports was to encourage students to reason about, use, and operate on specific benchmark angles as well as to encourage the integration of number and space.

In the design of Pilot 2 IG2, we privileged only feedback. Upon clicking the check button, the measure of the angle the child constructed on a given trial was displayed. This provided the children with the opportunity to reflect upon how the measure of the angle they constructed compared to the desired angle measure (cf. Jaehnig & Miller, 2007).

Discussion of Findings

Our findings indicate that the interventions had an effect on students' angle constructions. In this section we provide a representative sample of quotes to illustrate each of our three main findings.

First, we found that interacting with dynamic and static angle models increased third and fourth grade students' recognition and use of 30-degree benchmarks. Most of the third and fourth grade students did not mention the 30-degree benchmarks during the first eight trials, but by Trial 9, many of these young students in Pilot 1 and Pilot 2 IG1 were referring to them when they described what they were thinking about during or after angle construction.

- Trial 10, 80: "[While creating] 30, 60, that's about 80. [After check] I noticed that I was right, because since the 90 degree angle is um, is up I kind of noticed that if its 80 you might need to make it a little more slanted." (Bob, Grade 4, Pilot 1)
- Trial 10, 80: "I was, um, pretty close. [What were you thinking about when you made that?] Uh, well, it showed us the lines. So I knew it- one was 30, one was 60, and I knew it wouldn't be all the way– all the way to 90 but it's be a little less than 90 but a little more than 60." (Erin, Grade 3, Pilot 2 IG1)
- Trial 15, 120: "I was counting by 30s by like 30, 60, 90." (Charlie, Grade 3, Pilot 2 IG1)

Second, we found that by asking students what they noticed after the check button was clicked as part of our interview protocol, we prompted students to compare the desired and constructed angles numerically and spatially. Although this occurred for students in Pilot 2 IG2, it was more pronounced for students in Pilot 1 and Pilot 2 IG1. To illustrate, we provide four quotes below. We found the first two (from students in Pilot 1 and Pilot 2 IG1) to be qualitatively different from the last two (from student in Pilot 2 IG2).

- Trial 10, 80: "Well, I remember how last time, um, if I- I moved a lot but it was only just this much to get ta- just um like a tiny bit to get to like 74 from 70, [Mmm so 70] like 71 or something, so I didn't want to go too far from 60 because I know that's about 20 away." (Genny, Grade 3, Pilot 2 IG1)
- Trial 14, 70: "I had to make it just a little bit smaller in order to make it 70 degrees and like down here where I tried to make a 40 degree angle and I gave it too much space." (Frank, Grade 5, Pilot 1)
- Trial 11, 30: "28 degrees. [What do you think?] Good."(Pam, Grade 3, Pilot 2 IG2)
- Trial 12, 40: "35. 5 off. That's good still." (Billy, Grade 3, Pilot 2 IG2)

Third, we found that most students appeared to benefit from paired trials. Some students reasoned about a unit of 10 degrees and fractional parts of the benchmark created wedge. These students' quantitative reasoning was a bit more advanced than the students who could only reason about a little more or a little less than the benchmark rays, as illustrated by their quotes below in which they utilized specific numeric relationships (e.g., difference between the measure of the benchmark angle and the measure of the desired angle).

- Trial 14, 70: "So like 90, so that like is 60,…okay. Um, 'cuz 70 is, well obviously, uh 10 degrees larger than 60." (Amber, Grade 5, Pilot 1, created 71.97 degree angle)
- Trial 17, 150: "That, um, like the, um, the arrow, it went a little bit past it because, um, 120 is only 30 away and it seems like 30's long but it's not." (Eric, Grade 3, Pilot 1, created 139.01 degree angle)

One benefit of the microgenetic method is to have dense observations during a change to document that change. Hence, after we documented some shifts, we dug deeper to explore how one student's explanations and actions changed from trial to trial. Oscar, a fourth grade student in Pilot 1, exhibited improvement in his approximations for 10 degrees across the four sets of paired trials and clearly articulated how he learned from the previous trials. After constructing an angle that measured 86.36 degrees (for a desired angle measure of 80 degrees), Oscar explained, "I needed to go a little further back. I put, I was trying to measure about a little before the 90-degree angle." On this trial (Trial 10), Oscar correctly reasoned quantitatively about the benchmark angle measure and the desired angle measure (i.e., that 80 degrees is less than 90 degrees); however, he did not identify how much less (i.e., difference between the measure of the benchmark angle and the measure of the desired angle). On the next off-benchmark trial (Trial 12), Oscar constructed an angle that measured 47.84 degrees (for a desired angle measure of 40 degrees). When asked what he was thinking about when constructed his angle, he said, "It goes just a few shades away from the 30 degree angle, mmm, not that much." Although Oscar's constructed angle and explanation indicate he knew that a 40-degree angle was larger than a 30-degree angle, his reference to "not that much" supports an inference that he realized his attempt to create an angle that was 10 degrees more than the benchmark angle was too large. We take his explanation as evidence that Oscar had identified a 10-degree angle as a mental unit, and we predicted that his next attempt to create an angle 10 degrees more than the benchmark angle would be improved.

Based on these prior experiences, Oscar made an adjustment to his mental unit of 10-degrees on the next two off-benchmark trials (Trials 14 and 16). On Trial 14, Oscar constructed a 69.85-degree angle for a desired angle measure of 70 degrees. After checking the measure of his constructed angle, he said,

> It's like the other question...When it was asking the 40-degree, um, I went a little far; I was right there so then I kinda realized on a few more problems it was about like 80 and stuff like that—that I didn't need to go as far as I was going.

We take this as evidence that Oscar is reflecting on relating number to space—that is, his image of what a 10-degree angle looks like was too large in previous trials (Trials 10 and 12), so he had to make it smaller on this trial.

On the last off-benchmark trial (Trial 16), Oscar constructed a 110.91-degree angle for a desired angle of 110 degrees. When asked to explain his thinking before getting feedback, he said,

> I think this one is going to be right 'cuz…90 degrees. Because um, kinda of like the 60 degree and stuff, I knew that I had to go a little short, shorter because it was only 10 degrees higher than, I mean the 40 degrees… I only need it to go about right there. So then on the 90 degrees it was 20 degrees um higher than the 90 degrees so I went probably about another half of the way, right there [points to location on the screen with the cursor].

Oscar's explanation indicates that he was again reflecting on what a 10-degree angle looks like. However, how he used that 10-degree angle as a unit can be interpreted in two different ways. One interpretation is that he thought about a 20-degree angle as composed of two 10-degree angles—10 more than 90 degrees and then 10 more than that. Another interpretation is that he thought about a 30-degree angle as 10-degrees and 20-degrees and then a 10-degree angle as half of a 20-degree angle—10 more than 90 degrees (100 degrees) and then half way between 120 and 100 is 110 (i.e., "about another half of the way"). Although we expected to see Oscar use the benchmark angle of 120 degrees and think about 10 degrees less than the 120-degree angle, Oscar showed flexibility in his thinking by starting with the benchmark angle of 90 degrees to approximate a 110-degree angle. Regardless, Oscar exhibited improvement in his approximations for 10 degrees and his reasoning about angles.

Educational Importance of the Research

The experimental interventions enacted in Geometer's Sketchpad were designed to provide opportunities for students to engage with dynamic and static angle models. The results from the two pilot studies extend the previous literature on children's reasoning about angles and angle measurement. Specifically, our results suggest providing children with opportunities to reason about angles as multiples of 30 (e.g., a 120-degrees angle as four 30-degree angles) and as partitionings of 30s (e.g., a 40-degree angle as a 30-degree angle plus one-third of another 30-degree angle) has the potential to support children's recognition and use of 30-degree benchmarks. Our study also indicates that asking students what they noticed after receiving feedback on their angle constructions (after the check button was clicked) prompted students to compare the desired and constructed angles numerically and spatially.

Although we do not have evidence that the participants in this study were able to reason about angle measure by comparing the fraction of the circular arc and the circle's circumference, we do have evidence that the students were constructing their own knowledge and adapting their thinking. As illustrated by Oscar's explanations discussed in the results section, Oscar and others were adapting their thinking based on their experiences with the interventions. Specifically, Oscar's repeated attempts to approximate 10 degrees when constructing angles that were designed to be 10 more or 10 less than a benchmark angle helped him adapt this thinking and connect his numerical reasoning (i.e., 10 more degrees) to his spatial reasoning (i.e., what 10 more degrees looks like). Hence, his perception and interpretation of his experiences on these trials allowed him to describe an angle as something that can be quantified. The paired trials appeared to be an important component of the interventions in that it gave students opportunities to receive feedback on their quantitative reasoning. Thus we recommend future studies include more paired trials of 10 more or 10 less than a benchmark angle, interviewing more students from each grade level, including more trials, and parsing out the differences between groups of students with the benchmark rays and without the benchmark rays.

Endnotes

[i] Although other researchers have considered the sequencing of static images to be dynamic angle situations (e.g., Clements, Battista, Sarama, & Swaminathan, 1996; Devichi & Munier, 2013), we argue that to be truly dynamic, the sequencing of these static images needs to be more continuous.

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HOW SPATIAL REASONING AND NUMERICAL REASONING ARE RELATED IN GEOMETRIC MEASUREMENT

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The positive correlation between spatial ability and mathematical ability has been well-documented, but not well-understood. Examining student work in spatial situations that require numerical operations provides us with insight into this elusive connection. Drawing on student work with angle, length, volume, and area, we examine the ways in which students associate numerical operations with their spatial structurings of objects. We find that for students to correctly coordinate their spatial structurings and numerical operations, their solution methods must satisfy basic properties of measurement functions. We illustrate this claim by providing examples in which students successfully and unsuccessfully employ spatial-numerical linked structurings.

Keywords: Cognition, Geometry and Geometrical and Spatial Thinking, Technology

Numerous studies have found that spatial ability and mathematical ability are positively correlated (Mix et al., 2016). But specifying the exact nature of the connection between these abilities has been elusive, with much research in this area focused on understanding correlations between specific spatial skills (e.g., as measured by visualization and form perception tests) and mathematical performance (Mix et al., 2016). In this paper, we seek to precisely specify the spatial-mathematical connections in geometric measurement—a content area for which numerical and spatial reasoning must be properly coordinated. Indeed, de Hevia and Spelke claim that the human mind is "predisposed to treat number and space as related" (2010, p. 659). And researchers in mathematics education argue that understanding relationships between numerical and spatial reasoning is fundamental to developing a full understanding of geometric and measurement reasoning (Clements & Battista, 1992). However, although a great deal of research has investigated how students represent numbers on number lines (Gunderson, et al., 2012), in geometric measurement, numerous situations arise that are more complex than envisioning numbers on number lines. We have investigated numerous instances of these more complex situations, and in this paper, we analyze these situations to more fully understand the nature and properties of the connection between numerical and spatial reasoning in geometric measurement.

Theoretical Framework

Measurement Properties

For spatial reasoning and numerical reasoning to be properly connected in geometric measurement, certain basic properties of measurement functions must be followed, as described by Krantz: “When measuring some attribute of a class of objects or events, we associate numbers … with objects in such a way that properties of the attribute are faithfully represented as numerical properties” (1971, p. 1). That is, if *M* is the function that assigns measurement values to objects—*M(a)* is the measure of object *a*—then, consistent with Krantz et al. and basic axioms for geometric measurement (Moise, 1963), *M* satisfies the following properties:

1. If object *a* and object *b* are congruent, then $M(a) = M(b)$.
2. Object *a* is spatially larger than object *b* if and only if $M(a) > M(b)$.
3. If we join-without-intersection object *a* and object *b*, then $M(a \;\; join \;\; b) = M(a) + M(b)$.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

4. Given n copies of congruent and non-overlapping unit-measure objects a_1 ... a_n:

$$If \bigcup_{i=1}^{n} a_i \cong b, then\ nM(a_1) = M(b).$$

These properties justify the measurement iteration process in which we determine measure by iterating a unit measure to "cover" the object being measured with no gaps or overlaps. If, however, there are gaps in a unit-measure covering so that it is a proper subset of the object being measured, then Property 2 implies that the measure of the covering will be less than the measure of the object. If there are overlaps, then Property 3 is not satisfied, so we cannot count/add the unit measures to find the measure of the object.

Spatial-Numerical Linked Structuring

Beyond the basic measurement properties, linking spatial and numerical reasoning in geometric measurement requires use of what we call *spatial-numerical linked structuring* (SNLS). *Spatial structuring* is the mental act of constructing a spatial organization or form for an object or set of objects, imagined or real (Battista, 1999, 2007, 2008; Battista et al., 1998; Battista & Clements, 1996). *Numerical structuring* is the mental act of constructing an organization or form for a set of computations. A correct *spatial-numerical linked structuring* is a coordinated process in which a numerical measurement operation is performed along with a linked spatial structuring in a way that is consistent with the above measurement properties. Incorrect student enumeration is generally based on SNLS that violates at least one of the measurement properties. Note that each measurement property expresses a spatial-numerical linked structuring. For instance, putting one angle inside another to decide which is bigger spatially organizes the two angles with respect to each other. In this paper, we give examples of correct and incorrect spatial-numerical linked structuring.

Methods and Data Sources

The data we analyze comes from individual interviews and one-on-one teaching experiments with elementary and middle school students from several NSF-funded geometry projects awarded to the first author.

Sample Results and Discussion

To illustrate our results and analysis, we provide examples of spatial-numerical linked structuring for angle, length, volume and area. The examples (a) describe student actions, (b) discuss what students did, and (c) interpret what students did using the spatial-numerical linked structuring conceptual framework.

Spatial-Numerical Linked Structuring for Angles

For the computer-presented problem in Figure 1, KS employed several spatial-numerical linked structurings to find the angle that rotates the green point onto the red point.

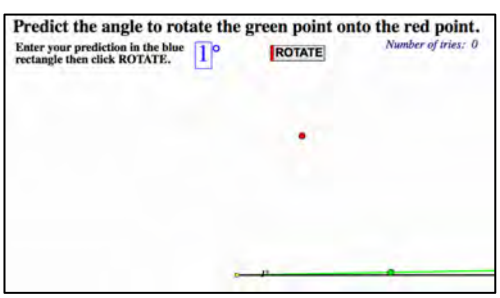


Figure 1.

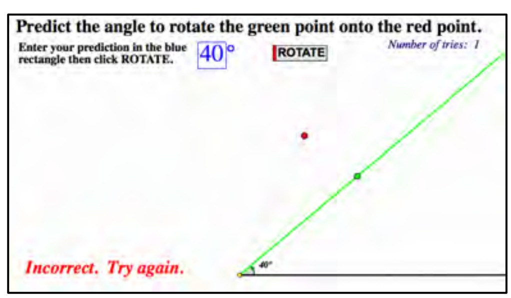


Figure 2.

KS: I think it may be 40 [enters 40; green ray rotates to the 40° position Figure 2].
Int: So what are you thinking?
KS: So if this is 40 [angle in Figure 2], I may have to go up maybe *20 more*.
Int: Okay, why 20 more?
KS: Cause, if this was 40 [pointing at the interior of the green 40° angle], then half of it is this [pointing to the interior of the angle between 40° and the target angle; enters 60°; Figure 3].

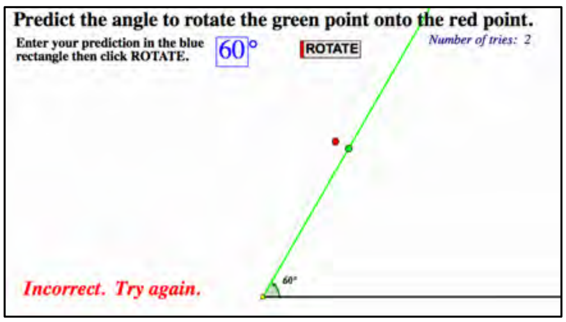


Figure 3.

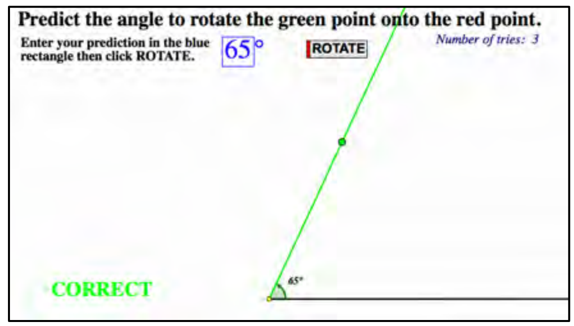
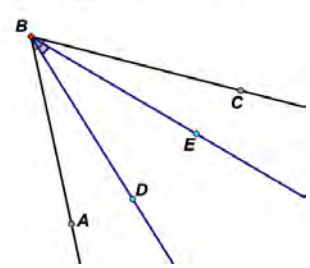


Figure 4.

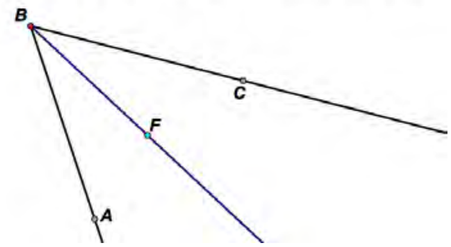
Int: Very close, what are you thinking?
KS: Hum. So maybe with the other [computer page showing 5° iterations of a ray] it shows that they were really close together, so maybe it'd be 65 [enters 65°; Figure 4].

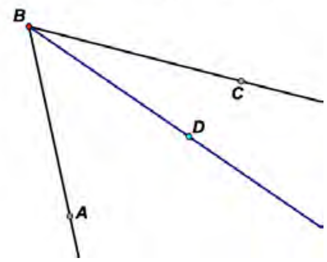
Discussion. KS used a sequence of spatial-numerical linked structurings (SNLSs) for solving this problem (see Table 1). After viewing the result of her first estimate, which is quite a bit off, KS reasoned that her original estimate was too small. This is an example of SNLS 1, in which KS recognized the smaller-than spatial relationship between the angle she made and the target angle. KS then, using SNLS 2, spatially compared the angle between the green ray and the black ray from her 40° estimate as half of the 40° angle. Then, using SNLS 3, she added 20° to 40° to produce a second estimate of 60°. Finally, in her third estimate, KS used SNLS 4 followed by SNLS 3 to recall a previously viewed 5° angle and add that to her estimate of 60°.

Table 1: Definitions of Types of Angle Spatial-Numerical Linked Structuring

SNLS 1. [Bigger Angle ⇔ Greater Measure; Property 2]: If Angle X is bigger than Angle Y, then the measure of Angle X is greater than the measure of Angle Y. Definition/Spatial Structuring: Angle X is spatially "bigger" than Angle Y if the angles have the same vertex and Angle Y fits in the interior of Angle X.	**∠ABC is bigger than ∠DBE.** 

<table>
<tr><td>SNLS 2. [One-half angle ⇔ One-half measure] If Angle X is one-half of Angle Y, then the measure of Angle X equals one-half of the measure of Angle Y.

Definition: Angle X is one-half of Angle Y if the initial side of Angle X coincides with the initial side of Angle Y and the terminal side of Angle X is in the interior of Angle Y and is halfway toward the terminal side of Angle Y.</td><td>∠ABF is half of ∠ABC. ∠CBF is half of ∠ABC.

</td></tr>
<tr><td>SNLS 3. [Add angles ⇔ Add measures; Property 3] If Angle X equals Angle Y plus Angle Z, then the measure of Angle X equals the measure of Angle Y plus the measure of Angle Z.

Definition: If point D is in the interior of Angle ABC, then Angle ABC equals Angle ABD "plus" [joined with] Angle DBC.</td><td>∠ABC equals ∠ABD plus ∠DBC.

</td></tr>
<tr><td colspan="2">SNLS 4. [Compare perceived angle to recalled angle; Property 1] The student compares a perceived angle to the recalled visual image of a previously seen angle, and says that the two angles are congruent so their measures are equal.</td></tr>
</table>

Spatial-Numerical Linked Structuring for Length

To examine the way students use spatial-numerical linked structuring with length, we consider a student's work in a computer golf game (Figure 5). Students "putt" a ball by entering a distance and angle. When students click the *PUTT* button, the ball travels to the right the entered distance, then arcs around counterclockwise as it sweeps out the entered angle, which is a multiple of 5° (Figure 6). Students receive visual feedback on each of their estimates until they determine a correct putt angle and distance. SJ is doing the problem in Figure 5.

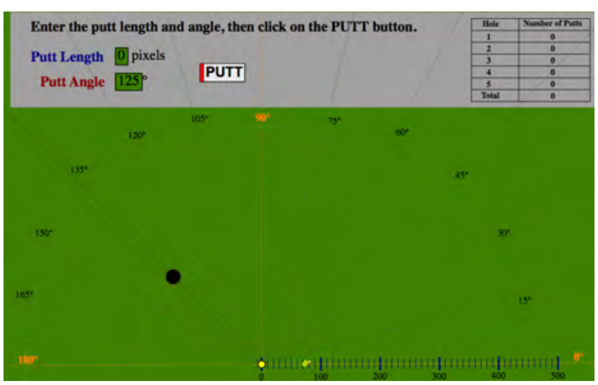


Figure 5.

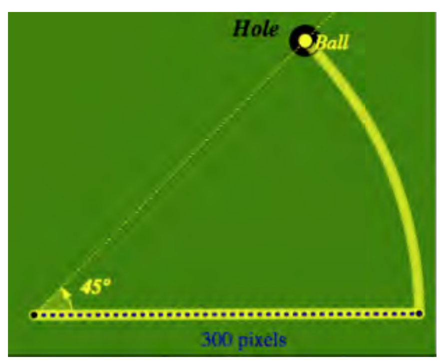


Figure 6.

Example 1:

SJ: [Pointing along hash marks 0-140 on the number line with the cursor] These lines are the pixels right?

Int: Yep. So this [pointing with a finger] is 100 pixels. That's 200 [pointing]. So they might be counting by, what do you think, in those little ones [points to hash marks between 100 and 200 on the number line]?

SJ: 25s?

Int: So let's see. If this is 100 [pointing to 100]. That'd be 125 [pointing to 110], 150 [pointing to 120], 175 [pointing to 130], 200 [pointing to 140].

SJ: Aw, never mind.

Int: So what do you think?

SJ: 10, 15, [points along hash marks 110 to 190 on the number line] 45. No [goes back to 110 on the number line]. *Oh, tens!*

Discussion: SJ understood that each hash mark represents the same amount of space (Measurement Property 1), but she could not immediately determine the correct numerical value for the distance associated with the space between each hash mark. When she estimated 25 as the distance, the Interviewer iterated by 25 starting at the landmark for 100 so that SJ recognized that the numerical value of 25 for each hash mark was too large. After choosing a smaller value of 5 and realizing it was too small, she correctly concluded that 10 was the distance between each hash mark. This is an example of a student using the measurement properties and the iteration spatial-numerical linked structuring to develop an understanding of the coordinate system inscription embedded in the game. The mistakes she made with the 25 and 5 estimates for hash-mark values seem to arise from not coordinating the number iterations of the hash-mark interval with the beginning and end values of the 100-to-200 interval. In essence, she violated Measurement Property 4.

Example 2:

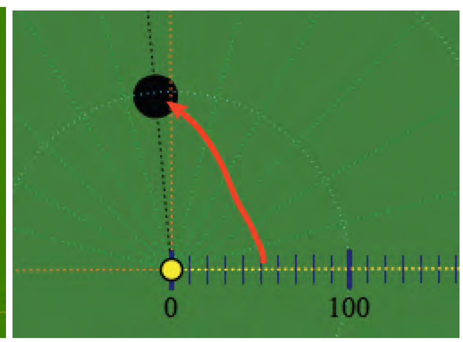


Figure 7. **Figure 8.**

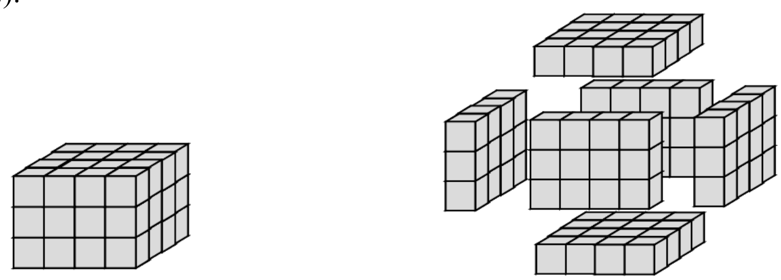
SJ: [For the problem in Figure 7] Okay. This one is probably going to be 50 [points the cursor at 50 on the number line]. Because like 10, 20, 30, 40, 50 [counting on the 10-50 hash marks with the cursor]. Here's the 50 [moves from 50 towards the hole; Figure 8]. Maybe even 60.

Discussion: In this example, SJ's spatial structuring of the rotation path of the ball is incorrect. Because of this incorrect spatial structuring, her numerical choice for the length of the putt was incorrect—her spatial structuring violated Measurement Property 2. Importantly, note that SJ does not understand the meaning of the distance-arc inscriptions for the embedded coordinate system. Because of her incorrect structuring of a point rotation, she does not recognize that every point on a distance arc is the same distance from the origin as the reference measurement on the number line.

Similar to many elementary students using rectangular coordinates (Battista, 2007; Sarama et al., 2003), SJ does not conceptualize the spatial-structural metric properties of the coordinate system. In order to accurately recall spatial relations, students must abstract not pictures but mental models that have encoded spatial properties of objects (e.g., Hegarty & Kozhevnikov, 1999).

Spatial-Numerical Linked Structuring for Volume

In the complex context of enumerating unit cubes in rectangular boxes, students must link their numeric structuring to their spatial structuring. For instance, consider the following example (Battista, 2004, 2012).

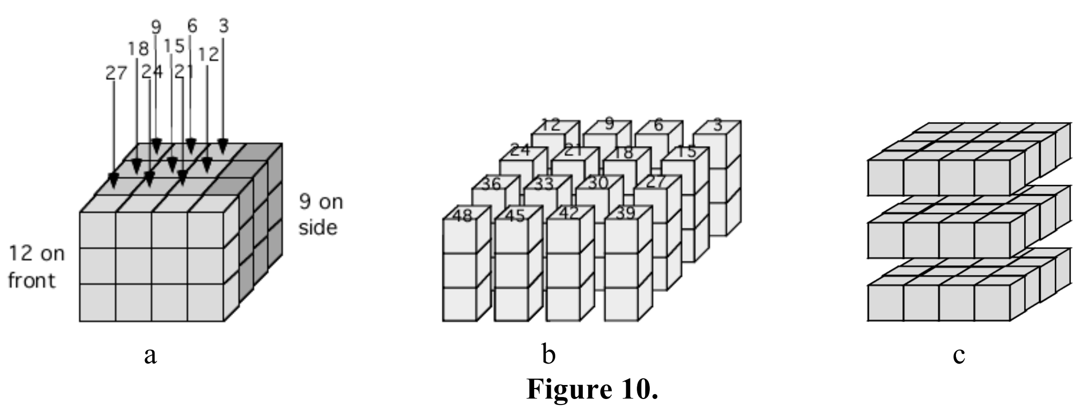


a b

Figure 9.

For the building shown in 9a, Fred counted based on the spatial structuring shown in 9b. He said that there are 12 cubes on the front, then immediately said there must be 12 on the back; he counted 16 on the top, and immediately said there must be 16 on the bottom; finally, he counted 12 cubes on the right side, then immediately said there must be 12 on the left side. He then added these numbers. Fred's numerical structuring of *12 + 12 + 16 + 16 + 12 + 12* corresponded to his spatial structuring of *(front + back) + (top + bottom) + (right side + left side)*. So his spatial structuring of the building into composite units of cubes violated Measurement Property 3—the cubes that he double-counted occupied the same space.

Below we see two alternative SNLSs for the same cube building. On the left, the spatial structuring of *front + what's left on right side + (9 columns of 3)* corresponds to the numerical structuring of *12 + 9 + (repeat 9 times counting 3 cubes in a column)*. In Figure 10b, we see a column spatial structuring that a student numerically structured as *3, 6, 9...45, 48*. Another spatial structuring is horizontal layers (Figure 10c) which students variously structure numerically as *16 + 16 + 16*, *3 x 16*, or *skip counting 16, 32 48*.



a b c

Figure 10.

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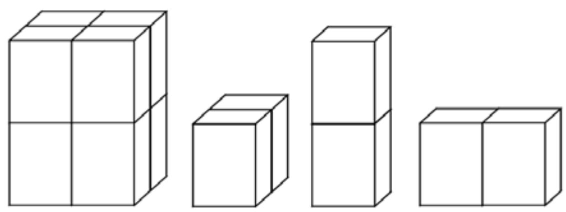
Note that, unlike the first SNLS in Figure 9b, the last three SNLSs produce correct answers. Given that there are multiple correct SNLSs for this cube building enumeration task, part of SNLS reasoning is consideration of enumeration efficiency. The SNLS in Figure 10a is correct but too cumbersome to be efficient and too unwieldy for large arrays. The SNLS in Figure 10b could be conceptualized in terms of 3 cubes in each column times 4 columns in a horizontal row times 4 horizontal rows, leading to the standard volume formula, as could the layer structuring SNLS (Figure 10c). So part of SNLS reasoning is metacognitive consideration of enumeration efficiency. Furthermore, using SNLS reasoning to make sense of the volume formula illustrates how SNLS reasoning can be used for generalization, not just enumeration.

SNLS reasoning as sense making for volume. The next example further illustrates how SNLS reasoning can be used to make sense of geometric measurement problems that deal with generalizations rather than enumeration. Consider the following problem (Battista, 2012). *The dimensions of a box are 3 cm by 2 cm by 4 cm. Give the dimensions of a box that has twice the volume.* The most common error that students make on this problem is to multiply all three dimensions by 2. SNLS reasoning can help students understand why the numerical structuring of *multiply all the dimensions by 2* is incorrect and what correct numerical structurings are possible. For instance, Figure 11a shows that doubling all the dimensions of a *3 cm by 2 cm by 4 cm box* gives 8 times the original box volume, whereas Figures 11b, c, d show that doubling any one of the dimensions of the box doubles its volume.

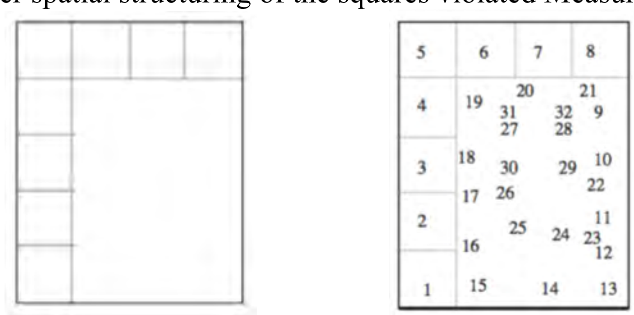


a b c d

Figure 11.

Spatial-Numerical Linked Structuring for Area

The ability to mentally construct an accurate spatial structure for rectangular arrays is a critical reasoning process for students determining area. But this process is surprisingly difficult for students to construct (Battista, et al., 1998). For example, student CS was asked to determine the number of squares required to completely cover the inside of the rectangle in Figure 12a (Battista, et al., 1998). CS counted in a non-random way as shown in Figure 12b. She counted the pre-drawn squares first, then she counted 9, 10, 11, 12, 13 down the right side and an equivalent number (15, 16, 17, 18, 19) up; overall counting in a clockwise spiral (Battista, et al., 1998). Because of the overlapping positions of CS' squares, her spatial structuring of the squares violated Measurement Property 3.



a b

Figure 12.

Significance

In addition to helping us untangle the complicated nature of students' coordination of spatial and numerical reasoning, this research helps us decompose the basic mental processes that students use in geometric measurement. It therefore helps us understand, for one content area, more precisely how spatial reasoning is related to numerical reasoning in geometry, which in turn helps us start penetrating why spatial reasoning has been found to be related to mathematical reasoning in so many correlational studies.

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TIME AS A MEASURE: ELEMENTARY STUDENTS POSITIONING THE HANDS OF AN ANALOG CLOCK

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Time is an area of measurement that is difficult for children. This interview study addresses the question: What are children's solution approaches to position the hands of an analog clock? To explore this, we investigated problem solving when using a clock manipulative with mechanically linked hands. We compare overall success rates among students in Grades 2 (n=24) and 4 (n=24) in positioning hour versus minute hands. We then present a qualitative analysis of solution approach for both hour and minute hands. Results indicate successful students may use the linked hands without overt consideration of the measurement structure of the clock.

Objective

Children have difficulty with the topic of time (Earnest, 2017; Kamii & Russell, 2012; Williams, 2012). Despite the fact that this topic is a staple of early grades mathematics instruction (NGA Center & CCSSO, 2010), little empirical research exists that documents problem solving in the context of common tools for telling time. Because time measure underlies mathematics of change in later grades as well as serving as an independent variable for STEM-related investigations, the present study seeks to reveal how younger students make meaning of units and unit relations on an analog clock, a prevalent cultural tool for time. In the present analysis, we investigate children's strategies for positioning hands on an analog clock to indicate particular times. The objective of our investigation is to reveal patterns across children's strategies and, because time is an area of measure, to consider how different strategies reflect measurement ideas related to unit and scale (Lehrer, Jaslow, & Curtis, 2003).

Theoretical Framework

We frame our study with two lenses. First, we consider mathematical aspects of time units as related to children's developing *theory of measure* (Lehrer et al., 2003). Second, we consider the mediating role of the clock manipulative itself. First, children develop a theory of measure through everyday examinations of the attributes of objects or events (Lehrer et al., 2003). In investigating the world in such a way, children gradually attend to such attributes as length, area, weight, and duration, each of which may be measured formally or informally. As children compare and contrast and, eventually, quantify such attributes, they grapple with mathematical ideas particular to measure (Lehrer et al., 2003). These ideas include conceptions of unit (such as the need for identical or equal units) as well as conceptions of scale (such as any point serving as an origin or zero-point).

As a measurement tool, the analog clock features 12 equal hour intervals, with the numeral 12 marking both a zero-point and ending point depending on that to which a user attends. As with any standard tool for measure, the analog clock provides equal intervals that are arranged end-to-end without gaps or overlaps. Of course, the clock does not represent only hours; rather, the same 12 intervals reflect both minutes and seconds. As in other areas of measure, individuals may draw upon tools for measure in procedural ways unrelated to their mathematical properties (Moore, 2013; Stephans & Clement, 2003). One concern in the present study was not only to identify students' approaches as they position the hands, but to understand them in relation to ideas related to unit and scale.

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Second, the present study positions thinking and learning as inextricably linked to cultural practices (Cole, 1996; Earnest, 2015, 2017; Sfard, 2008), with conventional tools (i.e., a digital or analog clock) serving a mediating role in problem solving. Analog and digital clocks represent time and its properties in different ways, with the analog clock's intervals of time translating duration into spatial distance (Lakoff & Nuñez, 2000; Williams, 2012). Digital time provides a precise time to the minute without reflecting part-whole relations of minutes and hours. The digital time 2:50, for example, provides a quick and precise numeric representation of time. In contrast, for the analog clock's hour hand to show 2:50, one may interpret its position as not just showing the "2" as with digital notation, but its displacement from 2:00 to 2:50 as well as the length remaining length for the ten minutes from 2:50 to 3:00. Our study involved a particular clock manipulative featuring mechanically linked hands; based on our perspective of thinking and learning, this material property is consequential to children's solution approaches.

Related Research

Classrooms in the United States typically feature classroom clock manipulatives for teaching time, though the functionality of clock features—and how unit relations are thereby supported—vary. Of two common clock manipulatives, one features mechanically linked hands such that movement in one hand provokes the proportional shift in the hour hand; this specific clock is the focus of our study below. For example, on a clock displaying 7:00, if one were to move the minute hand clockwise to show 7:30, the hour hand would proportionally move as well (see Figure 1). With this tool, a user may note the proportional shift in hand movement; alternatively, one may not attend to this particular feature at all, as such proportional movement is not dependent on the user's intentions in hand positioning. A different clock manipulative features independent hands, such that a user must deliberately position each of the two hands to indicate a particular time. On this clock, given 7:00, if one were to move the minute hand clockwise to reflect 7:30, the hour hand would remain at the 7 and thereby reflect a time that does not exist in our system (Figure 1b). In particular, we seek to understand if one manipulative is more helpful for children in terms of connections to measurement (not necessarily just in terms of accuracy).

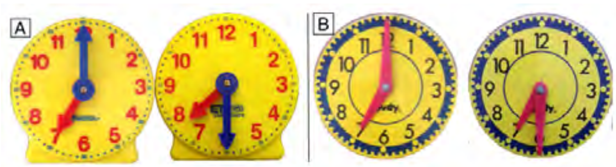

Figure 1. Manipulatives with (a) mechanically linked hands or (b) independent hands.

Given our concerns related to a theory of measure together with how tools mediate thinking and communication, our work has investigated problem solving related to time in the context of the two clocks in Figure 1. A recent analysis revealed that elementary students performed differently as a result of the clock available to them (Earnest, 2017), and in particular students were more successful when the hands were mechanically linked. To understand why students may have performed more poorly when the clock featured independent hands (Figure 1b), we further analyzed students' performances with this specific tool. We found that students' incorrect approaches often did not overtly reflect concerns for unit and/or a continuous scale (Earnest, Gonzales, & Plant, 2017). One such approach included treating intervals as containers; for example, treating the 2-3 interval as representing a container for the 2 o'clock hour (see also, Williams, 2012). In another approach, students matched a number from the digital time (i.e., the 2 of 2:50) to the numeral on the clock (i.e., positioning the hour hand on the 2).

Although less common, approaches reflecting concern for unit and scale often led to success (Earnest et al., 2017). One approach involved treating the two hands—functionally independent on the clock manipulative (Figure 1b)—as coordinated; for example, to show 2:50, one student explained how the hour hand would be in the 2-3 interval but close to the 3 because the minute hand was only ten minutes from the top of the hour. Other students identified a zero-point on the clock. For example, to position the minute hand for 2:50, students began with the 12 as a zero-point and counted by 5s to reach the accurate position; such a concern for zero-point has been identified as a key idea related to scale (Lehrer et al., 2003). Overall, students were statistically more successful with the minute hand as compared to the hour hand, which, because hands were not linked on the manipulative, each student had to deliberately position.

Research Questions

Our prior analysis focused on a manipulative with independent hands (Figure 1b), yet we found students were more successful overall when using the clock with mechanically linked hands (see Figure 1a) (Earnest, 2017; Earnest et al., 2017). We wondered if the success among students using linked hands reflected different solution pathways for students as compared to those using independent hands. If so, results could illuminate how the linked hands support disciplinary ideas related to time. Alternatively, the linked property of the hands may support pathways towards accurate hand placement through mechanically accomplishing this mathematical work on behalf of the user.

The present study investigates the questions: Are children more successful positioning one hand over the other? And, what are the solution approaches children apply to position each hand? In particular, we were concerned with how such strategies reflected treatments of unit and interval consistent with geometric measure.

Methods

Participants included students in Grades 2 (n = 24) and 4 (n = 24) from six elementary schools in diverse areas (urban and rural) of western Massachusetts. All schools were identified as having a high percentage of children from low-income families. Interviews were conducted in 2015. Grade 2 students were selected because standards indicate children in this grade have already mastered time to the hour and half hour and are currently working on time at the 5 minutes (NGA Center & CCSSO, 2010). Grade 4 students were selected because, according to standards, time concepts including elapsed time have been mastered in prior grades, and their performances therefore illuminate any persisting differences in performance on problems involving time.

Based on an assessment administered to a larger group of students in the six focal schools, we identified a range of students with permission in each classroom and assigned them to one of three clock conditions; our focus in this paper is on the condition featuring an analog clock with mechanically linked hands (Figure 1a; see Earnest et al., 2017, for a similar analysis involving an analog clock with independent hands).

Interviews lasted approximately 30 minutes. Our analysis here focuses on seven particular tasks specific to positioning hands on the clock, as these tasks reveal students' treatments of unit and interval (see Earnest, 2017, for all interview tasks). Hand Positioning tasks were designed based on prior literature (Kamii & Russell, 2012; Williams, 2012) and ongoing piloting of tasks. The seven times included in tasks were: time to the hour (7:00), time to the half hour (4:30), time on the first half of the clock (10:10), time on the second half of the clock (2:50), and time with a minute value less than 10 (9:03), along with two relative time tasks using hour units only (half past 11, quarter past 8). To present tasks, the interviewer provided the clock positioned to an unrelated time and asked the student to show the target time (e.g., "Show me what 2:50 looks like on this clock."). The interviewer also turned over a card on which the same question was printed (Figure 2). Once the student positioned the hands, the interviewer asked that student to explain her/his thinking.

Show 7:00 on the clock.	Show 4:30 on the clock.	Show 10:10 on the clock.	Show 2:50 on the clock.	Show 9:03 on the clock.	Show *half past 11* on the clock.	Show *quarter past 8* on the clock

Figure 2. Playing cards for administering the Hand Positioning tasks.

We conducted both a quantitative and qualitative analysis. First, using video and transcript, we coded each hand position separately as correct (1) or incorrect (0) for placement, enabling a comparison in performance between the two hands. To do so, we identified an interval for the two hands for each problem (i.e., 4.3-4.7 for the hour hand for 4:30), outside of which a response was considered incorrect. All responses were double coded, and any discrepancies in hand positioning accuracy were resolved in team meetings. Second, we open coded video and transcripts using the constant comparison method (Corbin & Strauss, 2008). Based on rounds of coding to identify particular solution strategies in data, we generated a codebook. Three coders then double-coded all data, with any discrepancies discussed until reaching consensus on a final code.

Analysis and Results

The analysis is presented in two parts. We first present quantitative results speaking to whether students had similar success at positioning hour and minute hands. Following this, we present a qualitative analysis to identify strategies students applied.

Performances on Hand Positioning Tasks

We first compared performances for hour and minute hands: Were students more successful at one hand over the other? Means and standard deviations are provided in Table 1, and in general mean performances out of 7 problems show that students were quite successful. A Two (Hand) ✕ Two (Grade) repeated measures analysis of variance (ANOVA) revealed a main effect for Hand, $F(1, 46) = 11.560$, $p = .001$, with better performance with the minute hand as compared to the hour hand. A main effect also emerged for grade, $F(1, 46) = 7.676$, $p = .008$, with Grade 4 students outperforming Grade 2 students. There was no significant Hand ✕ Grade interaction, suggesting that the discrepancy in performance was roughly equivalent across grades ($p = .168$). A post hoc Tukey's HSD showed that the difference for Grade 4 students in hand positioning was not significant ($p = .135$) and with a low effect size ($d = .194$) (Cohen, 1988). For Grade 2 students, the difference was found to be significant ($t(23) = 3.14$, $p = .005$), yet with a low to medium ($d = .338$) effect size, suggesting only a low or low to moderate practical significance of the result.

Table 1: Means and Standard Deviations for Each Grade for Correct Performance on the Seven Hand Positioning Tasks

	Grade 2	Grade 4
	Mean (SD)	Mean (SD)
Hour Hand	5.04 (1.681)	6.17 (1.129)
Minute Hand	5.54 (1.250)	6.37 (0.924)

This finding is different than the comparable analysis for the analog clock with independent hands, for which both grades were more successful with the minute hand and both with a large effect size (Earnest et al., 2017). In the present analysis, when using the clock with linked hands, there was little detectable difference in students' positioning of hour versus minute hands. Given our interest in children's approaches to showing time on the clock, we question how the linked hands may differently support such accuracy and whether such success owes to conceptual understandings students applied or, alternatively, the mathematical achievements underlying the manipulative's functionality. To address this, we turn to our second question; given the high level of success with this clock, what are students' strategies when positioning the hands, and how are such strategies related to key ideas within measurement?

Children's Strategies to Position Hour and Minute Hands

In this section, we provide an overview of solution codes that emerged in our analysis for the seven Hand Positioning tasks (Figure 2). The role of qualitative data analysis involving children's strategies is to further contextualize performance results above that suggests there was little difference in the challenge of placing the two hands and high overall accuracy in hand positioning. We first present the six codes that emerged from analysis of video and transcript. After this overview, we present our analysis of strategies for each hand across all Hand Positioning tasks in both grades among the 48 students using the clock with mechanically linked hands.

Our analysis of the 48 students' solutions resulted in six strategy codes (Table 1): Container, Number Matching, Hand as Lever, Number as Floor, Origin, and Coordination, with idiosyncratic or unclear strategies coded Other. Table 1 features code names with examples as well as the frequencies across the 672 possible instances (7 problems with 2 hands per problem for 48 students) and, given all instances for just that code, the percent correct for each of the two hands. We first consider four strategies that (in our determination) did not overtly relate to unit or scale followed by two additional strategies that reflected some aspect of these measurement ideas.

First, 50 responses were coded as Container. Consistent with Williams (2012), children treated a particular interval as a container, with any point in that interval the same as any other point (see examples in Table 2). Second, Number Matching refers to a strategy to match the number from the time in the prompt with a number on the clock (e.g., "It's 2:50, so the hour hand goes on the 2.". At times, this included the application of a fact that remained unexplained in the interview (e.g., "I know 6 is 30"). Third, Hand as Lever involved children treating the focal hand as a mechanism to move the other hand; only nine students applied such a strategy, and in all cases this involved treating the minute hand as a lever to move the hour hand. Fourth, Number as Floor involved students finding the position of the hour hand for the top of the hour (e.g., when solving for 2:50, first finding 2:00 so the hour hand points to 2) and then applying continued movement to the minute hand clockwise resulting in further movement of the hour hand. In these cases, after finding time to the hour, students focused exclusively on positioning the minute hand without any overt further consideration of the hour hand. Unlike Number Matching, for which students' goals involved positioning the hour hand as close as possible to the target number (e.g., indicating 1:50 on the clock when matching the hour hand as close to 2 as possible), students found the top of the hour and then relied on the functionality of the linked hands. This occurred 65 times in interviews.

We identified two strategies related to key ideas of measurement (Lehrer et al., 2003): Origin and Coordination. Origin involved strategies in which a student identified a zero-point on the clock—typically the 12—with the target position as a path from that starting point. Unlike Number as Floor, which involved finding the top of the hour to locate the hour hand and then (based on available data) turning their attention to the minute hand, Origin involved a starting point with an explicit follow-up strategy (see example in Table 2). This code was applied 76 times, yet we note that it was applied only for the minute hand and not once to the hour hand. In almost all applications (96.1%) of this strategy, students were accurate. We also coded strategies as Coordination (n = 105), which involved treating the position of the focal hand as dependent or related to the position of the other hand. In doing so, such cases involved the proportional relationship of hours and minutes. These two measurement strategies were more successful than the prior four strategies, yet arose less frequently in our data.

Table 2: Strategy Codes for Positioning Hour and Minute Hands (N = 672)

Strategy (frequency)	Example
Container (n = 50)	*To position the hour hand for 4:30.* "It's 4, so the hour hand is in the 4-space.
Number Matching (n = 263)	*To position the minute hand for 4:30.* "I know 6 is 30."
Hand as Lever (n = 9)	*To position the minute hand for half past 11.* "I placed this here because I wanted this hour hand to be between 11 and 12."
Number as Floor (n = 65)	*To position the hour hand for 2:50.* "I first found 2:00, and then I knew that the hour hand would be after the 2."
Origin (n = 76)	*To position the minute hand for 4:30.* "I started here [at 12] and counted by 5s until I got to 30.
Coordination (n = 105)	*To position the hour hand for 2:50*. "I know this goes there because it's close to the 3, and the minute hand is only 10 minutes away from 3:00."
Other (n = 104)	Idiosyncratic or did not respond.

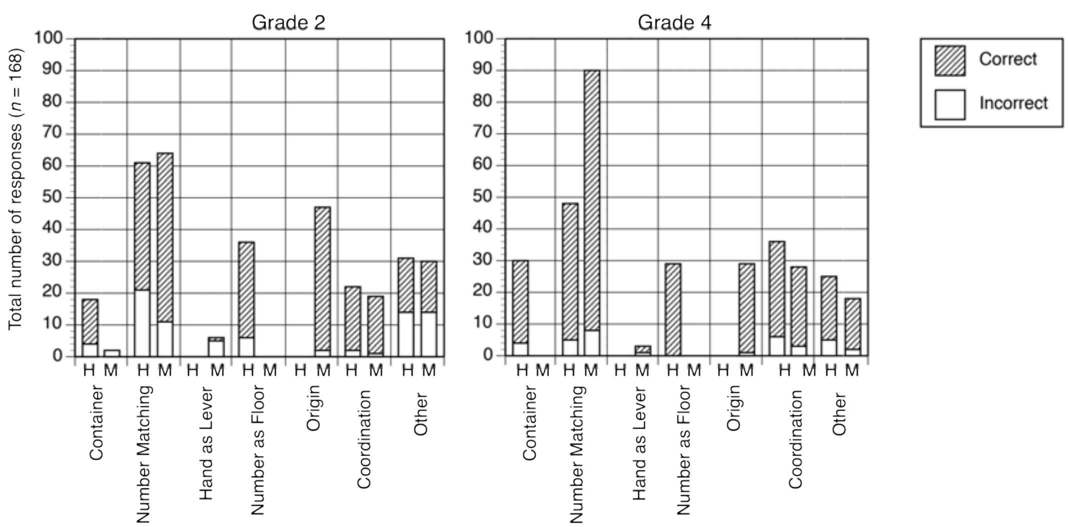

Figure 3. Strategy use for hand positioning when using clock with linked hands.

Figure 3 displays bar graphs for each grade to indicate frequencies of particular strategies for each hand along with whether such uses were correct or incorrect. Despite quantitative results above indicating little difference in success between the two hands, the strategies behind their positioning were often different. In particular, students' application of Container and Number as Floor were almost always for the hour hand. Although we would expect in general that strategies reflecting unit and scale would lead to greater accuracy (see Earnest et al., 2017), the material properties of the tool might have accomplished important mathematical work on behalf of the users applying Container, Number Matching, Hand as Lever, or Number as Floor strategies. We further note that the Origin strategy was applied exclusively to the minute hand. Such a result leads us to question whether, for children, the hour hand has an obvious zero-point like the minute hand does. We also note here that Hand as Lever was employed only in the context of relative times (i.e., half past 11) and, although we did not consider this to reflect unit or scale, any problem in which a student's minute hand position was coded as Hand as Lever received a Coordination code for the hour hand.

Concluding Remarks

With limited existing research focusing on children's understanding of time, we contend that the results of the present analysis are an indication that children may be developing an understanding of the inner workings of the clock—specifically how it reflects units and unit relations—in ways that are unrelated to mathematical properties of unit and scale. Further data is required to examine the extensiveness of these implications. Based on available data, if we were to look at overall performances among students using the clock with linked hands (Table 1), we may have concluded that this clock is a useful and productive manipulative for children to learn about time; however, our present analysis suggests that the material properties of the tool may be doing some important mathematical work on behalf of students.

Broadly speaking, what are our instructional goals related to time, particularly given that digital clocks are pervasive? We contend that instruction ought to move beyond procedures of clock-reading. Considering the role of time as a parameter in later mathematics, children ought to engage more deeply with the underlying meaning of clock features related to unit and scale. This research identifies a potential change in route with respect to how the field has framed learning and instruction related to time. We contend that more research is necessary in the area of how children come to understand time and common representations and tools for time. Although clocks are certainly pervasive in culture, results of this study underscore that children's ideas of time may be unrelated to the mathematical ways in which we measure it.

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EXPLORATIONS OF VOLUME IN A GESTURE-BASED VIRTUAL MATHEMATICS LABORATORY

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A room-scale virtual-reality environment was used to investigate students' conceptions of the volume of a pyramid. Participants controlled the virtual environment with a gesture-based interface that converted movements of their hands into actions on mathematical figures. Two students in graduate programs leading to certification in secondary science education investigated how the volume of a pyramid is affected by horizontal (i.e., shearing) or vertical (i.e., elongation) movements of its apex. Participants' actions within the environment were analyzed using the conceptions-knowing-concept (cK¢) *model of student conceptions. Both participants used an analogy of volume to surface area and area to perimeter to make sense of the effects of the shearing operator.*

Keywords: Geometry and Geometrical and Spatial Thinking, Technology, Teacher Education-Preservice

Introduction

New modes of interacting with virtual objects (Hwang & Hu, 2013; Kaufman 2011) can facilitate new ways of engaging with mathematical ideas (Abrahamson & Raúl Sánchez-García, 2016). Students today have a considerable advantage over the ancients: Instead of making do with crude, fixed diagrams sketched in sand, they can hold virtual mathematical objects in their hands and use intuitive, gestural interfaces (Goodman, Seymour, & Anderson, 2016; Zuckerman & Gal-Oz, 2013) to translate their instincts to explore different shapes (Hwang & Hu, 2013) into dynamic investigations of geometric structure (Duval, 2014; Sinclair & Bruce, 2015).

In this study, we report on two graduate students' explorations of the volume a pyramid in a virtual mathematics laboratory. We asked: How do students use virtual operators (e.g., the ability to dynamically elongate or shear a figure) to make and test conjectures about the volume of a pyramid? We use the cK¢ model (Balacheff & Gaudin, 2010) to describe two cases of students' explorations in the volume laboratory.

Theoretical Framework

Design of the Environment

The HandWaver environment (Dimmel & Bock, 2017) we developed was designed to create a space where learners could train their dimensional deconstruction skills (Duval, 2014)—the capacity to see representations of geometric objects as figures that can be resolved into components rather than as whole shapes (Laborde, 2008)—by using their hands to act directly on mathematical objects. The virtual tools were designed to provide various operators (Balacheff & Gaudin, 2010) participants could use to solve problems, such as that of relating the volume of the pyramid to the volume of a unit cube. For example, users could modify the (x,y,z) position of the apex of a pyramid by pinching and moving it in space, thereby changing the height of the pyramid by elongating (z-axis) or shearing (x-, y-axes).

The Conceptions-Knowing-Concept (cK¢) Model

Balacheff and Gaudin (2010) define a conception as a quadruplet (P,R,L,Σ): a set of problems (P), a set of operators (R), a representation system (L), and a control structure (Σ) (Balacheff & Gaudin, 2010; Balacheff, 2013). In this model, conceptions are not mental entities that belong to particular learners but are rather emergent, observable states of dynamic equilibrium between learners and their actions in an environment (i.e., a milieu). A conception is a "system of relationships" between the four components of the model. The set of problems for our study pertained to exploring the volume of a pyramid. A prototypical problem (Balacheff, 2013) was: How does the volume of the pyramid change as the position of its apex changes?

The environment provided various operators for participants to work on this problem. In addition to the elongating and shearing operators (Dimmel & Bock, 2017), the environment allowed users to constrain the elongate and shear operators by locking the apex of the pyramid in space along axes determined by the user. For example, users could lock the *z* coordinate of the apex, allowing them to move the apex in an *xy* plane without changing its height. Alternatively, users could lock its *xy* position and vary the pyramid's height while keeping its apex in its current vertical position. We report below on how participants used the shear and elongate operators to investigate volume.

Method and Participants

We investigated how pre-service teachers explored the volume of pyramids in the environment by conducting task-based interviews using a semi-structured protocol. Participants were asked to describe their thinking out loud (Fonteyn, Kuipers, & Grobe, 1993) as they navigated the environment and worked on the tasks. Participants were visually immersed in room-scale virtual reality for the duration of the interview. There was at least 2.5 meters x 3 meters of clear floor space for participants to move. All manipulatives were contained within the 2.5-meter x 3-meter virtual space. However, the virtual environment was unbounded such that participants could see to the horizon. Participants were recruited from a cohort of graduate students who were completing a Master's of Science in Teaching program. The two participants were given pseudonyms: Abe and Brendan.

Data

A 60-minute interview was conducted with each participant. A participant's first-person view of the virtual world was displayed and screen recorded on a monitor that was visible to the researchers throughout the interview. A third-person view of the participant navigating the virtual environment was also screen captured. A separate audio recording captured the conversation between the investigators and the participant.

Analysis and Results

After conducting three recorded interviews, each interview was transcribed and partitioned into episodes by task, differentiated by the operators common to each task (González & Herbst, 2009). Transcriptions included references to direction and geometric figures indicated by hand gestures. We consider here a task where participants investigated the volume of the pyramids by manipulating its apex. Episodes were partitioned by the type of manipulation (vertical or horizontal) of the apex during the participants' inquiry. We differentiated horizontal manipulation (shearing) from vertical manipulation (elongating).

Episodes of Exploring Volume

At the beginning of the task, Abe stated that he believed "moving the apex increases the volume" when the apex is restricted to vertical movement (elongating). As Abe began to experiment with horizontal motion, shearing the pyramid, he claimed that "[the surface area is] probably balancing out, so the total volume isn't changing, it's just redistributing where the volume is." After he was asked what tool exists or could be built that would increase his confidence in his claim, Abe first suggested that if "each side of the pyramid could be a different color, then…you might be able to visualize the changing of the area." Abe then suggested applying a "cube grid" (a graph-paper material seen on the unit cube) to each face. The measurement tools Abe wanted were absent from the environment by design, to discourage empirical discussions of volume (Herbst, 2005).

Reaching a Contradiction

The investigators prompted Abe to investigate the changes of the heights of the triangles and to use greater than or less than statements if measurement was not possible. Abe then attempted to use relative changes in the perimeter of each face of the pyramid to gauge relative changes in area. Abe acknowledged that the length of each edge of the square base was equal and constant when the apex of the pyramid was sheared. Abe used the changes in lengths of the two remaining sides to determine if a face was increasing or decreasing in area. Abe concluded that the net surface area was increasing, thus the volume was increasing. However, Abe stated that "It just doesn't feel like it is," and that it "seems weird to...say that ...the volume of [the sheared pyramid] is larger." The interview concluded shortly after Abe reached this contradiction, so we were unable to observe Abe attempt to resolve the contradiction.

Connection to Perimeter and Area

Brendan used an argument that related area to perimeter as he explored the relationship between surface area and volume. Brendan suggested that there is a relationship between the structure of perimeter and area, and surface area and volume. He appeared to be using the connection between perimeter and area as a control to work on the problem of describing how volume is affected by shearing or elongating. This aspect of his control structure could be described as follows: Volume is the 3D analog of area and surface area is the 3D analog of perimeter. So, if a change in perimeter implies a change in area, then a change in surface area implies a change in volume.

This control is a component of a conception of volume equality that could be described as follows:

- *Problem (P):* How is the volume of a pyramid affected when the position of its apex changes?
- *Operators (R):* Elongate (xy axis lock) or shear (z-axis lock) the pyramid. Use one's gaze or one's position in the environment to make a visual comparison.
- *Representation (L):* A dynamic pyramid in an immersive virtual space.
- *Control Structure (Σ):* Analogy between the relationships of volume to surface area and area to perimeter.

Conclusions & Scholarly Significance

We anticipated participants would use an available cross section tool (not described) to effect side-by-side comparisons of sheared and non-sheared pyramids of equal height. Participants' uses of the analogy between perimeter/area and surface area/volume as a control on the shearing operator was unexpected. The participants that used this control did not recognize that the perimeter of a figure can increase while its area remains constant and that the surface area of a figure can increase while its volume remains constant. Even though participants arrived at a conception of volume we had not intended, the environment we designed facilitated participants' use of dimensional deconstruction to reason about plane and solid figures. The use of physical manipulatives has previously been proposed to address conceptions in 3-D geometry, particularly in investigations of shearing (Fischman & McMurran, 2011). Our study highlights how virtual manipulatives in immersive spaces can be used to explore how learners reason about volume.

Our study is significant because mathematics education is at a technological crossroads. The virtual reality technology we used to conduct this study is commercially available, inexpensive, and will be as ubiquitous as mobile phones in the coming decade. How will classroom instruction adapt to strategically harness the vast potential of these new systems of representation and interaction to positively affect student experiences in mathematics? This is an urgent question. Studies like ours demonstrate what the potential of these new instructional technologies look like in practice.

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MIDDLE AND HIGH SCHOOL STUDENT UNDERSTANDING OF HEIGHT OF A TRIANGLE

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This study focuses on geometry, one of the under researched studies in mathematics education. Based off national and international tests, it is also the least understood areas of mathematics. Geometry teaching and learning should be at the center of reform policies. To contribute to understanding of this issue, in this study we conducted clinical interviews with 25 students from 13 middle and high school students. The results reveal misconceptions that are likely to hinder development of geometric understanding.

Keywords: Geometrical and Spatial Thinking

Objective

Educators contend that Geometry is one of the critical areas in mathematics education. It is foundational to higher-level mathematics, to understanding space in our world, and many other science disciplines. Several studies have shown student difficulties with geometry. Usiskin (1982) after a study with 2700 students on geometric thinking concluded "half the students who enroll in a proof oriented course experience very little or no success with proof" (p. 99). These difficulties have persisted over decades. Several studies (E.g., Stylianides & Silver, 2014), and achievement tests such as NAEP and TIMMS have shown geometry to be the weakest area for students compared to algebra, number, data, and chance. Secondary school mathematics teachers also report students' difficulties with high school geometry.

In light of these difficulties, standards for mathematics teaching (ex. Common Core States Standards, 2011) are targeting deeper and connected ways of understanding geometry from elementary school. In these standards, six grade students should be solving geometry problems that include volume and area. Seventh grade geometry standards include constructing geometric figures, and eighth grade students should be learning Pythagorean theorem, and congruence and similarity of geometric figures. One of the underlying concepts that affect depth of understanding in the geometry domains in the standards is the concept of height, for instance height of a triangle. Understanding height of a triangle pulls from understanding what a triangle is and feeds into conceptual understanding of area, volume, Pythagorean theorem, and congruence and similarity.

Clearly, it is critical to explore students' understanding of height of a triangle. Moreover, there is scarcity of literature on students' geometric thinking after implementation of Common Core 2010 standards. Furthermore, Bergstrom and Zhang (2016) in their meta-analysis of studies on geometric thinking and interventions reported that "compared to the rich literature on numerical instruction, research investigating students' development of geometric thinking is rather limited" (p.2). For these reasons, the objective of this study is to explore how middle and high school students understand height of a triangle.

VanHiele Theory of Geometric Thinking

This study is informed by Van Hiele theory of geometric thinking. This theory was developed in response to the need for improving secondary students geometric thinking (Van Hiele, 1959) and has been used as a model for geometry curriculum (Burger & Shaughnessy, 1989). Drawing from Jean Piaget's theory on cognitive development and other theories, Van Hiele theory organizes geometric thinking into five levels namely visualization, analysis, abstraction, rigor, and deduction.

At visualization stage, students focus on the orientation of geometric shapes. Their focus when identifying shapes is not the explicit properties of the shapes. Even for those beginning to use properties, the properties used are very imprecise as they compare drawings, sort, or characterize shapes. For example "a child recognizes a rectangle by its form and a rectangle seems different to him than a square" (Van Hiele,1959, p.62). The vocabulary for figures is present, but the definitions for such vocabulary are not understood.

Analysis stage follows visualization stage. At this stage, students analyze the shapes, discern properties, and are even able to make generalizations about those properties. Given the properties, students can draw the corresponding shapes. However, they are not able to see relationships between those properties. They are still unable to see a square as a rectangle (Van Hiele, 1986). Once children reach abstraction level, they are able to make informal deductions about properties and see their relationships. These informal deductions allow children to identify interrelations of properties both within and between shapes, enabling them to identify squares as rectangles. In the later levels, students are able to work with axioms and write formal proofs, and later work within abstract non-Euclidian geometry.

Methods

Data were collected from 13 female and 12 male students from 13 schools from Midwestern US. Ten participants were in grade 6, 7, and 8 while the rest were either freshmen or sophomores in high school. These students had participated in a one-week college residential camp conducting research in STEM fields. Students were given written questions that asked them to define height of a triangle, draw heights to given triangle bases, and write how many heights each triangle may have. Clinical interviews (Piaget, 1983) were conducted with each student to understand students' reasoning for their responses. The interviews were video recorded and lasted 30 minutes on average.

A line by line coding of the interview transcripts and student worksheets was conducted in Nvivo to identify themes for developing categories of student reasoning. NVivo coding helped to check how robust the themes were and to identify relationships between themes, and between verbal and written reasoning. Multiple coders built in the trustworthiness of the analysis (Creswell, 2007). The results are presented in the following section.

Results

Defining Height of a Triangle

From our sample, only three students who had just finished 10th grade were able to define height correctly and could identify heights that were either horizontal or vertical without changing the orientation of the triangle. For the rest of the sample, height of a triangle was how visually tall a triangle is. Some explained that a triangle can have up to three heights, but needed to rotate the triangle to explain that the height would change depending on which leg was the base. Some differences were observed between grades.

None of the ten students who had just finished sixth and seventh grade were able to correctly define height. From the interview and written transcripts, to identify height of a triangle, one simply needs to pick one corner of a triangle with a line (leg) that looks taller, and follow that leg to the top corner. They interpreted height as how visibly tall a triangle is as judged from "the bottom" of that triangle. When asked what happens to the height when the triangle is rotated, they believed that there is a need to consider the bottom line again and choose the longest line to the top. Only four of these sixth and seventh graders believed a triangle could have up to 3 heights while the others believed the maximum number is 2 or 1. Notably, this group of participants did not use any formal vocabulary during the interview or in their written work.

Eighth graders were using formal vocabulary (ex. hypotenuse, base). Although their definitions of height were incorrect and also viewed height as how tall a triangle is, they were able to talk of heights meeting the base, and that height did not necessarily have to be one of the legs. In general, the ninth and tenth graders had similar understanding but also talked of other factors such as the type of triangles. They tended to default to discussing height of a right-angled triangle as being the opposite of a hypotenuse, but their understanding was limited to right-angled triangles where the right angle was on one of the bottom vertices.

Drawing Height of a Triangle

Participants were also asked to draw a height to a given base *b.* Analysis of the heights they drew correlated with their description of height during the interview. There were four common misconceptions for the students who seemed confident in their definitions and drawing of heights.

The most common misconception was believing that the height of a triangle is the leg that looks the tallest. Students with this misconception, in line with their definitions of triangle, always used one of the legs that looked taller from the bottom to the top to draw the height. Because they believed that rotating the triangle would change the base and therefore which side looked the tallest, they were convinced that a triangle has three possible heights. However, they were confused when prompted to think whether *every* triangle would always have three heights. They thought number of heights depended on the relative sizes on the legs. It became very difficult for them to give a convincing argument on determinants of number of heights of a triangle.

A second misconception was that the height always bisects the base. With this misconception, students always drew a height starting from the middle of one of the legs. Some perceived heights were from the given bases and others from the bottom leg. They did not necessarily go to the vertex opposite the base, some legs were drawn from the mid point of the perceived base and to middle of the triangle, and others left randomly outside the triangle.

Another misconception was that the only triangles with heights or whose heights can be identified are right-angled triangles whereby the leg opposite the hypotenuse is the height. Interestingly, students with this view easily identified the height of right-angled triangle when the height was vertical and on the left, but almost always never identified such edges as heights when the edges did not look 'vertical enough' even on triangles labeled as right-angled.

Lastly, students had a misconception that the height of a triangle can only be drawn if the triangle measurements are known. The measurements that students mentioned were the formula for area of a triangle (A= b x h), angle measurements, and types of a triangle (ex. isosceles, acute, equilateral, e.t.c). These students focused on trying to recall computational geometry ideas and trying to figure out how they could use those ideas to draw heights. They discussed triangle properties at length but could not draw any height because drawing heights without measurements included in the triangles did not make sense to them.

Only three students were able to draw heights for any given triangle correctly, and their discussion was consistent with the mathematical definition of height. These students had just finished 10^{th} grade at different high schools.

Significance

Based off these data, students' reasoning with height of a triangle is within visualization and analysis levels. From different theoretical perspectives, development of geometric thinking is sequential; students cannot achieve higher-level thinking without the basic understanding. Any high level mathematics such as calculus builds on geometry concepts some of which are heights of figures. To reason with measurements in our world, for instance volume, require understanding height of objects. Students who do not fully comprehend the height of a triangle lack mathematical empowerment. The results of this study show students are not well prepared to do high school geometry or college mathematics. By sixth grade, geometry domains require understanding of height of a triangle and yet in our sample only three students could define and draw height of a triangle. This calls for rethinking geometry curriculum, and/or pedagogy.

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MOTIVATING THE CARTESIAN PLANE: USING ONE POINT TO REPRESENT TWO POINTS

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The Cartesian plane is often used as a representational tool in high school mathematics and beyond. Often, the Cartesian plane is taken as a given, and little research has examined students' constructions of the Cartesian plane. We present data from a teaching session focused on two ninth-grade students' activities during the Ant Farm Task—a task we designed to motivate students' construction of the Cartesian plane. We describe three elements of the students' constructive activity that we view as critical for constructing the Cartesian plane.

Keywords: Geometry and Geometrical and Spatial Thinking, High School Education, Cognition

Using the Cartesian plane, students are expected to represent and reason about various mathematical ideas (e.g., characteristics of geometrical shapes, functions, etc.). Consider the following statement in the Common Core State Standards for Mathematics (CCSSM) regarding the Cartesian plane:

> Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. (NGACBP & CCSSO, 2010, p. 38)

Descriptions of the Cartesian plane like the one above are common in textbooks and other curricular materials. In short, students are given the rules of "generating" a Cartesian plane and plotting points within it. We are aware of no curricular materials designed for high school students that attend to students' conceptions of coordinate systems or why we construct and use the Cartesian plane in this manner. In this report, we present how two ninth-grade students described the location of two points using a single point by constructing a Cartesian-like coordinate system. More specifically, we will present the Ant Farm Task, analyze the two students' strategies in the task, and consider educational implications.

Theoretical Framework

Based on our conceptual analysis (Thompson, 2008), we distinguish between two different uses of coordinate systems: spatial coordination and quantitative coordination (Lee, 2016; Lee & Hardison, 2017). Spatial coordination refers to the use of coordinate systems to re-present space by establishing frames of reference to locate points within the space (e.g., a map). Quantitative coordination refers to the use of coordinate systems to coordinate sets of quantities in a representational space. We are unaware of any curricular material which address our distinction between these two uses of coordinate systems, including the CCSSM.

Saldanha and Thompson (1998) explained thinking covariationally as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). Often, curves in the Cartesian plane are used as static representations of the mental covariation outlined by Saldanha and Thompson. Previously, we have argued that spatial coordinations are necessary (i.e, a barrier) if Cartesian coordinate systems are to be productive tools for students' quantitative coordination and subsequent covariational reasoning (Authors, year). In this paper, we present data and analysis of students' constructive activities in a task we view as a bridge between spatial coordination and quantitative coordination. We propose three features of students' constructive activities that may be critical for fostering students' transition from utilizing one number line to represent a single quantity to developing the Cartesian coordinate system to represent covariation in two quantities.

Methods

We conducted a yearlong teaching experiment (Steffe & Thompson, 2000) with ninth-grade students Kaylee and Morgan to investigate their constructions of coordinate systems (Author, year). The first and second authors served as the primary teacher-researcher and witness respectively for the teaching experiment. We collected video recordings and student work from each of the 20–25 minute teaching episodes. We conducted both on-going and retrospective analyses and oriented our work in modeling students' constructive activities (Steffe & Thompson). In this report, we discuss a single task: The Ant Farm Task (AFT). The goal of AFT was to engender students' creation of a space that would allow them to describe the location of two points simultaneously, with one possible solution being the standard Cartesian system. Providing the students with two plastic tubes that represent ant farms, we asked students to pretend there were two ants: one moving around in each tube. The task was accompanied by a model of this situation in a dynamic geometry environment (DGE) as shown in Figure 1.

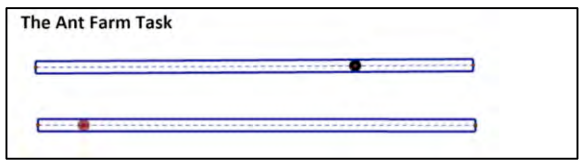

Figure 1. Ant Farm Task dynamic geometry environment model.

We asked the students to imagine the computer screen to be the floor of a room on which two ant farms—long, thin rectangles within the DGE—were resting. Each ant farm contained a single ant—a point that moved along the longest segment connecting midpoints of opposite sides of each rectangle (the dashed lines in Figure 1). The rectangular-shaped models could be translated and rotated in the DGE by dragging the end points. The ants' movement was designed so it appeared to move haphazardly along the interior of the tube. Additionally, in the DGE, there was an action button that allowed students to stop/activate the motion of the ants and one to hide/show the ants in each tube to encourage students to attend to both static and varying locations of the points. We asked the students to devise a way to represent the location of both ants using a single point so that if we were to hide both ants, they would be able to use their new point to determine the location of the two hidden ants. We presented the situation absent of any quantitative measurements (e.g., an ant's distance from the end of the tube).

Results

As the students worked together, Kaylee first suggested connecting the two ants and marking the midpoint of that line segment as their point, as shown in Figure 2a. As such, Kaylee constructed a new point outside of the ant farms in two-dimensional space to coordinate the location of both points simultaneously. Morgan animated the ants to explore if Kaylee's idea would work for any location of the ants. As the ants moved, Morgan attended to the line segment and midpoint Kaylee constructed and claimed, "This only shows where the middle of [the ants] are. It doesn't actually show us where they are." We consider Morgan's remark to indicate that she viewed Kaylee's solution as insufficient for determining the location of both ants

Moments later, Morgan suggested positioning the tubes perpendicularly, with one of the ants at the intersection, as shown in Figure 2b. The two ants were still connected by a line segment which

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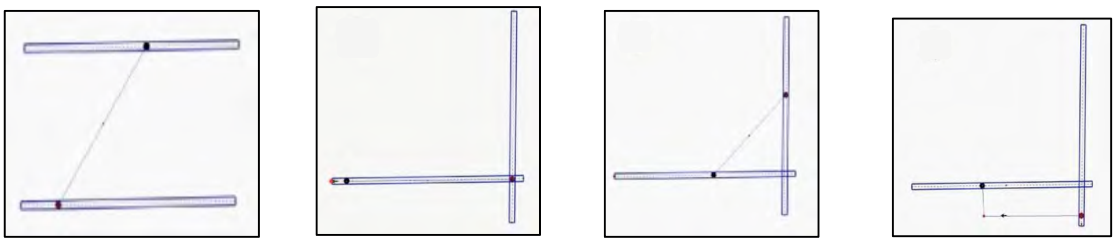
contained its midpoint (Figure 2c). Morgan then animated the ants and observed various "triangles" formed by the two ant farms and the line segment connecting the two ants (e.g., one instance captured in Figure 2c). Morgan explained that she thought of "axes, like a *x* and *y*" and triangles formed by the axes with the connected segment. The researcher hid the two ants and the connecting segment, leaving only the midpoint left on the screen, and asked the students to anticipate the locations of the two ants based on the visible midpoint. Testing several different locations, Kaylee and Morgan concluded that the midpoint of the line segment connecting the two ants was insufficient for locating both hidden ants by noting that multiple lines could pass through this midpoint. Although we consider their midpoint coordination valid, Morgan and Kaylee did not find their midpoint sufficient for reliably locating the two hidden ants.



(a) (b) (c) (d)

Figure 2. Kaylee and Morgan's activities in the dynamic geometry environment. (a) Kaylee connects two ant points and marks midpoint (b) Morgan re-positions the two rectangles (c) One instance of perpendicular rectangles with midpoint coordination (d) Kaylee's box coordination.

After discarding their midpoint coordination, Kaylee and Morgan sat in silence for approximately 40 seconds until Kaylee came up with the idea to make a rectangular "box" as shown in Figure 2d. Kaylee said, "I was thinking like putting a point right here so I can make a box," as she indicated the point that would be the intersection of lines perpendicular to the tubes through the ants. Then she made a rectangular motion connecting the two ants to this point and the intersection of the tubes. In subsequent activities, Kaylee and Morgan tested their approach by moving, hiding, and predicting the locations of the ants. Both students appeared very surprised when their predictions were accurate. In checking each prediction, they revealed the ants using the hide/show button and both exclaimed, "Wooo!" "What?!" "Gosh!" as they saw the ants were located where they predicted.

Next, Morgan posed the question, "But wait, what if we move this down more?" as she pointed to the vertically positioned tube and moved her finger as if to drag the tube downwards. She continued, "Would it change our answers?" Kaylee responded, "No, because it's still in line with it," as she moved the vertically positioned tube up and down. They were able to see the point in the plane move in accordance with the movement of the tube. When the researcher asked if that will work no matter where the two tubes were, Morgan explained using the plastic tubes, "Say these are the two tubes and make sure they're crossing somewhere and then wherever that dot is, it will go." Through their activities, the students had spontaneously constructed a (non- quantitative) Cartesian-like coordinate plane in order to coordinate the location of two points using a single point.

Discussion

Although Morgan and Kaylee had formal instruction in using the Cartesian plane, devising a system to coordinate the location of two points using a single point appeared novel to the students and neither student associated the activity with the Cartesian plane during the teaching session. Morgan described the configuration of tubes as similar to *x* and *y* axes; however, other than this figurative association, neither student mentioned the way in which points are coordinated in the

Cartesian plane. Furthermore, we consider the task non-trivial for the students as they spent 24 minutes working on the task before reaching what they considered to be a satisfactory solution. In a later teaching session, one student explained how in school they were told “this happens and that happens” on the coordinate plane but that “they never really gave us real world examples like [the Ant Farm Task].” Looking across the activities of Kaylee and Morgan in AFT, we highlight three features of their activities that we see as critical for fostering students’ transition from utilizing one number line to represent a single quantity to developing the Cartesian coordinate system to represent covariation in two quantities.

First, the students’ use of space outside of the ant farms suggests that the Cartesian coordination of two quantities represented on independent number lines requires an *awareness of two-dimensional space* outside of the one-dimensional number lines. Second, the students’ *attention to variability* in both ants’ locations motivated a particular spatial coordination (e.g., perpendicular lines through the ants) that would enable holding a sustained image of two locations simultaneously (Saldanha & Thompson, 1998) for arbitrary positions of the ants.

Having students move the tubes themselves, rather than pre-constructing them perpendicularly on the screen, and having the ants move haphazardly allowed the exploration to be novel for the students. Finally, the students’ decision to reorient the ant farms in the dynamic geometry environment suggests that developing the Cartesian coordinate system is supported by students *viewing number lines as objects* that can be rotated and repositioned in two-dimensional space.

To modify AFT to provide students with an opportunity to engage in quantitative coordination, we recommend asking students to represent each ant’s distance from one end of its ant farm via a variable point on a number line; we would then ask students to represent these variable points on two independent number lines using a single point. AFT and the findings from this study can provide insight for mathematics educators and researchers who are interested in engendering productive meanings for the Cartesian coordinate system.

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PUTTING OUR BODIES ON THE LINE: MATHEMATIZING ENSEMBLE PERFORMACES

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All humans have bodies. Our bodies profoundly affect our perspectives and understandings of the world. Today's schools neglect the body as a resource for mathematical thinking and learning. Recently this changed and researchers are putting the body back into the equation. Recent research explores changes in the modality and scale of mathematical activity, putting the power of paper and pencil into whole-body coordinated movements in open spaces. This study designs and studies learning in new forms of mathematical activity that involve multiple, moving bodies. This paper reports on the first phase of a two phase project in which we use interviews with experts in a variety of disciplinary fields, asking people to find and make sense of emergent mathematics in choreographed, ensemble performances from the opening ceremony of the 2016 Rio Olympic Games, to design and study experimental mathematics learning.

Keywords: Cognition, Geometry and Geometrical and Spatial Thinking

Introduction

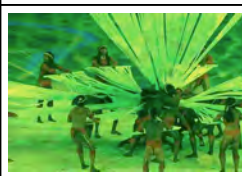
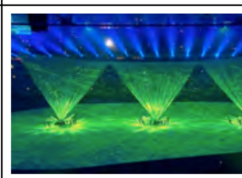
Our bodies profoundly affect our perspectives and understandings of the world. However, today's schools neglect the body as a resource for mathematical thinking and learning. Recently researchers are putting the body back into the equation (Lakoff & Núñez, 2000; Stevens & Hall, 1998). Research on embodied mathematics learning has shown multiple affordances of gesture use (Goldin-Meadow, 2005), but these investigations tend to focus on stationary, individual bodies with an emphasis on hand and arm movements. More recent research explores changes in the modality and scale of mathematical activity, putting the power of paper and pencil into whole body coordinated movements in open spaces (Hall et al., 2014; Ma, 2016). This study investigates learning in new forms of mathematical activity that involve multiple, moving bodies. In this paper, we share initial findings from interviews with experts from various disciplines, asked to find and make sense of emergent mathematics in choreographed, ensemble prop-based performances from the opening ceremony of the 2016 Rio Olympic Games (Table 1) when they are given similar props to think with. Findings from these interviews will inform future designs and studies of experimental mathematics teaching and learning at the crossroads of embodied ensemble learning and mathematics.

Background and Framework

During the opening ceremony of the 2016 Rio Olympic games, billions of people worldwide watched over 6,000 performers execute large-scale choreographed routines with dynamic geometric forms that were viewed at local and global scales (Table 1). In this emerging genre of theatrical production, viewers are invited to engage with a performance that is mathematically complex in design and appearance (Lakoff & Núñez, 2000; Ma, 2016) and are able to see close up views of the local organization of bodies as well as aerial shots that show the global organization of bodies in space. We argue that these performances hold rich relations to mathematics. The performer-plus-prop ensemble covers the enormous stage, exhibiting tessellation after tessellation; local organization becomes global in emergent complex systems; performers turn straight chords into curved cones. However, we do not know, whether or how viewers, particular professionals with different disciplinary backgrounds (e.g., a choreographer or mathematician), might make sense of these performances mathematically. Thus, the first phase of this project examines how people with different disciplinary backgrounds find patterns, articulate how they see these patterns being assembled in the performances (e.g., repeating shapes as tessellations), and use their own bodies (as well as other representational forms such as written diagrams and the use of embodied props) to demonstrate and communicate their understandings. Selections from the opening ceremony of the Rio games provide rich material for probing these understandings, since the video record involves hundreds of performers, who create complex dynamic formations by using their bodies and other tools (e.g., folding and rotating square panels and moving cords hanging from above into cones and triangular prisms).

Our focus in these interviews is (1) to understand a new theatrical genre composed of units of props plus bodies, which expresses something important in the present cultural moment, (2) explore what is compelling for viewers, and (3) to ask participants to use their own most "powerful tools" (including mathematics, if they find it relevant) for interpreting these performances, probing how participants make sense of their own role within the local organization of bodies-with-props in relation to global orders of repeated groups of bodies and props in space.

Table 1: Local and Global Orders in Opening Ceremony Clips

Name of Routine	Local Order	Global Order	Props Plus Bodies
Solar Panel People			The square solar panels plus the four performers that hold each one at a time afford different local folding symmetries and grid like global formations and tessellations
Spaghetti			The cords plus the performers holding them are able to create cones, where the cone point expands to a dorsal line suspended across the center of the performance space

Methods

In these interviews, participants are shown two clips from the opening ceremony of the 2016 Rio games (Table 1). After each clip, participants are engaged in a dialogue guided by the researcher. This discussion follows a line of inquiry structured by three primary questions. Participants are first asked to describe what they saw or heard in the clip. They are then asked what they think performers would need to know or be able to do to in order to create the performance. Finally they are asked how they would describe the performance mathematically. Participants are given control of the computer so that they can rewind to any part of the clip and rewatch for further, iterative analysis (Hall & Stevens, 2016; Jordan & Henderson, 1995). They are also supplied with paper, pencils and similar props to think with such as a square made of silver mylar to reenact aspects of the performances and test their own thoughts on how these performances are enacted. Preliminary interview sessions have been conducted at the focus group level (about 10 participants) and at the smaller scale of one and two interviewees at a time.

Through iterative analysis of interview recordings, we are developing grounded theoretical categories to describe similarities and differences in how participants make sense of these spectacular choreographed performances, both as a form of cultural expression (including the expression of mathematics) and as something that people learn to do together (i.e., as embodied, ensemble performances). We are also interested in exploring the bodies of the performers and participants as a *technology* to enact mathematical forms and processes, something that the performers are clearly doing and that interview participants may explore with their own bodies.

These performances and viewing sessions provide an opportunity to "restore" whole bodies as a sensible topic and medium for understanding things from a mathematical perspective. In this sense, local orders and how these "parts" add up to emergent "wholes" in the performance *necessarily* involve bodies, moving in coordination together. We also want to investigate the use of "props" that support both part and whole aspects of ensemble performance (i.e., performances that require many, coordinated bodies) as a technological tool for mathematics thinking and learning because they constrain the movement of bodies in ways that are able to systematically organize bodies in time and space. These performances thus involve *tools* that performing bodies manipulate as props to express visually regular motion at the scale of "parts" (or local orders). Assembling local orders at a broader scale (e.g., creating a matrix with either a static of moving structure for the entire field), and how these forms are made depends equally on technologies.

Preliminary Analysis and Findings

Thus far, participants have noticed two important aspects of these performances: (1) the narrative affordances of these performances and (2) how they relate to mathematical descriptions of them.

Many participants seem to expect different narrative elements in these performances and their sense making comes from either interpreting these expected narrative elements or dealing with an absence of these narrative elements. For several participants latching onto narrative elements and symbolic representations in these performances seems to be one way that they can start to make sense of these performances. These participants may begin by identifying familiar elements of the performance and the enable them to construct meaning in the performance. Alternatively, they may discard any narrative meaning and view the performance as pure entertainment, either when the performance lacks a clear narrative structure or the narrative elements are ambiguous. Both interpreting expected narrative elements and attempting to make sense of the absence of narrative elements has led some participants to question the relevance of mathematically describing these performances and has led others to eagerly explore mathematical descriptions. While some participants were resistant to describing the performances mathematically, claiming that mathematics was not relevant, others were able to share mathematical elements that they saw in the performance such as recognizable two and three dimensional shapes, symmetries, isometries, and wave patterns.

As we have begun to conduct these interviews we have also noticed the importance of getting participants to think with props similar to those used in the clips shown. Participants in our first focus groups were hesitant to engage with these props, but through encouraging our smaller interview groups and explorations ourselves, we have found that allowing participants to stand up and move under the constraints of these props has proven to be an effective means of getting people to explore the complexity of simple performance elements such as maintaining alignment with respect to multiple moving bodies, moving while keeping a piece of mylar taut, and folding a square into a right triangle. Participants have shared how the coordination of multiple bodies in space is deceivingly deceptive and very complex in execution.

In addition, asking performers to move under the constraints of props has also brought in their ideas about how time and rhythm and marked time are used as another tool to organize bodies in space. For example when one group was working with the square prop and trying to figure out how the performers in the Solar Panel People clip were able to fold a square into a right isosceles triangle, they discovered that if they stepped in synch to an agreed upon rhythm they were able to traverse equal distances and thus come together at the same time. Then if they took the same number of steps backwards, to the beat, they would form a taut triangle at the same time as well (Figure 1). The first image in Figure 1, shows remnants of the participant in the upper left corner moving his right arm up and down as he counts "1, 2, 3, 4" before the group moves on the same rhythm into the center as depicted in the second image in Figure 1. He embodies the rhythm in his arm before embodying it in his feet.

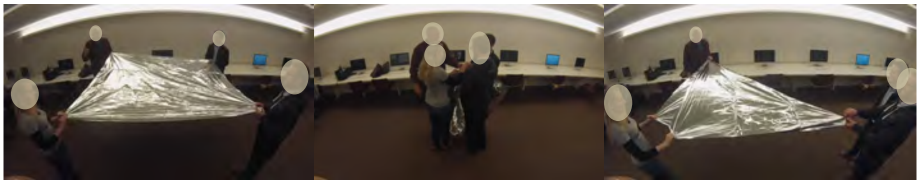


Figure 1. Quartet of participants folding a Mylar square into a triangle.

Preliminary Discussion and Conclusion

Our examination of these interview and analysis sessions has developed our understanding of how people use different resources to make sense of choreographed performances. It has also highlighted the role that mathematics plays in understanding the constituent structures of these performances. We have begun to see a relationship between recognizable narrative elements and mathematical elements to these performances. We have also identified affordances in having participants physically put their bodies on the line to reenact different elements from the performance, allowing them to get a better understanding of the choreographic rules at the local level. The number of participants in these interviews, however, does not afford many of our participants without performance experience with ways to think about how their local movements would need to be organized to produce visible global orders such as a grid. Nonetheless, this project is important for understanding how highly visible cultural events are (or can be) understood mathematically (i.e., extending how mathematics forms a significant component of culturally powerful and meaningful events), and also for understanding how human bodies can be used as a medium for creating and expressing these structures.

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A TEACHER'S GEOMETRIC CONCEPTUALIZATION AND REASONING IN TERMS OF VARIANCE AND INVARIANCE IN DYNAMIC ENVIRONMENT

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Purpose and Background

Developing sophisticated and abstract spatial thinking about figures, in terms of variance and invariance underlying mathematical structures that represent concepts and relationships between them, are critical for learning and teaching of mathematics at all levels (e.g Hadas, Hershkowitz, & Schwarz, 2000; Hegedus and Moreno-Armella, 2011; Sinclair, Pimm, & Skelin, 2012). This study uses a dynamic environment (Geometer's Sketchpad® (Jackiw, 1991, 2009) for iPad - Sketchpad® Explorer) to examine: (a) what conceptualizations and reasoning about the concept of circle, an in-service mathematics teacher demonstrate, in terms of variance and invariance?, (b) what are the key elements of such conceptualizations?

Framework & Methods

Invariant Property (IP) denotes to geometrical properties that remain unaltered when transformations on the geometrical object are enacted (e.g. Hadas et al., 2000). Data were collected from 90-minutes videotaped task-based interview with one certified 9th-11th grades teacher with five years of teaching experience from the U.S. (David - pseudonym). Task 1 was designed to explore ways to enable the existing of a circle, that passes through A and B and its center is C, by dragging point C (A and B are fixed). Point C leaves a trace that can be deleted by clicking a button (see Figure 1 that presents David's response to Task 1).

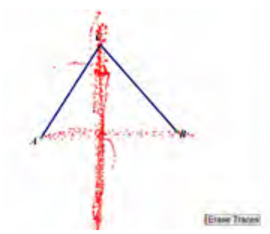

Figure 1: Task 1

Results

David offered different conceptualizations and reasoning of multiple circles. David offered to reflect point C with respect to AB and to keep the distance of C from AB equal and to keep AC and CB as equal distances (invariant properties) so that there would be exist a desired circle. Later on in the interview, David focused on continuous variation to conceptualize and to reason that under translation of point C along the perpendicular bisector of AB (maintaining dragging) it is possible to generalize that there are infinity many circles that exist and these circles are of different sizes.

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BERTIN'S RIGHT (ABOUT) ANGLE MEASURE: WE DON'T NEED TO BASE DEGREES ON 360

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Keywords: Geometry and Geometrical and Spatial Thinking, Cognition, High School Education

The fourth-grade Common Core State Standards for Mathematics describe a one degree angle as "an angle that turns through 1/360 of a circle." Degrees are often introduced in this manner in terms of a 360-unit composite. In this poster, I question this normative approach to developing degrees as a unit of angular measure and illustrate that students can develop coherent conceptions of degrees without leveraging a 360-unit composite. I draw on data from a yearlong teaching experiment (Steffe & Thompson, 2000), which was informed by the principles of quantitative reasoning (Thompson, 2011) and focused on students' understandings of angle measure. Here, I focus on Bertin, a ninth-grade student who participated in 15 thirty-minute sessions, and his ways of reasoning with two tasks (making a one-degree angle and the partitioned plane) involving degrees as a unit of angular measure.

When I asked Bertin how he'd make a one degree angle, he explained, "… if you get a 90-degree angle, you can divide that into nine. So it would be like 10 degrees each…and then you can divide each one of those into 10." In his explanation, Bertin indicated that he recursively partitioned a mental re-presentation of a right angle to produce 90 one-degree angles. Bertin's method for producing these one-degree angles incorporated a three-levels-of-units structure (i.e., a right angle partitioned into 9 ten-degree units, each of which contained 10 one-degree units).

Later, I presented Bertin with *The Partitioned Plane*—a dynamic geometry sketch wherein Bertin could set a whole-number parameter, n, to partition the screen into n equiangular parts. When I asked Bertin to produce one-degree angles, Bertin quickly set $n = 1$ and explained this gave "almost a straight line; I guess there's like one, small little degree somewhere in there." Moments later, I asked Bertin to set n to produce 90-degree angles. In sequence, Bertin set $n = 10, 2, 6, 7, 5$, and finally 4. Bertin explained that he settled on $n = 4$ because it looked like the edge of a square. When tasked with producing a 90-degree angle, Bertin did not anticipate a split of the plane into four equiangular parts. Instead, he resorted to a trial-and-error approach and stopped when he reached a perceptual configuration that was like "a square or a rectangle." Bertin's actions within the partitioned plane sketch indicate that he did not view the plane as a 360-unit composite angle that could be partitioned to produce other angles.

In conclusion, students can construct degrees as a unit of angular measure by taking a right angle as a 90-unit angular composite without viewing the plane as a 360-unit angular composite. Additional studies are needed to determine the prevalence of this conception of degree measure.

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PRE-SERVICE TEACHERS' UNDERSTANDING OF GEOMETRIC REFLECTIONS

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Keywords: Geometry and Geometrical and Spatial Thinking, Teacher Education-Preservice

In the mathematics education literature, there is little research specifically on how learners understand geometric reflections. Recent studies, such as Yanik (2006, 2011), have investigated PTs' learning and understanding of geometric translations in a dynamic geometry environment. His findings revealed that PTs have motion view based on their understanding of parameters and domain because use of dynamic geometric software (DGS) (e.g., GeoGebra, Geometer Sketchpad) may have some limitations for promoting PTs' understanding of geometric reflections as a mapping view. For instance, when they click on the "reflect about line" to perform a reflection, it is possible that they think of the line of reflection as a tool rather than a geometrical object. In other words, when students use features of dynamic geometric software such as the dragging modality, they may focus on as physical representations of the figures (e.g., movement of figures). Taken together, these empirical studies show that learners usually have a motion view of geometric reflections, which prevents their development of a mapping view. To help students develop a mapping view of reflection as a transformation, teachers need to know what factors are effective in facilitating the students' transition from motion view to mapping view (Hollebrands, 2003; Yanik, 2006).

The purpose of my study is to explore and investigate pre-service teachers' (PTs') understanding of the geometric reflections. To do this, I aimed to develop a model, a genetic decomposition of PTs to identify how schema development occurs between motion understanding and mapping understanding, what factors are effective to facilitate this development. To conduct this investigation, I created an environment in which I can explore PTs' schema development on geometric reflections paper-pencil. I will use clinical interviews to collect my data since it is difficult to observe PTs' understanding with direct observation. To explore PTs' understanding, I will use the Action-Process-Object-Schema (APOS) theoretical perspective as the conceptual framework. The research questions that will guide this study are as follows:

1. How does PTs' motion view of geometric reflections develop into a mapping view?
2. What factors facilitate PTs' motion view of geometric reflections into a mapping view?

Addressing the research questions of this study will be useful for teaching geometric reflections in the K-12 mathematics curriculum. More importantly, the findings and conclusions of this study will answer questions raised by the research community in regard to how motion view evolves and generate mapping view.

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GEOMETRY REPRESENTATIONS: WHICH ONES AND WHY DO TEACHERS USE THEM?

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Understanding representations and their place in the high school mathematics curriculum is vital to the study of geometry. Duval (2006) examined representations in a theoretical manner dividing them into four categories of a two-by-two matrix of discursive/non-discursive and mono-functional/multi-functional. Ainsworth (2006) devises a more practical approach in her DeFT framework. Pimm (2002) explored representations in a different manner, more specific for math education than Ainsworth; his descriptions of the many different representations are detailed and exhaustive.

In the presnt study, I follow Pimm's (2002) example, and list and analyze the representations presented in high school geometry classrooms. I observed five public school teachers in the Northeastern US during regular lessons on various content in geometry. After the observations, teachers shared their reasons for choosing the representations during recorded interviews. I analyzed the observations and the interviews looking for how often teacher used representations and common reasons for choosing them.

The teachers used many of the representations Pimm (2002) described like algebra, language, and coordinate grid. From the observations and interviews, I divided the geometry representations into thirteen categories: (1) spoken language, (2) written language, (3) gestures, (4) sketches/diagrams, (5) symbols in statements (e.g. $\angle A \cong \angle B$), (6) numerical, (7) algebraic expressions, (8) graphs on coordinate grid, (9) constructions with compass, straightedge, and other tools (10) physical objects, including manipulatives, (11) tables, (12) construction in dynamic geometry software, and (13) animations. Some representations were used more often than others; for example, spoken language and gestures were used by all teachers during all lessons, while animations and physical objects rarely and by only some teachers. Teachers had a hard time explaining the reasons for using certain representations. During the interviews, they often said that they did not think about it, that it was in the textbook, that it motivated students, that it was what the content required at the time, or that it was spontaneous.

Such results require more studies on representations and how they are taught in geometry. It seems that teachers do not have the background to discuss representations and to determine which ones are most useful in teaching certain concepts. They often rely on the textbook, spontaneity, or prior experience, and not on theory, which needs to be taught to pre-service teachers.

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LYDIA'S CIRCLE CONCEPT: THE INTERSECTION OF FIGURATIVE THOUGHT AND COVARIATIONAL REASONING

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We present data from a semester-long teaching experiment (Steffe & Thompson, 2000) examining preservice secondary teachers' covariational reasoning (Saldanha & Thompson, 1998). We use Piaget's (2001) distinction between figurative thought and operative thought to characterize the circle concept of one participant, Lydia. We define a figurative circle concept as an individual's ability to recognize or re-present a circle as a static form. In contrast, an operative circle concept includes an individual's mental image of a segment rotating about a fixed endpoint while the other endpoint traces out a circular path. We analyze Lydia's activities in two tasks—Going Around Gainesville (GAG) and Where Did They Go? (WDTG). In GAG, Lydia observed an animation of a car driving along a path from Atlanta to Tampa; a semi-circular portion of the path was centered about the city of Gainesville. We asked Lydia to create a graph relating the car's total distance traveled to the car's distance from Gainesville during the trip. In WDTG, we provided Lydia with a Cartesian graph displaying a car's distances from two cities, A and B. We asked Lydia to produce a path that the car could traverse to satisfy the graph.

Lydia exhibited a figurative circle concept in both tasks. Upon recognizing a semi-circular path in GAG, she drew in radii before concluding that the distance from Gainesville was invariant; this invariance was not available to her without carrying out the sensorimotor activity. As Lydia considered a horizontal segment on the WDTG Cartesian graph, she recalled that the curved path in GAG resulted in an invariant distance; however, the invariant distance did not necessitate a circular path in WDTG for Lydia. Instead, she relied on drawing particular curves and judging viability based on perceptual instantiations of the relevant distances. We hypothesize that an operative circle concept might have supported Lydia's covariational reasoning on these two tasks.

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MOVING BEYOND THE MORE A–MORE B CONCEPTION OF THE RELATIONSHIP BETWEEN AREA AND PERIMETER

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Keywords: Geometrical And Spatial Thinking, Middle School Education

Area and perimeter are important geometrical measurement concepts in school curricula. They build upon the ideas of linear measurement in the early grades and serve as the basis of topics such as fractions and multiplicative reasoning in the middle grades as well as the 3-D geometrical measurements in the upper grades. Prior studies have found the relationship between the area and perimeter to be a challenging topic (Battista, 2007). Often, students believe that an increase in area for a given shape results in an increase in perimeter of that shape, or vice versa. Tirosh and Stavy (1999) suggested that this misconception is rooted in the More A–More B intuitive rule. The goal of this study was to explore the visual and intuitive reasoning behind seventh-grade students' conceptions of area and perimeter. Particularly, this study sought to identify the concept definitions and images (Tall and Vinner, 1981) middle school students had about area and perimeter that could be used to build rich tasks that can further their development.

This study was conducted in a rural school district in a Midwestern town. The participants included 17 seventh-grade students. Each student was interviewed individually at the school, for 45 to 60 minutes, on a variety of measurement topics, including a task that asked students to find ways to cut a piece a piece off while making the perimeter bigger, stay the same or smaller. While the majority (13 out of 17) of the students initially believed that the shape with smaller area would have a smaller perimeter, 12 of them came up with some examples of shapes that would have smaller areas but either the same or a longer perimeter. Their examples and the reasoning provide a much richer set of concept definitions and images than have been documented in previous studies. For example, the idea that more turns or more segments would create a longer perimeter generated similar concept images for many students, but some students created a longer perimeter, while others did not.

Based on the concept definitions and concept images identified by this study, I proposed an instructional sequence to help students get a deeper understanding of the concepts of area, perimeter, and their relationship. Future study is needed to examine the effectiveness of such activities in helping students build a more solid understanding of area, perimeter, and their relationship. It would also be interesting to see if the effect of this sequence of activities can be transferred to the relationship between surface area and volume.

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Chapter 5

Inservice Teacher Education/Professional Development

Research Reports

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Brief Research Reports

Posters

AFFORDANCES OF THE CULTURAL INQUIRY PROCESS IN BUILDING SECONDARY MATHEMATICS TEACHERS' CAPACITY FOR CULTURAL RESPONSIVENESS

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Over the last couple of decades, there has been a growing call for teachers to become more responsive to the increasing cultural diversity of students as a means of improving students' experiences in school and their learning outcomes. Challenges exist in working with secondary mathematics teachers due to the common belief that math is culture-free and the lack of images of culturally responsive teaching in secondary mathematics classrooms. In this research, we explored the affordances of the Cultural Inquiry Process project in building inservice secondary mathematics teachers' capacity for cultural responsiveness.

Keywords: Equity and Diversity, Teacher Beliefs, Teacher Education-Inservice/Professional Development

Over the last couple of decades, there has been a growing call for teachers to become more responsive to the increasing cultural diversity of students as a means of improving students' experiences in school and their learning outcomes (Banks, 2001; Gay, 2010; Nieto, 2004). Studying culturally responsive teaching specifically within secondary mathematics education is important because mathematics education is its own cultural system (Nasir, Hand, & Taylor, 2008) and understanding that system is necessary for the types of awareness teachers need to be culturally responsive. In addition, supporting secondary mathematics teachers to be culturally responsive holds particular challenges. Mathematics is often seen as "culture-free" (Bishop, 1988), which can make the role of culture in the teaching and learning of mathematics more difficult to envision, especially as compared to language arts and social studies content. Related to this, there are few images of what culturally responsive teaching looks like in mathematics classrooms (Leonard, Napp, & Adeleke, 2009), which may contribute to the struggles mathematics teachers have in operationalizing cultural responsiveness in their practice (Morrison, Robbins, & Rose, 2008). Furthermore, secondary students often have particularly negative attitudes towards and low engagement with mathematics (Nardi & Steward, 2003), which can make it more challenging for teachers to interest students in the content.

In this study, we explored the nature of teachers' perspective changes that resulted from their completion of the Cultural Inquiry Process (Jacob, Johnson, Finley, Gurski, & Lavine 1996) project. The goals of the study were to describe teachers' perspective changes, relate these changes to how the teachers engaged with their CIP project, and analyze how this engagement may reflect a process of learning to be culturally responsive.

Rationale for Implementing the CIP Project

The purpose of the CIP project is "to broaden teachers' understanding of culturally diverse students and to maximize these students' success" (Jacob et al., 1996, p. 30). In the CIP project, teachers select a student whose behavior is puzzling, hypothesize potential cultural influences on the student, gather information about the student, design an intervention to support the student, and evaluate the intervention outcomes. Once the project is completed, the teacher turns in a report describing their work and reflecting on the experience.

We chose to use the CIP project with teachers because it is a form of action research. Action research has generally been established as a viable means of supporting teachers in improving their practice, understanding better themselves and their students, and developing dispositions to continue studying their practice (Zeichner, 1993). Also, because action research is situated in the teacher's setting, teachers can generate images of what a particular change looks like. Furthermore, action research can support teachers in co-developing their beliefs and practices. While debate exists whether to prioritize building teachers' beliefs or behaviors, Gay (2010) argues that, "The more important issue is that examining beliefs and attitudes about cultural diversity, *along with* developing cognitive knowledge and pedagogical skills, are included as essential elements of teacher education" (p. 151).

Besides being a form of action research, we valued the CIP focus on exploring cultural influences on students' engagement in school. Jacob suggests that teachers in the United States tend to develop tacit knowledge of students in terms of psychology, thus leaving teachers less equipped to understand students in terms of culture. The goal of the CIP project is to "provide teachers with new ideas and approaches they can use in the future in culturally diverse classrooms" (Jacob et al., 1996, p. 32). We also valued the CIP's focus on specific students, which as Jacob et al. (1996) argue, provides an opportunity for teachers to consider culture at a personal level rather than relying on cultural generalities. Lastly, we appreciated the CIP website (http://cehdclass.gmu.edu/cip/g/gs/gs-top.htm) created by Jacob to support teachers in conducting the CIP.

Theoretical Perspectives

We drew on two theoretical perspectives for our research.

Cultural Responsiveness

In order to analyze the degree to which teachers' perspective changes aligned with being culturally responsive, we used a framework we developed in related research (Parker, Bartell, & Novak, 2015). We view cultural responsiveness as dispositions grounded in cultural awareness to work to know, understand and support the engagement and learning of all students. Culturally responsive teachers, then, work to understand students' cultures and backgrounds and using such knowledge to support students' learning and cultural competence. Additionally, these teachers develop supportive student-teacher relationships based on culturally responsive care, have positive attitudes toward students' knowledge and experiences (i.e., reject deficit perspectives), and hold high expectations for student learning and achievement.

Critical Perspective

Another disposition scholars have identified as important to cultural responsiveness is being able and willing to employ a critical perspective to surface, question, and, when appropriate, change the normative beliefs and practices that may influence students' engagement and success in school (Bartolome, 2004; Valencia, 2010). Bartolome (2004) has described two aspects of this critical perspective: political clarity and ideological clarity. Political clarity includes being able to understand the influence of the broader culture, such as political, economic, and social variables, on subordinated students' academic performance. Ideological clarity includes being able to explore the ways in which one's own beliefs uncritically reflect those of the broader culture. With political and ideological clarity, teachers are able to understand the role of culture in creating a status quo and to make visible the beliefs and practices embedded in that status quo as a means of questioning how normative beliefs and practices may be limiting some students' access to school success.

In order to analyze how teachers' engagement with the CIP project might support teachers in developing cultural responsive dispositions, we analyzed the degree to which the teachers were able to surface, question, and ultimately change their beliefs and practices around normative practices when these practices appeared to be hindering students' success in school. We focused on examining how teachers critically examined the beliefs and practices related to their expectations for student behavior since perspective changes teachers described in their projects mostly related to their expectations of student behavior. This likely occurred because the project involved teachers selecting students about whom they were puzzled and this puzzlement invariably involved why students were not behaving in desirable ways, with "desirable" being a value embedded in school culture. Attending to student expectations is important because it is an aspect of school culture that commonly influence student success - particularly underserved students (Lane, Wehby, & Cooley, 2006; Valencia, 2010). To understand whether teachers were able to enact the process of exploring normative practices with respect to their influence on student success, our data analysis involved juxtaposing teachers' perspective changes with their self-described normative beliefs and practices related to their expectations for students' behavior. Finally, to understand the nature of the teachers' perspective changes, we analyzed the perspective changes against our framework for cultural responsiveness.

Data Collection and Analysis

The course that served as the context for this study was called *Culture in the Math Classroom* (CIMC). It was a required course that was part of a graduate program for practicing secondary mathematics teachers at a public doctoral-granting university in the Rocky Mountain region. The data we collected were 58 CIP projects submitted by teachers over four semesters of the CIMC course. The mathematics teaching experience of the teachers ranged from 2 to 22 years, with an average of 7 years. Almost all of the teachers identified as white. About 60% were women and 40% were men. The teachers taught in suburban or rural schools that typically had 27-35% minority students. Hispanic students were the largest minority group in most schools. The schools had 7-10% of students classified as English language learners and between 26-43% classified as low-socioeconomic status.

Our research questions were:

1. How did teachers' perspective shifts relate to their normative beliefs and practices about student expectations?
2. To what extent did the teachers' perspective shifts indicate a building of capacity to engage in culturally responsive pedagogy?

For data analysis, we drew on narrative inquiry, which is a set of tools and perspectives based on the idea that it is possible to explore how people make sense of their experiences through the stories they tell (Clandinin & Connelly, 2000). We chose narrative inquiry because 1) we wanted to understand teachers' perspective shifts in terms of their experiences conducting the CIP project and 2) the CIP project reports typically were written as stories.

The first step in analyzing the project was to identify projects with valid perspective changes. To do this, we included in the data analysis pool only those projects in which the teachers 1) followed the CIP project guidelines, 2) wrote reflections that appeared to align with their project activities, and 3) included in their reflections a description of a perspective change likely to influence their practice. Thirty of the 58 projects met all three of these criteria. The focus students in the 30 projects consisted of 12 girls and 18 boys. Four of the students were identified as white and in 12 cases, the student's race or ethnicity was not specified by the teacher, likely because the focus student was white. In 14 projects, the teachers identified their focus student as being of mixed race/ethnicity (N=3), Hispanic (N=8), African-American (N=1), or Native American (N=2).

The next step in the data analysis was to identify the student expectations the teachers' perspective changes related to. In 28 of the 30 projects, the teachers explored one or more of the following student expectations:

- Students should seek help if they are struggling (N=8).

- Students should complete their homework (*N*=14).
- Students should engage in class discourse (*N*=8).
- Students should value learning mathematics (*N*=6).

For each student expectation, the set of projects involving this expectation were analyzed to discern the normative beliefs and practices the teachers associated with the expectation, the types of interventions the teachers implemented to address their focus student's behavior, and the nature of the perspective changes the teachers described.

Normative Beliefs and Practices about Student Expectations and Perspective Shifts

For each of the four normative student expectations we identified in the teachers' CIP projects, we illustrate the beliefs and practices associated with them by describing how the teachers identified the expectation, why they believed the expectation was important, and how they typically responded to students who did not comply with the expectation. We then describe the types of interventions the teachers implemented and the perspective shifts the teachers had with respect to the student expectation.

Students Should Seek Help if They Are Struggling

Eight projects focused on the expectation that students should seek help if they are struggling. Half of these teachers mentioned that they thought the struggling student should seek help because not seeking help was limiting the student's success. Seven of the teachers provided some indications of how they would normally respond to struggling students who did not actively seek help. Four teachers suggested they would have responded in a limited way. For example, one teacher said he would invite students in for assistance outside class, but if they did not take him up on his invitation, he would not pursue the issue further. Another teacher said she would help the student pass the course, but would not think more broadly in terms of getting the student support outside of class. Statements made by three teachers indicated they would not actively respond to the student. For example, one teacher wrote that initially he had assumed that his focus student did not care about learning. Another teacher wrote that at the outset of the project he did not know how to support shy, low-achieving students.

The intervention teachers often selected for their focus student involved additional tutoring outside class. Sometimes the teacher did the tutoring and sometimes another tutor was found. Another intervention was changing seating arrangements in class so the student was more comfortable talking to at least one classmate. Other interventions were particular to the focus student's needs. For example, one teacher wanted to build the focus student's confidence in discussing mathematics and arranged for the focus student to be a tutor.

Themes in the teachers' perspective shifts included 1) the teacher needed to be proactive in finding additional supports for the struggling student and 2) in order to provide appropriate supports, the teacher needed to communicate with the student to understand the student's perspectives, issues, and goals.

Students Should Complete their Homework

Students needing to complete their homework was addressed in almost half (*N*=14) of the analysis pool projects. Seven of these teachers focused exclusively on supporting their student to complete homework and another five teachers focused on homework as well as the student's participation in class discourse. Typically, homework completion, and sometimes correctness, was part of the course grade, so when students did not complete their homework, their course grade suffered. However, the teachers believed that doing homework improved a student's learning of the mathematics, and this in turn, affected how well students performed on assessments, which also affected the course grade. The majority of teachers (*N*=10) made statements suggesting they did not initially have a way of responding to students who do not complete their homework. Four of these teachers attributed a student not completing homework to characteristics of the student and/or their family.

Teachers created different types of interventions to support their focus student with completing homework. Six teachers focused on the student feeling more comfortable and successful in class, such as by interacting with the student more often. Four teachers focused on helping the student be more organized to know what the homework assignments were, when they were due, and what materials were needed. Two teachers found time for their student to start homework in school. Two teachers changed their homework policies to accommodate their students. One of these teachers decided to grade only the homework the student completed in class, as long as the student did well on assessments. The other teacher disallowed partial credit for late homework for the focus student as he admitted he postponed doing homework if he knew he had some leeway.

Teachers indicated that they learned that a student not completing homework was not necessarily because the student was lazy or did not value their learning, but instead may be due to psychological or logistical issues that could be addressed at school. One teacher found out his focus student had experienced a trauma the previous year for which she had kept silent and therefore had not received any support. Once that was addressed, her school engagement, including homework completion, improved. Besides the two teachers who elected to change their homework policy, two other teachers questioned how they were approaching what they assigned for homework and even the necessity of homework. While these teachers decided to retain homework in the short run, they indicated they thought that homework merited further consideration. Most of the other teachers indicated that they could influence a student's tendency to complete homework by building a better relationship with the student so that the student felt cared for by the teacher and so the teacher better understood the student's needs.

Students Should Engage in Classroom Discourse

Eight teachers expressed concern that their focus student did not engage appropriately in classroom discourse, such as contributing to whole class discussions or engaging productively with group members. Only one teacher focused exclusively on this normative expectation. Other teachers lumped this with one or more other behaviors, such as not completing homework. Two teachers described reasons for students to engage in classroom discourse other than it generally supporting student learning. One teacher explained that the curriculum she used included an expectation that students discuss their ideas with each other. The other teacher cited research that reported good social skills were valued by employers. The teacher believed that high school was a good time for students to develop these skills. None of the teachers indicated ways in which they would normally respond to students who did not engage in classroom discourse. One teacher noted, "As a staff we have focused on the students that are culturally different for more visual reasons (e.g., race, clothing, etc.) but we haven't focused on the other students who are culturally different for less obvious reasons such as beliefs about social interactions."

Teachers' interventions for addressing this normative expectation included redirecting off-task behavior, changing the student's seat assignment, assigning competence to the student (Horn, 2012), speaking directly with the student about what behavior is desired and why, providing positive reinforcement for the desired behavior, and developing a closer relationship with the student.

Teachers' reflections focused on what they had learned about their ability to influence the classroom culture, which in turn could influence student engagement.

Students Should Value Learning Mathematics

Instead of focusing on the student's behaviors, six teachers ended up attempting to address their student's lack of interest in learning mathematics. In all but one case, the teachers' puzzlements were focused on their students' behaviors, but after speaking with their students during data collection, they decided to address their students' motivation to learn mathematics. These teachers did not provide much detail on why the students should value learning school mathematics, although, in the context of the project, there was an implicit assumption that students valuing learning mathematics would lead them to be more successful in class and even in completing high school. The teachers did not indicate they had particular ways of addressing their students' lack of interest in learning mathematics. One reason for this is captured in a teacher's question: "How do you reach a student who does not see value in mathematics?" which suggests readily available methods of approaching this issue were not known.

All of these teachers included as part or as the entirety of their intervention creating one or more lessons they believed incorporated context that would have more meaning for their focus student and possibly the class as a whole. In their project reflections, all six teachers indicated they felt more empowered to influence students' interest in mathematics by introducing context in their instruction that related better to students' lives.

Perspective Shifts and Cultural Responsiveness

The normative expectations teachers focused on in their projects were behaviors and attitudes teachers believed students should comply with to be successful in school. Teachers who reflected on their own beliefs and lived experiences found the normative expectations so obviously appropriate that they had a hard time imagining why someone would not follow them, except if the person were somehow "faulty," such as being lazy or not caring about their education. When the teachers attributed non-compliance to a student's inherent characteristics, the teachers typically did not feel able to address the mismatch between students' behaviors and what was expected of them.

While the teachers tended to focus on one, or at most two, of the four identified normative expectations of students, the teachers typically expressed perspective shifts in more general terms about how to approach supporting students. We found that overall, teachers' perspective shifts aligned with some aspect of the following narrative:

> I need to be proactive in knowing my students' backgrounds, perspectives on school, and goals. This process of knowing my students allows me to understand how to meet my students' needs and helps me to build better relationships with my students. Often, my assumptions about the student, such as that they do not care about their education or are lazy, are incorrect. Instead, school policies and the classroom culture can be negatively impacting students' engagement. Working on making the classroom a safe place, creating lessons that are meaningful to students, and implementing supports outside the classroom are within my power as a teacher.

No teacher expressed a perspective shift that encompassed the entirety of this narrative. However, most teachers emphasized one or two elements of the overall shift. This narrative contrasts with the teachers' normative beliefs and practices as it expresses:

- A stronger sense of responsibility for getting to know their students and responding to their needs
- An awareness that their assumptions about their students' perspectives and capabilities might not be accurate
- More self-confidence in their ability to design effective interventions for students

- More open-mindedness in considering the need to change their practices or school policy to better support students

The perspective shifts indicated by the teachers suggest they built their capacity for being culturally responsive in that teachers were more willing to take up the work of understanding and supporting students, less likely to engage in deficit thinking, and more confident and open in their thinking about ways to support students' school engagement and achievement.

In terms of developing a critical perspective, the teachers in this study appeared to exhibit more ideological clarity than political clarity in that they questioned their personal normative beliefs and practices related to the expectations they held for students, but did not reflect on the influence of the broader school and societal cultures. This could be a function of the CIP project or the way it was facilitated. It could also be representative of the way in which people's cultural awareness evolves "from a self-centered state to identification with society and eventually to the larger global community" (McAllister & Irvine, p. 18, 2000).

Discussion

The nature of the perspective changes teachers expressed aligned with being culturally responsive in that teachers indicated an increased willingness and confidence in reaching out to, understanding, and supporting students they initially find puzzling. We cannot make definitive claims about what aspects of the CIP project might be responsible for the ways teachers engaged with and learned from the project, but we do believe that the project characteristics we identified in our rationale to implement the CIP are likely mechanisms. Another factor might have been teachers' propensity to be empathetic towards their students. Teachers sometimes wrote about their experiences with their students in moving and passionate ways. Goodman's (2000) research suggests that empathy is often one of the orientations that can move people from privileged groups to take action towards social justice. It is possible many of the teachers in the CIMC course had this orientation, which contributed to their engagement with and learning from the CIP project. Understanding what teachers may bring to the work of building cultural awareness can support teacher educators in conceptualizing teachers as "competent learners who bring rich resources to their learning" (Lowenstein, 2009, p. 187).

Additional factors may have influenced teacher engagement with the CIP project. First, during the project implementation, the instructors offered feedback to the teachers at several points, which may have also influenced teacher thinking. Second was their participation in the CIMC course. The course content and other activities may have supported the development of knowledge and skills that pre-disposed teachers to explore the role of culture more successfully than they would have otherwise. The beginning of the course delved into theoretical foundations by asking: What is culture? How can examining students' home culture influence their experiences at school? Is mathematics culture-free? What are the central tenets of culturally relevant pedagogy? The remainder of the course explored issues involving culture and student learning including student motivation, status and small group work, language in the classroom, the purpose of mathematics education, and teaching math for social justice. Teachers also completed projects analyzing case studies, exploring an aspect of their local community, and implementing a change in their instruction to improve student motivation.

A limitation of our work is that we did not follow the teachers into their schools and classrooms after they completed the course, so we cannot describe what teachers brought into their ongoing practice. An area of further exploration would be to explore the impacts on teachers' cultural awareness and cultural responsiveness in the projects where teachers did not indicate a perspective change. In half the projects, though, teachers' engagement with the CIP project indicated they created experiences in their practice that modeled being culturally responsive and developed their own perceptions about how this would influence their practice going forward. The CIP project, then, appears to have affordances for building the capacity of secondary mathematics teachers for being culturally responsive.

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CHARACTERIZING IMPACTS OF ONLINE PROFESSIONAL DEVELOPMENT ON TEACHERS' BELIEFS AND PERSPECTIVES ABOUT TEACHING STATISTICS

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With online learning becoming a more viable option for teachers to develop their expertise, our report shares one such effort focused on improving the teaching of statistics. We share design principles and learning opportunities, as well as discuss specific impacts evident in classroom teachers' course activity concerning changes to their beliefs and perspectives about statistics. Specific course experiences that served as triggers for critical reflection are discussed.

Keywords: Teacher Education-Inservice/Professional Development, Teacher Beliefs

Statistics has gained a prominent place in middle and high school curricula through the National Council of Teachers of Mathematics (2000), Common Core State Standards (National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010), and recommendations endorsed by the American Statistical Association (Franklin et al., 2007; Franklin et al., 2015). Professional development (PD) for secondary teachers to develop their statistical content and pedagogy are being offered across the country, typically on a small local scale, and these often include focused evaluation and research efforts to document impacts. However, the need for preparing teachers to teach statistics is much bigger than what can be addressed with small local programs. In this paper, we discuss a way of leveraging the internet to assist in a solution that is free, open access, and can reach many more teachers across geographic boundaries (Kim, 2014). With an online solution at a much larger scale, methods for examining impacts must also evolve. We offer a glimpse at one effort to use course participants' online activity, forum discussions, and self-reported changes on surveys as a way to measure impact.

For a "massive" and "open" course, there are many design challenges to meet the needs of participants with varied backgrounds. Massive Open Online Courses (MOOCs) are designed and delivered in a variety of ways, depending on learning goals for participants, to serve different target populations and provide diverse experiences for learners. In recognizing the potential for MOOCs to serve as large-scale PD, some are crossing local boundaries to design MOOCs specifically for Educators (MOOC-Eds, Kleiman, Wolf, & Frye, 2014). Those that engage in and study impacts of professional development for mathematics and statistics teachers must consider how this new frontier can potentially assist in developing teachers' content understanding and pedagogical strategies for improving practice, and forming global communities of educators. To contribute to the synergistic discussion needed at this crossroad, our focused question for this report is:

> *How can the experiences in an online professional development impact participants' perspectives about the nature of statistics and teaching statistics? Which resources and experiences in the course seem to influence any changes in perspectives?*

Literature and Framework

Beliefs and perspectives about statistics include a teacher's ideas about the nature of statistics, about oneself as a learner of statistics, and about the classroom context and goals for students' learning statistics (Gal, Ginsburg, & Schau, 1997; Pierce & Chick, 2011; Eichler, 2011). Certain beliefs would likely lead to different teaching practices. For example, if a teacher believes that statistics is a way of quantifying data and that the many procedures available in statistics for computing measures lead to such quantification, his or her teaching practices may favor a focus on statistical procedures and have less emphasis on the context of the data, the process of ensuring good data is collected and available (sampling methods), and making claims about data that are uncertain in nature (Pierce & Chick, 2011). Eichler (2011) further discusses how the focus of teachers' intended curriculum in statistics can be considered on a continuum from traditionalists (focused on procedures absent of context), to those wanting students to be prepared to use statistics in everyday life (focused on engaging in an investigative process that is tightly connected to contexts of real data). A goal in statistics teacher PD is to move teachers along this continuum towards a focus on investigative processes, which requires impacting teachers' beliefs about the nature of statistics and learning goals for students related to statistics.

Professional development that includes accessible, personalized, and self-directed elements can provide increased opportunities for sustained, collaborative and meaningful work among teachers that can affect their knowledge, beliefs and practice (e.g., Vrasidas & Zembylas, 2004). Researchers have found that online professional development (OPD) that addresses the varied needs and abilities of its participants can be effective in changing teachers' instructional practice (e.g., Renninger et al., 2011; Yang & Liu, 2004). Designers of OPD should be especially mindful that activities are meaningful, accessible and relevant so that participants can apply their learning to their educational context (Ginsburg, Gray, & Levin, 2004; Vrasidas & Zembylas, 2004).

Just as communities of practice can exist in face-to-face PD, OPD should facilitate development of an online community of practice (CoP). Researchers have highlighted benefits of such communities that are not always afforded in traditional face-to-face PD. For example, Mackey and Evans (2011) argued that online CoPs provide members with "extended access to resources and expertise beyond the immediate school environment" (p. 11), thereby offering ongoing PD and the potential for increased application in their classroom. Designers of OPD should build infrastructure to support such communities across geographic and time zone boundaries. Asynchronous discussion forums, for example, provide opportunities for participants to reflect on practice, exchange ideas, and discuss ways to improve on their own schedules with colleagues with whom they may not otherwise interact (Treacy, Kleiman, & Peterson, 2002).

While making changes in teachers' statistics teaching practices and ultimately changing students' learning of statistics is a major goal, we are guided by the integrated model for PD proposed by Clark and Hollingsworth (2002). In this model, they represent the change process for teachers through PD as being one that includes reflection and enactment among an external domain and a teacher's professional world that includes domains of personal, practice, and consequence. The external domain includes information and resources often experienced through a PD, including interactions with others. In our study the external domain includes the resources in the OPD and the discussions with others in forums. The personal domain includes one's knowledge beliefs and attitudes. The practice domain includes any professional experimentation, with content or instructional strategies, and the domain of consequence is concerned with salient outcomes that result in practice. Because of the massive size of our OPD about teaching statistics, we are most concerned with the reflections and enactments between the external domain (experiences and resources in the OPD) and the reflections and enactments we can discern concerning their beliefs and perspectives about statistics and teaching statistics. Though some teachers may be able to engage in professional experimentation during the course, this is hard to examine given everyone's different curriculum and timing of when statistics units may be taught. To aid us in considering how PD experiences may have an impact on teachers' beliefs and perspectives related to statistics, we draw upon Mezirow's (2009) theory of transformational learning in adult education. Specifically, we are interested in what stimulus in the OPD (external domain) may act as triggers to evoke dilemmas (or cognitive dissonance) for teachers where they question their understandings or perspectives that have been formed from prior experiences.

Online Professional Development Context

The MOOC-Ed effort at the Friday Institute for Educational Innovation includes several courses built using research-based design principles of effective PD and online learning (Kleiman, Wolf, & Frye, 2014) that emphasize: (a) self-directed learning, (b) peer-supported learning, (c) job-connected learning, and (d) learning from multiple voices. One such course, *Teaching Statistics Through Data Investigations* [TSDI], aimed to have participants think about statistics teaching and learning in ways likely different from their current practices and past experiences. The course did not focus on a particular grade band or specific statistical content. A major goal was for teachers to consider statistics as an investigative process, promote statistical habits of mind, and view learning statistics from a developmental perspective.

The TSDI course consisted of an orientation unit and five units (http://friday.institute/tsdi). The course was open for about 15 weeks to allow for flexibility for participants to engage while managing their busy professional lives. Units began with an Introduction video of the instructor highlighting critical aspects of teaching and learning statistics in the unit. The Essentials included materials to read or watch. The design principle of learning from multiple voices guided the decision to include many videos of Expert Panel discussions with the instructor and three experts in statistics education. Multiple voices were also present in many classroom videos with teachers and students working on statistics tasks using various technology tools, as well as, animated illustrations of real students' work were created (using tools like Go Animate or Powtoon) that represented students' statistical reasoning and use of technology tools.

Self-directed and job-connected learning opportunities included Dive Into Data experiences in each unit for participants to use a variety of free technology tools (e.g., Gapminder, Tuva, CODAP, GeoGebra simulations) or import data into their own data analysis tool (e.g., Fathom, StatCrunch). These experiences allowed teachers to use tools accessible in their schools and connected them to relevant and free sources of data that can be useful in their lessons. For example, in Unit 4, the Dive Into Data used the Census at Schools website and asked teachers to download data and engage in statistical investigation. Extensions in each unit include extra resources (e.g., data sets, lesson plans, brief articles, java applets, additional videos) and provide self-directed opportunities to explore resources that may be useful in their educational context.

Peer-supported learning is a cornerstone of the MOOC-Ed experience. Since participants are geographically dispersed, it is important to provide focused and ample opportunities for them to connect with and support one another in learning and applying the material in the course. Each unit contains two discussion forums: 1) a forum focused on discussing a specific pedagogical investigation about aspects of teaching statistics (e.g., analyzing statistics tasks, considering students' approaches to statistics tasks through video clips), and 2) a forum where participants start their own discussions about unit materials or other ideas related to teaching statistics.

Building upon an existing framework (GAISE, Franklin et al., 2007), the development team incorporated recent research on students' statistical thinking and productive statistical habits of mind into a new framework, Students' Approaches to Statistical Investigations [SASI]. There were several learning opportunities for participants to develop an understanding of its importance and potential ways it can influence teaching. The diagram in Figure 1 shows the investigative cycle, reasoning in each phase can occur at three levels of sophistication, and productive habits of mind. Two brief PDF documents described statistical habits of mind and the framework. In a video, the instructor illustrated the framework using student work from research, and another video featured an expert illustrating development of the concept of mean across three levels. Participants could watch two animated illustrations of students' approaches to an investigation using different levels of statistical sophistication and then discuss students' work. More details about the design of the TSDI course can be read in Lee and Stangl (2015).

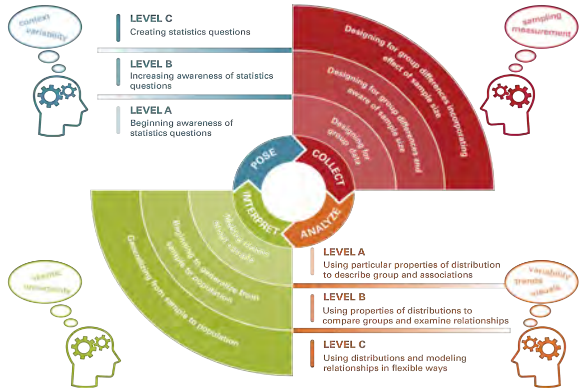

Figure 1. Framework for supporting students' approaches to statistical investigations.

Methods

Participants

Though the course is offered several times per year, we focus here on Fall 2015, the second run of the course. The course was advertised through many different organizations (NCTM, ASA, CAUSEweb), social media posts, state-level leaders in mathematics education, and personal contacts. Enrollment was free and open. For the purpose of this paper, we are only interested in how course experiences could be impacting beliefs and perspectives of K-12 classroom teachers. From registration (n=827), we discerned demographic characteristics and focused on those participants self-classified as classroom teachers (n=489). The classroom teachers resided in 46 states and 29 countries, with majority in the US (n=380) and New Zealand (n=48). The majority were female (67.5%) and 72.8% had a master's degree or above. Their years of experience in education were evenly distributed, creating a diverse community with varied teaching experiences that impact their starting perspectives and growth opportunities. Of those 489 self-identified classroom teachers, we confirmed 412 were actively working in K-12 contexts (e.g., some classified themselves as classroom teachers but taught community college).

Data Sources and Analysis Methods

Aside from registration data, four other data sources were used. Course activity was tracked through click logs that allowed us to examine trends in participants' engagement with material. Qualitative data was collected from three sources: 1) posts in discussion forums (two per unit, for a total of 10 forums), 2) open-ended responses to end-of-unit and end-of-course surveys, and 3) a follow-up survey sent 6 months after the course to inquire about how participants may have applied their learning and what they considered the most impactful ideas from the course.

All registration and click log data was merged and displayed in a dashboard in Tableau that allowed investigators to visualize participants' engagement over time and with certain types of resources. The dashboard facilitated the ability to filter by role of classroom teacher, so that we could examine and report on trends of the 489 participants classified as classroom teachers. Descriptive statistics and graphical displays were used to examine engagement patterns. All discussion forum log data was filtered to only include posts from those classified as classroom teachers. Because our research question was focused on how K-12 classroom teachers' beliefs and perspectives may be impacted by course experiences, it was important to maintain information about which activity and unit the discussion forum was embedded in, but not information about all posts from all participants. For the results reported in this paper, we are not discussing the impacts of particular ideas posted by participants or the social networks that formed in the forums as an indication of a community of practice (this is part of a larger study). The 977 discussion forum posts were analyzed using open coding guided by our focus on change in beliefs and perspectives related to teaching statistics. Each post was considered a unit of analysis and we were specifically coding instances where teachers self-reported changes or shifts in their beliefs and perspectives. The coded posts were sorted until four themes emerged. Within the themes, the posts were re-examined and tagged for evidence of what seemed to be triggering the change in perspective. We documented which triggers were most prevalent and only kept triggers associated with many instances of impacts on perspectives and beliefs, and discarded those only occurring once or twice. These themes and triggers were then used to examine responses on the end-of-units, end-of-course, and follow-up surveys. While we were looking for confirming and disconfirming evidence of themes and triggers, disconfirming evidence was not evident and no new themes or triggers were documented.

Results

Engagement

When registration opened, participants could enroll and engage in the Orientation unit that included an overview video, survey to self-assess their confidence in teaching confidence, and an introduction forum. Each unit opened in weekly intervals for 5 weeks, with earlier units always accessible when later units opened. This allowed for participants to start and engage in course material on their own time and pace, which is part of the self-directed design principle. Once Unit 5 opened, the entire course remained active for seven more weeks. After the course closed, participants could still access course material and read discussion forums in a read-only format (no new posts allowed in forums). In this way, the course site remained an open resource.

Overall, a majority (n=370) of enrolled classroom teachers engaged in various aspects of the course. Thus, we use 370 as the number of classroom teachers who began the course. With respect to accessing the course by units, the greatest number of classroom teachers accessed Unit 1 followed by Unit 2. In Unit 1, 293 classroom teachers engaged, but by Unit 5, only about 25% (n=92) of the classroom teachers that began the course had accessed material. The number of classroom teachers accessing the course in Units 3-5 was relatively the same, indicating that most classroom teachers who engaged through Unit 3 finished the course in its entirety.

The most accessed resources were discussion forums and videos (instructor videos, expert panel, and classroom and student work videos). 206 of the classroom teachers who began the course (56%) posted to a forum. The frequency of posts per teacher who engaged in forums is a skewed distribution (Figure 2), with 57% of teachers posting 1-3 times (typically in Units 1-2), 38% posting 4-14 times across several units, and 11 very active teachers posting 15-45 times. The levels of engagement in discussion forums and videos were highest in Units 1 through 3. Teachers' highest level of engagement with videos was in Unit 3 where 93 teachers took advantage of the video-based learning opportunities related to the SASI framework.

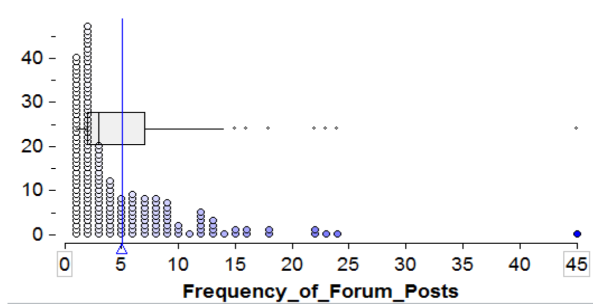

Figure 2. Frequency of posts per teacher across all discussion forums.

Impact on Perspectives and Beliefs

Looking carefully at themes from our coding, we saw four major ideas emerge related to how teachers' beliefs and perspectives may have changed:

1. engaging in statistics is more than computations and procedures and should include investigative cycles and habits of mind;
2. engaging in statistics is enhanced by the use of dynamic technology;
3. engaging in statistics requires real (and messy) data; and,
4. statistical thinking develops along a continuum.

Due to space limitations, we only elaborate on the first and last with examples from teachers.

We noticed a shift begin in Unit 1 with participants thinking about statistics as *more than computations and procedures*. This was expanded by posts in later units and evident in the survey responses. There were two aspects to this shift in perspective. The first seemed to be a realization that the statistics they experienced and that they tend to teach was too focused on procedures, and that this focus was not aligned with what they were experiencing in TSDI. For example, a teacher started a discussion thread detailing a dilemma because of points made in a video by statistics education experts in Unit 2. The extensive post began as:

> I had a "lightbulb moment". Although I have been teaching HS math for 24 years, I have never actually taught "statistics" as defined by the members of the expert panel. I have taught units that I THOUGHT were statistics, but I was merely providing students with a few mathematical tools that statisiticians [sic] can use (e.g. finding a mean, making a histogram, calculating a standard deviation, etc.)...(female, 24 yrs experience)

Twelve participants joined that discussion, 10 of which were teachers. They echoed that they were "guilty" of teaching statistics this way and that their own prior experiences in learning statistics treated the subject in a procedural manner. Similar discussions and replies about this issue were also started by several others. Teachers also recognized that attending and engaging in all parts of an investigation would give students opportunities to make sense of how statistics is used to answer questions and how important data collection (or experimental design) is to the process. Many admitted they spent little time on this with students and aimed to improve.

Related to the final idea that emerged, teachers seemed to realize that statistical thinking and understanding develops across a continuum and that they can use this to think about instructional decisions and assessment of students. "The idea of the 4 process cycle and the different levels for different ages of each process, has helped me lot. I understand more and feel I am a better teacher to my students" (female, 15 yrs experience). Considering statistics as developing across levels seemed to impact many teachers. For example, after commenting on students' work in a video and describing what levels she thought students may be working at, a teacher (10 yrs experience) noted, "with the SASI framework, I like how it never mentions age or grade level. I feel it's a continuum that students, depending on the context, can move back and forth between. If they get to a harder problem, they may not know how to exactly collect the data without bias and ensuring randomness. But with an easier experiment, that may be more obvious to them."

Triggers for Dilemmas

Four elements emerged as often cited for triggering critical reflection that had impacts on beliefs and perspectives about statistics and teaching statistics. By far, the *SASI framework* (and associated documents and multimedia) was the most dominant trigger for change. The *expert panel video discussions* and the *videos of students and teachers* engaged in statistics tasks were also dominate triggers to assist teachers in reconsidering their prior experiences in learning and teaching statistics, and help them envision a different outcome for their students if they change their practices. The *use of technology for visualizing data with real data* that are multivariable and “messy” was an additional trigger that seemed to impact perspectives. The technology experiences directly influenced their ideas that engaging in statistics is enhanced by using dynamic technology tools and real world messy data. These triggers came from learning opportunities that included videos of students and teachers engaged with messy data using technology, discussions in expert panel videos, and opportunities to Dive into Data. Two frequently referenced experiences were using the Gapminder tool, and engaging with Census at School. When asked the most valuable thing they learned on a follow-up survey, a teacher responded as follows, with triggers bolded that may have sparked her learning.

> “The most valuable aspect of the MOOC was obtaining **resources for the improved use of technology** to make instruction come to life and be more meaningful to students. I was able to **see the statistical process in action** and now have an idea of **what it should look like in the classroom**.” (female, 19 yrs experience)

Discussion

One of the challenges in designing OPD for teachers is identifying how to leverage stimuli that has the potential to act as triggers to impact teachers’ beliefs about teaching. For those who are at a crossroads facing this challenge, whether face-to-face or online, our identification of triggers can provide guidance as they embark on PD efforts for teachers. While we have no evidence (yet) that teachers experiences in a brief OPD in teaching statistics has impacted actual teaching practices and students’ learning, our research indicates that the purposeful design elements of the course were successful in causing critical reflection through certain triggers. Many teachers appear to have moved along the continuum described by Eicher (2011) towards beliefs that we should engage students in doing statistics through investigations, not merely teach them mathematical tools to apply to numbers devoid of context.

Teachers are attracted to and can make sense of how frameworks apply to their practice and within the context of their learning environments. In addition, they learn a lot from expert opinions (beyond just a single PD instructor), as well as from the voices and experiences of other teachers with whom they collaborate with as part of the course. These voices act as additional resources outside of their physical school environment (Mackey & Evans, 2011) to impact their perspectives about statistics and teaching statistics. In accordance with other researchers, the discussion forums indeed provided opportunities for critical reflection about teaching statistics.

Teachers also learn a lot about what it means to engage in statistics, by doing it, as well as from examining students’ thinking. Is any of this a big surprise? Perhaps not to experienced teacher educators. However, the key is to include these types of learning opportunities and potential triggers in professional development that occurs online, whether it is to a local group or massive and open to teachers around the world. Our research also supports the idea that online professional development that emphasizes: (a) self-directed learning, (b) peer-supported learning, (c) job-connected learning, and (d) learning from multiple voices can be effective for areas in mathematics education (e.g., teaching statistics) that need wide-scale efforts.

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IMPROVING KNOWLEDGE OF ALGEBRAIC LEARNING PROGRESSIONS THROUGH PROFESSIONAL LEARNING IN COLLABORATIVE VERTICAL TEAMS

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The following research report documents a professional development that focused on promoting understanding of algebraic learning progressions across vertical teams of elementary and secondary teachers. Analyzing quantitative data from the pre and post MKT assessment and qualitative data from lesson reflections and final course reflections, revealed multiple outcomes for teachers. More specifically, elementary teachers had lower MKT than the secondary teachers in both pre and post assessment, and teachers exhibited a significant increase in their MKT as a result of the program. Further, all teachers gained more in depth knowledge about the learning progression of algebra through collaboration in vertical teams and revealed a focus on improving pedagogy through the use of high leverage teaching practices.

Keywords: Algebra and Algebraic Thinking, Learning Trajectories (or Progressions), Mathematical Knowledge for Teaching, Teacher Education-Inservice/Professional Development

States and districts often have to realign their curriculum when standards are revamped. In fact, forty-two states, the District of Columbia, four territories, and the Department of Defense Education Activity have adopted the Common Core State Standards and had to undergo realignment with their curricular standards (NGACBP & CCSSI, 2010). Although our state did not adopt the Common Core State Standards, the state standards were revised and realigned with some standards expanded to bring more rigor, depth, and breadth to the learning objectives. As district leaders, mathematics educators, and teacher leaders worked with classroom teachers to unpack the standards, we designed professional development that focused on helping teachers learn the mathematics concepts more deeply by mapping out the learning progressions that will guide the sequence of concepts crucial in building mathematical understanding. The importance of this knowledge for teachers leads to a critical, practical question: what professional development experiences are necessary for teachers to develop an understanding of the learning progression in algebraic thinking?

This study is aimed at examining the ways in which the designed professional development can engage teachers in deepening their understanding of algebra. Further, we wished to explore how engaging in vertical teams for collaborative planning of lessons can help teachers at different grade levels better understand students' learning progression of algebraic thinking.

Theoretical Framework

In the math community, the term "learning progression" has been used to describe the vertical articulation of standards with an emphasis on conceptual understanding (Confrey, 2012; Confrey, Maloney, & Corley, 2014; Wilson, Mojica, & Confrey, 2013). Understanding of learning progressions is important for teachers; they serve as the guidepost for analyzing student learning and tailoring their teaching sequence (Wilson et al., 2013). A research-based learning progression also informs how mathematical content knowledge and conceptual understanding for students develop over time.

Understanding How Vertical Knowledge of the Curriculum Contributes to Teachers' Mathematical Knowledge for Teaching

Mathematics knowledge for teaching (Hill, Sleep, Lewis & Ball, 2007) includes understanding of general content but also having domain specific knowledge of students. Teachers with mathematical knowledge for teaching must have an extensive and complex set of knowledge and skills that facilitates student learning across the learning progressions so that they learn the structure and relationship of students' understandings about a particular mathematical concept, teach specific strategies to elicit student thinking, strategically evaluate students' responses, and move the mathematical agenda forward (Wilson et al., 2013).

An example of the importance of learning progressions is found in the research on developing algebraic reasoning in the earlier grades through problem solving which requires depth of understanding. Blanton and Kaput (2008) reported that teachers become better at teaching algebraic reasoning when the teachers' own mathematical knowledge and understanding was increased and their algebra "eyes and ears" allowed them to bring out the algebraic reasoning while looking at student work and listening to their discussions and questions. To know what to look and listen for in the classroom, teachers must have a deep and profound understanding of the foundational concepts for algebraic reasoning through patterning, relations, functions, and representations using algebraic symbols and utilizing mathematical models to represent relationships (NCTM, 2000).

The knowledge of learning progressions is vertical knowledge. Vertical knowledge includes "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Shulman, 1986, p. 10). This vertical knowledge can be supported through knowledge of learning trajectories and vertical teaming by teachers. Confrey states that using learning trajectories, teacher can "plan their instruction based on how student learning progresses. An added strength of a learning trajectories approach is that it emphasizes why each teacher, at each grade level along the way, has a critical role to play in each student's mathematical development" (Confrey, 2012, p. 3). However, teachers do not have frequent opportunities to work with teachers from different grade bands. Understanding the learning progression across grade levels requires the collaboration of teachers through meaningful vertical articulation and PD.

Context: Detailing Our PD Design and Activities

The designers and instructors of this project included a university mathematics educator, a mathematician, teacher leaders from the districts, and doctoral students in mathematics education. This PD was conducted as year one of a three-year Mathematics Science Partnership grant focused on algebraic thinking and proportional reasoning during the transition years from fifth grade into high school. We based this project's design on the current research and needs in mathematics education with a specific attention to creating opportunities for vertical articulation focused on algebraic learning progressions as students move into high school algebra. We considered all the core features of effective mathematics PD which includes having content as the focus, being sustained over time, collective participation of teachers working together on issues central to instruction, and focus on instructional materials that teachers use in their classrooms (Desimone, 2009).

To focus teachers' work with content, we used NCTM's standards (2000, p. 39) and explored the algebraic learning progression beginning with recursive patterns, representing mathematics situations with quantitative relationships, multiple representations of functions, including numeric, graphic, and symbolic, since the representational fluency develops a deeper understanding of functions (Moschkovich, Schoenfeld, & Arcavi, 1993). Weekly activities in content-based class sessions included modeled lessons using rich tasks combined with in-depth conversation about both the algebra content and the pedagogical strategies employed by facilitators. In addition, a centerpiece of the PD experience was having teachers learn about the high leverage practices outlined in the *Principles to Action* (NCMT, 2014), and asking teacher teams to select a goal that they wanted to focus on for improving their practice. Many of the teacher teams selected goals of facilitating meaningful discourse and posing purposeful questions. In addition, we shared Smith and Stein's (2011) *5 Practices for Orchestrating Productive Mathematics Discussions* with the PD participants and guided their use of the five practices in the planning and implementation of their lessons.

We conducted two iterations, each with a different cohort of teachers, of the content-focused course which included either a follow up Lesson Study or participation in a video-based teacher study group in school based vertical teams. We focused on engaging teachers in active learning through algebraic problem solving tasks and exploring pedagogical strategies. Our goal was to deepen teachers' algebraic thinking, encourage vertical articulation, and develop a productive disposition towards teaching through problem solving. The follow-up sessions were designed to provide teachers with continued support in professional learning implementing algebraic content, as well as providing opportunities for vertical articulation between and among grades levels.

Methods

For our study, we used a mixed methods approach to examine the outcome of the PD focused on learning progressions and vertical teaming of teachers. The quantitative research focused on examining teachers' content knowledge and the qualitative research focused on professional growth as identified by teachers' reflections.

Participants

The data were collected from the teachers who participated in the two cohorts. A total of 54 teachers participated in the study, 23 of whom were in Cohort 1 and 31 in Cohort 2. Most teachers worked in public schools (N=51 from seven school districts). A total of 35 schools were represented with 21 schools being represented by one teacher, nine schools by two teachers, and five schools by three teachers. The teachers also held various positions at their schools: elementary school teachers (N=22), secondary teachers (N=25), special education teachers (N=3), ELL teachers (N=1), and coaches (N=1). Two teachers did not report their positions. Specifically, the participants taught Grade 2 (N=1), Grade 3 (N=2), Grade 4 (N=3), Grade 5 (N=11), Grade 6 (N=11), Grade 7 (N=9), Grade 8 (N=3), and Grade 9 (N=6). Eight teachers did not indicate which grade they taught. On average, the teachers (N=52) had 9.42 years of teaching experience (SD=7.42) with a minimum of 0 and a maximum of 30 years.

Data Sources

Quantitative data. Prior to the beginning of the PD program, the Mathematics Knowledge for Teaching (MKT) of the participating teachers was assessed (Learning Mathematics for Teaching, 2007). Specifically, they completed the 2007 Middle School Patterns Functions and Algebra – Content Knowledge instrument, administered in two forms (N=24 for Form A; N=30 for Form B). At the end of the program, the same assessment was administered again (N=30 for Form A; N=24 for Form B). All teachers completed different forms for the pre and posttest. The forms were assigned to the teachers randomly. Due to non-equivalence of the test forms, teachers' raw test scores were converted into IRT scores, which were used in the analysis.

Qualitative data. The qualitative data sources included final course reflections, teacher reflections from lessons, video clips from the research lessons. To delve in deeper into the nature of the collaborative exchange, we collected a final course reflection, that the researchers created for the second PD cohort, which asked teachers to reflect on the nature of the vertical professional learning and how the focus on an instructional practice by the team contributed to their professional learning.

We chose one of the video-based teacher study groups as a case that provided us the best opportunity to learn about how team members exchanged professional knowledge within the video lesson analysis.

Research Questions

1. What differences did we see in content knowledge of teachers at different grade levels?
   - Do the two cohorts of teachers differ in their MKT over the PD program?
   - Do elementary and secondary teachers differ in their MKT over the PD program?
2. What did teachers identify as areas of professional growth from their vertical teacher study groups focused on algebraic thinking and enhancing their instructional practices?
   - How did vertical teaming enhance teacher's planning and instruction?
   - What is the nature of the collaborative learning that teachers identified through their work with peer study groups?

Data Analysis

The quantitative data were statistically analyzed using IBM SPSS Statistics 22. To begin analyzing the themes in the qualitative data, we systematically analyzed the data by developing initial codes and used the method of axial coding to find categories in such a way that drew emerging themes (Miles, Huberman, & Saldaña, 2013). To verify and compare recurring themes and categories, the research team worked individually on coding the documents before comparing preliminary codes in order to agree upon recurring themes from the reflections. Dedoose, an internet-based data management tool (Dedoose Version 6.2.7) was used to code and analyze the data. Initial codes reflected specific teacher practices and actions (e.g., posing purposeful questions, learning content from peers, anticipating student responses). As codes were categorized, several main themes emerged: improved pedagogical practice due to instructional strategies, improved content knowledge; supporting student thinking in algebra; improved pedagogical practice resulting from collaboration, and gaining vertical knowledge of content from peer collaboration.

Results

Quantitative Results

The descriptive statistics for the pre and posttest IRT scores are presented in Table 1. In addition to the total, descriptive statistics by Cohort and Position at school are also presented. Due to the sample size restrictions, only elementary and secondary positions are considered (secondary teachers include both middle and high school teachers).

To further explore teachers' MKT, we conducted a comparative analysis for the cohorts and teachers' positions (a Bonferroni correction was employed). First, to determine if there was a difference between the two cohorts on the pre and posttest, we conducted independent-samples t-tests. The results showed that the cohorts did not differ in MKT either on the pretest *($t(52)=0.741$, $p=0.462$)* or on the posttest *($t(52)=0.428$, $p=0.670$)*. Second, to determine if there was a difference between the pre and posttest for the two cohorts, we conducted paired-samples t-tests. The results indicated that teachers in Cohort 2 scored higher on the posttest than on the pretest *($t(30)=-2.546$, $p=0.016$)*; however, no difference was observed between the pre and posttest scores for Cohort 1 *($t(22)=-1.219$, $p=0.236$)*.

Next, to determine if there was a difference between elementary and secondary teachers on the pre and posttest, we conducted independent-samples t-tests. The results showed that the elementary teachers scored lower than secondary teachers on the both pre *($t(45)=-3.948$, $p=0.000$)* and posttest *($t(41.811)=-3.785$, $p=0.001$)*. Finally, to determine if there was a difference between the pre and posttest for the elementary and secondary teachers, we conducted paired-samples t-tests. The results indicated no difference between the pre and posttest scores of either elementary *(t(21)=-2.248, p=0.035)* or secondary *(t(24)=-1.060, p=0.300)* teachers.

Table 1. Descriptive Statistics for the Pre and Posttest of MKT

Measure	Mean (SD)				
	Total (N=54)	By Cohort		By Position	
		Cohort 1 (N=23)	Cohort 2 (N=31)	Elementary (N=22)	Secondary (N=25)
Pretest	-0.049 (1.000)	0.068 (0.918)	-0.137 (1.063)	-0.532 (0.749)	0.497 (1.000)
Posttest	0.185 (0.917)	0.248 (0.936)	0.139 (0.915)	-0.244 (0.632)	0.645 (0.962)

Research question #1a. First, we aimed to determine whether the two cohorts differed in MKT over time. To answer this research question, we conducted the two-way mixed design ANOVA with Cohort (Cohort 1 and Cohort 2) as a between subject factor and Time (pre and post) as a within subject factor. The results indicated the main effect of Time *(F(1, 52)=6.502, p=0.014)*, i.e., teachers' scores, averaged across cohorts, were higher on the posttest than on the pretest. However, there was no main effect of Cohort *(F(1,52)=0.393, p=0.534)*, i.e., teachers' scores, averaged across time, did not differ between the cohorts. Additionally, no interaction effect of Time and Cohort was observed *(F(1, 52)=0.289, p=0.593)*. Thus, considering these results and comparisons above, we decided to combine the two cohorts for the further analysis.

Research question #1b. Next, we aimed to determine whether elementary and secondary teachers differed in MKT over time. To answer this research question, we conducted the two-way mixed design ANOVA with Position (elementary and secondary) as a between subject factor and Time (pre and post) as a within subject factor. The results indicated the main effect of Position *(F(1, 45)=17.084, p=0.000)*, i.e., elementary teachers had lower scores on MKT, averaged across time, than secondary teachers. The comparisons above also suggest that elementary teachers had lower MKT on both pre and posttest. Additionally, there was the main effect of Time *(F(1, 45)=5.187, p=0.028)*, indicating that teachers' scores, averaged across positions, were higher on the posttest than on the pretest. However, according to the comparisons above, when the teachers are split into elementary and secondary levels, the effect of time does not hold for either level. This finding is also consistent with the absence of the interaction effect of Time and Cohort *(F(1, 45)=0.532, p=0.470).* Larger sample sizes may be needed to determine differences within the levels (i.e., elementary vs. secondary).

Qualitative Results

While the MKT results showed that teachers overall made gains in their content knowledge as a result of the PD, they also reported, universally, through the qualitative data that the PD led to improvement in their understanding of algebraic content and pedagogy for math instruction. We will next examine reflections of the teachers from the second cohort to identify which particular benefits the teachers gained from the PD.

Vertical teaming and its impact on teacher's planning and instruction of algebra across the learning progression. All of the second cohort teachers who completed the final reflection indicated that they learned from their peers and found added value in that learning because the teams were made up of teachers from multiple levels (i.e., elementary and secondary). Barbara, an algebra teacher, described the benefit of working in vertical teams:

> There really should be more collaboration between elementary, middle, and high school. I loved hearing from the elementary teachers because of all their use of manipulatives and their different approaches can also work for high school and be helpful, especially for low level students.

By working together in vertical teams, the teachers built a broader foundation of both personal understanding of content and knowledge of how students at different levels (both in age and readiness) approach algebraic thinking. Ralph, a 5th grade teacher stated:

> The wide-range of grade levels that are taught within our group served as a catalyst for a deeper understanding of the levels of reasoning and problem-solving students are at and what can be achieved during those levels and what can be and needs to be done for those that may be behind developmentally.

Finally, teachers indicated that they had a firmer grasp of the learning progression necessary for students to succeed in algebraic thinking at all levels. Rebecca, a 4th grade teacher shared:

> I really was able to open my eyes to how the ideas that start in early elementary are so foundational to how students think when they get to their algebra class. The connections that can be built through patterns and being able to recognize them is so valuable.

Collaborative professional learning through teacher study groups. Consistent with the quantitative results, almost all participants (30/32) specifically indicated that working with their peers improved their knowledge of math content. More than half (18/32 total teachers) identified that they learned math content from their colleagues: elementary teachers learned to see equations and formal algebra from their secondary colleagues (10/18 elementary teachers) and secondary teachers learned to utilize manipulatives and visual models from their elementary colleagues (8/14 secondary teachers). In fact, most teachers indicated that working with peers expanded their knowledge of diverse strategies and the use of varied representations when solving problems (29/32). Jan, an 8th grade math teacher elucidates this idea:

> Being able to see how others approach problem-solving was very enlightening. It was so interesting to me to see that there are so many different ways of solving problems. I was never taught and haven't felt comfortable using manipulatives, etc., but I really appreciated seeing this strategy being used. Now when I look at a problem, I am able to see different ways of approaching it.

Teachers also learned to improve their pedagogy through collaboration with their peers. Half the teachers made specific comments that their pedagogical practice improved due specifically to their interactions with their peers (16/32). Elizabeth, a 5th grade teacher, stated, "I liked brainstorming with my group members and thinking outside of my box. I realized that anticipating the questions that students might ask prior to the lesson is very helpful!" Further, Jocelyn, a middle school math teacher reflected that, "Working with the group helped frame the plan of what ideas to share and how to make those mathematical connections between the students' strategies and the 'Big Ideas'."

While teachers' reflections offered insights into their thinking, their video discussions allowed us to view their practice. We selected one case study that we felt demonstrated notable exchanges among team members in regards to the benefits of collaboration. The team was composed of five teachers, ranging from 2nd grade to 8th grade algebra. The study group chose a growing toothpick pattern task which they modified for implementation at each grade level. We coded the commentaries from the collaborating teachers on each video. These exchanges revealed ways in which peers exchanged their knowledge for teaching and assisted one another. One of the major themes was how peers validated each teacher's instructional practices.

Ann, a 4th grade teacher presented the Hexagon Pattern. A peer-teacher, Karen, commented that Ann did not jump in to tell a student she had the wrong number of toothpicks for two hexagons. Instead, Ann allowed the student to discover the error when building the pattern.

> *Karen:* I love how you didn't originally tell her that 12 was wrong, she noticed it was wrong when she got up to place the toothpicks!

Another peer-teacher, Holly, viewing Ann's lesson commented on Ann's use of posing purposeful question to follow up on the student's discovery:

> *Holly:* Purposeful questions - "If we know that hexagons have 6 sides and we are building 2 then why do we have 11 toothpicks instead of 12?" Great job!

In another notable exchange, two teachers share the challenge of knowing and gauging the right balance between allowing students to experience "productive struggle" and knowing when to ask a probing question while commenting on a lesson taught in Micki's classroom.

> *Micki:* Another moment where a purposeful question would have been helpful. This student got stuck and I gave her time to think, believing I was supporting productive struggle, but I should have left her with a question to help move her thinking along.
> *Sara:* Sometimes it is really hard to find that balance of when to let them keep going with the productive struggle and when would it be better to give them a purposeful question to help them along. Remember hindsight is 20/20 and it is why we should always reflect on our lessons. If you do another activity with your class before the end of the year, you will know this student needs a little more support.

Conclusions and Implications

For several decades, the mathematics education community has focused their attention on the importance of the *Mathematical Knowledge for Teaching* (Hill et al, 2007) that defines the deep, broad, and well connected knowledge that is needed to decompose and unpack content to make it comprehensible by students. In addition to conceptual understanding of the knowledge needed for teaching, teachers must have command of high leverage teaching practices. Our PD design focused on the task of exposing teachers to the complexity of ambitious teaching. To examine the benefits of participation in our PD, we conducted a mixed-methods study.

With our quantitative analysis, we found that teachers were able to gain significant knowledge during the PD course. Our qualitative results supported the increase in teachers' content knowledge and also indicated learning benefits beyond the content knowledge; the teachers content scores did not tell the whole story. In particular, perhaps the most encouraging result from our study is the notion that PD utilizing collaborative, vertical teams of teachers can contribute to teachers' professional learning as they examine their practice and work to improve their pedagogical skills. The evidence also suggested that validation from their peers supports teachers' continued growth. As we consider Synergy at the Crossroads, mathematics educators and researchers may consider providing more opportunities for teachers to work in vertical professional learning communities focused on understanding the mathematics learning progressions in future PD offerings to increase teachers' MKT and improve their practice.

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MATHEMATICS TEACHERS' TAKE-AWAYS FROM MORNING MATH PROBLEMS IN A LONG-TERM PROFESSIONAL DEVELOPMENT PROJECT

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Considering the role of mathematics-focused professional development programs in improving teachers' content knowledge and quality of teaching, we provided teachers opportunities for dealing with mathematics problems and positioning themselves as students in a large-scale long-term professional development (PD) project. In this proposal, we aimed to understand the impact of engaging in morning math problems on teachers in terms of their mathematical understanding and teaching practices. Both written work and interviews showed that solving open-ended problems helped teachers better understand the mathematics content and students' challenges as they solve problems; thus, suggested an effective means of PD for teachers.

Keywords: Teacher Education- Inservice/Professional Development, Problem Solving

As many teacher education researchers highlighted, mathematics-focused professional developments play a central role in efforts to improve teachers' knowledge base (Ball, 1990; Hill, 2007; Moss, 2006). Through encountering mathematics problems and positioning participating teachers as students, we sought to improve not only the teachers' mathematical content knowledge but also pedagogical knowledge, pedagogical content knowledge, and beliefs about what it means to 'do' mathematics. Considering this role, we aimed to provide teachers opportunities of dealing with mathematics problems and positioning themselves as students in a large-scale long-term professional development (PD) project. Each professional development session started by asking teachers to work on authentic and challenging mathematics problems. We also interviewed teachers at the end of the project to learn their thoughts about morning math sessions, the nature of the math problems they worked on, and what they learned about mathematics and mathematics teachers. We particularly focused on the following research questions:

1. What do teachers think about the role of morning math sessions on their improvement as mathematics teachers?
   1.1. What did teachers gain in terms of mathematical content from morning math sessions?
   1.2. What did teachers gain in terms of mathematics teaching from morning math sessions?

We found these questions significant to investigate to better understand the role of teachers' solving math problems and experiencing student position as a means of professional development. Thus, this study links to the conference theme, *Synergy at the Crossroads: Future Directions for Theory, Research, and Practice*, in that it introduces a promising component of professional development for mathematics teachers, discusses the role of morning math sessions in improving teachers' mathematics teaching practices, and makes suggestions for future directions to develop more effective professional development sessions for mathematics teachers.

Theoretical Framework

Teacher Knowledge and Role of PD in Teachers' Knowledge Development

Over the last 40 years, understanding what teachers need to know has become one of the most important concerns in the field of education (Cochran-Smith & Lytle, 1999). While some studies have focused on the knowledge that teachers need to know as professionals (Grossman & Richert, 1988; Shulman, 1987), others have aimed to understand the knowledge that teachers need to know for the practice of teaching (Hiebert, Gallimore, & Stigler, 2002). Cochran-Smith and Lytle (1999) took this distinction between professional and practitioner knowledge further and suggested three conceptions of knowledge: (1) knowledge-*for*-practice, (2), knowledge-*in*-practice & (3) knowledge-*of*-practice. Among these three conceptions, knowledge-*for*-practice referred to the formal knowledge that teachers gained in teacher education and professional development programs. In this vein, in the mid-1980s, Shulman (1987) had proposed seven categories of teacher knowledge: (i) content knowledge; (ii) general pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter; (iii) curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers; (iv) pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding; (v) knowledge of learners and their characteristics; (vi) knowledge of educational contexts, ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; and (vii) knowledge of educational ends, purposes, and values, and their philosophical and historical grounds. (p. 8)

Shulman argued that these categories constituted a teacher knowledge base which was supported by both theoretical and empirical sources of knowledge. Shulman's perspective viewed the teacher as a trained professional who could learn about subject matter, curriculum, educational philosophy and history and as an active member of a scholarly community, who could pursue and help others pursue intellectual development.

Understanding those categories of teachers' content knowledge provides a strong basis for designing effective teacher education and professional development opportunities. Especially, mathematics-focused professional developments play a central role in efforts to improve teachers' knowledge base (Ball, 1990). As argued by Moss (2006, p.97), "In order to encourage their students' mathematical thinking, teachers must be able to appreciate and evaluate the reasonableness of their thinking. However, to be able to do this, they must have for themselves a deeper understanding of mathematics." Thus, providing teachers opportunities of evaluating their understanding of mathematics is important for teacher development.

Five Practices for Orchestrating Productive Mathematics Discussions

The PD sessions at the focus of this study were designed using the five practices for orchestrating productive mathematics discussions (Stein & Smith, 2011); therefore, PD trainers demonstrated these five practices during morning math sessions. Stein and Smith (2011) developed these five practices to help teachers design and implement lessons involving mathematically rich discussions and enhancing students' mathematical understanding. Stein and Smith (2011) summarized these five practices as follows:

1. anticipating likely student responses to challenging mathematical tasks;
2. monitoring students' actual responses to the tasks (while students work on the tasks in pairs or small groups);
3. selecting particular students to present their mathematical work during the whole-class discussion;
4. sequencing the student responses that will be displayed in a specific order; and
5. connecting different students' responses and connecting the responses to key mathematical ideas (p. 8).

In addition to those, Stein and Smith (2011) proposed that setting instructional goals and selecting appropriate tasks could be viewed as the practice zero, which had to be ensured before five practices. It is important to note that these practices do not serve as a manual but suggest an effective way for teachers to characterize their work of orchestrating student-centered discussion by ensuring they make sense of students' work and connect students' thinking to the big mathematical ideas.

Mode of Inquiry

Professional Development Project and Participants

Approximately twenty-two elementary teachers from across three school districts participated in the two-year professional development program focused on preparing mathematics teacher leaders. The professional development program consisted of monthly sessions and intensive summer PD workshops. All sessions were focused on engaging teachers' in activities to develop their teachers' mathematical knowledge for teaching and leadership skills. They were involved in 16 monthly 8-hour workshops and 80 hours (10 days) of PD over two summers for a total of 26 workshops days. The sessions involved in a range of activities (e.g., video discussions, math content sessions, rehearsals etc.) and teachers were engaged in coaching sessions (McGatha, 2009) with the professional developers. In this study, we focus on the activities involved in one regular PD session we called *Morning Math*. During each Morning Math session, teachers were given a problem (or two) to solve. These problems were selected to not resemble typical textbook tasks; rather, they represented true problems for the teachers.

Data

In this proposal, we focused only on teachers' work on morning math problems during two summer PD sessions; a five-day PD in June 2015 and a five-day PD in July 2015. In particular, we examined teachers' work on four mathematics problems in June PD and five mathematics problems. The names of the problems that teachers worked during each PD session are given in the table below.

Table 1: Morning Math Problems

Name of the Math Problem	PD Session
Darts Remainder 4 Milk Chocolates Painted Cubes Rocket Science	June PD
Marbles Gum Drops The Sheep Activity Coin Sums	July PD

Although teachers worked on these problems in groups, they recorded their thinking on provided worksheets, and regularly recorded their work on large flipchart paper to share with peers. Furthermore, their group discussions were video recorded. In addition, teachers were individually interviewed. Thus, data set also consisted of transcriptions of teachers' interviews.

Data Analysis Process

In the initial round of data analysis, teachers' written work on math problems were analyzed based on content analysis for understanding the range of solutions (Neuendorf, 2016). Then, the video records of group working were analyzed using more focused coding about the ways in which teachers approached these problems. In the final round of data analysis, the teachers' interviews were examined using thematic analysis to understand whether teachers thought the morning math sessions impacted their practice as mathematics teachers (Clarke & Braun, 2013).

Findings

In this proposal, we only present teachers' work on the Milk Chocolates problem. To introduce the Milk Chocolates problem, we showed a box of chocolates as shown in Figure 1. We also provided a model of this chocolate box and asked whether they could find the number of milk chocolates in the box without counting. Moreover, we explained to the teachers that we were not only interested in the correct answer to the problem but also the ways in which they got the answer. Therefore, we asked them to find the answer in as many different ways as they could and record a numerical expression that modeled their thinking.

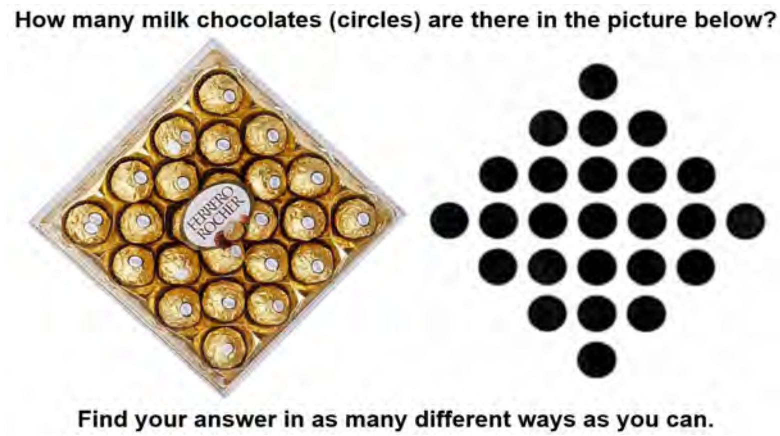

Figure 1. Milk Chocolates Problem (adapted from Balka & Hull, 2012).

Teachers' Content-Related Take-Aways

Teachers worked in groups of 2-3 and produced fifteen different solution methods as shown in Figure 2 below. As demonstrated by these solutions, teachers developed different ways of counting the chocolates in the box. When they started to work on this problem, they had not anticipated that there would be fifteen different solutions. As they solved the problem, we observed that they were changing their perspectives: "Ohh, okay. There might be a couple other ways to get the answer." As the discussion ensued and additional solutions were shared, they were very excited and engaged as they saw many different ways that their colleagues shared on the board.

This is also evident in their interviews that most of the teachers mentioned about more than one solution method when they were asked about morning math problems. The following excerpts illustrate this issue:

> Teacher A: I just learned different strategies during morning math and I learned, you know, not to give up and continue to keep trying.
>
> Teacher B: Well, there's definitely more than one strategy. One way to think about something that everybody takes a different way, and, um, it seems that when you work in groups, um, with the teacher walking around and kind of, like, monitoring.
>
> Teacher C: Just how many different ways people approach things. Because I try to think-- well, I seem to think kind of 1, 2, 3 stepwise. And then people are pulling out all this other stuff that I've never even dreamed of thinking of. So it's good to see all the different ways that people approach things.

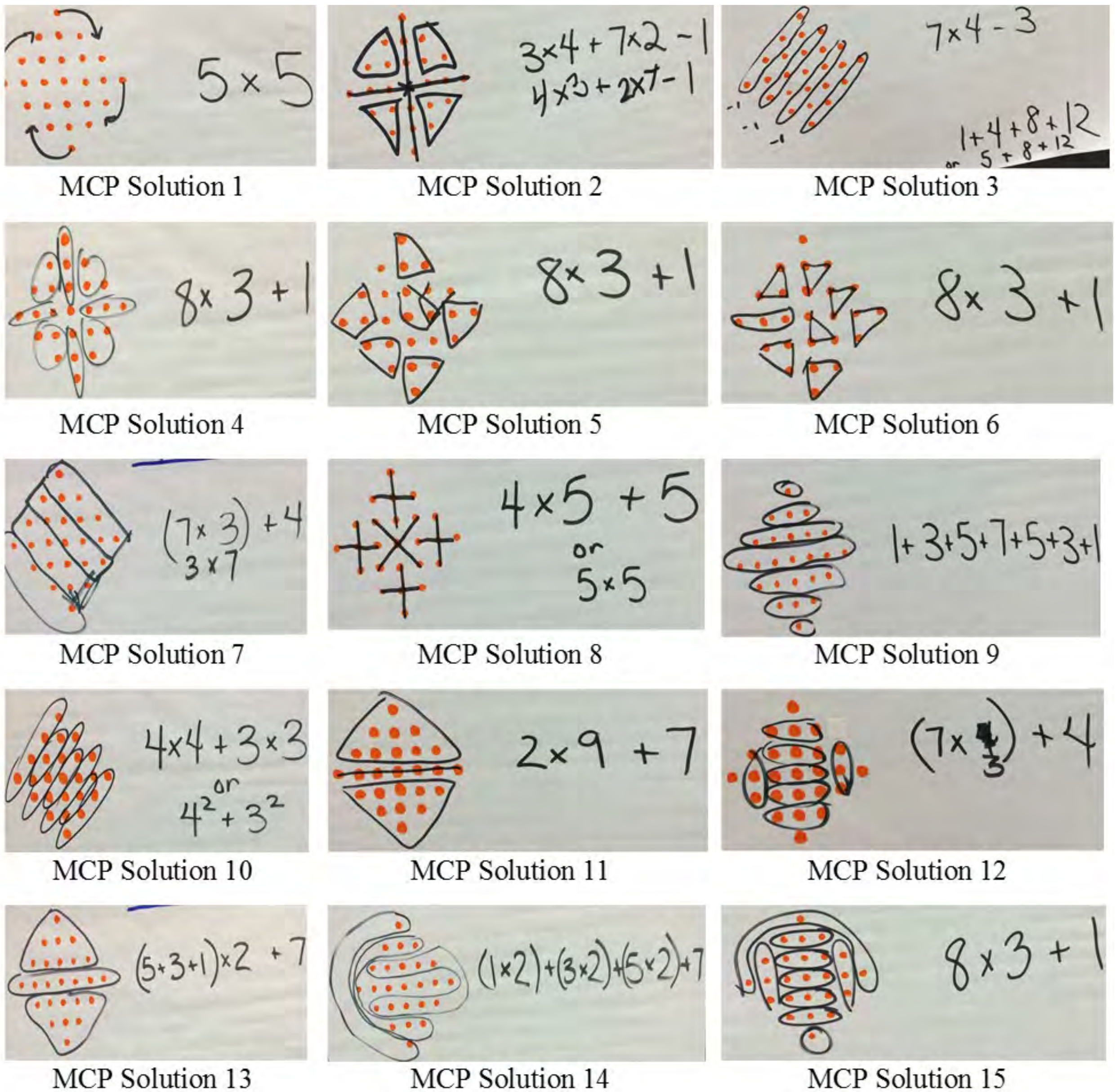

Figure 2. Teachers' solutions on Milk Chocolates Problem (MCP).

Like many other teachers, these three teachers highlighted different solution methods that were developed by other teachers and that they would not have anticipated otherwise. After sharing fifteen solutions to the milk chocolates problem, we asked teachers an extension question:

- What if the size of the box changes?
- Which method would you use to find the number of chocolates in these boxes (see Figure 3 below)?

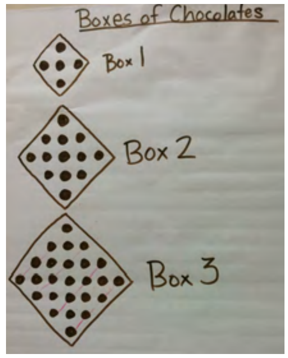

Figure 3. Milk Chocolates Problem Extension Part 1 (adapted from Goldenberg et al., 2015).

- If the pattern (shown in figure 3) continued, how could you find the number of chocolates in the box of any size? (i.e., Milk Chocolates Problem Extension Part 2)

In this extension part, teachers picked the method they thought to be more efficient to find the number of chocolates in Box n. It was challenging because not all of the solutions shown in Figure 2 led to a more general solution. Figure 4 presents four of the fifteen methods that teachers utilized to develop algebraic expressions.

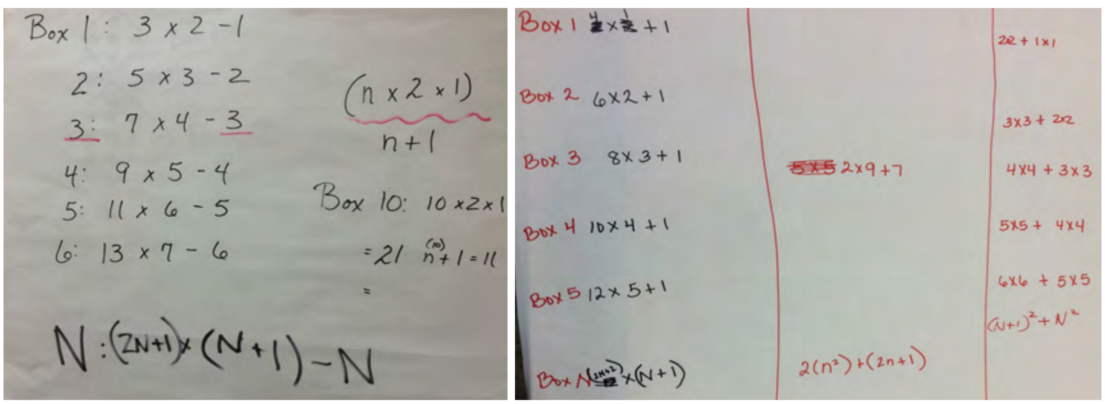

Figure 4. Teachers' Solution to the Milk Chocolates Extension Problem

After this work was shared by teachers, they engaged in a discussion about how they found different algebraic expressions and whether each of these expressions was a different one. During this discussion, teachers reached a conclusion that each of the methods showed them a pattern in counting, but these patterns resulted with the same simplified algebraic expression (i.e., $2n^2+2n+1$). Similarly, teachers had not thought that they would reach an algebraic expression with one unknown at the end of this problem, and they did, in fact, find the same expression even though they developed different solution methods.

Teachers' Teaching Practices-Related Take-Aways

During the interviews, teachers mentioned about morning math problems in relation to the 'selecting' and 'sequencing' practices:

Teacher D: Well, we worked on--there's usually--definitely there's more than one way to answer the problems and all of them and the way that he [*PD trainer*] just placed them is all--generally, the concept--the overall concept that he's trying to get everybody to see, the real wow moment, he saves for the end, like when there's a formula, he saves them for the end, so he's working simplest to most complex is what it looks like in, uh, showing the students work and that is a good strategy that I feel I could benefit from.

Teacher E: Well, it really has helped me to, um, well, learn that, uh, there's different ways that people think about a problem and that, um, you have to help, uh, persevere through some of the problems because they really help you to really think and make connections to other, um, other math concepts that you weren't even thinking at first.

Those teachers described that the morning math problem sessions went beyond only showing that multiple solutions existed. The multiple solutions were discussed, compared with one another and through engaging in the 'connecting' practice, wisely connected to a mathematical big idea. In fact, the open-ended nature of the problem (i.e. involving more than one answer), the potential for multiple solutions, and the opportunity to build understanding of an underlying math idea were three characteristics named by teachers as they describe the nature of the morning math problems and the quality of math problems in general.

Another important role of the morning math was that it required teachers to engage in productive struggle and that perseverance is important in the problem-solving process. The excerpt of a teacher below illustrates the sentiments of several teachers' in terms of the impact of morning math sessions in that the struggle and perseverance were necessary for learning new mathematical ideas.

Teacher F: Uh, I've learned--actually one, it's taking me back to, like, what children experience and how they have to persevere and just struggle through and go back--and go back to all those ideas and teachings that you learn from the past. So, it brought me back to that and understanding what they have to experience. … And then also, with the right coaching, with the right assistance through those tests, I think that that's been a really big eye-opener, um, made me think past first-grade math.

Interviewer: (laughs) and that's good or bad or…

Teacher F: Oh, that's good because, I mean, I like Math so I try to refresh and keep myself as, I want to say up-to-date as I can. Because not too long ago, I was looking at some Algebra II books and just, you know, just for the sake of time, just messing around and just refreshing it. … knowing how to persevere and understanding what children feel. … I think it makes anyone a better teacher because the more you see it, the more you understand that person's experiences, the better you're able to actually help them through their experiences.

Having first-hand experience of what students were experiencing while learning new mathematical knowledge helped teachers think more about how they could support students differently; thus, this experience helped them to improve their understandings and skills for "monitoring" practice. As many teachers also pointed out, one of the ways of enhancing student learning is by selecting good problems which are challenging but attainable and which will allow making real-life connections. In this vein, we could observe that teachers understood the importance of the practice zero, "setting goals and selecting tasks." To sum up, morning math sessions of the PD not only provided teachers direct experience with mathematical ideas but also demonstrated the ways in which such problems enhance students' learning.

Conclusions

Our aim in this particular study was to understand the role of morning math PD sessions on teachers' understanding of mathematics content and mathematics teaching. As illustrated with the data shared in this proposal, we argue that providing teachers direct experiences with mathematics problems is important for them to learn mathematical ideas conceptually, to understand the challenges that students experience as they learn, to appreciate the effectiveness of challenging, open-ended, and sense-making problems in learning mathematics, and to understand the role of cooperative learning in teaching mathematics. These findings are important for teacher educators and teacher education researchers since they indicate morning math problem-solving sessions as an effective means of professional development for mathematics teachers. However, the role of morning math sessions did not solely result from teachers' solving *any* mathematics problems. In fact, the problems which connected with real life situations, which had multiple solutions and sometimes multiple answers, and which were challenging but attainable, help teachers improve themselves in terms of mathematical content and mathematics teaching skills, particularly around five practices for orchestrating productive mathematics discussions because teachers also observed PD trainers as they monitored their work, selected and sequenced the solution methods, and connected with the big mathematical idea (Stein & Smith, 2011). By presenting teachers' take-aways from morning math sessions of this PD project, we provide insight to teacher educators and teacher education researchers for improving professional development sessions designed for mathematics teachers, which we expect to result in improvement in teachers' mathematics teaching practices.

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NEGOTIATING THE ESSENTIAL TENSION OF TEACHER COMMUNITIES IN A STATEWIDE MATH TEACHERS' CIRCLE

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Math Teachers' Circles (MTCs) bring math teachers and university mathematicians together to engage in collaborative mathematical activity. Currently there are over 110 MTCs across 40 states. A key claim is that MTCs are "communities of practice." However, to date there has been no research to substantiate this claim. In this paper, we explore the ways in which participants in an MTC negotiate aspects of community formation.

Keywords: Teacher Education-Inservice/Professional Development, Teacher Beliefs

Founded in 2006 by the American Institute of Mathematics, Math Teachers' Circles (MTCs; www.mathteacherscircle.org) bring K-12 math teachers and research mathematicians together to engage in collaborative mathematical activity. Unlike traditional professional development, which tends to foreground pedagogical practice, MTCs focus on engaging participants in mathematical activity. Notably, the model:

> emphasizes developing teachers' understanding of and ability to engage in the practice of mathematics, particularly mathematical problem solving, in the context of significant mathematical content. The core activity of MTCs is regular meetings focused on mathematical exploration, led by mathematicians or co-led by mathematicians and teachers (White, Donaldson, Hodge, & Ruff, 2013, pp. 3-4).

MTCs have expanded rapidly, and currently, there are over 110 MTCs in 40 states. As MTCs have expanded across the country, a small amount of research has begun to explore MTCs as a form of professional development for teachers. One significant finding is that MTCs can increase teachers' mathematical knowledge for teaching (White et al., 2013). This is an important result, as mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008; Hill & Ball, 2009) is associated with effective math teaching (Hill, Rowan, & Ball, 2005). Further, surveys of MTC participants have suggested that teachers who participate in MTCs begin to identify more strongly as mathematicians (Fernandes, Koehler, & Reiter, 2011; White & Donaldson, 2011).

Finally, an often-stated claim is that MTCs are *communities of practice* that support sustained teacher learning. For teachers, communities of practice help to support intellectual renewal and provide a sustained venue for new learning (Grossman, Wineburg, & Woolworth, 2001). However, there is currently no research-based evidence to support the claim that MTCs are—or develop into—communities of practice. This is important because communities are not created by fiat, and not all groups of teachers are communities of practice in the way that the term has been used in the anthropological literature (e.g., Lave & Wenger, 1991; Wenger, 1998).

Given the importance of communities of practice to teacher professional development, it is crucial to understand the ways in which MTCs are—or are not, or develop into—communities of practice. In this paper, we explore the ways in which participants in an MTC negotiate aspects of community formation.

Conceptual Framework

A community of practice is defined by three features: mutual engagement, joint enterprise, and shared repertoire (Wenger, 1998). Mutual engagement refers to the requirement that participants jointly participate in the practice(s) that binds and defines the community. Joint enterprise refers to the purpose of the community. Shared repertoire refers to the objects that are naturalized in the community—those objects that are so natural to members so as to be taken-for-granted, but which may seem foreign or strange to outsiders (Bowker & Star, 1999).

In this paper, we pay particular attention to the negotiation of joint enterprise by focusing on the *essential tension in teacher communities*: the tension between focusing on pedagogical practice on the one hand, and engaging in subject-matter disciplinary practices on the other (Grossman et al., 2001). This is an important consideration with respect to MTCs. Primarily, MTCs are meant to engage participants in mathematical practice. The improvement of pedagogical practices is not a "core" activity (White, et al., 2013). However, Grossman et al. (2001) contend that both foci are essential elements in the joint enterprise of a teacher community:

> We contend that these two foci of teacher learning must be "brought into relation" in any successful attempt to create and sustain teacher intellectual community... Teacher community must be equally concerned with student learning and with teacher learning. (p. 952)

Grossman et al. (2001) suggest that the negotiation of the essential tension will go through three ordered stages as a group develops into a community. A "beginning" group demonstrates a lack of agreement around whether the joint enterprise ought to be one focus or the other, and there is often opposition tension between the two foci. An "evolving" group maintains the opposition between the foci, but begins to demonstrate a willingness to allow different people to pursue different foci. Finally, a "mature" community holds the two foci in productive relation, recognizing that "teacher learning and student learning are fundamentally intertwined" (Grossman et al., 2001, p. 988).

Research Questions

On the one hand, pedagogical practice is officially backgrounded in MTCs so as to maintain a focus on engagement in disciplinary practice. On the other hand, "for a group of teachers to emerge as a professional community, the well-being of students must be central" (Grossman et al., 2001, p. 951). This makes us wonder, even if the stated goal of an MTC is to engage participants in mathematical activity, *what actually happens when a group of math teachers gets together to do mathematics?* Do teachers simply engage in mathematical activity? Do they focus on pedagogy? Or some combination? Our study is the first to employ anthropological methods to answer anthropological questions about math teachers' circles—in particular, just *what is* the joint enterprise, as it is negotiated by participants? Our research questions are:

1. In what ways, if at all, are the two foci of the essential tension—mathematical activity and pedagogical practice— manifested in MTCs?
2. When pedagogical practice is invoked, how is it treated by participants?

Materials and Methods of Analysis

Our data come from the initial gatherings of a newly-inaugurated statewide MTC. The gatherings include two after-school gatherings from each of five state-wide locations and a 3-day "summer retreat." These were the first gatherings for the statewide MTC, although two locations had previously hosted MTC gatherings.

Local gatherings were facilitated by "lead teams" composed of 3-5 local teachers and university mathematicians. These lead teams attended a group training session facilitated by the American Institute of Mathematics, the organization that created and currently disseminates the MTC model. The lead teams designed and conducted their gatherings independently of each other. The summer retreat was organized and facilitated by the coordinators of the statewide MTC, four of whom are also authors of this paper (Peck, Erickson, Roscoe, and Wu).

Gatherings were organized around "activities"—mathematical problems that participants worked on groups of 3-6 people, followed by large group discussions of the problem. There was 1 activity in each of the 10 local gatherings, and 9 activities in the summer retreat. In all, the data encompass 11 gatherings and 19 activities. Across all sites, there were 177 participants: approximately 80% were practicing teachers (20% elementary, 30% middle school, 30% high school), approximately 10% were post-secondary mathematics faculty, and approximately 10% were pre-service teachers (these percentages are approximate because there are some participants for whom we do not have demographic data).

Communities develop via engagement in joint activity. Participants interact with each other and with artifacts, and through this interaction norms of engagement, joint practices, and a shared repertoire emerges; a community develops and people become part of it (Bowker & Star, 1999; Dean, 2005; Lave & Wenger, 1991). Because community development occurs in interaction, we used video and audio recorders to capture the naturally-occurring interactions of participants as they engaged in activity during the gatherings. For each of the 19 activities, we have video and audio recordings of 2-6 problem-solving groups. Additional data include:

- Participants' notebooks from the summer retreat. Participants used these notebooks for jottings and work space during the retreat. They also used the notebooks to provide written responses to a series of reflection prompts at the end of the retreat.
- Interviews with 10 participants from the summer retreat. This represents a selective sample of all participants. We invited all participants to be interviewed. From the set who agreed to be interviewed, we chose interviewees selectively in order to achieve a diverse sample with respect to gender self-identification, level taught (elementary, middle, high), and region of the state.

Our initial analysis focused on the recordings of MTC activities. We used a cyclical data analysis method, which relied on both inductive and deductive approaches (Miles, Huberman, & Saldaña, 2014). First, we developed a list of deductive codes based on our conceptual framework. We then engaged in the following process for each activity. Members of the research team each watched/listened to a different group engaging in the same activity. The team member created a content log (Maxwell, 2013) of the recording and coded the log according to the codebook, allowing new codes to emerge from the data. We then met to discuss our observations and coding. We refined our codebook and then used the refined codebook to code the next activity. We proceeded in this fashion, with inductive codes emerging from the data and subsequently undergoing refinement, for all 19 activities.

We used these coded content logs to identify key segments in which participants negotiated the two foci of the essential tension (mathematical activity and pedagogical practice). We transcribed these key segments and analyzed them using multi-modal interaction analysis (Erickson, 1992; Goodwin & Heritage, 1990; Streeck, 2009).

We employed a similar procedure to analyze the participants' notebooks and interviews.

Findings

We present our findings organized around our two research questions.

RQ 1: In What Ways, if at all, Are the Two Poles of the Essential Tension Manifested in MTCs?

Perhaps unsurprisingly, we found that the majority of activity in MTCs involved engaging in disciplinary (mathematical) practices. Pedagogical concerns occupied less than 5% of the "official" activity. We gloss an activity as "official" if it was introduced by the facilitators as the focal activity of the group.

Pedagogical concerns were sometimes explicitly backgrounded by facilitators. For example, in the introduction to the one of the initial spring gatherings, a facilitator explained the goals for the gathering:

> You shouldn't feel like there's any expectation to be walking away this evening with anything other than a good feeling, alright? We're not trying to prove anything, this is just for us. We're not trying to say, "and now, fourth-grade math achievement will go up because-" ((laughter)). That has nothing to do with it. You see, we're just- what we're trying to do is just, be a group that likes mathematics.

In this turn, the facilitator invokes both pedagogical concerns ("fourth-grade math achievement") and disciplinary activity ("be a group that likes mathematics"). The turn is designed such that the two foci are put into opposition with each other. This can be seen in the use of the adverbs "not" and "just" to modify the verb "trying" in the second half of the turn ("*not* trying," and "*just* trying"). In particular, the use of the word "not" in "we're not trying to say" negates the pedagogical focus. This is reinforced with the exclusionary "just" in reference to mathematical activity.

This finding—that the primary activity of an MTC is mathematical, not pedagogical, activity—is not surprising, given that engagement in mathematical activity is the explicit purpose of MTCs. Furthermore, the finding that the two foci were treated oppositionally is also not surprising due to the "beginning" nature of these teacher groups.

However, we also found that pedagogical concerns were invoked in multiple, interesting—and sometimes surprising—ways during MTCs.

Pedagogy was rarely the official topic of activity. Most commonly, if pedagogy was the official activity of an MTC gathering, it happened in the final phase of the gathering. This phase was framed as a "reflection" time, and pedagogy was a topic for reflection. For example, at the end of the 3-day summer retreat, the group met all together, and reflected on what it meant to *do* mathematics, based on their experience in the summer retreat. After the group generated a list of attributes associated with doing mathematics, the facilitator (Peck) referenced the list and said:

> So if we think about all this stuff that doing mathematics <u>is</u>, um, take a moment and write just a couple ideas about how you might incorporate some of this into your classroom.

Both the design of this turn, and the subsequent uptake by participants provide evidence that pedagogy has become the official activity. By employing the imperative mood (*take* a moment and *write* just a couple of ideas…), and applying it to pedagogical concerns (…about how you might incorporate some of this into your classrooms), the facilitator signals that the official activity is now related to pedagogy. Participants' uptake confirms this. After the facilitator's turn ended, participants began writing and there is silence on the video and audio recordings. Analysis of participants' notebooks confirms that each response involves pedagogy.

Notice also how this turn brings pedagogy into a productive relationship with mathematical activity. Rather than treating pedagogy as separate from the mathematical activity of the retreat, the facilitator, through the use of the word "incorporate," suggests that the mathematical activity of the retreat can productively be brought to bear on classroom pedagogy.

More commonly, pedagogy came up informally in relation to the concurrent mathematical activity. As participants engaged in mathematical activity, they often related the activity to pedagogy. For example, the strip of dialog in Table 1 occurred while participants were exploring the question, "can any number be written using only powers of 2?" The three participants, Amy (5-6th grade teacher), Diane (3rd grade teacher), and Patty (7-12th grade math teacher) discuss the mathematical question in turns 1-17. In turns 18-20, they transition to a pedagogical discussion related to the mathematical activity, which they continue for the remainder of the strip.

Table 1: Doing Mathematics and Talking Pedagogy

Turn	Speaker	Talk[1]
1	Amy:	I think, I mean-
2	Diane:	Well we have to be able to because how else- That's how binary works.
3	Amy:	How else could we-
4	Diane:	They have to-
5	Patty:	Make every number?
6	Diane:	Yeah, binary's gonna work every time.
7	Patty:	mmm-hmm
	((Amy looks at notebook, where she has written a list of powers of 2.))	
8	Amy:	How do you get 127?
9	Diane:	((points to notebook)) There's your two, 16 – ((moves finger along notebook, where Amy has written successive powers of 2)) That's gonna be… one hundred… [twenty seven!
10	Amy:	[twenty seven! Okay…
	((Talk continues in this fashion, for turns 11-17, with Amy suggesting an number, and Diane showing how to make the number)).	
18	Amy:	Two:::: f::::- ((smiling))
19	Diane:	You little pain in the butt! ((laughing))
20	Amy:	Hmmm… I'm trying to think like a f-
21	Diane:	Trying to think like a sixth-grader?
22	Amy:	Yes!
23	Diane:	They are difficult little critters, but they're adorable!
24	Amy:	"well what if you want to do this? What if you want to do this?"
25	Diane:	"So tell me how. What's the pattern you're seeing?"
26	Patty:	So what grade do you start doing these problems?
27	Amy:	Binary?
28	Patty:	No-
29	Diane:	Exponents!
30	Amy:	[Oh exponents
31	Patty:	[Yeah, just- just exponents?
32	Amy:	Fifth- fifth grade.

33	Patty:	Well, we talk about squaring (crosstalk) not- not square roots-
34	Diane:	Just squared?
35	Patty:	Just squared. So I [sh-
36	Amy:	[I don't even talk about square roots, until-
37	Patty:	So I show the kids that notation ((draws a "2" in the air)), and I talk to them about how if it's 54 square units, I show them how you can write 54 units squared ((draws "2" in the air)) [with a-.
38	Amy:	[yeah
39	Patty:	with a- with an exponent two… so what grade do you start talking about, "what does that exponent MEAN?" ((draws 2 in the air)) and-
40	Amy:	I do fifth and sixth grade, and in fifth grade I introduce it to them-
41	Patty:	Okay.

[1] In general, talk is transcribed using standard punctuation, so that a comma denotes a short pause, a period denotes a longer pause after a falling intonation, and a question mark denotes a pause after a rising intonation. Ellipses ... denote a long pause. Underline denotes vocal emphasis, co:::lons denote a drawn-out sound, and a hyphen- denotes a restart. Vertically-aligned open brackets [denote overlapping speech. ((Double parentheses)) denote non-vocal action.

Because participants often invoked pedagogy even when it was not the "official" topic, we found that participants were engaged in pedagogical conversations or activity approximately 15% of the time—3 times more than that which was accounted for in the official activity.

RQ 2: When Pedagogical Practice Is Invoked, How Is It Treated by Participants?

We found that pedagogical practice was treated as a normative topic of discussion in MTCs. The strip in Table 1 is representative of this. Notice the framing of the turns where pedagogy is first evoked, and the response to these turns:

20	Amy:	Hmmm… I'm trying to think like a f-
21	Diane:	Trying to think like a sixth-grader?
22	Amy:	Yes!
23	Diane:	They are difficult little critters, but they're adorable!
24	Amy:	"well what if you want to do this? What if you want to do this?"
25	Diane:	"So tell me how. What's the pattern you're seeing?"
26	Patty:	So what grade do you start doing these problems?
27	Amy:	Binary?
28	Patty:	No-
29	Diane:	Exponents!
30	Amy:	[Oh exponents

In particular, notice the absence of an account for why pedagogy is being introduced. Neither Amy nor Patty provides a rationale for why they are introducing pedagogy, and subsequent turns do not hold them to account for such an introduction. Together the design of turns 20-30 can be taken as evidence that for these participants, pedagogy is normative topic of discussion (consider how these turns would be designed differently in a situation where pedagogy was not normative, say at an adultleague softball game). This finding is somewhat surprising, considering the "official" framing of MTCs as primarily focused on engaging in disciplinary practice.

Even though pedagogy was a normative topic, the way that it interacted with disciplinary practices varied. In some cases, these two foci were treated oppositionally, as would be expected in a "beginning" group like the ones that we studied. The first facilitator quote given above is one example of this. A second example comes from the reflections of participants in the summer retreat, one of whom wrote,

> Some of the activities were good, but others were not helpful. I guess I was looking for more options to take back to my classroom.

Using the word "but," the participant contrasts "good" activities with those that were "not helpful." She goes on to identify "helpful" activities as those that could be used in the classroom. This comes even after the participant discussed how much she had learned about doing mathematics from the activities. This suggests that, for this participant, "engaging in disciplinary activity" and "improving pedagogical practice" are two *separate* foci.

However, we also found multiple times where the two foci were held in "productive relation," which would be evidence of a more mature community. The strip of talk in Table 1 is one example of this, where the content of the activity spurred a conversation about pedagogy related to that content.

We also see evidence for the "productive relation" in participants' reflections. For example, a different participant reflected:

> Working together to solve problems reminds me how important and fun it is, and I need to do that as much as possible in my class!

Here the participant uses a different conjunction: "and" instead of "but." In doing so, she brings disciplinary activity ("working together to solve problems") into a productive relation with pedagogy ("I need to do that… in my class").

Conclusion and Significance

Math Teachers' Circles have exploded in popularity since their introduction in 2006. A key claim is that MTCs constitute *communities of practice.* However, this claim has not been subjected to analytical scrutiny. In this paper, we take a step towards such scrutiny by employing anthropological methods to analyze the ways that participants in MTCs negotiate their joint enterprise.

We found evidence that both disciplinary practice *and* pedagogical practice are part of the joint enterprise. This is a surprising finding because the groups are "officially" framed around disciplinary practice only, and are represented as such in the published literature (e.g., the excerpt from White et al, 2013, in the introduction). This supports Grossman et al.'s contention that teacher communities must include a focus on pedagogical practice.

We also found that, in negotiating the joint enterprise, MTC groups display hallmarks of both "beginning groups" and "mature communities." This complicates the claim that all MTCs are communities of practice: our findings suggest that a beginning MTC, at least, may not be a mature community of practice. Participants are still negotiating the essential tension, and some participants struggle to hold the two foci in productive relation.

However, what is perhaps our most striking finding is that, most often, when pedagogy was invoked, it was treated as normative by participants. Most of the time, when the "essential tension" manifested itself, there was no *tension* at all. This finding complicates the model of Grossman et al. (2001) in which communities must go through ordered stages of oppositional tension between mathematical activity and pedagogical practice, before they can hold the two in productive relation. This finding should be explored and elaborated in future research.

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STEM AND MODEL-ELICITING ACTIVITIES: RESPONSIVE PROFESSIONAL DEVELOPMENT FOR K-8 MATHEMATICS COACHES

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This research highlights a university-school division collaboration to pilot a professional development framework for integrating STEM in K-8 mathematics classrooms. The university researchers worked with mathematics coaches to construct a realistic and reasonable vision of STEM integration built upon the design principles of model-eliciting activities (MEAs). Analysis of participant reflections after they experienced two MEAs as learners in mixed grade-level teams suggests an evolving conceptualization of STEM integration with an explicit connectedness to mathematics content. Mathematics coaches valued the potential for MEAs to provide multiple entry points to open-ended problem solving, but they articulated a sense of vulnerability as they contemplated the challenges of time and teacher buy-in within the contextual realities of curriculum pacing and standardized test preparation.

Keywords: Teacher Education-In-service/Professional Development, Modeling, Instructional Activities and Practices, Problem Solving

Introduction

School divisions across the United States have "embraced" the slogan of STEM (Bybee, 2010), but there is limited evidence of theoretical frameworks for the design and development of sustainable STEM integration in K-12 education [National Academy of Engineering (NAE) & National Research Council Committee (NRC), 2014]. Mathematics may be relegated to a supporting role in STEM integration (Fitzallen, 2015) when it is characterized as the calculations or the data representations in science classrooms, technology labs, or outside-of-school programs. To bring mathematics content to the forefront in STEM integration, designers must attend to "learning goals and learning progressions" within mathematics (NAE & NRC, 2014, p. 148) and avoid a dilution of mathematics content (Shaughnessy, 2013). Without specific connections to mathematics content, teachers may perceive STEM integration as an additional instructional requirement that is placed on top of the existing curriculum (Wang, Moore, Roehrig, & Park, 2011).

The university researchers reflected upon these challenges as they collaborated with a school division to develop a STEM integration professional development (PD) framework for mathematics coaches. They hypothesized that model-eliciting activities (MEAs) offer a vehicle for *reasonable and realistic* STEM integration in K-8 mathematics classrooms by providing open-ended problems within a client-driven, real-life context. With mathematics coaching support and explicit connectedness to content, teachers can use MEAs to engage students in both collaborative mathematical thinking and productive engineering design processes.

In this paper, we describe preliminary findings from our exploratory study during a STEM integration PD initiative in a mid-Atlantic school division. The mathematics coaches experienced two MEAs as learners and collaborated to envision MEAs as an instructional vehicle for integrating STEM within the bounds of their coaching contexts.

Conceptual Framework

STEM integration should offer students and teachers the opportunity to engage in "real-world, rigorous, and relevant learning experiences" (Vasquez, Sneider, & Comer, 2013). For many mathematics teachers, this conceptualization of STEM integration is so distant from their daily understandings of content, curriculum, and pacing that implementation becomes unrealistic. STEM may be perceived as a project or activity that is ancillary to content-specific instruction and may become an unintended barrier to richer opportunities to learn. Although reform curricula have emphasized the need for problem-solving opportunities in mathematics, many teachers still perceive problem solving as an elite activity, accessible only to students who have mastered essential mathematical skills and formulas (Crespo, 2003). This perception impedes conceptualization of mathematics content instruction through STEM activities. In addition, mathematics students in underperforming populations are often denied access to opportunities to practice 21st century skills as they are instead provided with remediation and extra support to gain computational fluency in preparation for standardized assessments.

Drawing from Stohlmann's (2013) definition of STEM integration as "an effort for mathematics teachers to use the engineering design process as the structure for students to learn mathematical content along with science concepts through technology-infused activities," the researchers designed a PD structure that would support teachers in incorporating MEAs within existing mathematics curriculum. The research team's goals were to: 1) explore participant understanding of mathematics within STEM integration; 2) use hands-on MEA experiences to elicit a more accessible classroom implementation of STEM aligned with coach and teacher beliefs; and 3) build coaching capacity to use MEA design principles to develop STEM tasks with a specific focus on keeping the mathematics content explicit.

MEAs were originally conceptualized as a device to help mathematics education researchers elicit student modeling and to develop expertise about cognition and problem-solving behavior. They have since become a tool that can also be used to help teachers and students develop their own competencies (Hamilton, Lesh, Lester, & Brilleslyper, 2008). MEAs support the learning of mathematics within STEM by integrating other content areas found inside and outside of mathematics, by encouraging learning through discovery, and by promoting problem-solving dispositions (Magiera, 2013) with a specific focus on accessibility to varied learning styles and experiences (Stohlmann, 2013). MEAs offer teachers a contextual and content-focused lens on student mathematical thinking (Chamberlin & Moon, 2008; Stohlmann, 2013) that yields explicit evidence of student learning that is needed to ensure the mathematical rigor in STEM integration.

Because educator expertise may be the "key factor" in STEM integration (NAE & NRC, 2014, p. 115), PD is needed to support teachers who did not learn mathematics in STEM contexts to build a working knowledge of what STEM integration can look like (Stinson, Harkness, Meyer, and Stallworth., 2009). PD that is both "site-based" and "curriculum-linked" (Penuel, Fishman, Yamaguchi, & Gallagher, 2007, p. 928) is theorized to improve teacher enactment of reform-oriented instruction, and prior research has shown that ongoing mentoring and support (p. 124) and teacher collaboration (p. 125) increase the likelihood of successful STEM integration (NAE & NRC, 2014). The university researcher-school division partnership offered a bridge between research with MEAs and the practical demands of existing curriculum and standards. As the division mathematics coaches experienced MEAs as learners, they could begin to conceptualize STEM integration built upon engineering design and real-world contexts with an explicit connectedness to mathematics content. The research question which guided this study was as follows: How do iterative experiences with MEAs during a university-facilitated PD help mathematics coaches to envision STEM integration in K-8 classrooms?

Methodology

Design-based implementation research (DBIR) is an emerging methodology in which stakeholders are committed to iteratively developing an educational innovation with a goal of broader and sustainable impact (Penuel, Fishman, Cheng, & Sabelli, 2011). This collaborative PD connected research on MEAs which "engage learners in productive mathematical thinking" (Hamilton, et al., 2008, p. 5) to STEM integration efforts within one school division. The university-school division partnership drew upon studies of specific enactments of MEAs as tools in mathematics and engineering education and theorized a STEM PD structure to explicitly focus on instructional coaching and mathematics content. The team worked to construct coaching expertise that schools would need to broadly implement MEAs as a vehicle for STEM integration.

Setting and Participants

The university researchers and the division mathematics supervisor collaborated to create a longitudinal PD structure in response to a district directive to integrate quarterly STEM tasks at each grade level. Within this school division, 73% of students were traditionally underserved and 53% were economically disadvantaged students (State Department of Education, 2015). The division had five elementary schools, each with Title I designation, one intermediate school, one middle school, and one high school. The supervisor sought to leverage the university's mathematics education and instructional coaching expertise to bring mathematics to the forefront of STEM integration. She allocated time for mathematics coaches from the seven K-8 schools to explore and design MEAs with university facilitator support during their monthly academic year coaching meetings.

The Professional Development Context

The university researchers developed resources aligned with the partnership's PD goals and piloted a four-day summer institute for eight aspiring STEM teacher leaders to engage with MEAs and plan for use within their classrooms. Two of the participants were mathematics coaches. The summer institute participants explored their perspectives on STEM, engaged with MEAs as learners, evaluated the affordances and challenges of implementing MEAs in their classrooms, and adapted existing curricular materials and online resources to design MEA instructional materials for their schools. The university researchers drew upon participant reflections and questions from the summer institute as they designed ongoing PD for the mathematics coaches during monthly academic year meetings. The evolving PD structure was purposefully adapted to support coaches in seeing the possibilities of MEAs, first through the eyes of students, then as teachers, and finally as coaches.

Month 1. The university researchers asked the coaches to reflect on the meaning of STEM in order to situate their existing understandings before introducing MEAs. The definition of MEAs offered by Maiorca and Stohlmann (2014) provided an accessible set of four design features (open-ended, client-driven, mathematics similar to real-life, and engineering design process) to help coaches as they began to build a working knowledge of model eliciting. These practitioner-friendly constructs were a crucial bridge from the researcher language of MEA design principles (Hamilton et al., 2008) to coach envisioning of MEAs in classrooms.

Teams of coaches engaged with the Survivor MEA (Maiorca & Stohlmann, 2014) to explore the affordances and challenges of MEAs from a student perspective. Although the Survivor MEA offered an engaging, relatable context for the participants as they experienced their first mathematics-focused STEM integration task, the research team observed that the physical construction of the weather-resistant shelter model became the primary focus of iterating and refining. The coaches required additional hands-on experiences to see the potential for explicit mathematics content instruction within this type of engineering activity.

Month 2. Before the coaches engaged in a second hands-on MEA experience, they needed time to make sense of MEAs and connect them to their own K-8 educational contexts. The university researchers offered a task adaptation template based upon the four MEA design features to support coaches in thinking critically about MEAs and in exploring online resources. This space for exploration allowed the coaches to purposefully think about not only the teachers they would support and the classrooms in which they would pilot MEAs, but also to build important contextual knowledge that would prepare them for their second hands-on experience.

Month 3. The university researchers selected the Pelican Colonies MEA (see Figure 1) as a second hands-on experience because the mathematics content could be more flexibly connected to K-8 standards and there were more varied opportunities for use of manipulatives. The coaches were purposefully grouped to offer mixed levels of content expertise and grade-level experience. The goal of this second exploration was to allow the coaches an opportunity to not only analyze their own collaborative engagement with the MEA as learners but also to reflect on transferring these ideas to their coaching practice.

Data Collection

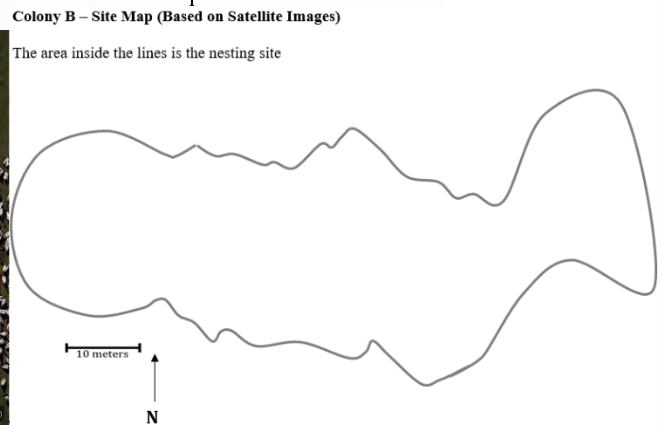
Multiple qualitative data sources were used to characterize the envisioning and enactment of MEAs as vehicles for STEM integration. Written reflections were centered on the participants' perceptions and understandings of STEM integration and the affordances and challenges of using MEAs in mathematics teaching and coaching. Additionally, discussions during and after MEA problem-solving experiences were audio recorded to capture the dynamic nature of conversations and preserve the integrity of the participants' experiences and perceptions. Finally, PD artifacts, mathematics task adaptation charts, and modified curriculum materials provided evidence of the evolution of participant thinking on specific strengths of MEAs and the contextual challenges of implementing MEAs.

Data Analysis

Data gathered was qualitatively analyzed to inform real-time changes in the PD structure and to support longitudinal evaluations of changing conceptualizations of STEM integration. The university researchers examined participant writing prompts at the end of each session as they modified each iteration of the PD to maximize stakeholder involvement and negotiation in design decisions (Fishman, Penuel, Cheng & Sabelli, 2013; Penuel et al., 2011). Group discussions and written reflections provided evidence of beliefs about the potential role of MEAs in mathematics instruction and the emerging challenges of using MEAs to integrate STEM in their classroom contexts. Sample codes that emerged from the analysis of audio recordings and reflection prompts included: reasonable, realistic, teacher buy-in, appreciation of PD design, need for collaboration, student readiness, and time. The reduction of all codes into categories led to the emergence of themes which illuminated the role of PD design decisions in both evoking and alleviating coaches' concerns about introducing and supporting MEA enactments.

Pelican Colonies MEA - Excerpts from Client Memorandum

"The U.S. Fish and Wildlife Service needs a procedure to estimate the number of nests at each pelican colony...We are enlisting your team's help to create a procedure that will allow us to estimate the number of nests in a pelican colony, based on the photograph that shows a sample of the colony, and a map that shows the size and the shape of the entire site."



MEA Design Features	Affordances of Pelican MEA for PD for Mathematics Coaches
Client-Driven	Request from U.S. Fish and Wildlife Service; Context activated by one-page newspaper article; connection to scientific research
Open-Ended	Variety of relevant manipulatives: ruler, tape measure, small beans, transparencies, and markers
Mathematics Similar to Real Life	Multiple K-8 entry points: Multiplication; decomposing area of a polygon into rectangles; measurement; estimation; ratios and proportions; unit rate; random sampling and inferential statistics
Engineering Design Process	Iterative refining and testing

Figure 1. Pelican Colonies MEA Summary and Design Features (Adapted from http://wordpress.unlvcoe.net/wordpress/wp-content/uploads/2013/01/Pelican-Colonies-MEA-Teacher-Materials.pdf.

Findings

Because mathematics coaches have varying school contexts and administrator expectations, the university researchers needed to be responsive to the changing needs of the participants as they engaged in a collaborative journey toward envisioning MEAs as a vehicle for STEM integration with a specific place for K-8 mathematics content. The PD design decisions to provide time and space for peer exploration of MEA resources and to offer a second hands-on experience with an MEA were critical moments in the collaborative journey toward STEM integration in mathematics classrooms. While coaches expressed enthusiasm about the problem-solving potential of MEAs, they also became more fully aware of the pedagogical and leadership challenges they would face in bringing MEAs to classrooms.

Affordances of a Second Hands-On Experience

As the coaches discussed their experiences with the Survivor MEA during Month 1, they initially focused on the relatability of the context to their lives. They also adopted a teacher lens as they discussed the possibilities of tailoring the context, introducing mathematical language, and completing the written component of the task outside of mathematics instructional time. However, there was no organic discussion of possible application of grade-level content. One of the university researchers needed to prompt the coaches to specifically think about the mathematics content within this MEA.

In contrast, the coaches' second hands-on experience with the Pelican Colonies MEA during Month 3 elicited deeper mathematical thinking that aligned with familiar standards. The centrality and potential flexibility of mathematics application elicited by the Pelican Colonies MEA encouraged the coaches to draw on their content knowledge and past teaching experiences. Mathematics coaches looked at engagement with the MEA through a different lens than Month 1 as they initiated discussions on the multiple mathematical entry points and their grade-level expectations. In Group 1, the district math supervisor acknowledged that her problem-solving approach was consistent with her secondary math experiences. "I thought unit rate, others on the team thought ratio." The other two K-6 mathematics coaches in her group connected to content by drawing on their most recent classroom teaching responsibilities. "Amy said 60 x 19 because that's what we would do in 3rd grade, while I was still thinking ratio because the last grade I taught was middle school." (Jill, K-6 mathematics coach)

Because the previous monthly PD sessions built working knowledge of MEA design features and task design, coaches were able to reflect on the MEA from a teaching perspective centering on mathematical implementation. Yet they also maintained an important attentiveness to client-driven, open-ended, real-world aspects of the MEA. Although the coaches in Group 2 did not progress as far into the task as Group 1, they engaged in an engineering design process. "We did not finish because we took time to think, which the kids need to do" (Mia, 6-8 mathematics coach). Mia also wondered if middle school students would question the purpose of the MEA. "Why are we trying to save the pelicans?" Anticipating potential student responses to the MEA context was essential for the mathematics coaches to think about the future enactments in K-8 classrooms. "Real world to kids is something they can activate - something they can be interested in doing. In middle school it's hard to get them engaged to think something is important."

The university researchers had selected the Pelican Colonies MEA because of its broad potential for K-8 mathematical thinking and for use of manipulatives. The mathematics was more accessible to the coaches than it had been with the Survivor MEA, and the coaches responded by discussing the explicit use of mathematics content without specific prompting by the university researchers.

Coaching Challenges

During Month 3, the coaches transitioned from acting as students to anticipating the unique demands of implementing MEAs in their own schools. Their critical thinking about the potential of MEAs to improve student opportunities to learn was accompanied by a vulnerability they would feel as they introduced MEAs to K-8 mathematics teachers and students. They shared their anxieties about advocating for the use of a resource that they were still exploring.

> Because I am new to MEAs, I feel like I will be entering uncharted territory where I will no longer be the 'knowledgeable other' or expert... It's hard to walk into someone else's classroom. If [the experience] is perceived as a waste, what is the fall out for that? (Jill, K-6 mathematics coach)

The concern about consequences of failure extended beyond elementary contexts. Mia, the middle school mathematics coach, worried that an unsuccessful MEA could harm her ongoing efforts "to get kids to do more than the teachers." She further wondered how teachers would see MEAs connecting to math, specifically standardized tests and other accountability measures.

> The math piece of it is where I think the teachers will struggle as much as the kids do when we talk about there not being [one] answer. They are at that level where they think there should be [one] answer or [the task] needs to look like the [standardized test] questions. (Mia, 6-8 mathematics coach)

Finally, the coaches articulated concern that students would not be able to make the important "connection between what they are doing and the content that they know" (Jess, K-6 mathematics coach). They also speculated that teachers would assume that the MEA would be too hard for their students.

While the coaches expressed vulnerability with respect to student and teacher readiness, they believed that opportunities to collaborate with other coaches and teachers could build their confidence in addressing these challenges. In her written reflection at the end of the Month 3 PD, Jill described her planned collaboration with another mathematics coach, a STEM coach, and a 3rd grade teacher to "help build buy-in with other teachers, provide more adult support with students, and increase excitement/engagement of the students." Three of the coaches expressed a need for more time to collaborate to develop their MEAs. These needs were shared with district mathematics supervisor as the university-school division partnership planned future directions for the STEM integration PD.

Conclusions

The construction of a *reasonable* and *realistic* STEM orientation for teachers is critical as the education community looks toward connecting STEM integration and mathematics learning. At the beginning, the coaches were challenged to reevaluate their conceptualizations of STEM from prior PD experiences. The university facilitators engaged in real-time iterations of planning, reflecting, and revising as they gauged participant perceptions and responded to participant challenges to connect research to practice. As the coaches iteratively engaged with MEAs, they envisioned multiple entry points within these problem-solving structures with respect to grade-level content and student readiness.

School contexts, administrator expectations, and assessment-driven cultures must inform the ongoing negotiation of STEM implementation. The challenges that the coaches articulated in bringing their MEA designs to the classroom is consistent with prior research on the need for ongoing school-based support. Coaches, teachers, and researchers will continue to engage in this STEM integration design and development process as they reflect upon prototype MEA enactments and redesign resources for wider implementation. Their shared investment in realizing classroom and school STEM integration capacity with a specific focus on mathematics outcomes and coaching contributions will offer a model for STEM integration that challenges "one-size-fits-all" PD, defines a new role for mathematics coaches and teachers as STEM instructional leaders, and promotes meaningful readiness for STEM citizenship and careers for their students. The district supervisor articulated the student-centered possibilities of MEAs that continue to motivate the work of the team. "The kids are acting as mathematicians instead of learning about math."

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TEACHER PERCEPTIONS ABOUT VALUE AND INFLUENCE OF PROFESSIONAL DEVELOPMENT

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We used a situative perspective to examine teachers' perceptions of a professional development intervention that integrated lesson study, video clubs, and animation discussions. The analysis of interviews with the five geometry teachers who participated in the intervention during two consecutive years showed three characteristics of professional development that were valuable: designated time to collaborate, focus on student mathematical thinking, and use of animations to represent practice. A fourth characteristic, accountability for implementation, is also discussed. The findings have implications for designing professional development, because participants cited links between their experiences and changes in their practice.

Keywords: Teacher Education-Inservice/Professional Development, Geometry and Geometrical and Spatial Thinking, High School Education

Perspectives

Putnam and Borko (2000) cited the importance of creating situative learning experiences for inservice teachers and described aspects of such experiences that promote growth in teacher learning and practice. In a situated learning experience, teachers engage with others (e.g., in a discourse community) both within their own classrooms and outside their classrooms (Putnam & Borko, 2000). Although familiar contexts may help teachers make connections to their daily practice, there are limitations to professional development that occurs exclusively in a teacher's own classroom. Comfortable habits, supported by a local culture, are difficult to break. Discussions with others may serve as disruptors to entrenched patterns of behavior which may then spark reflection and change. Putnam and Borko described how discourse communities can help teachers face the risks entailed in making meaningful change. When a group explores new materials and strategies in a forum that draws on different perspectives and expertise, practice may become the subject of critical reflection. As a result, teachers may be empowered by knowledge drawn from and trust in the group to try new ideas that were previously seen as too unfamiliar and, thus, risky to use with students.

Lesson study (Lewis, Perry, & Murata, 2006) is one example of a professional development model that is both classroom-centered, and considers teaching and learning as objects of reflection by a community of practitioners. Hiebert, Gallimore, and Stigler (2002) concurred that effective professional development should be centered in the classroom, part of a long-term collaborative process, and focused on student learning and curricula. These perspectives frame our investigation of teachers' perceptions about valuable characteristics of professional development.

Purpose and Research Questions

The purpose of this study was to determine what teachers valued about a professional development experience that was designed to create a situative learning experience. In other words, we looked at ways that teachers might justify the value of different aspects of such an experience, including components of that experience that were both centered on their own practice, yet occurred away from their classroom in the context of a discourse community. Given that such situative learning experiences have been shown to induce powerful teacher learning, we examined the extent to which teachers reported such learning and the factors to which they attributed that learning. The following research questions guided our analysis:

1. Which aspects of the professional development intervention did teachers find most valuable and how did they justify that value?
2. What changes did teachers note in their practice and to what factors did they attribute those changes?

Methods

The teachers in this study were participants in a larger study focused on promoting teacher noticing and use of students' prior knowledge to inform lesson design, implementation, and reflection on implementation (González, Deal, & Skultety, 2016). The five participants were all high school geometry teachers, with 4–26 years of teaching experience, who taught in high-need schools in the Midwestern United States. Teachers participated in two iterations of a lesson study process. The teachers met in 3-hour monthly sessions that teachers called study groups. Each year, participants watched and discussed animated, cartoon depictions of several versions of a geometry lesson (i.e., animations; Chazan & Herbst, 2012), collaboratively planned and implemented a lesson on the same topic, watched and discussed videos of their own students participating in the planned lesson (in a video club; van Es, Tunney, Goldsmith, & Seago, 2014), revised the lesson they developed, and repeated the process.

The lessons topics (i.e., dilations and perpendicular bisectors) were predetermined by the research team. Because the focus of the lesson development and analysis was on identifying and building upon students' prior knowledge during instruction, the focus of study group discussion was on student understanding, rather than on the teacher actions during lesson implementation. Participants constituted a discourse community, as they worked together to plan the lessons and analyze video clips of students' work during the lessons.

The data for this paper are teacher self-reports from 20–60 minute individual interviews that the first author conducted at the end of each of the two years of the professional development experience. Teachers were asked to describe strengths and weaknesses of the study group, their perceptions about the goals of the study group, and the relative value of each aspect of the study group (e.g., animations, collaborative planning, analysis of student thinking). Participants were also asked to comment on whether the experience had an impact on their teaching.

The first author audio recorded and transcribed the interviews, using pseudonyms for each teacher. Transcripts were analyzed for common themes mentioned by participants using a grounded theory approach (Corbin & Strauss, 2008). In the first iteration, we used an open coding process, identifying any significant aspects that the participants noted, regardless of the question that they were addressing. In a second iteration, we looked more closely for participants' justifications for significant aspects of the professional development and examples participants provided of how their practice changed as a result of the professional development.

Results

The resulting themes from the open coding process are listed in Table 1, by teacher and year. Themes are grouped in broad categories: (a) aspects of the professional development that were significant for teachers in their own right (e.g., designated time to collaborate, forum for reflection, authentic to my curriculum or students, focus on one thing over time), (b) outcomes of the professional development (e.g., change in teaching practices), and (c) mechanisms that were inherent in the professional development that may have led to perceived outcomes (e.g., focus on students' mathematical thinking, focus on students' prior knowledge, accountability for implementation, use of animations to represent practice). Students' prior knowledge and students' mathematical thinking were related, and, indeed overlapped on occasion, but teachers were more likely to refer to students' prior knowledge when discussing lesson planning and students' mathematical thinking when describing making sense of student ideas retrospectively (e.g., when reflecting on videos).

Table 1: Significant Aspects of the Professional Development by Teacher and Year

Theme	Year	Teacher 1	Teacher 2	Teacher 3	Teacher 4	Teacher 5
Designated time to collaborate	1	X	X	X	X	X
	2		X	X	X	X
Authentic to my curriculum or students	1	X	X			X
	2		X			
Focus on one thing over time	1	X			X	
	2			X	X	
Forum for reflection	1					
	2			X	X	X
Change in teaching practices	1	X	X	X	X	X
	2	X	X	X	X	X
Focus on students' mathematical thinking	1	X	X	X	X	X
	2	X	X	X	X	X
Use of animations to represent practice	1	X	X	X	X	X
	2		X		X	
Students' prior knowledge	1	X		X		X
	2	X		X		X
Accountability for implementation	1			X	X	X
	2					X

Note: Themes addressed in this paper are in bold. An "X" in the cell at the intersection of that teacher and the year within a theme, indicated that the theme was mentioned by that teacher at least once during the interview.

There were two themes that were mentioned both years by every teacher: (a) change in teaching practices (an outcome of their participation) and (b) focus on student mathematical thinking (a mechanism inherent in the learning experience that may have contributed to the outcome). A third theme, designated time to collaborate, was mentioned by every teacher in year 1, and by all but one teacher in year 2. A fourth theme, use of animations to represent practice, was also a potential mechanism for allowing change to occur, and was noted by every teacher in year 1 and reiterated by three of the teachers in year 2. We discuss each theme in more detail below and provide examples of teacher utterances that are representative of how teachers expressed each idea. Although a fifth theme, accountability for implementation of the new lesson, was not mentioned by all teachers, the three who did mention it were adamant about the importance of this factor in accounting for the effectiveness of the professional development. Results are organized by the two research questions.

Significant Aspects of the Professional Development

Teachers uniformly valued a focus on student mathematical thinking, and one justification for that choice was its role in sparking change in practice. There was also broad agreement on the value of designated time to collaborate, although justifications were a bit more diffuse. The use of animations was frequently cited and valued for its effect on the process.

Focus on students' mathematical thinking. The teachers named a focus on students' thinking as a beneficial aspect both years, and they attributed that value to video analysis, more so than to the animations. Teachers justified that a focus on student mathematical thinking was significant due to three main factors: (a) focusing on the student's role in instruction was novel to them, and it was something they did not believe was a focus of their teacher preparation programs; (b) listening to student thinking helped teachers understand why students respond as they do so that they can act on that information; and (c) knowledge of student thinking sparked changes in their practice. In the excerpts below, teachers' justifications are highlighted (in italics) within their claims about the significance of a focus on student mathematical thinking. The year during which the claim was made is identified after the teacher number (e.g., year 1 is represented as Y1). Further details about changes in practice are detailed in a subsequent section.

Teacher 1 (Y1): But I think the discussions that helped me the most were when we broke down what the kids did… That to me was huge, because, like I said, I've never thought about that part of it. *I've never thought about the kids being part of the process*. I thought about "I'm the teacher, I know everything." You know, so I thought that was the biggest thing there. When we really analyzed the kid's thought process that was huge.

Teacher 2 (Y1): So, [I think the goal of the study group was] to help support me as a teacher *to create those [problem-based tasks]* and then from there then look at the student thinking to help *create better tasks* or help improve that specific task.

Teacher 4 (Y1): The beneficial aspect is when we're sitting around the table and we're analyzing, so we're trying to get into the students' shoes and try to figure out what they're thinking during it. And then, it's not just how I think they're thinking, but I get to hear everyone else's thinking that they're [the students are] thinking. So it kind of *broadens my perspective of what I'm thinking about students' thinking*.

Teacher 1 (Y2): And then if you go to the discussion on the student thinking, that was such a *foreign concept*. Like I said before, nobody talks about that. You're the teacher. I'm going to impart my wisdom on you and you're going to absorb it in like a sponge. And, unfortunately, that's the way a lot of education classes were. You were taught in the way the professors were taught in the way they were taught in the way they were taught. And it was all just teacher is the expert; they lecture; you get the material; you test over the material; you move on. It was never, *there was never discussions about, "Well why did this kid do this first?* Let's look at their paper. Let's look at the their steps."

Teacher 3 (Y2): But it's still good to hear how students are thinking differently. So then *as a teacher I can be aware of how my students might approach problems*. So, that can either *change how I teach it* or just when I'm working with them and I see them doing this I can think, "*Oh, maybe this is how they're thinking I need to redirect them that way*."

Teacher 5 (Y2): I feel like as a professional, we, other than this study group, we've never necessarily just sat down and been like, "Let's look at student work and try to think of what they were thinking?" You never get time to practice that…. And generally you're just thinking Johnny's crazy and has no idea what he's doing. *But maybe secretly he just has a different frame of reference. And what he's saying actually makes sense to him.* And in his frame of reference makes sense. But *you're so clueless to his frame of reference that you just think he just doesn't make sense*. And I feel like that misunderstanding between student and teacher, is sometimes what turns kids off of math altogether.

Designated time to collaborate. All teachers indicated that they valued having a designated time to collaborate with other professionals, specifically geometry teachers. The teachers justifications were (a) they were able to see other teachers in action via excerpts of video-recorded lessons that were shared during the study group, and (b) they provided an opportunity to share ideas. In the excerpts below, teachers' justifications are highlighted (in italics) within their claims about the value of teacher collaboration. At times, teachers attributed their growth to collaborations, specifically, and these instances are highlighted, as well.

Teacher 1 (Y1): Because to me that's the biggest advantage of things like this is when you *get to steal ideas* from people that are specifically doing what you're doing.

Teacher 2 (Y1): I just think overall it was a really great experience, especially being able to talk to other teachers that are teaching geometry but in other buildings, I think was really valuable.

Teacher 3 (Y1): That's how I feel like *I have been able to grow the most professionally is just through the collaboration* with other professionals.

Teacher 4 (Y1): I feel like having to talk about specifics like as far as teaching geometry on like a regular basis has been really good for me and keeping me kind of like inspired, to fix things and do more and whatever. Because I feel like a lot of times when you are kind of like on your own, you kind of get in a rut and you start teaching the same things and you kind of no one really stirs the pot or makes you think, you know, unless you're like super self-motivated, like, to do that [laughter]. So it's kind of nice to be able to get together with professionals who are not necessarily at my school but are teaching the same concepts, same standards. And be able to *bounce ideas off* and *see what other people are doing* has been very good for … the teaching act. It's … kept a lot of my stuff fresh.

Teacher 5 (Y2): But then it was also *giving each other ideas* of … you know because it's the same problem they did in their class, and they probably ran into the same misconception. But they handled it differently than I did. And it was *nice to see different people handling them in different ways*. And I feel like that's part of practicing is seeing it done different ways, but you never get to see each other's classrooms like that. So, it was just really neat to see.

Use of animations to represent practice. Teachers noted two reasons that animations were valuable: (a) it was more comfortable to critique the practice of an animated, cartoon image of a teacher, rather than a real teacher, because the anonymity of the teacher created a safe space for honest dialogue; and (b) there were fewer distractions (e.g., background noise, student offhand comments, or off-task behavior) than would be inherent in a video of real people.

Teacher 1 (Y1): At first I was like, "Are you kidding me?" But I think the good thing about it was you weren't looking at a specific real person. It made you focus, *it made me focus on the content*. Because it was a cartoon setting and you knew it was scripted. So, you weren't looking at "Okay, how did the teacher say this?" You were looking at what did they say and how did the kids receive it and what did they say. So, *it really made me focus on what was being taught and what was being heard and what was being learned*, rather than "What kids were talking in this corner?" and "What was the teacher doing?" and that kind of thing.

Teacher 4 (Y1): I think those are really helpful because I think what happens there is the focus is… you're not like … as a teacher or as a classroom *you don't feel at all defensive because it's not like... you're not being analyzed*. You're analyzing a student's thinking. So, it's easier to be more open and share that information.

Teacher 5 (Y1): You know, you're looking at it and it's a cartoon guy. So you don't feel, *you feel very free to kind of like have criticism of it.* You know what I mean? It's like, you don't feel like your watching your buddy teach and you're like, "Why did you do that? That makes no sense." You know what I mean? So it's like… *it's not as personal*.

Teacher 2 (Y2): Well first, the [animated] vignettes, I think, provided a good opportunity. We didn't use them as much. But, it provides *a good opportunity to look at something without the bias of certain groups of students or looking at the teacher* or … It kind of *takes away more of the personal aspect* of it, when *you're just looking at the vignettes*. So I think that was kind of nice. Especially as would like at the beginning when we were kind of getting to know each other or starting to kind of feel out what this process whole… was all about.

Accountability for implementation. Although only three of the teachers addressed this issue, we found it compelling and have included it here. Teachers stated that a significant aspect of the professional development was the fact that there was accountability for implementing a practice (i.e., teaching the collaboratively developed lesson) that was a product of the study group. Often, this claim was made in response to a question in which teachers were asked to compare the study group with other professional development experiences. Teachers justified this claim by stating that being forced to attempt something new meant that they could not easily ignore the ideas that arose in the study group, as they might otherwise do after a professional development.

Teacher 3 (Y1): So, like, and the fact that it's like I'm held accountable for it. It's not like I'm sitting in some like even a full day, a full professional development day maybe by my district or whatever. And like we could come up with some good ideas, like "we could do this and that." And maybe we'll throw something in the air, but sometimes like once teaching…Like once the year starts, once the… *we kind of can fall back into that same old grind*.

Teacher 5 (Y1): So it's like very much like, "Here's some skills, now we're going to put them into practice. And, I'm coming to your school on Tuesday to see it being done." And you're like "Okay." So *it like forces you to like really do things*. Whereas a lot of PD is like, "They paid me to do this, here's a bunch of stuff, and I'm never going to see you again. So, use it or don't use it. I don't care." And then it's like not as effective. There's no follow up."

Teacher 5 (Y2): And then they hold you accountable *because they're coming to your room with cameras*! So you can't just tell them you're going to do it and then go and not do it. You, like, *you gotta do it*!

Changes in Teaching Practices

Teachers identified several specific changes in practice, even though study group facilitators focused specifically on helping teachers analyze students' prior knowledge, and not on any of the participants' teaching practices. Changes in practice that were cited were (a) increased focus on how to launch a problem (b) increased focus on effective implementation of wait time, including anticipating student responses and formulating next steps; and (c) increased skill in implementing problem-based tasks or discovery activities. Launches were mentioned more often by teachers in year 1, which was the year during which they were a focus of the study group. Although a common launch was developed during group planning, during implementation, teachers tailored their launches to their own talents and knowledge base or to what they perceived would be more motivating for their students. The excerpts illustrate the types of changes teachers reported in their practices.

Teacher 1 (Y1): One of the things that I think I've taken from this is that, we talked about a lot, kind of like *launching a problem*, and kind of how you'd build up a problem at the beginning. And, often times, I just kind of roll through, "Alright let's go, here's the paper, here's the materials, make it happen." Where I don't necessarily think about how to build that up or how to get those kids involved or kind of hook the students. And so, *that piece, kind of made me think overall, in my other courses, too, how can I try to get the hook involved more*.

Teacher 1 (Y1): I think it goes back to thinking about what kids bring to the table. It made me stop and think about that when I was planning things, when I was presenting things, when I was doing my *wait time*, y'know. "*Don't just ask a question and wait. Think about their response, and formulate the next questions.*" So, I think just forcing me to think about what kids bring to the table was the biggest positive.

Teacher 5 (Y1): I feel like it's *made me more thoughtful in how I go about presenting topics*. Especially something like in the first time. Like, *how do we discover this*? *How should I present it*? What's the *launch* gonna look like?

Teacher 2 (Y2): And then I think I'm starting to do a better job of kind of *managing that bigger group, bigger task ideas*. And then, I want to do more of just creating more of these task oriented. So that's my goal. I'm not there yet…. I think what I really appreciated and I think will benefit most from my teaching is really *looking at if it's more of these hefty problems*, like what are the *anticipated responses*? And then kind of game planning how I might do that.

Teacher 4 (Y2): I mean, so *you are thinking about the way the kids are thinking about it which changes the way that you're writing the lesson*.

Discussion

Teachers identified several significant aspects of the learning experiences, situated both in their classrooms and in a context removed from those classrooms, that were consistent with claims in the literature about effective professional development. Specifically, teachers valued the opportunity to participate in a discourse community (Putnam & Borko, 2000), especially when members of that community were teaching peers. Teachers' justifications about the value of those discourse communities were fairly general (e.g., share ideas, seeing other teachers in action), and they may not have recognized the affordances of collaborating in that way. Putnam and Borko (2000) claimed that having the support of a discourse community can support the risk taking that is necessary changing practice. Although none of the teachers made explicit claims connecting their changes in practice to the discourse community, they did claim to have made such changes. The teachers did argue that accountability requirements of participation in the study group enabled them to take the risk of implementing a lesson that they may have otherwise left unimplemented. Perhaps this need for accountability could also be characterized as a disruptor to the teachers' existing practice (Putnam & Borko, 2000).

The teachers stated that they valued a focus on student thinking, rather than on teaching. Teachers noted that understanding students' logic (even if it was not correct) was an impetus for changes in teaching practice. Although this may seem counterintuitive, creating the need for change by seeing classroom events through the students' eyes was a more powerful motivator, and perhaps less intimidating than having a direct focus on the actions of the teacher. As one teacher implied, there is a tendency to be defensive, when one's work is the object of scrutiny. Thus, protecting teachers from a perceived vulnerability, by limiting discussions of teaching to those based on animated representations, encouraged dialogue and may have built the trust that was needed to sustain interactions later in the study group process, when videos from participating teachers classroom were analyzed.

Finally, the changes in teaching practice reported underscored the value of a focus on students' mathematical thinking in the study group. The changes reported (e.g., focus on the lesson's launch, increasing wait time, or listening carefully to what students say before responding) were indicative of an increased interest in student-centered learning. Thus, a focus on making sense of students' prior knowledge, encouraged teachers to reflect on how to modify their practice to find more opportunities to listen to students.

Overall, identifying characteristics of professional development programs that teachers find valuable is important because teachers may persevere in long-term, or time-consuming professional development if they see the inherent value. In our case, the teachers established the importance of having authentic experiences connected to their students and their curriculum. At the same time, the opportunities to collaborate with teachers from other schools and districts prompted them to discuss important problems of teaching. Resources used in the program such as the animations served as important vehicles for helping the teachers extend and connect their knowledge of their context (e.g., the geometry curriculum and their students) and their knowledge of teaching. The teachers' analyses of student mathematical thinking opened the door to examination of their own practices and moved them closer to the goal of effecting robust mathematical understandings in their students.

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ATTITUDES/BELIEFS OF EARLY CAREER SECONDARY MATHEMATICS TEACHERS IN REGARD TO THEIR SUPPORT SYSTEMS

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This study describes the data from a series of national surveys of mathematics teachers in their first three years of teaching. The initial, pilot survey was created by a team of researchers and educators from 13 universities and four K-12 school districts and involved a year long study of 41 teachers. The final survey, devised as a result of data gleaned from the pilot, was administered in December of 2016, with 141 participants responding. The main objective of the final survey was to gather information about how these teachers perceive they were being supported and inform initiatives aimed at improving teacher retention rates. The survey focuses on what types of professional activities and communities in which teachers are participating, their perceptions of these activities, and how practice is influenced. Additional questions focused on administrative and university support, job satisfaction, and anticipated longevity in the field.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs; Teacher Education-Inservice/Professional Development; Teacher Education-Preservice

Significance/Purpose of Study

We are in the midst of a crisis in math teacher education that has a critical effect on how prepared our students are to be successful in high school, college and beyond. Half of all teachers leave the profession within the first five years, and this rate is highest for math positions in high poverty schools (Goldring et al, 2014; Fantilli & McDougall, 2009). Furthermore, with half of all current teachers in the U.S. retiring in the next five years (Foster, 2010) and enrollment in many teacher preparation programs declining, the teacher turnover is expected to cost $7.3 billion annually (National Math + Science Initiative, 2013).

Theoretical Framework

Research has defined key components of a more cohesive and effective system of mathematics teacher preparation and development that facilitates teacher growth and retention. Our theory of action focuses on one of these components: ensuring early-career mathematics teachers have high-quality, content-specific professional support. Targeted mentoring by experienced teachers, for example, especially as part of a long-term arc of teacher learning, has a dramatic impact on a new teacher's beliefs, practices, effectiveness, reflectiveness, satisfaction, and likelihood to stay in the profession (Ingersoll & Strong, 2011; Oh, Ankers, Llamas, & Tomyoy, 2005; Ronfeldt & Reininger, 2012; Walkington, 2005), and ultimately affects student achievement (Strong, 2006; Strong, Villar, & Fletcher, 2008). Effective support is especially critical and impactful in a teacher's pre-service year before entering the classroom full-time, and in his or her first few years of teaching (EdSource, 2014). Additionally, when teachers participate actively in professional learning communities, the likelihood they remain in the teaching profession and become more effective at teaching mathematics increases (Fulton, Yoon, & Lee, 2005).

Methodology

In 2016, STRIDES researchers designed a pilot survey of early-career secondary mathematics teachers to identify potential mismatches between the kinds of support research has shown are important for early-career teachers and the support early-career teachers are actually receiving. The researchers distilled three change ideas with which they developed a revised survey and an early set of interventions. These ideas were:

1. Build professional learning communities (PLCs) of early-career mathematics teachers.
2. Ensure early-career mathematics teachers and mathematics teacher PLCs have effective mentors.
3. Bolster administrative support for early-career mathematics teachers and teacher PLCs.

The STRIDES researchers developed the “Reflection on Professional Activities” survey in iterative cycles of survey design, implementation, and data analysis, including a year-long pilot with 41 early-career mathematics teachers. The survey was designed to gather semi-annual information about the three change ideas that would support the researchers implementing the change ideas more effectively.

Participants

The pre-service and early career teacher participants were solicited via electronic means by MTEP partnership university faculty. Twelve percent of the respondents designated themselves as pre-service teachers, twenty-six percent in their first year, twenty-six percent in their second year and thirty-six percent in their third year of teaching. An overwhelming number (94%) of these teachers were serving in public secondary schools in rural (13%), suburban (32%), and urban (23%) settings, and teaching in a full range of classes from 6th grade general math through calculus. The schools they were serving were characterized by their teachers as low-SES (26%) and high-SES (9%). Most (72%) of the pre-service and early career teachers surveyed stated that between five and twenty percent of the students they were teaching had Individual Educational Plans (IEPs), fifty-nine percent stated that between five and twenty percent of their students were designated as English Language Learners (ELLs) and fifty-five percent of them reported that between forty to one hundred percent of their students qualified for free and/or reduced lunch. The map below shows the locations of the teachers who participated in this study.

Findings

Several questions were posed to participants that asked them to select their answers from a given list of choices. These questions focused on career choice, job satisfaction, anticipated longevity in the field, and estimated weekly time spent on teaching, planning, and professional activities. Other questions asked participants to choose professional activities in which they had recently participated and select to what extent those activities impacted their enthusiasm for teaching. The final quantitative questions asked participants who supports them, the depth of that support, and what type of support they get from those people.

Initial data analysis shows that most of the pre-service or early career teachers surveyed (81%) "certainly" or "probably" would become a teacher "if (they) could go back and start college again" and nearly half (46%) of them would remain in teaching "as long as (they) were able". The majority of teachers (53%) say that they spent between one and two hours a week involved in professional learning activities and another eighteen percent say they spend three to five hours a week. Teachers reported spending a large chunk of their weekly time planning with colleagues (56% spend 1-2 hours, another 27% say 3-5 hours) and another large amount of time planning alone (35% spend 6-10 hours, another 30% say 3-5 hours). Sixty three percent of respondents claim to spend more than 20 hours a week teaching. With regard to professional learning activities that "increase their enthusiasm for teaching mathematics", working/communicating with a mentor or coach rated the highest among all choices (83% responding that it was either moderately or very influential). In terms of support from administrators in a variety of areas (curriculum, classroom management, course assignments, assessment, instruction, collaboration and affirmation), the respondents relied to a much larger degree on those who were on-site (principals and assistant principals) rather than university professors and district office personnel. The three graphs below depict some of our findings.

When teachers were asked what they wish was different about their job, the words that were mentioned the most frequently were support, non-teaching duties, class size, collaboration, student behavior, and pay. Participants also reported that the most used online forums for professional use were blogs and Twitter. When asked to describe a professional learning opportunity the teachers participated in that had a positive effect on their ability to facilitate student learning, teachers described a huge variety of activities, some of which were embedded in formal or informal professional learning communities. For example, one teacher described a community of first-year teachers the school district created, and many described their departments as versions of PLCs that supported their teaching practice. Few teachers described formal professional learning communities created or facilitated by their university teacher preparation program or school district, but many teachers reported access to mentors or coaches (e.g., “I am currently participating in an Algebra 1 blended learning pilot being conducted by the state department. I meet bi-weekly with an coach online to discuss teaching practices and strategies.”). The “positive effect” of the support teachers described tended to be either related to a change in a planning or teaching strategy, move or structure (e.g., [I learned] “the regal look - a look you give your students when you want them to be quiet”) or to feeling generally “prepared” to teach. Few teachers described changes in student learning related to changes in teaching supported by professional learning communities, mentors, or administrators.

Lastly, participants were asked to describe how a professional learning activity increased their enthusiasm for teaching mathematics. The majority of the responses fell into two categories. The first category is teachers who are inspired and rejuvenated by working with someone who has an infectious, enthusiastic attitude about teaching. Participants described listening to a presenter who was like this, reading testimonials online, or working with colleagues such as math coaches or department members who have this passion for teaching. For example, one participant wrote, “I participated in a conference call with colleagues from across the nation. Every time we chat, I am reminded of why I love to do what I do. Their enthusiasm definitely rubs off on me.” The second category of responses was enthusiasm gained by trying out new ideas in the classroom and seeing their students succeed with them. Teachers learned of new activities via conferences, blogs, meetings with colleagues and various other ways and described how invigorating it was to customize these ideas to fit the needs of their students, try them on using their own teaching style, and see their students succeed.

Conclusions

The MTEP STRIDES team is not comfortable at this time to draw conclusions or suggest possible interventions based on this initial data. Our one-year study is only halfway complete, with the second survey administration set to happen in April of 2017. Once these same teachers complete the survey a second time, detailing how they were supported for a full school year early on in their career, the research team will meet in the summer to fully analyze the data and propose interventions. These targeted interventions will be implemented during the 2017-2018 school year with a goal to expand meaningful support for early career mathematics teachers and to increase teacher retention at large, serving as proof of principle for more wide-scale efforts.

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CONNECTING TEACHERS' BUY-INTO PROFESSIONAL DEVELOPMENT WITH CLASSROOM HABITS AND PRACTICES

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While professional development (PD) provides an opportunity for teachers to cultivate skills that are consistent with best practices in the field, it is their buy-into the PD that ultimately determines the effectiveness of the PD. We examined how teacher buy-in affected the classroom habits and practice of four elementary teachers who took part in a district wide PD. Using baseline and first-year implementation video recordings, in conjunction with frameworks for discourse analysis, cognitive demand, and tools built specifically to measure PD implementation, we found that varying combinations of teachers' beliefs served as a mitigating factor for PD implementation.

Keywords: Teacher Education-Inservice/ Professional Development, Teacher Beliefs

In this report, we explore the effect of teachers' buy-in for a high-quality, sustained, district-wide professional development (PD), Mathematics Studio PD (Foreman, 2013), on improving their classroom habits and practices. Systematic change requires coordination and cooperation between the system (school and PD program) and the participants (teachers). Without high buy-in, teachers will likely implement little of what they learn in even the strongest of PD programs. We present four divergent cases to illustrate the relationship between the exhibited level of buy-in and how it affected their mathematics teaching practice in their elementary classrooms.

Background and Theoretical Framing

Field-endorsed best practices for PD often exist at the program level with recommendations like "intensive, ongoing, and connected to practice; focuses on the teaching and learning of specific academic content; is connected to other school initiatives; and builds strong working relationships among teachers" (Darling-Hammond et. al, 2009, p. 5). We challenge that program level recommendations are insufficient without looking at individual participating teachers. As PD represents an appeal to change, the inclination of a teacher to making said changes in their teaching practice is an important factor in the success of the PD. We capture this inclination using the construct of *buy-in* from the management and leadership field (Thomson et al., 1999). We adopt Thomson et al.'s two types of buy-in: *intellectual* and *emotional*, where intellectual captures the degree of understanding and emotional the degree of commitment. We treat belief alignment between teacher and PD as an (intellectual) indicator of buy-in, and seeing a need for change as an (emotional) indicator of buy-in.

Teacher Beliefs and Classroom Practice

To address belief alignment, we both identified teacher beliefs from their discussion contributions in PD sessions and explored related factors of their classroom practice. In this context, our focus is on beliefs about mathematics, teaching, and learning. The principles underlying the PD focus on mathematics as a sense-making activity where are all students are capable of deep engagement in meaning-making via justifying and generalizing. To explore belief relationships and their classroom practice we used cognitive demand and patterns of discourse. Henningsen and Stein

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

(1997) defined cognitive demand as, “The kind of thinking processes entailed in solving the task as announced by the teacher and the thinking processes in which students engage” (p. 529). When teachers engage students in high cognitive demand tasks, it is an implicit reflection of a belief that students can do highly demanding mathematics and that mathematics is richly connected (Wilhelm, 2014). A second way beliefs may manifest in observable classroom actions can be seen in patterns of discourse. We leverage Scott, Mortimer, and Aguiar’s (2005) interaction and authority framework to address the balance of student and teacher engagement in doing mathematics. In this report, we focus on the authority dimension where discourse is classified as *authoritative* or *dialogic.* An authoritative classroom has only one acceptable solution path and correct answer versus a dialogic classroom allows for multiple solution paths.

Critical Components and Measuring Fidelity of PD Implementation

We also examined teacher’s classrooms for explicit implementation of the PD measured as *degree of implementation* to capture “the extent of change that has occurred at some particular time toward full, appropriate use of the target innovation” (Scheirer & Rezmovic, 1983, p. 601). We analyzed the *critical components* (O’Donnell, 2008) of our PD and developed a classroom observation tool, *The Mathematically Productive Habits and Routines* (MPHR) to measure the implementation of the PD components in classrooms (see Melhuish & Thanheiser, 2017).

Methods

Data for this project was taken from a large-scale study aimed at discerning the efficacy of a 3-year PD program in an urban school district in the Northwestern United States. Our data consist of classroom video recordings (two lessons before PD and two lessons after PD), as well as video recordings and detailed field notes from five PD sessions across the year at two schools.

Identifying Teacher Buy-In

Researchers observed and video-recorded all PD sessions taking detailed field notes which were analyzed using thematic analysis (Braun & Clarke, 2006). The themes were informed by the need to identify important factors that relate to the efficacy of the PD program. We identified four case study teachers to further analyze. They were selected based upon their variations in terms of belief alignment and perceived need to grow.

Analyzing Classroom Change

Each year, two lessons were recorded for all participating teachers. For our case study teachers, we focus on their baseline videos (prior to any PD) and their year 1 videos (after a year of PD). To facilitate in the process of scoring and coding, each video was segmented into episodes; each episode representing a portion of the lesson where the curricular goal/aim was consistent throughout. Each episode was then scored and coded according to the frameworks for the discourse analysis and cognitive demand analysis (i.e. 1-memorization, 2-procedures w/out connections, 3-procedures w/ connections, 4-doing math). Each lesson was given an overall degree of PD implementation score based on the MPHR.

Results & Discussion

In this section, we provide an overview of our four case study teachers and focus more extensively on our most extreme cases: Cora and John. The buy-in level was based on two factors: perceived need to grow in teaching practice and belief alignment with the PD. A summary of the four cases in terms of: (1) 2 factor buy-in, (2) belief and classroom practice alignment, and (3) PD Implementation can be found in table 1. For a more nuanced discussion of their buy-in see Fasteen, Melhuish, and Thanheiser (2015).

Table 1: Degree of Implementation Growth and PD Buy-In for Case Study Teachers

Case Teacher	John (Low)	Nina (Mid)	Kim (Mid)	Cora (High)
Belief Alignment with PD	No	No	Yes	Yes
Need to Grow in Practice	No	Yes	No	Yes
Beliefs Aligned with Classroom Practice	Yes	Yes	Inconsistent	No
PD Implementation	No	Yes	No	Yes

Case 1 & 4: John (Low-level buy-in) & Cora (High Buy-In)

Cora and John were at opposite end of their careers. John was preparing to retire while Cora was in her second year of teaching. During the PD, Cora displayed indicators of high-level emotional and intellectual buy-in while John displayed low levels of both.

Baseline lessons. Prior to involvement with our PD, Cora's classes had a high number of student contributions, but the tasks were often low-demand (see Table 2). Her lessons tended to include majority authoritative discussions. In John's baseline lessons, his class had minimal student interaction with most interaction consisting of pro forma call and response leaning heavily authoritative. The task demand was low with heavy focus on procedures (see Table 2). John's traditional beliefs aligned with his classroom practice. In contrast, Cora's beliefs that students are capable and that mathematics is a rich subject was reflected only in her students having opportunities to contribute while the mathematics remained procedural.

Table 2: Cora & John's Lessons in Terms of Cognitive Demand and Discourse

Lesson / Teacher	Cognitive Demand (% of time High)		Authority (% of time Dialogic)	
	Cora	John	Cora	John
Baseline 1	Varied (40%)	Low (0%)	Authoritative (32%)	Authoritative (0%)
Baseline 2	Low (12%)	Low (12%)	Authoritative (31%)	Authoritative (32%)
Post-PD 1	Varied (40%)	Low (0%)	Dialogic (72%)	Authoritative (12%)
Post-PD 2	High (85%)	Low (13%)	Dialogic (85%)	Authoritative (31%)

After one year of PD. After involvement with the PD, Cora's classroom came into closer alignment with her beliefs. The level of cognitive demand increased. The discourse moved from authoritative to largely dialogic reflecting the acceptance and discussion of multiple strategies and viewpoints. The nature of John's class changed little after the PD. His lessons remained predominately low cognitive demand and authoritative in nature (see Table 2). Cora's implementation of the PD rose after a year of sustained support. This growth reflects her students engaging in mathematical habits of mind and interaction and her use of teaching habits and teaching routines. The tools provided through the PD may have allowed Cora's beliefs and classroom actions to more closely align. As John had low buy-in for the PD, and had beliefs that may limit growth both in terms of his own need to grow, student capabilities, and the nature of mathematics, his degree of implementation score did not rise despite a year of PD.

Conclusion

A teacher's beliefs and disposition towards the subject area, learning, and their own practice play an important factor promoting teacher change through PD. We use the buy-in construct to explore alignment or misalignment of these beliefs and the PD's principles. The literature has established that teacher beliefs and classroom actions are related, but the relationship is often complex. Our cases illustrate some of the complexities. Cora's case is particularly compelling as she has aligned beliefs (and subsequently high buy-in to the PD), but prior to the PD intervention, the beliefs alone were insufficient to promote high level reasoning in her mathematics classroom. When provided with the tools, Cora's classroom became more in-line with her beliefs. John, who did not perceive a need to grow, implemented little work from the PD into his teaching. Cora and John each represent very different types of teachers that may participate in PD. As providers of development and researchers on innovation, attending to beliefs and belief-alignment in classroom actions, may provide a starting ground for addressing the variance in individual PD participants.

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CREATING SPACES FOR STATEWIDE TEACHER COLLABORATION: EMERGING PRACTICES IN VIRTUAL SESSIONS DESIGNED TO SUPPORT TEACHERS IN THE IMPLEMENTATION OF NEW STANDARDS

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This study explores mathematics teachers' participation in a series of virtual sessions focused on implementation of new state mathematics standards and promoting more equitable learning opportunities for students. Our findings indicate that teachers' preferred the use of text-based chat features for interactions. Further, their emerging participation practices focused on task implementation, student work, and mathematical content.

Keywords: Teacher Education-Inservice/Professional Development, Technology, Standards

Introduction and Background

Mathematics education researchers have built a strong body of research that continues to move the fields of both research and teaching forward; however, a divide continues to exist between research and practice (Battista, 2007; Cai, et al., 2017) and issues of implementation at scale (Cobb & Jackson, 2011). Responding to this challenge, researchers and funding agencies are encouraging research-practice partnerships where researchers and practitioners work together to iteratively design and research problems of practice (Kane, 2016; Penuel & Farrell, in press).

As part of a statewide research-practice partnership between a state education agency, district leaders, teachers, and mathematics education researchers at several institutions, our work draws upon Design-Based Implementation Research (DBIR) (Fishman et al., 2013) as an approach to facilitate the design of intervention efforts related to new state mathematics content standards and efforts to promote more equitable learning opportunities for students. These efforts include many spaces for engagement, one of which is a weekly virtual session designed to be "just in time" regarding the new standards and related mathematics topics addressed in each of the three integrated and sequenced high school mathematics courses, Math 1, Math 2, and Math 3.

Theoretical Perspective

Central to a DBIR approach are key principles of drawing upon multiple stakeholders' perspectives; collaborative and iterative design; a commitment to developing both theory and knowledge; and developing sustainable practices (Fishman et al., 2013). The focus of this paper is the design-research efforts related to the nature of mathematics teachers' participation in the virtual sessions. We conceptualize virtual sessions as an affinity space (Gee, 2005) where teachers come together to engage in the common activity of sense making around statewide implementation of new mathematics content standards in ways that promote more equitable teaching and learning. Viewing the virtual sessions as an affinity space is helpful because it frames both the design of the space and the ways in which participants interact with both the space itself and with one another.

Fundamental to affinity spaces are the constructs of what Gee (2005) calls a generator, portals, and internal and external grammar. A generator represents the focus of the space; portals represent the various ways in which participants can engage in the space; internal grammar represents the design of the space; and external grammar represents the ways in which participants behave and interact. In this study, the generator is a statewide effort to support teachers in student-centered implementation of new mathematics content standards through weekly virtual sessions. The portals for this space include registration (which provides access to documents to be shared in the session), attending, chatting using a text-based feature, and speaking aloud. Using a DBIR approach to this work, we iteratively design the internal grammar of the space while studying the external grammar of the ways in which participants engage within the space. Specifically, our research is guided by the questions: (1) *In what ways do teachers use the portals for participation within the space?* and (2) *What is the nature of teachers' participation evidenced by their external grammar within the space?*

Design of the Affinity Space

The online platform used to conduct the virtual sessions, *GoToWebinar* (https://www.gotomeeting.com/webinar), was the contracted platform of the state agency partner. Only teachers were invited to take part in the space, as we hoped to create a safe place for open discussions. The platform allows for participants to chat via a text feature and speak aloud for those with microphones. The sessions ran approximately one hour each and were offered the first three Thursdays of each month in the order of Math 1, Math 2, and Math 3. The design of the sessions was initially informed by our initial learning conjectures and joint design work with teacher and state agency partners, and was continuously revised based on our ongoing analysis and teacher feedback within the space and through surveys. Over the first few iterations a "typical" session design emerged. Sessions were typically hosted by two teachers/leaders with a focus on implementation of a task related to the mathematics standards of focus for that month and included samples of student work.

Methods

Qualitative methods were used to understand the nature of mathematics teachers' participation in the Math 1, Math 2 and Math 3 virtual sessions. Participants in this study included teachers across the state who taught Math 1, Math 2, or Math 3. All teachers were invited to participate through email announcements and at state, regional, and district level face to face meetings. Twelve virtual sessions were held and approximately 350 teachers participated at least once over the course of the semester. Participants represented 70 school districts (out of 115) spread across the state, with many being from rural areas.

Data for this study includes text of participants use of the text-based chat feature and video recordings of virtual sessions. Video recordings were viewed by members of the research team and all participants' talk was transcribed verbatim. To understand the ways in which teachers were participating, frequencies related to portal use were calculated. To understand the nature of teachers' participation, all chat and verbal transcripts were open coded to identify themes. We used a constant comparative method (Strauss & Corbin, 1998) to identify emerging categories and refine these categories as they contrasted with data.

Findings

The ways in which teachers participated in the virtual sessions varied from month to month and across courses. There were four portals within the virtual session platform: 1) *registering* provided access to the tasks that were the focus of each session; 2) *attending* in which participants could listen to (and read) the interactions of others; 3) *chatting* in which participants could interact through text; and 4) *speaking* aloud during the session. A summary of the use of each portal appears in Table 1. Attendance in the sessions decreased each month across all three courses, with the Math 1 sessions having significantly higher attendance than the other two courses. The most common mode of interaction during the sessions was the use of the chat feature. This suggests that the chat feature was the easiest or most comfortable mode of interacting within the session platform. In addition, the amount of interaction among participants increased, as is seen by the increased trend of growth in the ratio of attendees to total chats until the drop off in attendance in December.

Table 1: Virtual Session Portal Use

	Sept			Oct			Nov			Dec		
Math Course	1	2	3	1	2	3	1	2	3	1	2	3
registered	255	162	182	216	136	133	99	32	30	42	28	18
attended	132	74	77	61	49	38	43	11	11	16	15	7
total chats	19	63	82	74	52	38	45	8	4	69	27	3
total talks	n/a	10	6	14	4	14	2	4	1	0	2	0

Analysis of participants' chat text and transcribed talk revealed that interactions fell into the following categories: *general about students* (i.e. comments about students in general, not specific to the context of the task being discussed), *general about teaching* (i.e., about teaching unrelated to the task being discussed), *general about standards tools* (i.e., about other project spaces and tools), *task discussion* (i.e., focused on the mathematics of the task itself), *pedagogical decisions related to task* (i.e., focused on suggested pedagogical decisions, shared pedagogical ideas, or pedagogical questions related to the task of focus), *student thinking* (i.e., analysis of student work or anticipation of student thinking), and *technical issues* (technological issues with the virtual session platform). Below we discuss the most commonly used codes to illustrate the nature of the participants' discussions. All examples are pulled from the text-chat data as they tended to be more succinct.

Focusing on Task Implementation (43.3% of all quotations)

When participants discussed pedagogical decisions related to the implementation of the task their contributions included questions about implementing the task or suggesting questions to pose to students (e.g., "I might ask: Do you see a pattern with the change in time leaking?"). However, the majority of their contributions regarding pedagogical decisions related to sharing ideas with one another for implementing the task. For example, when multiple examples of student work had been shared, a teacher used the chat feature to conjecture an instructional decision to support students' mathematical engagement in the task, stating, "The student whose work is currently shown should be paired with the previous student. They should discuss how their tables are connected." Additionally, while listening to one teacher sharing a task another teacher chatted, "This is a good place to discuss inverse relationships."

Focusing on Student Thinking (23.4% of all quotations)

When discussing student thinking participants tended to anticipate student thinking when presented with an instructional task. For example, "I think students may struggle with the three variables (price, # of boxes, and revenue) when they are only used to an input/output, and identifying which is the input and output." In addition, when presented with student work samples the participants showed evidence of critical analysis of that work. For example, as a sample of student work was shared one teacher noted, "This table doesn't completely show understanding of the repeated multiplication aspect of exponential functions, just that they know they are multiplying answers by 3."

Focusing on Mathematics (21.9% of all quotations)

Each session was designed around a specific mathematical task that allowed for a focus on mathematics related to the new standards. Within the virtual sessions participants discussed ways that the task addressed the new standards (e.g. “It seems to fit nicely with the set of G.CO group. Ideas about transformations, congruence, similarity, changes in area and space.”) and characteristics/purpose of the task itself (e.g. “I think that the task really allows students with different methods to solve the problem. They can use the properties studied. They can use models. They can use tools such as rulers, protractors, and folding paper.”). Furthermore, in each session there was evidence of discussion around the specific mathematics related to the task. This included asking other participants for help with the mathematics (e.g., “Can you re-explain how you developed the quadratic to solve for the min?”) and sharing their own solution strategies (e.g., “I see that the rate of change in the table is not constant, but it has linear scatter plot. So, use the model to see if the y-intercept is 400?”).

Conclusion

The purpose of the virtual sessions was to provide a space for mathematics teachers across a state to engage with new state mathematics standards and a pedagogy centered on student thinking. The virtual sessions were successful in that approximately 350 teachers chose to enter the space at least once. These teachers were mostly from rural areas of the state, which is important as they often noted that they did not have others to interact with at their schools. While it is promising that participants’ interactions focused on important instructional issues, we are also encouraged by the mathematical nature of their interactions. Teachers not only discussed mathematics, but also some went as far to admit uncertainty and ask for help. Overall, we feel as if the design of the space is meeting teachers’ needs, however, the drop off in attendance and focus on the use of the chat feature suggests that revisions are needed.

Creating spaces in which teachers feel safe to take mathematical and pedagogical risks is important for teachers as they work toward implementing mathematics standards in ways that provide more equitable learning opportunities for students. As we consider the conference theme, “Synergy at the Crossroads”, we are confident that research-practice partnerships can provide a productive and fruitful space for researchers and practitioners to work together on challenging issues of implementation at scale and the development of theories of implementation.

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DEVELOPING FORMATIVE ASSESSMENT TOOLS AND ROUTINES FOR ADDITIVE REASONING

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This early-stage design and development project combines formative assessment, learning trajectories, and professional development to improve mathematics teaching and learning in grades K-3. Formative assessment items relating to addition, subtraction, and number have been developed based on current research, learning progressions, and the Common Core State Standards. These items are being piloted by teachers and validated to determine the utility of items in eliciting student thinking and informing instruction. Early analyses of piloted items have revealed the effectiveness of items in revealing a range of student strategies and levels of understanding, as well as challenges in capturing the thinking of very young learners.

Keywords: Assessment and Evaluation, Elementary School Education, Learning Trajectories (or Progressions), Number Concepts and Operations

Introduction

This project is an early-stage design and development study extending the Ongoing Assessment Project (OGAP), a project that combines formative assessment, learning trajectories, and professional development to improve mathematics teaching and learning in elementary and middle school. OGAP has previously focused on multiplicative reasoning, proportional reasoning, and fractions for grades 3-8; the goal of the current project is to broaden this focus to include formative assessment tools and routines for addition, subtraction, and number for grades K-3. These tools and routines include: (1) a collection of formative assessment items that vary by characteristics such as problem context, situation, number complexity, and representation of operation; (2) a set of frameworks for addition, subtraction, and number that illustrate a progression of student strategies that increase in depth of understanding and sophistication of strategies; and (3) grade-level professional learning communities (PLCs), in which teachers collaborate to examine student work on formative assessment items, analyze students' strategies, and determine next instructional steps based on the learning trajectory frameworks. The OGAP formative assessment system lies at the crossroads of theory and practice by distilling research in early number and operations into illustrative frameworks and formative assessment items that allow teachers to make research-based instructional decisions on an ongoing basis. This paper will focus on the formative assessment item bank development and validation process.

Conceptual Framework

Classroom formative assessment is a process in which information is gathered regularly from students to provide feedback and inform the adjustment of teaching and learning (Black & William, 1998). The formative assessment item bank development and validation process employed in this project is shown in Figure 1.

The item bank development process began with a review of the research on the three areas of focus for the project—addition, subtraction, and number (e.g., Baroody, 1987; Carpenter, 1985; Clements & Sarama, 2009; Cross, Woods, & Schweingruber, 2009; Fuson, 1988, 1992; Ginsburg, 1983). The existing research, learning progressions, and common core standards on these topics were utilized to create an item bank blueprint. The item bank blueprint includes content, concepts, problem structures, number complexity, strategies, and representations for the three topics across grades K-3.

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Frameworks illustrating the learning progressions for addition, subtraction, and number were also created based on the research and standards review.

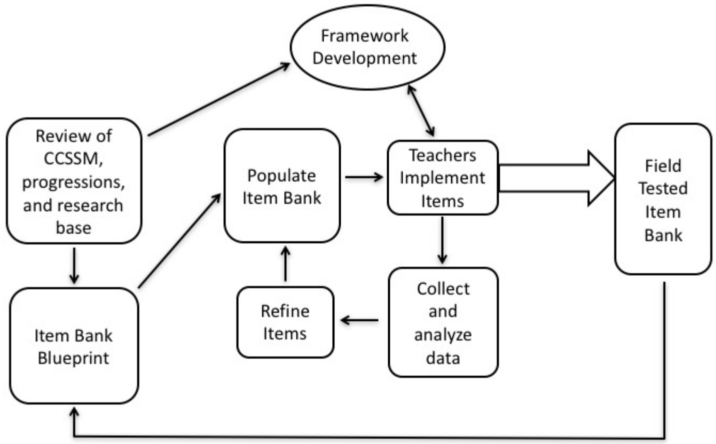

Figure 1. Formative assessment item bank development and validation process.

Formative assessment items were developed based on the item bank blueprint. The items are written tasks that can be administered to a whole group, small group, or individual. Items were created to address each of the characteristics identified in the item bank blueprint for the three content areas. Typical developmental level of a child in the grade in which a topic is taught was considered when writing problems. For example, a kindergarten counting problem would be written in simple, brief language and may contain a counting situation that could be represented easily with a small number of objects or a drawing, while a third grade problem could contain an addition situation involving distance that would lend itself to a number line representation.

Items were then distributed to teachers to implement in classrooms. Teachers collected their students' work and used the framework to sort the work by strategy use. In PLCs, teachers collaboratively sorted student work using the framework and determined next instructional steps based on the strategies and understandings demonstrated in the work.

The item validation employed in this project focuses on Stobart's (2012) recommended area of emphasis for validation in formative assessment: "Validity is based on the purpose(s) of an assessment and how effectively the interpretation and use of the results serve each purpose…effective formative assessment will do this and the 'threats to validity' are those things that get in the way" (p. 233). Through both PLCs and student work sorts conducted by the researchers, items were validated based on how students performed on the items, whether the items elicited a range of student strategies, and if the items provided information that would be useful to teachers in guiding future instruction. Sorting and discussing student work has informed both item refinement and framework development.

The first round of item development, validation, and refinement has been completed, and we will continue to validate and refine items in preparation for field testing of the fully developed item bank in a larger number of schools in year 2 (2017-2018). After field testing is complete, the item bank will be mapped back to the item bank blueprint to ensure that all characteristics and learning targets of the three content areas have been sufficiently addressed in the item bank.

Research Questions

The research questions guiding the development and validation of the formative assessment item bank are as follows: (a) What information is elicited by the formative assessment items? (b) What information is not revealed by the formative assessment items? (c) What characteristics make items useful for formative assessment?

Pilot Data Collection and Analysis

Formative assessment items have been piloted by 21 teachers in 5 public schools located in a city in the Mid-Atlantic region of the United States during year 1 of the project. Student work has been collected for 103 items. Items on addition, subtraction, and counting/number have been collected across grades K-3. Items have been piloted in two ways: in schools where teachers meet regularly in PLCs to sort and discuss items, and in schools where teachers are not implementing PLCs. In addition to the student work sorts that have been conducted in PLCs, the research team has been sorting student work to validate items and inform framework development. The data that have been collected in year 1 include student work on formative assessment items, PLC summary records, and researchers' notes on student work sorts. The data have been analyzed by reviewing the student work and sort records to determine whether students were able to answer the question, whether a range of strategies was elicited, and whether the student responses were useful for making subsequent instructional decisions.

Early Results from Item Bank Validation

In initial analyses of the data collected in year 1, two types of information elicited by the items have been identified: addition/subtraction/number strategies and concept understanding. On problems that can be solved in a variety of ways, such as an add to problem with a missing addend, a range of solution strategies was generated. These strategies can be mapped to the framework to determine subsequent instructional steps. Concept problems assess understanding of an idea relating to addition, subtraction, or number (e.g., the commutative property of addition). Though student strategies on these problems cannot be mapped to the framework, these problems reveal understanding—or lack thereof—of key concepts and properties.

Data analyses have also revealed that some information is not being elicited by the formative assessment items. Metacognition—or awareness of one's own thought process, self-regulation, and beliefs about mathematics—is a cognitive behavior that is developing in children (Schoenfeld, 1987), making it challenging for children to identify and represent mental solution strategies. When a single piece of student work contains multiple solution strategies, it may not be evident which strategy was used by the student to solve the problem. A particular challenge has revealed itself during piloting of items in kindergarten classrooms—for children who are in the process of learning to read, write, and draw, representing a solution strategy, or even writing an answer, can be a difficult task.

Three primary characteristics that make an item useful for formative assessment have been revealed through item bank validation. First, an item must draw out student thinking. For example, if an item can be answered easily with a known fact such as a double addition fact, the item may not generate much student thinking, whereas an item that incorporates a "near double" or "double +1" allows students to demonstrate whether they can use the known fact to solve a related problem, thereby calling for more complex thinking. Second, an item should generate a *range* of strategies. If most students in a class solve a problem similarly (e.g., most students accurately use the U.S. addition algorithm), the information obtained from the student work may not help the teacher to make instructional decisions, indicating that the item may not be an appropriate formative assessment item (or perhaps not appropriate for that grade level or time of year). On the other hand, if a problem generates a limited range of strategies that differ significantly in sophistication (e.g., an addition problem that generates only the counting all strategy and the addition algorithm), this may indicate that intermediate strategies that facilitate students' development of understanding of the topic should be introduced through classroom instruction. The framework plays an important role in helping teachers identify appropriate transitional strategies based on evidence of understanding present in the students' work. The third characteristic that makes an item useful for formative assessment is accessibility. The structure and number complexity of an item needs to be appropriate for the grade level and time of year, the item context should be familiar to students, and the language and vocabulary contained in the item needs to be clear and easily understood by students.

The Future of OGAP

We will continue to develop and validate items with the following questions in mind: (1) When a problem does not generate a range of student strategies, does this indicate that the problem is not valid and should not be used in the item bank? Does this indicate that the problem is not appropriate for the grade level/time of year/group of students but would produce a wider range of results in another context? Does this indicate a need for instruction on intermediate strategies so that students can move from using early counting strategies to transitional strategies and eventually to additive strategies? (2) Is there a way to collect written information from young children, or do we need to add a verbal component or observation checklist to our formative assessment items? We will then move from item development and validation to field testing, in which teachers will use formative assessment items in their daily instruction.

Acknowledgments

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EXAMINING THE EFFICACY OF SIDE-BY-SIDE COACHING FOR GROWING RESPONSIVE TEACHER-STUDENT INTERACTIONS IN ELEMENTARY CLASSROOMS

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Learning to respond to student thinking in the moment is challenging precisely because responsiveness cannot be scripted in advance. In this study, I situated teacher learning opportunities within teachers' daily instruction to support learning to respond during teacher-student interactions. This embedded professional development - side-by-side coaching - was found to have a significant positive impact on teachers' ability to respond to and advance students' mathematical thinking in the moment through the practice of conferring.

Keywords: Classroom Discourse, Teacher Education-In-Service/Professional Development

Ambitious teaching practices are difficult to learn because they often require responsiveness to student thinking in the moment that cannot be pre-planned. Professional learning opportunities often focus on learning in the absence of students, for example, in workshops or collaborative planning. But teachers need opportunities to learn to respond to student thinking as it emerges. In this study, I examine the effects of using side-by-side coaching, a professional development structure situated directly in teachers' practice, to grow teachers' capacity to respond to student thinking in the moment in ways that advance students' thinking.

Prior Literature

Responding to student thinking in the moment requires teachers to orchestrate a variety of skills, capacities, and knowledge. While teachers can prepare for such moments by anticipating student thinking in advance, they cannot plan how these instructional moments will unfold. In previous work (Munson, 2016, 2017), I have described a type of responsive instructional interaction – conferring – in which teachers and students work together to uncover and advance student thinking. Conferring requires two essential types of instructional work: *eliciting* student thinking and then *nudging* that thinking along a productive avenue. A nudge is an instructional response to the elicited student thinking which advances that thinking while maintaining sense-making. Nudging is challenging; teachers who start by eliciting student thinking do not always nudge that thinking forward. In a previous study of teachers who were learning to confer, I found that fully two-thirds of the time they began to confer by eliciting student thinking, they did not end up nudging. Others have similarly found that opening questions are easiest to learn, while advancing the instructional conversation beyond initial elicitation is far more challenging (Franke et al., 2009). However, when teachers did nudge, advancement of students' mathematical thinking, engagement in mathematical practices, or productive collaboration was evident in the discourse (Munson, 2016, 2017). Given the instructional advantage of nudging, how can teachers learn to engage in this responsive practice?

Teacher educators have developed pathways for teachers to learn in, through, and from teaching, where these complex moments of instruction emerge (e.g., Ball & Cohen, 1999; Lampert et al., 2013). Grounding teacher learning in the particulars of practice allows teachers to grapple with particular challenges, such as specific misconceptions or individual student needs, rather than speaking in generalities. One possible platform for locating teacher learning directly in practice is coaching. Gibbons and Cobb (in press) identified several "potentially productive coaching activities," one of which, co-teaching, is a fruitful activity situated in instruction. When the coach and teacher are side-by-side during instruction, opportunities to reflect, seek input, and make decisions can be

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

created. Further, both modeling and observing and offering feedback have been shown to be effective coaching activities. In side-by-side coaching, leadership of instruction can be handed off between the teacher and coach in the moment, rather than fixing each in the role of teacher and observer for an entire lesson, so that modeling and feedback can be embedded in the activity. Opportunities for teacher learning can then be as responsive as the instruction they aim to promote. In this study, I ask, can side-by-side coaching support teachers in learning to respond to student thinking when conferring?

Methods

Participants

The teacher participants included three elementary classroom teachers (grades 1, 2, and 4) from the same school in an urban area of northern California. The school serves predominantly bilingual Latino and Pacific Islander students from a low-income neighborhood. The teachers are part of a larger, teacher-initiated research-practice partnership, and were selected based on their interest in engaging in coaching and learning to confer. Through participant observation, the author served as the coach and shaped the coaching provided in partnership with the teachers.

The teachers all taught by launching each lesson with a task for which students were actively developing strategies. Students worked in partnerships for the bulk of the lesson, with the teacher supporting student work through conversations. Typically, lessons would end with a whole class discussion of the strategies or mathematical ideas that emerged during worktime.

Data Sources and Collection

Data was collected in three periods: pre-coaching, during coaching, and post-coaching. For the purposes of this analysis, I focus on comparing the pre- and post-coaching data. Pre-coaching, I videoed 4-6 math lessons, of approximately one hour each, over a two-week period for each participating teacher; post-coaching, I recorded a similar number of lessons for each teacher. The camera was positioned high and in one corner of the classroom. The teacher wore a lavalier microphone and a back-up audio recorder; the audio, thus, followed the teacher as she moved interacting with students.

Side-by-side Coaching Model

The coaching period for each teacher lasted approximately four weeks. During each week, the teacher and I conferred with students side-by-side for two days. Each teacher received 7-8 days of coaching in total. On each coaching day, I spoke with the teacher before each lesson about the students' mathematical goals and the teacher's own learning goals for the day. We negotiated how we wanted to work together based on these goals.

Coaching took place during the students' worktime. During this time, we would approach students at work to confer together. Conferring unfolded as co-facilitated by the teacher and the coach, with each taking the lead at different times during the interaction. During most interactions, I would pause the conversation at some point and ask the teacher to think aloud with me about what she was noticing about student work, how she interpreted what she noticed, and how she might respond. These time outs were brief and served to create space for reflection and decision-making in the moment. At times, the teacher would ask me to lead the interaction so she could observe. After each conferring interaction, we would reflect briefly on the course and outcome of the conversation and consider anything we could learn from it before moving on to confer with the next group. At the end of each lesson, the teacher and I would debrief the lesson overall and consider next steps for students and for the teacher's own learning.

Analysis

Video records of each lesson were analyzed in Studio Code, a qualitative video analysis software. In each pre- and post-coaching lesson, I identified conferring interactions based on the following criteria: (1) the interaction took place during student mathematics worktime; and (2) included elicitation of student thinking. Such interactions exclude exchanges regarding simple directions, materials, or matters of classroom management.

Each conferring interaction was then coded as either including a nudge or not. Nudges, a series of talk turns within conferring that advance student thinking, were defined based on prior research (Munson 2016, under review) as fulfilling four criteria: (1) initiated by the teacher to advance students' mathematical understanding, engagement in mathematical practices, or productive collaboration; (2) contingent on elicited student thinking; (3) taken up by students; and (4) maintained student ownership and sense-making of the work. If a conferring interaction contained a series of talk turns after eliciting student thinking that met these criteria, the interaction was coded as a *conferring interaction with a nudge*. All other conferring interactions were coded as *conferring interactions without a nudge*. These two types of interactions were then counted in the pre- and post-conferring periods for each teacher.

To determine the effects of coaching on getting to a nudge in conferring interactions, I analyzed the pre/post data using a linear mixed-effects regression (LMER) using the lmer4 package version 1.1-12 in R (Bates, Maechler, Bolker, & Walker, 2015). Interactions (n=105) were nested within teachers (n=3), with fixed slope and random intercepts to address the variation in initial teacher practice. Binary variables were coded as follows: condition: pre-coaching= -1, post-coaching = 1; interaction: conferring interaction without a nudge= 0, conferring interaction with a nudge= 1. I obtained a p-value for the regression coefficient using the lmerTest package version 2.0-33 in R (Kuznetsova, Brockhoff, & Bojesen, 2016).

Results

Conferring interactions that included a nudge, and thus advanced student thinking, significantly increased after coaching. The total number of conferring interactions in the pre- and post-coaching period varied by teacher, but a meaningful shift from interactions without a nudge to interactions with a nudge can be seen in Table 1.

Table 1. Number of Conferring Interactions With and Without a Nudge, By Teacher, Pre- and Post-Coaching

	Teacher 1		Teacher 2		Teacher 3	
	Pre	Post	Pre	Post	Pre	Post
Conferring Interactions (No Nudge)	21	0	9	3	15	2
Conferring Interactions (With Nudge)	1	7	5	17	15	10

Overall change can be seen by examining the proportions of conferring interactions with and without a nudge for each of the three teacher participants pre- and post-coaching, as shown in Figure 1. Condition (β=0.25, SE=0.04, t=5.79, p<0.001) had a significant effect on nudging. The slope indicates that the likelihood of conferring with a nudge after coaching increased by a full 50%. Increases in the rate of nudging were seen across the three teacher participants, as can be seen in Figure 1.

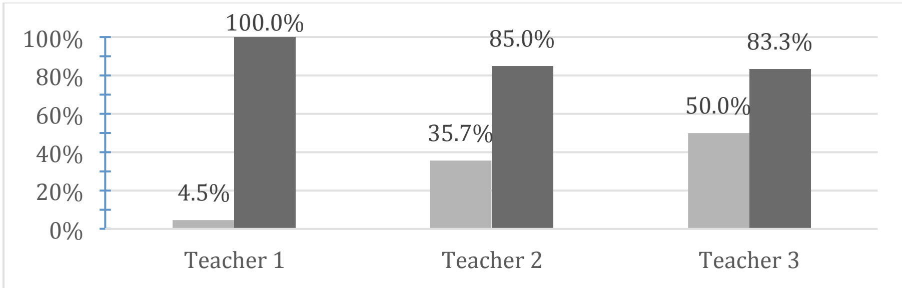

Figure 1. The proportion of conferring interactions that include a nudge, pre- (light gray) and post-coaching (dark gray) by teacher.

Significance

This study has implications for both research and practice by providing a promising intervention and by offering an underexamined way coaches can support teacher learning. Ambitious practices, particularly those demanding in-the-moment responsiveness such as conferring, are challenging for teachers to learn. Some strategies have shown promise in promoting such learning, typically outside of instructional time. This study shows one new pathway for embedding professional learning of responsive practice directly within teaching and adding to the potentially productive coaching activities (Gibbons & Cobb, in press).

How did side-by-side coaching achieve these effects? Ongoing analysis indicates that by surfacing teacher thinking in the moment, considering alternate interpretations together, and co-constructing a response, the coach and teacher can grow instruction as it happens. Additional research is needed to unpack how the coach and teacher co-constructed moments of teacher learning while engaged in the act of teaching. Side-by-side coaching is a promising avenue for future research in in-service teacher professional development of ambitious teaching practices.

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HOW URBAN MATHEMATICS TEACHER SELECTION, TRAINING AND INDUCTION AFFECT RETENTION

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Selective alternative teacher certification programs like the New York City Teaching Fellows (NYCTF) have trained thousands of mathematics teachers for urban school districts. This study draws survey and administrative data to examine the retention of 620 selective route mathematics teachers who entered teaching through the NYCTF program in 2006 or 2007. It uses logistic regression to estimate how teacher selection, training, and induction in NYCTF shapes mathematics teacher retention in New York City schools.

Keywords: Teacher Education-Inservice/Professional Development

Selective Alternative Teacher Certification Programs

Selective alternative teacher certification programs (ATCPs) like the New York City Teaching Fellows (NYCTF) actively recruit high-achieving graduates from the nation's most selective colleges and admit very few applicants (Clark, Chiang, Silva, McConnell, Sonnenfeld, Erbe, & Puma, 2013). These programs currently recruit thousands of teachers in dozens of U.S. districts In New York City alone since 2000, approximately three-quarters of all new secondary mathematics teachers in the city's public schools have entered through selective ATCPs. There is much to learn from selective ATCPs in spite of the mixed record (see Clark et al., 2013; Darling-Hammond, Holtzman, Gatlin, & Vasquez Heilig, 2005; Meagher & Brantlinger, 2011).

Research Methods

We adopt a teacher preparation program perspective to examine how mathematics teacher turnover is influenced by the major components of selective ATCPs, namely, their: (1) selection, (2) training, (3) first schools, and (4) induction. This perspective reflects research on teacher preparation by Humphrey and Wechsler (2007), Susan Moore Johnson and her colleagues (e.g., Johnson & Birkeland, 2003) and Richard Ingersoll (e.g., Ingersoll & May, 2012). Informed by this perspective, the study uses logistic regression to examine how the features of these program components shape mathematics teacher turnover in urban schools.

Research question. What is the association between SMTF turnover and their *selection* (i.e., teacher characteristics), *training,* and *induction* as part of NYCTF?

Participants. The study includes data on 620 prospective secondary mathematics Teaching Fellows (SMTFs) who entered teaching through NYCTF in either June of 2006 or 2007. SMTFs attended one of four NYCTF university partners for secondary mathematics certification.

Data. We linked data from three sources. *Survey data* on participant backgrounds and first-year induction support collected in three waves: 2006-07, 2007-08, and 2016. *Service history* data from the NYCDOE that provided information on how long individual SMTFs worked in their first school and also in the district through the spring of 2015. And, *first school data* (i.e., school climate and demographic data) for every school in the NYC school district is collected and published annually by New York state and the NYC Department of Education.

Logistic Regression. To explain how the major components of selective ATCPs affect secondary mathematics teachers district and school retention, we calculated the coefficients of four logistic regression models (Table 1). These models were estimated as:

$$ln\left(\frac{\hat{p}}{1-\hat{p}}\right) = \mathrm{B}_0 + \mathrm{B}_1\mathrm{X}_1 + \mathrm{B}_2\mathrm{X}_2 + \mathrm{B}_3\mathrm{X}_3 + \mathrm{B}_4\mathrm{X}_4$$

The coefficients B_i are in log-odds units and predict the retention outcome (i.e., the natural log of the odds ratio) for each participant provided measures of their teacher (selection) characteristics (X_1), university-based training (X_2), first school contexts (X_3), and induction (X_4).

Results

The descriptive statistics (not shown) show that the majority of the SMTFs were white, young, and graduated from a selective college and most were outsiders to the district (as judged by their high school). Fewer than one-in-six majored in mathematics or had completed the equivalent in coursework prior to beginning NYCTF. Approximately 26.8% of the SMTFs leave their first school within a year of starting teaching and, of these, about half (13.1%) move to another district school. Just over half have left the district within five years and less than 15% remain in their first school for eight years or more. These rates are much higher than national averages (Goldring, Taie & Riddles, 2014; Ingersoll & May, 2012).

Math teacher selection significantly shapes retention. Different SMTF characteristics became salient in explaining school retention in different Table 1 models. Graduates of the most selective colleges had the worst district retention. When compared with SMTFs from the most selective colleges, those SMTFs from other selective institutions were estimated to be 2.1 times more likely than those from the least selective colleges remain in the district beyond eight years. Where SMTFs went to high school (as students) was predictive of their district and school retention. Specifically, SMTFs who graduated from an NYC high school were estimated to be at least 1.5 times more likely than other SMTFs to remain in the district over time.

University training was predictive of SMTF retention. SMTFs trained at UnivB had the best estimated odds of retention. As shown in Brantlinger and Smith (2013), UnivB (and UnivD) training included a focus on mathematics teaching methods whereas UnivA and UnivC did not. Moreover, school characteristics like student race and poverty levels shape turnover, but school leadership also was impactful. In particular, SMTFs were estimated to be much more likely to return to placement schools with the most supportive principals and, surprisingly, less likely to remain in placement schools led by principals who were seen as particularly effective managers.

First-year induction also mattered. In particular, the quantity of professional interactions that SMTFs had with (1) mentors, (2) administrators, and (3) other math teachers during the first-year significantly influenced their school and district retention. For example, SMTFs who received the most mentoring (i.e., 13 professional interactions or more with their mentor during the first year) were estimated to be 9.1 times more likely to remain in their placement school for eight or more years than those who received least (i.e., 4 or fewer interactions during the first year) and 4.8 times more likely to remain in the district for eight or more years.

Discussion

This study illustrates how the major components of such programs *combine* to shape secondary mathematics teacher turnover in urban schools. We believe findings about secondary mathematics teacher training and selection are applicable to other programs that prepare mathematics teachers to teach in urban schools. The results also point to several things district and program leaders can do to improve the retention of secondary mathematics. In particular, the use of college selectivity as a screen for teacher selection, a hallmark of selective ATCPs, does not fare well in this study. Second, if retention is the goal, mathematics teacher training in selective ATCPs should include a greater focus on mathematics content and mathematics teaching methods. Third, the induction results have clear implications for how urban district and school leaders could provide better support to early-career teachers.

Table 1: Log Odds Estimates of SMTF Retention in NYC Public Schools

	District 8 Years	School 8 Years	School 3 Years	School 1 Year
Analytic Sample	456	456	456	456
Constant	0.139**	0.033**	0.219*	2.296
Entry Age 24-27 Years (21-23 Years)	1.314	1.505	1.370	2.388*
Entry Age 28+ Years (21-23 Years)	1.395	2.158*	1.177	0.559
Asian (White)	0.762	1.017	0.749	0.800
Black (White)	1.479	0.579	1.703	1.584
Hispanic (White)	0.515	1.829	1.021	0.752
Attended HS Near NYC (In NYC)	0.543*	1.479	0.769	0.394*
Attended HS Far (In NYC)	0.467**	0.558	0.841	0.651
Selective College (Most)	2.114**	1.526	1.181	1.599
Less Selective College (Most)	1.433	1.388	1.251	0.870
Training at UnivD (UnivB)	0.783	0.996	0.908	0.392
Training at UnivC (UnivB)	1.093	0.346^{Δ}	0.497	0.307*
Training at UnivA (UnivB)	1.349	1.447	0.860	0.190*
Certificate - High School (Middle)	0.940	0.880	0.727	0.330**
First School in Brooklyn (Bronx)	1.386	1.51	1.517	2.707
First School in Manhattan (Bronx)	0.976	1.447	1.46	1.135
First School in Queens (Bronx)	0.925	0.703	1.021	1.219
Student Attendance – Below 80%	1.080	0.203**	0.475*	0.531
Student Attendance – Above 94%	0.975	0.806	1.976	1.487
Free-Reduced Lunch – Below 61%	1.255	0.896	0.726	0.595
Free-Reduced Lunch – Above 96%	0.293**	0.147	0.328**	0.630
Under-Repres. Minority – Below 65%	1.700	3.671**	1.723	5.107**
Under-Repres. Minority – Above 98%	1.149	1.030	1.188	0.757
Teachers Not High Qual. – Below 6%	0.534	0.394	0.703	0.618
Teachers Not High Qual. – Above 27%	0.522	0.209*	0.397*	0.293**
Teacher Safety Respect Score - Low	0.58	1.795	0.769	0.634
Teacher Safety Respect Score - High	0.689	1.132	1.033	1.409
Principal Effective Manager - Low	1.418	2.237	3.840*	2.433
Principal Effective Manager - High	1.062	0.342	0.672	0.414
Principal Support Score - Low	0.993	0.356	0.295*	2.199
Principal Support Score - High	1.024	2.028	1.754	1.877
Induct: 5-12 Mentor Interactions (0-4)	3.007*	4.580	1.821	2.328^{Δ}
Induct: 13+ Mentor Interactions (0-4)	4.846**	9.101*	2.246	2.243
Induct: 5-12 Admin. Interactions (0-4)	1.432	0.483	1.868	1.591
Induct: 13+Administrator Ints. (0-4)	0.944	0.661	2.327	5.868**
Induct: Some Math Tcher Intrctns (Few)	0.871	1.009	0.834	1.723
Induct: Frequent Math Tchr Ints. (Few)	1.292	1.127	1.382	2.789*
Clustered Assignment (Individual)	0.987	1.058	1.048	0.770

(*p<0.05, **p<0.01)

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INCREASING COLLECTIVE ARGUMENTATION IN THE MATH CLASSROOM THROUGH SUSTAINED PROFESSIONAL DEVELOPMENT

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We share the case of one teacher engaged in professional development (PD) designed to improve collective argumentation. We present an analysis of two lessons in her classroom, one before and one after her engagement with the professional development. Findings show that the classrooms differ across both teacher support for collective argumentation (requesting ideas and elaboration vs. requesting facts and methods), and student contributions (justifications vs. procedures and facts).

Keywords: Teacher Education-Inservice/Professional Development, Instructional Activities and Practices, Classroom Discourse, Reasoning and Proof

Objective

In this paper, we explore the change in one teacher's classroom after participating in a professional development (PD), Mathematics *Studio PD* (Foreman, 2013), designed to improve collective argumentation in the classroom. More specifically we examine the question: "How does engagement in *Mathematics Studio PD* play out in one individual teacher's classroom?"

Background and Theoretical Framing

We leverage frameworks related to contributions from students and supportive questions and actions from teachers for collective argumentation to make sense of the totality of a lesson. The PD is designed to address these constructs using mathematically productive habits and routines. We begin by describing the underlying principles of the PD and describe each construct.

Underlying Principles of the *Studio PD*

The *Studio* PD advocates for student-centered classrooms where all students engage in and contribute to discourse that focuses on mathematical sense making, justifying, and generalizing mathematical ideas. A constructivist theory of learning (Von Glasersfeld, 1995) underlies these tenants where students are meant to engage in cognitively demanding tasks (Smith & Stein, 1998) providing opportunities for productive disequilibrium leading to deep mathematical learning. All students are viewed as capable mathematical thinkers with the PD's focus on growth mindset (Dweck, 2007). In this way, mathematics is not treated as a set of rules, but rather as an interconnected and logical structure (Hiebert, 1986) and the authority lies within the mathematics rather than the teacher or the textbook.

Teacher Support of Collective Argumentation.

Teachers support such mathematics by orchestrating the classroom discussion towards collective argumentation focused on justification and generalization. We use the construct of collective argumentation to describe discussions which "involve[s] multiple people arriving at a conclusion, often by consensus." (Conner, Singletary, Smith, Wagner, & Francisco, 2014, p. 401). Teachers facilitate collective argumentation through their questions (requests of action or information) and other supportive actions (directing, promoting, evaluating, informing, and repeating). The quality of these questions and support impacts the students' contributions to collective argumentation occurring in the classroom.

Contributions Types

We use the term contribution to define statements made by the students in support of collective argumentation. In the PD, student contributions are categorized into *procedures and facts (PF), justifying (J),* and *generalizing (G)* (Foreman, 2013) (see Figure 1 for a description of each category). To engage in meaningful mathematical discourse contributions should include justifications and/or generalizations.

PF USING PROCEDURES/FACTS	**J** JUSTIFYING	**G** GENERALIZING
No evidence of reasoning • Short answer to a direct question • Restating facts/statements/rules • Showing or asking for procedures *Uses meanings, definitions, properties, known math ideas to describe reasoning when:* • Explaining ideas & methods • Questioning to clarify • Noticing relationships/connections But ***doesn't show why*** the ideas/methods work	*Reasons with meanings of ideas, definitions, math properties, established generalizations to:* • Show why an idea/solution is true • Refute the validity of an idea • Give mathematical defense for an idea that was challenged	*Reasons with math properties, definitions, meanings of ideas, established generalizations, and mathematical relationships as the basis for:* • Making conjectures about what might happen in the general or special cases or • Justifying a conjecture about what will happen in the general or special cases

Figure 1. Contribution types.

Methods

The setting for this study is an elementary school in a mid-sized school district in the Pacific Northwest. This school has an enrollment of approximately 580 students with a 73% minority enrollment and 79% of children enrolled in free and reduced lunch. At this school 53% of 5th graders were meeting the math standards. The school is participating in a 3-year district-wide professional development program focused on improving instruction in mathematics. This PD uses the *Studio Model* of PD combined with summer workshops on best practices for teaching mathematics (Foreman 2013). Data collected includes 2 lessons videotaped at the end of each year, starting with a baseline video (Year 0) before engagement with PD as well as after the completion of each full year of the PD (Years, 1, 2, and 3). In addition, researchers observed and video recorded each PD session and took detailed field notes.

For this study, we focus on one fourth-grade teacher (Hannah – all names are pseudonyms) and analyze two lessons, one from before her engaging with the PD (Year 0) and one after (Year 3). We highlight the changes in her classroom and share some of Hannah's reflections throughout the PD to give insight into her engagement with the PD. Hannah was a participating teacher in the PD in Year 1, and the studio teacher in Years 2 and 3. Each lesson analyzed was transcribed and watched by two researchers multiple times.

To code student contributions and support for collective argumentation, talk turns supporting or contributing to collective argumentation were identified in the transcript. Each talk turn was coded as a direct contribution or question/supportive action. Direct contributions were coded as *procedures and facts, justification,* or *generalization* (see Figure 1). For example, a student working on the claim that 24/42 > ½ stated, "she divided 42 divided by two and she got 21. And since 24 is greater than 21, than it's over- the half. It's greater than half." This statement was coded as a justification as the student was "reasoning with meanings… of math properties" (Figure 1). Questions and other supportive actions were coded with the framework in Table 1. For example, the teacher asking a student "How do you write ten cents?" was coded as r*equesting a factual answer* as the request only

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

included a how. The teacher question “Why does it work mathematically?” was coded as *requesting elaboration* as it requested the student to elaborate further on their response, justifying their answer using mathematical reasoning. For supportive actions, a talk turn including the teacher statement “OK guys, let's see if they fixed it in the right way,” was coded as *evaluating* as it centered on the correctness of the mathematics.

Results & Discussion

Hannah’s lessons in Year 0 and Year 3 differed across the constructs listed above. Next, we discuss these observed changes and connect them to Hannah’s statements throughout the PD, illustrating her intentional engagement with the PD.

Collective Argumentation.

From Year 0 to Year 3 a shift occurred in terms of *teacher questions* and *supportive actions*, captured by the collective argumentation framework (Conner et al. 2014). In Year 0 most teacher questions focused on requesting facts (58%) or methods (21%). In Year 3 most of the teacher questions focused on requesting ideas (24%) or elaborations (58%) (see Table 1). In terms of *teacher supportive actions,* promoting actions increased (1% to 30%) while evaluating actions decreased (32% to 4%). Additionally, we saw an increase in informing actions (20% to 26%) and a decrease in repeating actions (24% to 5%) (see Table 1).

Table 1. Categorization of Teacher Questions and Teacher Supportive Actions

Teacher Questions			Teacher Supportive Actions		
	Year 0	Year 3		Year 0	Year 3
Requesting Fact	58%	3%	Directing Action	23%	35%
Requesting an Idea	4%	24%	Promoting Action	1%	30%
Requesting a Method	21%	8%	Evaluating Action	32%	4%
Requesting Elaboration	12%	58%	Informing Action	20%	26%
Requesting Evaluation	5%	7%	Repeating Action	24%	5%

The change in focus is correlated with changes in the quality of student contributions. In Year 0 most of those contributions were categorized as procedures and facts (96%) while in Year 3 42% were categorized as justification (see Table 2).

One of the foci of the PD is on questioning to research children’s mathematical thinking so the teacher can build on their understanding. At the beginning of her engagement with the PD, Hannah’s questioning did not model this focus. In the initial year she began as the studio teacher (Year 2, Studio 1) she reflected on questioning, stating the realization that “The questions are [asked] to give you [the teacher] ideas where they [the students] are at and not to teach them. That is something I never thought of.” In Year 2, Studio 3 Hannah responded to the prompt *What are key elements of your professional learning from today’s collaborative inquiry?* Her response included “Plan on asking specific questions during conferring [with the students] – research first and then advance their thinking.” In Year 3, Studio 2 Hannah responded to the prompt *What is it that you know about the HOM now that you didn’t know at the beginning of studio?* Hannah responded, “Pushing students to show their thinking rather than just having a correct answer.” Additionally, she shared that she “found it interesting because I started to use more visuals when I started training with the math studio model. Math Studio really brought more of the visual, justifying with the visual.”

Table 2. Categorization of Contributions from students (columns represent 100%)

Contributions	Year 0	Year 3
P/F	96%	58%
Justification	4%	42%
Generalization	0%	0%

Conclusions/Take-Away

In the context of this three-year PD, Hannah made significant changes, bringing her teaching in line with the goals and philosophy of the PD. Throughout the PD, Hannah's reflections, goals for next steps, and remarks made during the studio days captured her intentional implementation of this PD. These comments align with observed changes from Year 0 to Year 3. The focus in the classroom shifted from mostly focusing on procedures and facts to including justifications. Students were credited with (re)inventing mathematics and student strategies were shared with the class. Being able to justify was the ultimate authority. This change is exemplified in the following excerpt from the Year 3 lesson analyzed for this paper.

Hannah: How do you know that this is right?
Student: Because I am smart
Hannah: That is not math reasoning. Math reasoning is the authority in this classroom. *I am smart* does not tell me anything. *I am smart* tells me that you think too much of yourself. So mathematically why does this make sense? And what strategy did you use to solve it?
Hannah: [to S's partner] you hold him accountable to explain to you.

These changes in Hannah's teaching practices and in her students and their contributions are an inspiring example of changes that can occur in a long-term PD. Her example of growth illustrates the many strengths of this PD and informs teacher educators, PD providers, and school administration and leadership of the potential benefits of a PD of this nature.

Acknowledgments

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INTERACTION OF PROFESSIONAL DEVELOPMENT SUPPORT ON CO-TEACHING HIGH QUALITY MATHEMATICS TASKS

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We illustrate a professional development design focused on infusing high quality mathematics tasks within inclusive elementary mathematics classrooms. Our design supported teams of general education and special education teachers with integrating tasks and differentiation strategies into co-taught mathematics lessons to meet the unique needs of all students. We highlight the interaction between the professional development activities and the actions of the teachers. Analyzing, creating, and implementing tasks was the process used with teachers with thoughtful consideration for differentiation and co-teaching strategies.

Keywords: Teacher Education-Inservice/Professional Development, Equity and Diversity, Instructional Activities and Practices, Elementary School Education

Rigorous standards and a heterogeneous student population are characteristics in today's classrooms that make continued professional development essential for high quality mathematics teaching. National organizations expect equitable, accessible, and quality instruction for all students (e.g. NCTM, AMTE, CEC, TODOS), yet without professional development and on-going support teachers may struggle to meet the needs of students with varying instructional needs as noted in the persistent achievement gap between typically achieving students and students with identified disabilities (NCES, 2015). It is critical that general and special education teachers collaborate and develop a common vision for effective and equitable instruction that requires teachers to eliminate the "one size fits all" approach (Gregory & Chapman, 2002; Tomlinson, 2003). Meaningful learning of mathematics includes a shift from procedural instruction to instruction focused on understanding and reasoning. This instructional shift creates a crossroads of uncertainty, as teachers may not be prepared for changes in content and practice. Teachers are often dependent on textbook scripts and controlled formulas for solving problems, and their content knowledge may be lacking (Ma, 1999). To assist teachers with this shift, professional development and support of teachers' instructional practice are necessary.

Professional Development Design

To provide the needed support to establish and maintain effective collaboration between general and special education teachers, we designed a longitudinal professional development plan based on Guskey's (2000) recommendation for effective professional development that included infusing multiple formats of professional development delivery. Our design included both synchronous (face-to-face) and asynchronous (independent) opportunities with on-site coaching sessions to enhance instruction in inclusive elementary mathematics classrooms. Our hybrid approach addressed teachers' beliefs, misconceptions, and practices regarding high quality mathematics teaching, co-teaching, and differentiated instruction in order to positively influence instructional practices and student learning. Intentional planning based on teachers' needs, goals, and interests was a key component of the hybrid professional development, in lieu of a prescribed, one-dimensional design. The current description highlights the integration of high quality mathematics tasks into instruction, which was one aspect of the larger professional development project designed for co-teaching teams of general and special education teachers.

Teacher Application and Actions of High Quality Tasks

Recognizing the difficulty in the shift from typical problems to high quality tasks, planning was very deliberate within the professional development. High quality tasks require a high cognitive demand from students and are an essential staple in good mathematics teaching (Stein, Smith, Silver, & Henningson, 2001). Therefore, teachers must focus on building conceptual understanding through problem solving which requires providing an instructional environment to develop the behaviors presented in the Standards for Mathematical Practices (CCSSM, 2010), instead of using rote memorization tasks that only require procedural knowledge.

Differentiation of the task is also critical for successful implementation in inclusive mathematics classrooms. Differentiation can be thought of as a pedagogical approach in which teachers modify and adapt curricula, instructional practices, support, and assessments in order to maximize learning based on student needs (Algozzine & Anderson, 2007; Tomlinson et al., 2003). The progression of high quality task support within our professional development was carefully planned and scaffolded to ensure effective understanding and application. The sequence of support followed these steps: analyzing sample tasks, creating tasks from simple word problems, and implementing high quality tasks in a co-taught setting.

Analyzing Tasks

One of the first sample tasks presented to teachers was *Danny Dinosaur's Designer Coat* (see Figure 1). Teachers worked individually and then in teams (co-teaching partners: at least one general education and one special education teacher) to analyze and solve the *Danny Dinosaur's Designer Coat* task in ways that made sense to them; this included some teachers using manipulatives, such as color tiles, some drawing pictures or tables, and a few using symbolic representations. Group discussions on how the problem was solved provided opportunities for teachers to share ideas and learn from their peers. This discussion generated discourse around this open task with multiple solution paths.

Danny Dinosaur's Designer Coat
Danny Dinosaur's designer coat is missing 5 buttons. How many ways can you use blue and red buttons to fix is coat?

Figure 1. Examples of high quality mathematics task.

Next, we posed the question about how to differentiate the task to students with varying learning styles and ability levels. Teachers generated adaptations and modifications, and used differentiation strategies to implicate how this single task could be used in a variety of grade levels and settings. Teachers suggested: requiring a red button to be first, incorporating patterns, and changing the number of buttons from five to ten. Thinking ahead to more advanced mathematics, we explained how this task can be extended by asking how many different patterns of blue and red buttons can be created where the order matters, which builds the knowledge of permutations and combinations. Next, teachers brainstormed how to implement the task in a co-teaching setting. Teams discussed how the use of tasks would be put into action in their classrooms, with conversations about the roles of each adult, the flow of a lesson built around the task, as well as specific differentiation strategies to use with their students in their actual mathematics classrooms. By providing time for each team to discuss how they would use the *Danny the Dinosaur* task, teachers were able to relate the general instruction from the professional development sessions into a specific plan for classroom implementation.

Creating Tasks

Teachers were given examples of word problems and tasks to classify in order assist them in recognizing the differences between the two and the differences in cognitive demand required by students to solve them. Problems and tasks were selected from Stein et al., (2001, p. 9) classifying activity. Next, teachers selected one of the word problems from the classifying activity to revise and transform into a high quality task, focusing on the level of cognitive demand needed for the solving process. Teacher teams pasted the original task on chart paper and then wrote their newly created task. Once teams were finished and chart papers were posted around the room, teachers individually gave feedback on each teams' work (see Figure 2).

<table>
<tr><th colspan="2">Original Tasks</th></tr>
<tr><td>The cost of a sweater at J.C. Penney's was $45.00. At the "Day and Night" sale it was marked 30% off the original price. What was the price of the sweater during the sale? Explain the process you used to find the sale price.</td><td>Rylee donated $45 to Children's Miracle Network, Morgan and Katie each donated $12 a piece. How much more money did Rylee donate than Morgan and Katie?</td></tr>
<tr><th colspan="2">Revised Tasks</th></tr>
<tr><td>Erin wants to buy a red Christmas sweater on Black Friday. The sweater is originally $80, but Erin has only $50 to spend. She has 4 coupons in her pocket; she can only use 2. How many different combinations of coupons can Erin use to get her at, or below, her $50 budget?

$15 off purchas
30% off original price
20% off purchas
40% off</td><td>Rylee, Morgan, & Katie donated $75 together to the Children's Miracle Network. Each girl gave different amounts, but no girl gave more than $50.00. Use the table below to show possible amounts for each girl gave.

<table><tr><th></th><th colspan="4">Possibilities</th></tr><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><td>Rylee</td><td></td><td></td><td></td><td></td></tr><tr><td>Morgan</td><td></td><td></td><td></td><td></td></tr><tr><td>Katie</td><td></td><td></td><td></td><td></td></tr><tr><td>TOTAL</td><td></td><td></td><td></td><td></td></tr></table></td></tr>
<tr><th colspan="2">Feedback</th></tr>
<tr><td>• Good job! I love that students will have to think hard on this!
• Great application! Maybe include a chart to show the combinations vs. just asking how many ways.
• Good use of coupons.
• Awesome! Love the real life application!
• Wow! Nice job using % and $ off.</td><td>• Love the table.
• Nice use of High Quality Task.
• Excellent.
• Graphic organizer!
• I LOVE the table!
• Students have to work around limitations. Great job!
• Excellent due to different possibilities and diagram.</td></tr>
</table>

Figure 2. Creating tasks.

Implementing Tasks

As teams became more comfortable with analyzing and creating tasks, they began to implement tasks into their classroom instruction. A group of fourth-grade teachers (two general education and one special education teachers – two teams of teachers) developed and implemented a task to focus on addition, subtraction, and multiplicative thinking called "Chartable Giveaway." In this task, small groups of students were awarded different amounts of money to "help purchase toys for children in need." Students worked together to decide which toys (from a sheet designed with various toys and prices) to purchase and how many of each toy they wanted to purchase, while keeping in mind how much money they had spent and had left over to spend. Student groups recorded their selections on a teacher created table to show their decision-making process throughout the task. Differentiation was a key component in the successful implementation of this task. Students were purposefully grouped together to support one another and meet the needs of all learners. Student groups received monetary values ranging from $500 to $20,000. As the students worked, both the general education and special education teachers together monitored students' progress and provided prompts to support student thinking and learning. Monetary values were changed as needed. For instance, one group of students successfully completed working with $1,000 and was then given a larger amount of $5,000 in order to provide students with an extension. After the lesson, the teachers reflected on the successes implementation of the lesson, including positives about the mathematics instruction, differentiation strategies used, and co-teaching components. The teachers believed this was a highly successful lesson with all of their students and expressed that incorporating more task-based lessons would be a goal of their future instruction.

Summary

The implementation of this professional development design created a positive and visible interaction between the teacher learning and instructional practices creating a synergy between general education and special education teachers. This project took the theoretical model of best practices of professional development and put it into meaningful practice focused on the differentiation of high quality tasks in inclusive mathematics classrooms.

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LEADERS HELPING LEADERS: BUILDING LEADERSHIP CAPACITY TO SUPPORT STANDARDS IMPLEMENTATION

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This exploratory study examines how leaders across a state take up and engage in webinars designed to build leadership capacity to support teachers' implementation of new mathematics standards. Leaders Helping Leaders monthly webinars served as a space through which the community embodied mutual engagement, joint enterprise, and shared repertoire. Findings indicated that tensions among the design principles undergirding the webinars surfaced, raising questions about how to prioritize limited resources for continued support of leaders.

Keywords: Design Experiments, Policy Matters, Standards

Introduction

Statewide implementation of new mathematics standards is no small feat. Successful implementation that promotes mathematics teaching focused on students' thinking requires concerted efforts by classroom educators and educational leaders alike. Indeed, district leaders (Lawrence et al., 2011), principals (Desimone, 2000), and instructional coaches (Neufeld & Roper, 2003) are all critical to implementing and sustaining instructional change. This study addresses a gap in the literature about how to cultivate leadership capacity amongst district mathematics leaders, principals, and instructional coaches for implementation of new high school mathematics standards. Specifically, the North Carolina Collaborative for Mathematics Learning (NC^{2}ML) partnership described below used a design-based implementation research (DBIR) (Fishman et al., 2013) approach to iteratively design and study the implementation of capacity-building opportunities for leaders related to the new standards.

North Carolina Collaborative for Mathematics Learning

NC^{2}ML is a statewide research-practice partnership involving the state education agency, educational researchers from multiple universities, teachers, and leaders from across the state engaged in a complex research endeavor to examine and learn from the implementation of North Carolina's new high school mathematics standards. NC^{2}ML engages and supports multiple constituencies, including teachers, educational leaders, and parents/community members. This paper focuses on educational leaders and examines the ways in which they participate in an online community of practice (CoP) – a "space where learning takes place and [sic] where learning is the practice itself" (Savard, Lin, & Lamb, 2017, p. 42). This paper examines the use of monthly Leaders Helping Leaders (LHL) webinars and is guided by the questions: (1) *What can be learned from patterns of participation in LHL?* (2) *What are participants' perceptions of LHL?* and (3) *How do leaders take up and engage in a* CoP *around implementation of high school new mathematics standards?*

Conceptual Framework: Communities of Practice

Three dimensions of practice characterize what brings coherence to a CoP – mutual engagement, joint enterprise, and shared repertoire (Wenger, 1998). Mutual engagement signifies that communities are developed and maintained through engagement in a shared set of practices valued by a community. Respectively, joint enterprise and shared repertoire denote communal goals and viewpoints, and use of a shared set of tools. CoPs can work to solve problems of interest to the group, foster best practices, and develop skills; serve to build members' capacities and create and exchange knowledge; and are bound not by formal membership structures but rather by affinity.

Methodology

NC^{2}ML draws upon DBIR (Fishman et al., 2013) as an approach to facilitate the design of implementation efforts related to new state mathematics content standards. DBIR transcends traditional research-practice barriers by applying design-based principles to address and study problems of practice. The four core principles of DBIR include: 1) attention by teams of researchers and practitioners to persistent problems of practice; 2) use of iterative, collaborative design; 3) development of knowledge and theory through systematic inquiry around learning and implementation; and 4) emphasis on cultivating sustained change at scale. The focus of this study is on how educational leaders engage in a CoP within the Leaders Helping Leaders space.

Design of the Affinity Space

Leaders Helping Leaders (LHL) is a monthly virtual session hosted by the NC^{2}ML team, with agendas informed by an advisory board comprised of math leaders from across the state. LHL sessions were facilitated by members of NC^{2}ML and focused on three overarching objectives: 1) Provide support to math leaders as they support teachers; 2) Provide space for leaders to share and problem solve together (collaborate); and 3) Cultivate community amongst math leaders. LHL agendas were informed by the interests and stated needs of participants and guided by a consensual set of norms. LHL sessions typically consisted of a review of the norms, a community-building activity, and focused on a specific topic (e.g. parent resources for mathematics learning, tools to support teacher learning and instructional planning).

Methods and Analysis

The NC^{2}ML team analyzed three datasets for the September 2016, October 2016, and November 2016 LHL webinars: webinar analytics (registration, attendance, and participation data), post-webinar survey results, and webinar transcripts and chat comments. Participant perceptual data were collected via an online survey tool. The response rate for each survey was low: 18.5% in September (n = 10/54), 5.7% in October (n = 2/35), and 11.1% in November (n = 3/27). As such, it is important to keep in mind that these data are unlikely to be representative of the full population of LHL attendees. We conducted content analysis of transcript and chat data, leveraging open and deductive coding.

Findings

Participation Patterns: Leaders Helping Leaders Webinar Analytics

Overall, 116 leaders attended the sessions, representing 40 (34.8%) of the 115 districts across the state. Attendance fell each month from September (54) to October (35) and November (27). Active participation, defined as speaking or using the chat feature, was highest in September and inconsistent across months. High level activity, defined as speaking or using the chat feature multiple times, while also inconsistent across months, was highest in November, when attendance was lowest, raising the question whether settings with fewer participants allow more space for high level activity.

Attendance patterns across sessions. Of the 54 participants who attended in September, 30 (55.5%) did not return (Disengagers), while 11 (20.4%) attended all three fall sessions (Stayers). Seventeen participants were new in October and eight were new in November (Entrants). These data reflect the idea that even with a substantial number of Disengagers, the Stayers and Entrants fostered a viable CoP that can sustain over time, suggesting that the CoP can remain viable even as attendees differ.

Participant's Perceptions: Post-Webinar Survey Results

On a 1-5 Likert scale for four items regarding respondents' perceptions of the LHL webinars, means increased from the September session (3.6 across items) to October (5.0) and November (4.7). There are multiple possible explanations for the apparent increase in perceived quality from the first to second/third sessions: LHL sessions became more polished and fine-tuned with experience. Also, it appears that LHL webinars benefit from having a focused topic(s) that includes explicit resources/tools that leaders can use in their districts. Noteworthy is the disconnect between plans to attend future sessions and aforementioned data regarding high numbers of Disengagers versus Stayers. This suggests that the moniker Disengagers might be a misnomer. Perhaps it is less a desire or decision to disengage and more a function of other factors, such as schedule conflicts or competing demands on leader attention and time.

Open-ended survey items indicated that respondents find collaboration, networking, and the focused specific topics/tools most helpful. A few respondents indicated that providing input on future sessions would be helpful. Specifically, respondents would like future webinars to provide space for sharing ideas regarding implementation of the new standards, collectively identifying solutions to challenges, and finding ways to help teachers with standards and pacing guides. Overall, respondents have positive views of LHL webinars, plan to attend future webinars, prefer the live version, and seem to appreciate a focus topic/tool for each webinar.

How Leaders Take Up and Engage in a CoP: Webinar Dialogue

In this section, webinar dialogue – spoken comments and use of the chat feature – is examined using the concepts of mutual engagement, joint enterprise, and shared repertoire.

Mutual engagement. This construct involves ways in which participants engage with one another and interact. At the beginning of each webinar, facilitators shared a community-building prompt. In each webinar, 11-12 participants responded to these prompts. During the November session, 12 of 27 (44.4%) of participants responded.

Mutual engagement, however, also involves "implied competency" (Savard, et al., 2016), the notion that participants have expertise that allows for interactions that are thoughtful and meaningful. Participants have responded to prompts by facilitators that ask participants to share experience and examples from their school/district related to the topic. For example, during the October session, participants shared what they are doing to inform parents about the implementation of the new mathematics standards. For example, one participant shared:

> I love what some high school mathematics teachers are doing …creating videos to answer questions that students send them! We have teachers who are considering this concept in our district.

This example illustrates sharing out as well as considering implementing ideas, and it also represents the type of affirming comments that are common in LHL sessions. In response to the aforementioned comment, a participant responded, "great idea." These comments contribute to mutual engagement.

Joint Enterprise. Also common to the chat conversation are posts through which participants networked and collaborated with one another. For example, during the September session, one participant asked the group, "Is there anyone creating benchmarks that would be willing to co-author them with other districts??" Several other participants responded in the affirmative to this request. During the same session and based on the robust conversation amongst participants, the facilitator (a state department of education math coordinator) offered to send out an initial group email to those interested that could then be used as an informal listserv by which leaders could communicate directly with one another. Twenty-seven participants asked to be included on the email and continue to communicate with one another. In these ways, participants are engaging in joint enterprise: They collaborate on shared interests, learn from one another, and go beyond these to develop a collective sense of shared practice and work.

Shared repertoire. CoPs coalesce around tools, artifacts, ideas, and routines. A number of the tools discussed and used within LHL were created by collaborative writing teams facilitated by the state agency partner, including a pacing guide and curriculum guide that expands upon the mathematics standards, provides aligned mathematics tasks, and offers instructional ideas related to mathematics teaching and student learning. As previously detailed, some leaders from LHL worked together to create common assessments aligned to the new standards. During the October session, which focused on informing parents about the implementation of the new math standards and building bridges with parents, five participants chatted about the ideas they wanted to try in their own school/district, such as: "Looking forward to sharing these videos via Facebook – parents will check out Facebook." Their shared repertoire is a reflection of – as well as a means of – knowledge-sharing and learning within the CoP.

Conclusions

While the data collected thus far are limited, they provide valuable insights for iterative design through DBIR. First, while participation in LHL webinars fell markedly with each webinar, and participation rates have been uneven across sessions, these findings do not mean that the CoP is anemic. Wenger (1998) argues that participants may move in and out of the CoP through its porous periphery and move towards and away from the core at different points. The richness of the CoP itself can be maintained or even expanded. The data suggest that the leaders' CoP embodies the key constructs of mutual engagement, joint enterprise, and shared repertoire.

As we continue our efforts in supporting stakeholders in implementation of new mathematics standards, we are excited about the possibilities of research-practice partnerships and CoPs, and we see DBIR as a productive approach to jointly working on problems of implementation at scale and developing theories that lay the groundwork for future crossroads or intersections between theory, research, and practice.

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PROFESSIONAL COMPETENCIES THAT MATHEMATICS TEACHER EDUCATORS SHOULD HAVE: REFLECTIONS FROM A WORKSHOP

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As part of a larger project aiming to develop a framework of competencies for mathematics teacher educators by utilizing a Delphi approach, the purpose of this study was to investigate what a group of experts think about the competencies that mathematics teacher educators should possess. The data were collected in a workshop with 10 experts coming from mathematics teacher education programs and non-governmental organizations. Findings revealed a set of competencies about the works and identities of university-based mathematics teacher education faculty as teachers, researchers, and community leaders.

Keywords: Instructional Activities and Practices, Teacher Beliefs

Despite the efforts to define competencies of qualified teachers throughout the world, studies regarding the competencies of teacher educators who play a key role in training teachers and researchers are limited (Goodwin & Kosnik, 2013). In recent years, a few institutions throughout the world attempted to identify the competencies that qualified teacher educators should possess. While some of these efforts encompass the competencies for all teacher educators (e.g., Association of Teacher Education [ATE], 2008; Koster, Dengerink, Korthagen, & Lunenberg, 2008), some are focused on competencies within specific disciplines competencies (e.g., see Association of Mathematics Teacher Educators [AMTE], 2002 regarding qualifications for mathematics education doctoral students).

Given the importance of reflective inquiry in developing as a mathematics teacher educator, studies focusing on mathematics teacher educators' professional education and development are rare and those available are mostly self-studies of university-based teacher educators (e.g., Chauvot, 2009; Williams, Ritter, & Bullock, 2012). Lack of a framework for the professional knowledge base and competencies that mathematics teacher educators should possess might be useful for the education and professional development of teacher educators and to assess the effectiveness of teacher education programs. It would also be useful to identify the mathematics teacher educator competencies using a research based empirical approach. Thus, the purpose of this study was to investigate what a group of experts involving mathematics teacher educators and representatives of educational non-governmental organizations think about what competencies mathematics teacher educators should possess.

Even though the term "teacher educator" might refer to those who are involved in training prospective teachers and/or practicing teachers (Even, & Krainer, 2014; Krainer, Chapman, & Zaslavsky, 2014), in this study, it is conceptualized from a broader perspective as comprising those who also do research, train prospective teacher educators (i.e., graduate students), and are engaged in institutional and community work. Moreover, within this study the term "competencies" is used when referring to the knowledge, skills, attitudes, motives and personal characteristics that mathematics teacher educators should possess in order to act effectively in a particular situation (Spencer & Spencer, 1993).

Method

Context and the Participants of the Study

The study reported here is part of a larger project aiming to (*i*) identify the competencies that mathematics teacher educators in Turkey should possess; (*ii*) investigate the self-efficacies of mathematics teacher educators in meeting these competencies. The data reported in this paper were collected from a three-day workshop which was the first step of a four-step Delphi method (Landeta, 2006) utilized in order to develop a develop a framework of competencies for mathematics teacher educators. The participants of the workshop were 10 experts representing middle school and high school mathematics teacher education programs from six different universities and four different educational non-governmental organizations.

Data Collection and Analysis

Prior to the workshop and on the first day, the participants/experts were briefed about the larger project, the purpose of the workshop and what is expected from them during the workshop using situations and vignettes designed to help them think about the qualifications a competent mathematics teacher educator should have. The two overarching questions that the participants were asked to answer were as follows: “What duties and responsibilities do mathematics teacher educators have?” and “What knowledge, skills and attitudes do they need to have in order to fulfill these responsibilities?” The participants were divided into two groups and worked separately. In each group, a researcher acting as moderator and a research assistant for taking detailed notes of the deliberations were present. The groups worked to produce a draft list of competencies in the first two days. In the third day, the two groups joined together and made presentations of their draft competency lists. They further discussed the similarities and differences between the frameworks that came from the two groups.

For data analysis, in addition to field-notes, video and audio-recordings of the discussions in the two groups were coded and then compared in order to highlight the participants’ emerging thoughts about competency areas and competencies.

Results

Focusing on qualifications they would seek when hiring a mathematics education faculty for a hypothetical open position, Group-1 suggested the following competency areas: *Mathematical Knowledge and Skills; Knowledge for Teaching the Subject (i.e., Mathematics); Professional Development and Social Responsibility (Service); General Pedagogical Knowledge; Mentorship; General Knowledge; Attitudes and Values; Knowledge of Programs for Mathematics Teacher Education and Faculty Development*. Somewhat different from Group-1, Group-2 focused on duties and responsibilities of a university-based mathematics teacher educator and identified the following competency areas: *Teaching; Academic Work, and Service to Society*.

One of the major issues participants discussed during the workshop was the definition of a mathematics teacher educator. After intense debates about who a mathematics teacher educator is, it was decided that a mathematics teacher educator is someone who received his/her doctorate in mathematics education, have continuing scholarly work in mathematics education and/or teach courses on mathematics education. On the other hand, particularly in Group-2, one of the initial concerns was whether the competencies should aim for the minimum or a higher level of competencies. Through deliberations, they decided that the competencies should be descriptive of a mathematics teacher educator who is capable of educating qualified mathematics teachers.

Another issue that was the subject of long and dense debates in both groups was the extent of mathematical content knowledge that a mathematics teacher educator should have. Knowledge, skills, and attitudes related to mathematics and its teaching were discussed as essential components of mathematics teacher education competencies. In Group-1, it was decided that a mathematics teacher educator should at least have the knowledge and skills to be able to analyze mathematical concepts in the school mathematics curricula and their connections to the connected concepts and topics at the post-secondary level mathematics. On the other hand, Group-2's dilemma was whether a mathematics educator should be competent enough to be able to teach all of the undergraduate level mathematics courses. Their conclusion was that having such level of advanced mathematical knowledge is not realistic for all mathematics teacher educators and thus there needs to be a boundary depending on the similarities and differences between mathematicians and mathematics educators. Similar to Group-1, they adopted the idea that a mathematics educator should possess sufficient knowledge of mathematics in order to be able to make connections among concepts and topics in school mathematics. According to Group-2, a mathematics teacher educator must have deep knowledge of mathematics taught from kindergarten to the first year of university (K-13). They also argued that having experience in teaching mathematics would contribute to a deeper understanding of mathematics; thus, having teaching experience would be desirable and be considered as an indicator of mathematical content knowledge.

Another important discussion point was about who would be responsible for teaching mathematics (content courses) to the teachers. In Group-2, while some participants indicated that it is mathematicians' responsibility, others argued that it should be mathematics teacher educators' responsibility because the ultimate purpose in teaching mathematics to teachers is directly related to teaching how to teach the content. Group-2's major emphasis was on the fact that the nature of mathematical knowledge of a mathematics teacher educator should be different than that of a (research) mathematician in the sense that it should focus on how to teach it or help students learn it. This view led to another discussion that the knowledge about learning and understanding mathematics is as important as the knowledge of mathematics and content knowledge for teaching mathematics. Thus, the ability to produce knowledge regarding teaching and learning of mathematics should be an integral part of mathematics teacher educator competencies.

Another topic of discussion in Group-2 was about in-service teacher education. While some of the participants argued that in-service teacher education is a direct responsibility of mathematics teacher educators and thus it should go under the competency area "Teaching", others argued that such activities are not among official requirements and duties of the universities or related governmental bodies and thus it needs to be under "Service to the Society". After some discussion, even though all agreed that in-service mathematics teacher education is among the responsibilities of a mathematics teacher educator, it needs to be considered as part of "Service to the Society". A similar discussion was held in Group-1. They concluded that whether mathematics teacher educators feel responsible or not, they should be competent in designing and managing in service teacher training programs and they thought it fell under "Mentorship."

Discussion

Involving a group of professionals/experts into a process of active discussion, and establishing consensus using Delphi procedures with a focus on mathematics teacher educators' competencies make the present study different from the other studies focusing on standards and competencies of teacher educators. Findings showed that developing a framework of professional competencies for mathematics teacher educators is a challenging task and different parties would have their own priorities. Based on the findings of this study in the Turkish context, the priorities are addressed from the perspective of professional identities of university-based mathematics education faculty mainly as teachers and researchers. This implied that a balance between research and teaching is needed as part of being a competent mathematics teacher educator. Additionally, the extend of mathematical content knowledge that a mathematics teacher educator should have and who would be responsible for teaching mathematics (content) courses to prospective teachers seemed to be important in defining the duties and responsibilities of mathematics teacher educators and the competencies they need to have.

We believe that this study contributes to the international debate in professional development of (mathematics) teacher educators. It is important for other contexts for stimulating teacher educators towards reflection. It is considered that the results of this study, as part of developing a framework for mathematics teacher educator competencies, would provide mathematics teacher educators with opportunity to reflect on, serve as a guide for understanding the development of professional competencies, and provide insights into the nature of mathematics teacher educators' expertise.

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QUANTITATIVE LITERACY IN MIDDLE SCHOOL MATHEMATICS: A TEACHER'S IMPLEMENTATION

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This study draws on critical pedagogy to frame and interpret the investigation of a teacher's implementation of quantitative literacy in a middle school mathematics classroom after taking a graduate education course on numeracy. We provide a brief overview of elements of critical pedagogy and discuss how we used this theory to explore how a middle school teacher implemented quantitative literacy instruction in her classroom. Finally, we present the first theme, authentic application of mathematics concepts, which emerged from the data.

Keywords: Teacher Beliefs, Equity and Diversity, Instructional Activities and Practices

Theoretical Framework

Critical pedagogy was first used by Henry Giroux in 1983 (Wink, 2011) and involves identifying and naming power structures and examining the role schooling plays in reproducing hegemony (Allen & Rossatto, 2009). Critical pedagogies posit that the goal of education is to train every student to be a productive member of a democratic society (Erikson, 2016). This is accomplished through valuing students' lived experiences and voices (Cho, 2010). The goal for social institutions is to reform systems based on predetermined principles such as equality, democracy, emancipation, common good, social justice, and equal rights (Cho, 2010). Critical pedagogy closely relates to quantitative literacy (QL) instruction in middle school. Literacy is a form of power that "naturally breaks down barriers of time, space, and culture" (Wink, 2011, p. 70). Literacy goes beyond just reading and writing and includes multiple types of literacies, such as academic, functional, constructive, critical, financial, and quantitative (Wink, 2011). QL instruction empowers students in that it operates only in the context of real-world applications and, therefore, provides students the ability to draw upon their own real-world experiences to problem solve.

Background and Purpose

QL is different than typical school mathematics. School mathematics tends to focus on abstract, procedural knowledge with little problem-based, investigative thinking. In contrast, QL involves the use of mathematics in the context of real-world scenarios. QL is supported by the National Council of Teachers of Mathematics (NCTM, 2014) and the National Governors Association Center for Best Practices and Council of Chief State School Officers (2010), who have established national standards that advocate for use of "real-world" mathematics. However, "real-world" mathematics is typically implemented in the form of story problems that involve hypothetical stories rather than authentic tasks (Wiest, Higgins, & Frost, 2007). For true implementation of QL, students must apply fundamental mathematics to genuine, real-world, interdisciplinary situations (Yarnall & Ranney, 2017).

Unfortunately, many people do not fully understand how mathematics relates to the real world (Erikson, 2016). For example, pre-service teachers tend to exclude real-world information, such as tax in solving shopping problems, and thus preference non-realistic answers to real-world problems (Verschaffel, Greer, & DeCorte, 2000), indicating that pre-service teachers tend not to approach problems from a QL standpoint. Further, Gutstein (2006) found than many teachers do not appreciate the practical utility of the mathematics they teach. Moreover, in a survey of 62 teachers found that teachers reported giving real-world examples, rather than engaging students in tasks with real-world data (Gainsburg, 2008). Gainsburg recommends professional development to address teachers' lack of appropriate QL implementation.

Tunstall et al. (2016) supports this post-secondary need stating "the value of QL (QL) for college graduates is well documented…to empower students to feel confident in quantitative situations (p. 2). This lack of confidence, due to a lack of training and experience, for teachers might result in teachers limiting or omitting QL topics in their instruction. Thus, mathematics education for teachers should include exploration of QL to increase implementation of QL instruction in K-12 education (Garii & Okumu, 2008). Given the lack of teachers who implement QL and the connections between training/education and implementation, this study explored the implementation of QL in one teacher's classroom instruction.

Context and Methods

The teacher selected for this research was chosen because she was a middle school mathematics teacher who was enrolled in a QL course, Critical Numeracy Across the Curriculum (offered at a land grant university located in the western United States). She had taught mathematics for four years, was in her second year teaching middle school, and was in her first year teaching at her current school. The teacher had earned a bachelor's degree in discrete mathematics, a master's degree in applied mathematics, and a secondary teaching license program, and she was currently enrolled in a doctoral program in mathematics education. The teacher taught at a small parochial school with an enrollment of 288 students, including 35 eighth graders. The teacher's math class took place three days each week for a total of five hours. Twenty-two of the eighth graders were enrolled in algebra and were taught by this teacher. These students were 12 to 13 years old and included 12 female and 10 male students. Most were from middle-to-upper-class families, and all were Caucasian.

Initially, we conducted an interview with the teacher, asking her to describe QL. Additionally, the teacher was asked to identify ways she implements QL in the classroom. We then observed the classroom instruction three days a week for two months, conducting observations as described by Florio-Ruane (1999), mapping the classroom, making a classroom log, and collecting field notes. After the observations, we conducted a second interview to ask questions prompted by the observations. Additionally, lesson plans and assignments were collected and analyzed for consistency with the observations of teaching and the teacher's reports of quantitative-literacy implementation in the classroom.

We employed grounded theory for data analysis (Glaser & Strauss, 1967) and simultaneously collected and analyzed data (Lichtman, 2011). The analysis was a part of the research design with the coding of the first data set serving as a foundation for future data collection and analysis (Corbin & Strauss, 2008). We adjusted additional data collected based on analysis of the first data set (Corbin & Strauss, 2008) by deriving the interview questions on analysis of the initial data. We proceeded through the coding process by analyzing the data line by line and paragraph by paragraph and then coding the deconstructed fragments (Lichtman, 2011). Codes were compared, renamed, added, or deleted as we constantly compared them (Corbin & Strauss, 2008). We constructed conceptual categories based on multiple re-readings and re-coding of the data. "This 'checking back' is a method of confirming or disconfirming that ensured that the categories were grounded in the theory rather than 'flights of fancy' or pet ideas" (Lichtman, 2011, p. 63).

Results

The results are categorized into the following themes: authentic experience, connection to mathematics standards, teacher knowledge, and student engagement. This paper reports on the first theme, authentic experience.

Authentic Application of Mathematics Concepts

The teacher reported a goal of using mathematics authentically as her primary reason for implementing QL in her teaching. She stated that it was important that her students know how to research real data, represent it using graphs and charts, and analyze it. Further, she indicated that it was important for her students to know mathematics rules and have the ability to apply them in their daily mathematics encounters, saying, “I use QL in my class because it is important for students to be able to apply their mathematical skills to the real world. Applying their knowledge to solve a problem in a different context involves a deeper level of thought.”

This perception of authentic instruction was supported frequently in the teacher’s teaching. The teacher’s lessons, homework assignments, and assessments could be categorized into four levels:

1. Students research authentic data to analyze and respond to a question(s).
2. Students are provided authentic data to analyze and respond to a question(s).
3. Students use hypothetical data to respond to a question(s) set in a real-world context.
4. Students use no QL.

The majority of the lessons fell into categories 1-3. Lessons that were not based on QL occurred only on days that were dedicated to review for a state assessment and one day when technological problems prevented the planned lesson from being implemented.

The teacher had students use laptop computers to research authentic data and provided instruction on identifying reputable Web sources. Many lessons required students to find their own data to analyze. For example, in one lesson students were asked to apply their knowledge of graphing systems of equations to research the cost of two gym memberships, create equations for both, graph the system of equations, and write a conclusion regarding which membership they would choose and why. When the students were not researching their own data, they were frequently analyzing authentic data provided to them, such as data on the average unemployment rates and salaries of individuals with different levels of educational attainment.

The assessments and homework assignments, however, tell a different story. No assessment was given during this study that evaluated any QL skill or task. Further, the only required written QL homework assignments were incomplete class work that students were asked to complete as homework. Typical homework included worksheets of algebra problems. The students also had electronic homework on a class web page. Approximately one-quarter of the online assignments had a QL component, requiring the students to discuss or comment on the QL activities that were completed in class.

Discussion

These results indicate that taking a teacher education course in numeracy increased this teacher’s QL instruction in her middle school mathematics teaching. This supports the call for QL instruction in college (e.g., Tunstall et al., 2016) and, in particular, teacher education (e.g., Garii & Okumu, 2008). Further, the teacher asserted that the course was practical in terms of QL implementation, stating that she used complete lesson plans and lesson plan ideas directly from the course. This might help inform design of an effective teacher education course in QL. The role of a specific course in QL rather than increased requirements in mathematics is supported by Gutstein’s (2006) plea that teachers need to learn how to read and write the world with mathematics in addition to possessing both disciplinary mathematics content knowledge and pedagogical content knowledge.

Although the course provided the teacher who participated in this study with the resources to implement QL instruction in her classroom, she did not conduct QL assessments. When asked about this in a follow-up interview, the teacher indicated that she was not sure how to assess this knowledge and did not feel she needed to assess QL since it was not a required standard. Lack of assessment in QL is noted by Pugalee, Hartman, and Forrester (2008), who state, "Despite the importance of such skills, assessment of QL has not been a focus in education and there is a lack of assessment tools that specifically address these skills" (p. 35). There is both a need for research and development in the area of formalized QL assessment and a need for explicit instruction on informal classroom QL assessment in teacher education courses.

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SCAFFOLDING GENERATIVE FEEDBACK WITH TECHNOLOGY IN ONLINE PROFESSIONAL DEVELOPMENT

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This study examined 21 practicing teachers' participation in an online community-based professional development course. This study integrated a web-based assessment environment into the online course. The tool was designed to scaffold a virtual boundary encounter with the Math Forum. Teachers' interactions mediated by the tool and the discussion board (DB) was examined. Results indicate that the web-based tool scaffolded participants' development of feedback in qualitatively different ways than participants developed feedback in the DB. In the DB participants shared and compared ideas when providing feedback and scaffolded by the web-based tool participants challenged colleagues to refine their mathematical explanations.

Keywords: Teacher Education-Inservice/Professional Development, Technology

Introduction

Research and policy demand ambitious goals for mathematics instruction, which include calls for problem-based learning, peer-to-peer argumentation and formative assessment practices (NCTM, 2000). Professional development (PD) is crucial for achieving these goals and supporting teachers' instructional change. Community-based PD, in particular, provides context for the collective development of mathematical and pedagogical content knowledge, opportunities for teachers to reimagine their instructional practice, and has shown to support teachers' instructional change (Vescio, Ross, & Adams, 2008). There are shortcomings to school-based communities; for example, it is often difficult for teachers to fit community into their daily schedule and local school districts' norms for instruction typically do not align with those called for by research and policy. Online communities can address these shortcomings as teachers are not constrained to collaborate while at work and norms that emerge in alternative contexts are transferable into teachers' instruction (Vescio et al., 2008). There is a lack of research on teachers' participation in online communities. To fill this gap, the following research question was posed: How does a web-based assessment environment impact teachers' feedback practices in online community-based PD?

This study is framed by the conceptual framework communities of practice (Wenger, 1998) and conceptualizes mathematics teacher PD as participation in boundary encounters, which is an interaction between communities of practicing teachers and teacher educators (Sztajn, Wilson, Edgington, & Myers, 2014). Sztajn et al. (2014) regard this as a boundary encounter because teachers and teacher educators engage qualitatively different practices, particularly when engaging with students' ideas. This study extends the idea of mathematics teacher PD as a boundary encounter and conceptualizes of a *virtual boundary encounter,* which is a boundary encounter mediated by a web-based assessment environment designed to scaffold activities that are consistent with the practices of the Math Forum — a leading group for mathematics education in the United States (Shumar, 2009) — and enhance the process of developing feedback. The tool supports user's focus on the details of written mathematics work and grounding analysis within these details. In particular, the web-based environment includes design features that scaffold highlighting or "quoting" the details of colleagues' written mathematics work and then developing comments that are linked to these highlighted details. After participants engage these activities they share their highlights and corresponding comments with colleagues in the form of feedback. In this way, the assessment environment mediates participants' engagement in collective mathematical activity. These activities are also consistent with the Math Forum's evidence-based feedback practices. Thus, the tool is designed to act as a proxy for participation in PD with the Math Forum.

Methods

This study investigates 21 practicing teachers' participation in an online community-based PD course. The content-based course engages teachers in reasoning covariationally (e.g. see (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002)) about quantitative scenarios. The assessment environment was integrated into this course to mediate participants' interactions around their mathematics work. Participants also communicated about mathematics in a non-scaffolded environment, the course discussion board (DB). The analysis in this study examined teachers' interactions in these two environments. A grounded theory approach was applied through open and axial coding procedures and the iterative analysis of themes (Glaser & Strauss, 1999).

Findings

The analysis of participants' interactions on the course DB and EE found that teachers provided feedback to colleagues in qualitatively different ways according to the mediating environment. Mediated by the web-based tool, participants frequently *challenged* one another to refine their mathematical explanations. Mediated by the DB, participants *shared and compared* their thinking when providing feedback. The following analysis will briefly examine an example of feedback developed in each environment and then will discuss the significance of these results.

The examples are in the context of the class's examination of quantitative relationships. Our ongoing work indicates that the class was participating in an emerging social norm for mathematical activity in which we refer to as *explaining why,* that is it was becoming normative amongst the class to provide reasons for why functions graphs look a particular way (Matranga, 2017). The first example is a discussion on the DB, where a participant, Hank, developed an initial post where he argued that the graph that resulted from tracking the covariation between two quantities could not be categorized as a parabola. In doing so, Hank referenced the "geometric" definition of a parabola in order to explain why the function graph could not be regarded as a parabola. In response to this initial post, Ava and Cindy develop posts that primarily share and compare their thinking. Consider the two posts:

> Ava to Hank: I have never heard the geometric definition of a parabola... or maybe I have and its been a long time... anyway, thanks for sharing as it made me think about parabolas and gave me a deeper understanding!
>
> Cindy to Hank: Hank, I thought the way you explained and reasoned your answers was very clear and precise. Also in my response, I said I wasn't sure if it was correct to say that the rate of change was constant as the car moved. You did an excellent job explaining why this is. And what an interesting way to think about the graph!

In this example, Ava appears to have compared her thinking to Hank's when she noted that she never heard of the particular definition of a parabola Hank referenced. In this sense, Ava commented on the comparison of the extent of their mathematical knowledge. Cindy's post was similar as she shared information when she seemed to evaluate Hank's idea ("your explanation was clear and precise") as well as when she appeared to praise Hank's explanation ("You did an excellent job explaining"). Cindy also compared the correctness of her thinking with Hank's when she said, "in my response I wasn't sure if this was correct…" Taken together, while Hank initiated a thread that introduced and argument for why a parabola was insufficient for making sense of the quantitative scenario, Ava and Cindy *shared and compared* information as they praised Hank's explanation, made evaluative comments, compared the correctness of their responses, and compared the extent of their mathematical knowledge.

The next example illustrates the way in which participants developed feedback to their colleagues mediated by the web-based assessment environment. The example is an occasion where a participant does not explain why a function graph looks a particular way. The feedback is interpreted as an occasion of challenge as Paul pushes Nina to explain why. The feedback has a particular structure that is a result of the scaffolding of the web-based tool. In particular, the structure includes a particular detail of Nina's work and a comment that is connected to that particular detail. Consider the interaction where Nina attempted to explain why the sine function has a particular look:

Paul's selection from Nina's work: You wrote: This graph appears as it does because of the Unit Circle. Essentially as the values of sin(x) make their way around the circle, they start again at zero.

Paul's comment to Nina: ...and I wonder... if you could elaborate on this concept more. Why do the values start again at zero? Why does the graph have hills and valleys?

In the above example, Paul highlighted a particular aspect of Nina's work that does not explain why the graph of y=sin(x) has a particular look. While Nina noted, "the graph appears as it does *because*..." her reason lacks detail as well as specificity. In Paul's comment to Nina, he pushed Nina to explain why: "Why do the values start again at zero?" "Why does the graph have hills and valleys?" The first 'why question' asked Nina for more detail regarding something she explicitly said in her solution. The second 'why question' explicitly pushed Nina to explain why the graph has particular visual features ("hills and valleys"). Thus, it appears that Paul was challenging Nina to refine her mathematical explanation.

The above examples illustrate the difference in feedback according to the environment in which it was generated. Mediated by the DB, participants *shared and compared* information where their feedback did not contribute to the mathematics being discussed. Scaffolded by the web-based tool, participants developed feedback that challenged colleagues and was generative in regard to pushing colleagues to expand and refine their mathematical explanations. This finding indicates that the web-based assessment environment likely contributed to participants' engagement in more generative feedback practices. In particular, the web-based tool scaffolded the structure of participants' feedback, that is the way in which the tool scaffolded highlighting aspects of colleagues' work and providing comments linked to that highlighted detail. This structure was illustrated in the example above as the feedback essentially included a "quote" (e.g. Paul's selection) and a comment linked to that quote (e.g. Paul's comment). Thus, it appears that scaffolding the examination of colleagues' work by highlighting details and making comments on those details contributed to participants beginning to challenge their colleagues to refine their mathematical explanations.

Discussion

The above analysis showed that the web-based tool likely contributed to teachers' development of more generative feedback. Research indicates that in online asynchronous discussions, teachers overwhelmingly share and compare their ideas Yücel & Usluel, 2016; Zhang et al., 2017). Gunawardena, Lowe, and Anderson (1997) developed a framework for examining the social construction of knowledge in online discussions and argued that questioning and challenging is a more generative form of interaction than sharing and comparing information. This suggests that the web-based tool scaffolded participants' development of more generative feedback practices that broke down typical norms for teachers' interactions in online discussions. Teachers' engagement in more generative feedback has implications for the emergence of generative and productive online communities of teachers as well as the potential for such feedback practices to transfer into teachers' instruction.

Conclusion

This study conceptualized mathematics teacher PD as a virtual boundary encounter. At the core of the notion of a virtual boundary encounter is the design of technology that can scaffold activity consistent with a community's practices. The results of this study show promise for the potential of this conceptualization of mathematics teacher PD as participants used the tool for its designed use (e.g. they highlighted aspects of colleagues' work and made comments on these highlighted details) and this activity contributed to their development of more generative feedback. In this sense, it was teachers engaged in practice mediated by technology that impacted their feedback. This indicates the potential scalability of this design for PD as technology begins to expand the outreach of teacher educators through scaffolding participation in community's of teacher educator's practice. Future research is needed to further investigate the role of the web-based assessment environment in teachers' development of feedback.

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SECONDARY MATHEMATICS/ SCIENCE TEACHERS' CHALLENGES IN DESIGNING COGNITIVELY DEMANDING TASKS

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In this study, we examined the design of mathematical tasks at different levels of cognitive demand. There are four levels of cognitive demand by Stein et al., (2000) memorization, procedures without connections, procedures with connections, and doing mathematics. 55 secondary mathematics and science teachers were asked to design tasks at each level of cognitive demand and 13 were interviewed. We found that teachers had issues designing tasks at the higher levels. We also found the challenges of designing such tasks: teacher related challenges, students' related challenges, and selecting/modifying vs. designing tasks.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Inservice/Professional Development, Teacher knowledge

Objectives

Smith and Stein (1998) have argued that a selecting and creating good tasks are important to keep students engaged in the task and develop their problem-solving abilities. In this study, we examined the design of mathematical tasks at different levels of cognitive demand. We were interested in the teachers' creation of their own mathematical tasks to analyze their understanding of cognitively demanding tasks (CDT). This study addresses the following research questions: (a) How do secondary math and science teachers design tasks at different levels of cognitive demand?; (b) What are the main challenges secondary math and science teachers experience while designing tasks at different levels of cognitive demand?

Theoretical Framework

We used the construct of cognitive demand by Stein, Smith, Henningsen, and Silver (2000) to frame this study. Stein et al. (2000) have defined cognitive demand regarding mathematical tasks as "the kind and level of thinking required of students to successfully engage with and solve the task" (p. 11). There are four levels of cognitive demand by Stein et al., (2000) memorization, procedures without connections, procedures with connections, and doing mathematics. In this framework, the first two are considered a low-level and the last two are considered high-level of cognitive demands. Level 1 (memorization) is about reproducing formulas, rules, or facts without a procedure. For example, remembering a formula or a definition. Level 2 tasks (procedures without connection) are algorithmic such as substituting values in a formula. There are no connections to other concepts and does not require an explanation. Level 3 tasks (procedures with connections) require students to have a deeper understanding of the procedures to solve the task. Finally, the last and highest level of cognitive demand is level 4 (doing mathematics). This level is non-algorithmic and requires students to go beyond a procedure by exploring the concepts.

Modes of inquiry

Participants

This study was conducted as part of a larger project that took place during three years (2013-2015). It was a series of professional development workshops from the state-funded Teacher Quality grant. The workshops took place in a university located on the U.S.-Mexico border. The secondary school mathematics and science teachers met for three hours every other week as part of a professional development workshop. N=55 teachers from all the main districts in the region participated in the study.

Data Sources

First, teachers discussed in teams the meaning of cognitive demand, then they learned about the four levels of cognitive demand and had a group discussion about the topic. After that, teachers were asked to fill out a survey titled the cognitive demand survey. One part of the survey was aimed to measure whether teachers could design tasks at different levels of cognitive demand on topics relevant to secondary school mathematics and science curriculum. Individual interviews with a purposefully selected sub-sample of 13 teacher-participants were conducted. As part of the activities of the teacher quality grant, teachers had to develop a lesson to be presented to their peers. This activity was observed to assess the level of cognitive demand presented during their classroom. Similarly, as in the lesson during the workshop, teachers were observed during one class session to examine the level of the task presented in practice.

We conducted one semi-structured interview with each participant. We selected the participants for interviews based on the following criteria: a) teacher(s) who know different levels of cognitive demand but cannot implement CDT neither in the workshop nor in the classroom setting; b) teacher(s) who know the levels and can apply CDT in the workshop but not in the classroom setting; and c) teacher(s) who know the levels and can apply CDT in both the workshop and classroom settings. The interviews lasted in average approximately 35 minutes.

Data Analysis

The part of the cognitive demand survey about designing tasks at different levels was analyzed and graded by the researchers based on the following criteria: a score of 1 was assigned if the task was not designed at the desirable level or a teacher didn't provide any task, a score of 2 - if the designed task was partially correct, and a score of 3 - if the task was designed at the required level. Teachers were not provided with the total score on the survey in order to not impact their interview responses. The semi-structured interviews were coded to look for instances in which the teachers talked about the design of mathematical tasks. Emergent themes were extracted using linguistic analysis and meaning coding techniques (Kvale & Brinkmann, 2009) and placed into the following categories: teacher related challenges, students' related challenges, and selecting/ modifying vs. designing. Several meetings were held between the two researchers in order to reach consensus on all the codes from the interviews. All the names that appear in this study have been changed to pseudonyms.

Results

Table 1 shows the percentages of teachers rated on the tasks they designed at each level. As we can see from the table, the majority of the teachers were able to design tasks at level 1 and 2 but had issues with designing tasks at levels 3 and 4 with less than a quarter being able to design tasks at the high levels. In the case of the level, 3 task more tasks were rated as a 2 (47.3%) while for level 4 more tasks were rated as a 1 (51%).

In the designing survey, participants were given a topic, and then they had to design their tasks at each of the levels. Also, they were also required to provide a solution to the task they developed as well as an explanation of why they think that task is at that level. 83.6% of participants designed a task at level 1. Most of their tasks were about remembering a formula. Participants wrote as an explanation, "because it can be recalled without meaning or understanding" or "student doesn't need to apply any solution." Those who did not design a level 1 task created a level 2 task because it was more procedural than memorization. The majority of participants (94.5%) were able to design a task at level 2. One participant designed a task with a proportion and one missing value and then wrote:

"must find x using cross multiplication, then division, very procedural no connection." The few participants that did not design a level 2 task designed a task that was unfinished or left that part blank.

Less than a quarter of the participants (23.6%) correctly created a task at level 3. One participant created the following task "what is the maximum area of a rectangle if the perimeter is 20" and explained that it was a level 3 task because "it requires to use prior knowledge of area and perimeter." The following task was rated a level 2 task when the participant intended to write a level 3 task. "Find the amount needed to double or triple a recipe." Less than third of the participants (27.2%) were able to design a task at level 4 correctly. There were a few participants that created a level 2 or level 3 tasks instead of a level 4 task since the raters found them as more procedural. One participant that successfully designed a level 4 task on the topic of "Area of a triangle" created the following task "derive the formula for area of a triangle" and said, "they are doing mathematics because there must be deep understanding of concepts."

In the interviews, teachers stated that there are some challenges when designing tasks that are cognitively demanding. Through interview analysis, we identified the following emerging themes that addressed difficulties expressed by teachers: teacher related challenges, student related challenges, and selecting/modifying vs. designing tasks. The first emerging theme from the interviews was teacher related challenges because some teachers mentioned having issues understanding the higher levels of cognitive demand. Derek had challenges at designing a task at level 4 in the survey. When we talked to him during the interview, he confirmed this by saying

> Derek: the first three are probably really confident just the last one maybe not so much. I remember I was like yeah I can do the first three easy and ten the fourth one I took some more time to think about it.

Level 4 was challenging for him and took some time to create a task at that level. Another teacher expressed confusion between the levels because

> Monica: They're sort of, I think they are sort of confusing, like it could be a grey area with them it's not just ok this is level three, and this is level four, I think they can kind of, could overlap, so that's why I think that's where the confusion is.

Another emergent theme was addressing teachers' concern that the students are not ready to solve cognitively demanding tasks. Teachers feel that if they design a task that is at very high-level students will not be able to understand the task and therefore to solve it. Level 4 tasks have rigor and require students to discover mathematics and science concepts. Some teachers feel insecure engaging students in level 4 tasks claiming that students need guidance in working with this kind of tasks. For example, Anna said, said: "if students struggle with the concept a little bit as a teacher I feel like I want to give them all the tools they need and all the guidance instead of just kind of letting them work solving." Another teacher mentioned the level and language of the students and how she doesn't believe that they will be ready for a task that is too cognitively demanding,

> Isabel: Because my kids, if I bring an activity that is at a very high level… if you show a very difficult problem to a kid that is performing at a third-grade level and that English is not their primary language they probably won't do anything.

During the interviews and the workshops, the majority of the teachers expressed that they would rather select tasks that they want to implement in the classroom and then modify them instead of designing it. Cesar said in an interview, Cesar: I'm not that creative so coming up with my own stuff like I said most of the lesson that I have I borrowed from other teachers and modified, I am good at modifying things but actually creating my own, unique, individual lessons and problems, not as good at that.

Teachers said that they either take the tasks from the internet or other teachers and then they modify them to fit their classroom.

Table 1: Distribution of Teacher Responses on the Task Design Survey

Levels	L-1	L-2	L-3
Not correct	7.3%	3.6%	29.1%
Partially correct	9.1%	1.9%	47.3%
Correct	83.6%	94.5%	23.6%

Scholarly Significance

The significance of the study is twofold: from the scholarly lens, it attempts to address the issue of task design through the construct of teacher challenges, and from a practical perspective, it leads to understanding the issue of equal learning opportunities through cognitively demanding teacher preparation and professional development. More specifically, we engaged teachers in learning the four levels of cognitive demand and then designing their mathematical tasks at each level. Designing mathematical tasks is not an easy assignment for teachers. Most teachers had issues designing tasks at the higher levels of cognitive demand. Many created tasks that were at a procedural level instead of level 3 or 4. It is important to examine teacher's understanding of cognitive demand so we can better understand what kind of tasks they consider as cognitively demanding. Some teachers might think that a task requires more cognitive demand than what it requires. We have seen that there are different challenges for teachers to design mathematical tasks thus investigating these challenges further can help researchers understand the implementation of cognitively demanding tasks in mathematics and science classroom.

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SUPPORTS FOUND BENEFICIAL AND CHALLENGES FACED BY ADJUNCT INSTRUCTORS WHEN IMPLEMENTING A RESEARCH BASED CURRICULUM

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This report shares the initial results from research of a model designed to support Precalculus adjunct instructors. The model is based on the organization and coordination of Precalculus along with sustained professional development in the form of a professional learning community. The initial results for instructional practices and job satisfaction show that instructors found access to a course-coordinator and participation in a PLC beneficial.

Keywords: Post-Secondary Mathematics, Teacher Education-Inservice/Professional Development

Objectives

Research suggests that students' persistence in pursuing STEM degrees is influenced by their experiences in their first year mathematics courses (Hutcheson, Pampaka, & Williams 2011; Pampaka, Williams, Hutcheson, Davis, & Wake, 2012). In this regard, the quality of pedagogy can make a big difference in the retention of STEM students beyond beginning mathematics. Moreover, there is a growing trend that higher education institutions are increasingly employing part-time, non-tenure track faculty, such as adjunct instructors (Mason, 2009; Curtis, 2014). Between 2001 and 2011, the number of part-time faculty employed in degree-granting institutions increased by 35% (Snyder & Dillow, 2015). This trend points to a need to better understanding how institutional policies and practices, including the availability of professional development (PD), can improve part-time instructors' experiences (Kezar & Sam, 2013).

While there is a growing body of research aimed at providing PD for graduate teaching assistants (DeLong & Winter, 2001), much needs to be done with respect to the growing population of adjunct instructors (Austin & Sorcinelli, 2013). The current study focuses on the adjunct instructor population because of our own institution's increased reliance on this group, especially in our introductory mathematics courses. We are currently developing and refining a model to help adjunct instructors implement best practices for learning and instruction through professional development and course coordination. This model incorporates instructor supports backed by research and provides course coordination of Precalculus. Our course coordination includes a course coordinator, Precalculus tutors, a common pacing guide, syllabus, assessments and rubrics for all instructors. In addition, we provide a summer workshop for new Precalculus adjunct faculty and tutors, during which participants receive a comprehensive training on the adapted curriculum. This workshop is part of a larger PD effort that continues throughout the semester through online weekly meetings led by a full-time faculty member or graduate student. These meetings form the foundation of a professional learning community (PLC) focused on providing content and instructional support. Adjunct faculty highly regard such supports as they help improve teaching and integrate adjuncts into their institutions (Lyons, 2007; Bowers, 2013).

In this brief research report, we present some of the preliminary findings of our study where we investigate how the adjunct instructors view challenges and supports for this course.

Background

Our study is guided by research on policies and practices for supporting adjunct instructors. All faculty members are important to an institution and must be supported to fulfill their academic responsibilities (Gappa, Austin, & Trice, 2007). Training and development programs for part-time faculty are important because proper training and support can ensure that instructors improve their practice (Leslie & Gappa, 2002). Part-time instructors should feel involved and appreciated within the institutional community. The schedules of part-time instructors often do not allow them to interact with regular staff because of specific class schedules, classrooms in different buildings, and lack of proper office space within the department. In addition, a faculty member who is on campus to teach just one or two courses may have different professional growth interests than a full-time, tenure-track faculty member (Gappa et al., 2007).

To respond to the diverse needs of faculty members, many institutions have developed innovative approaches to PD. For example, some universities and colleges, in recognition of the time constraints, are providing online and in-person PD opportunities such as series of workshop sessions, one day retreats, and online programs (University of Central Florida); late-afternoon and early evening sessions including a light meal, a stipend and a certificate of completion (University of Louisville); or an orientation with mentoring and online PD (Ivy Tech Community College (ITCC)) (Lyons, 2007).

There is no single model that can fit the needs of all the institutions therefore institutions need to develop specific programs that cater to their own needs (Austin & Sorcinelli, 2013). New approaches to adjunct PD need to be established to flourish within organizations (Austin & Sorcinelli, 2013). One way of providing support and development to the faculty is a focus on collaboration both inside and outside the institution (Austin & Sorcinelli, 2013). Collaboration in the form of Professional Learning Communities (PLCs) has been proven to be a beneficial model for K-12 teachers, as teaching has been shown to be more effective, and student achievement improves, when teachers develop strong PLCs in their schools (Fulton & Britton, 2011). In an overview of the research on PLCs in higher education, Roth (2014) found that engagement from faculty in these communities also led to more effective teaching and improved student learning. In fact, all faculty members, regardless of their appointment type, can benefit from participating in PLCs (Gappa et al., 2007).

Methods

This brief research report presents the findings using a subset of data from a larger study. Our initiative aims to measure the impact of course coordination and support on adjunct mathematics instructors' knowledge, instructional practices, and job satisfaction. The transcription data from the interviews is being analyzing using thematic analysis. Through this analysis, we seek to discover emerging themes focused on changes in their class room practices, beliefs about teaching and learning mathematics, expectations of students, and persistent challenges of their role as adjunct instructors.

The study is taking place at a Ph.D. granting public institution in the northeastern United States. The current participants are 8 adjunct instructors implementing a research-based Precalculus curriculum. The instructors were interviewed in person at the beginning and end of the semester, and all the participating adjunct instructors had previously taught one or two Precalculus classes. The interview data has been initially analyzed and coded for emerging themes. This data will support results from our larger data sources including classroom observations, content assessment and belief survey to provide us with additional evidence of the impact of the course coordination and adjunct support. Our specific research questions for this report are as follows:

1. What challenges do adjunct instructors face when implementing a research-based Precalculus curriculum?
2. What supports do the adjunct instructors find more beneficial when implementing a researched-based curriculum?

Results

Based on our initial analysis, various themes emerged about the ways in which our adjuncts' instructional practices changed due to the challenges they faced as well as the supports they found beneficial. We hypothesize that these supports may lead to increased job satisfaction.

The reported challenges mostly focused on factors necessary to implement the curriculum well and the supports adjunct instructors would need in order to overcome these challenges. The instructors identified what specific supports were more beneficial to them, and reported back to us on how those supports and resources could be improved. In addition, the instructors' initial experiences with the curriculum allowed them to develop a plan of what changes they would need to make to their own instruction, pacing schedule, and other course planning.

Challenges

The first challenge stemmed from instructors' prior experience teaching Precalculus as either an adjunct instructor or high school teacher. Most of the adjunct faculty had several years of experience teaching Precalculus at the college level, with the exception of one instructor who was teaching the course for the first time. Some instructors were concerned about the fast pacing that covered the broad range of content in the new curriculum. They were afraid that as a result, their students may not be well-prepared to take the first Calculus course the following semester. The main reason behind such concern was the fact that the content they had covered when they had taught this course previously focused primarily on procedural techniques rather than conceptual understanding of the topics.

In addition, some instructors were not comfortable planning lessons and implementing the curriculum due to its novelty. Based on their past experiences, they had built a repertoire of techniques in terms of planning and delivering their lessons in class, and therefore had felt more confident in providing examples to the students because of this familiarity with the content. Many instructors were also concerned about bringing students onboard with the new curriculum, which focuses on conceptual learning and understanding. Based on their responses, it was clear that adjunct instructors had already started experiencing challenges in involving students in working through specific investigation modules throughout the course. This experience proved to be a source of dissatisfaction for the teachers both in the classroom and while planning for their lessons. They also found it challenging to communicate with students about the reasons the new curriculum required greater class participation.

Finally, structural issues like classroom size, set-up, traditional furniture and lecture style halls proved to be a challenge to implementing a new research-oriented Precalculus curriculum focusing on discourse.

Supports

Our second research question specifically talked about the supports that our adjunct instructors found beneficial when implementing the curriculum. While not surprising, we found it interesting that none of our adjunct instructors had ever received any form of mentoring as an adjunct instructor. While they may have received mentoring as a student teacher or during their first year teaching at the K-12 level, they had never been involved in any form of mentorship or professional development at the college level. They appreciated the online PLC meetings and having access to a course coordinator. The PLC meetings gave them a platform to learn about the ways in which the students should be guided through the investigations as well as a place to share their concerns and ask their questions. Similarly, having access to a course coordinator allowed them to share their concerns and get some direction. They found the common exams and the pacing guide that were designed by the course coordinator most beneficial. The exams and the pacing guide, as well as having a course coordinator as a go-to person opened up time for other activities like designing projects for students, developing online videos etc.

Our adjunct instructors also collaborated with each other in smaller informal groups and continued to engage in these collaborations along with the online PLC. The nature of these informal meetings was impacted by the curriculum as well as the discussions during the formal PLC meetings. The informal PLC conversations focused on content and pedagogy instead of logistical issues. This focus was a result of the logistical concerns being taken care of by the course coordinator.

Conclusions

The results presented here are part of a larger research effort to measure the impact of course coordination and support on adjunct mathematics instructors' knowledge, instructional practices, and job satisfaction. Our findings from the analysis of the initial interview data with the instructors focus on their instructional practices and job satisfaction. Moving forward we plan to extend our analysis to delve deeper into the ways in which supports, like weekly online meetings and access to a course coordinator, impacts our instructors. We are also interested in investigating how instructors' content knowledge might be impacted by these efforts. Our hope is that these findings will make a significant contribution towards the scant literature on supporting mathematics adjunct instructors.

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TEACHER LEARNING THROUGH PERFECTING A LESSON THROUGH CHINESE LESSON STUDY

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This study reports on learning through Chinese lesson study (LS) by three elementary school teachers in the US. Three cycles of LS were facilitated by knowledgeable others to make sense of comparison subtraction through problem solving. Data consisted of lesson plans, videotaped research lessons, post-lesson debriefings, and reflection reports. The lesson improved regarding: using number talks to activate relevant knowledge; smooth transfer from acting out to using manipulatives to symbolizing the comparison model; and student outcomes. Teachers voiced they grew professionally and gained knowledge about content, pedagogy and student learning.

Keywords: Number Concepts and Operation, Learning Trajectories, Problem Solving

Introduction

Implementing common core mathematics standards in classrooms has been a nationwide effort since 2010 (CCSSI, 2010). *Principles to Actions* (NCTM, 2014) describes eight practices for research-based, effective teaching. However, it is still a daunting task for many teachers to teach in this way. Lesson study (LS) can be effective in helping them implement CSSM (Lewis & Takahashi, 2013) and improve their learning and students' learning outcomes (Huang & Shimizu, 2016). A team of mathematics teacher educators and mathematics specialists has developed and implemented a hybrid model of professional development (PD) for K-6 teachers consisting of an extensive summer institute, demonstration lessons and coaching or LS. Twelve of 48 participants in a PD project voluntarily participated in LS. Four LS groups completed the cycle of LS: designing a lesson, teaching/observing the lesson, debriefing, re-teaching the lesson, and re-debriefing and reflection. This paper reports on one of the LS groups which focused on comparison subtraction at grade 1 to address the research questions: (1) How did the research lesson improve through the LS process? (2) What did teachers perceive of and learn from LS?

Literature and Theoretical Framework

Chinese Lesson Study

Lesson Study, a practice-focused, collaborative, student-focused PD model, has been recognized as powerful for improving teaching and promoting student learning (Huang & Shimizu, 2016; Lewis, 2016). Similar to Japanese LS structurally (Lewis, 2016), Chinese LS emphasizes repeated teaching of the research lesson and knowledgeable others' input (Huang & Han, 2015). In the context of Chinese LS, Han and Paine (2010) found that improving teaching of mathematics as deliberate practice gave teachers an opportunity to refine their core instructional practice. Thus, Chinese LS can provide a way to improve instruction and professional expertise.

Teaching Mathematics Based on Learning Trajectory and Variation Pedagogy

Two theoretical frameworks guided the design of and reflection about lessons throughout the LS: learning trajectory (Clements & Sarama, 2004) and variation pedagogy (Gu, Huang, & Gu, 2017).

Learning trajectories (LT) have been proposed as a foundation for classroom instruction (Clements & Sarama, 2004; Simon, 1995). A hypothetical LT is a pathway on which students might proceed as they advance learning toward the intended goals, which describes children's thinking and learning in a specific mathematical domain and a conjectured route through a set of instructional tasks (Clements & Sarama, 2004). Variation pedagogy (VP) arises from the Chinese mathematical teaching tradition, and focuses on using deliberate and systematic variation in mathematics tasks to help students develop new concepts and problem solving abilities (Gu et al., 2017). The key idea is to emphasize the importance of constructing patterns of variation and invariance to create necessary conditions for mathematics learning.

Teaching and Learning of Subtraction with Whole Numbers

According to CCSSM 1.OA. A.1 (CCSSI, 2010), first grade students should learn to use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart and comparing by using objects, drawings, and equations. Three models for interpreting subtraction are: taking away, missing addend, and comparison (Sowder, Sowder, & Nickerson, 2014; Van de Walle, Karp, & Bay-Williams, 2016). The first two are based on the part-part-whole model which can be presented visually using a part-part-whole mat (Van de Walle et al., 2016). The comparison model involves the difference between two distinct sets or quantities, and can be represented visually by counters or cubes. However, "it is not immediately clear to students how to associate either the addition or subtraction operation with a comparison situation" (Van de Walle et al., 2016, p. 177). The dominating part-part-whole model may negatively impact students' understanding of the comparison model of subtraction. Researchers (Van de Walle et al., 2016) have suggested that when discussing the difference between bars, asking "how many more" may help students to generate the subtraction equation. Research further suggests that using physical actions to match concrete manipulatives in a one-to-one fashion leads to visualized objects being matched in the same manner, and finally to the development of number sentences using mathematical symbols (Zhou & Lin, 2001). The studies reviewed suggest that teaching the comparison model of subtraction should follow a hypothetical LT with five levels: (1) physically acting out comparison subtraction (how many more or how many fewer) within a daily context; (2) using manipulatives (counters or cubes) to model comparison subtraction concretely; (3) drawing diagrams to represent the model visually; (4) creating mathematical equations; and (5) varying problems regarding different unknowns (i.e., difference, larger or small quantities).

Method

Lesson Study Group

The LS took place in a school system in a mid-size city in a southeastern state in the USA. An expert team of three mathematics educators, two from the mathematics department and one from the college of education at a large public university in the city, and one mathematics specialist from the school system oversaw all LS activities. This included facilitating lesson planning and post-lesson debriefing meetings and commenting on lesson plans. The LS group included three first grade teachers from three schools, similar in terms of social-economic status and academic achievement. Table 1 contains information on the teachers.

Table 1: Background Information of the Participating Teachers in the Lesson Study

Name	Gender	Highest Degree	Teaching experience	Licensure
Ms. Luna	F	Master of Arts – Reading Education	6-10	K-6
Ms. Schultz	F	Bachelor of Science – Early Childhood Education	0-5	K-3
Mr. Murphy	M	Master of Education	>20	K-4

Data Collection and Analysis

The LS included one lesson planning meeting, three research lessons on the same topic, and three post-lesson debriefings. After completion of the LS process, teachers submitted a reflection essay guided by several questions. The three lesson plans, videotaped research lessons (including student work) and debriefing meetings, and reflection essays constitute the data set for this study.

Based on the framework of LT and VP, the three research lessons (videotaped) were analyzed with a focus on whether the LT was reflected and how the tasks were aligned with VP. The effect of each lesson was evaluated based on classroom observation and examination of exit tickets. Debriefing meetings were analyzed to reveal factors leading to changes in the lessons. Reflection essays were analyzed using NVivo to ascertain teachers' perceptions of the LS process and what they learned. Results are presented in alignment with research questions.

Results

The Major Changes across the Research Lessons and Key Causal Factors

Each of the three teachers taught the research lesson once. The goal was to develop students' understanding of comparison subtraction through solving word problems. Immediately after each lesson, there was a debriefing session on strengths and weakness of the lesson and suggestions for changes. Table 2 has major changes from initial to final lesson and associated casual factors.

Table 2: Major Changes between the Initial Lesson and the Final Lesson

Phases	Initial lesson	Final lesson	Changes
Number talk (Activating relevant knowledge, leading to new topic)	11+7=18; 7+?=11 Focus: Part-part-whole and missing addend subtraction	5+2=7; 7-2=5; 5-3=2 Focus: How many more and subtraction equation	Activating relevant knowledge closely related to new topic
Acting out comparison (match up boy and girl)	Matchup physically (9 and 14)	Matchup physically (9 and 11). Modeling with linking cubes. Drawing diagrams to match up. Setting equations.	Transferring from physical to visual to symbolic representations
Modeling and symbolizing (using linking cubes, diagrams and equations)	Task 1: Word problem without a picture (13-8=5). Task 2: Word problem without a picture (7-3=4). Less appropriate tools.	Task 1: Word problem with a picture (7-4=3). Task 2: Word problem without picture (13-8=5). Using linking cubes, diagrams, and equation.	Progression from simple to complex. Used multiple represent tations. Removed part-part-whole mat.
Challenging task	Task 3: If A is more than B and A+B=15, find A and B.	N/A	Removed
Exit Ticket	N/A	Task: Word problem without picture. Request: use diagram number line, equation.	Added
Learning outcomes	15% mastered comparing 13 with 8	63% mastered comparing 15 with 7	

Participant Teachers' Perception of the Lesson Study and Their Perceived Learning

Each teacher expressed enthusiasm for the LS process and product as illustrated by Ms. Schultz, "I was proud of our work on the compare lessons and enjoyed learning and participating in watching these lessons change, evolve, and enhance the learning of the children involved in this project." The teachers perceived their teaching skills and knowledge of content, pedagogy, and student thinking increased. Mr. Murphy reflected, "over the course of the LS we were able to make the necessary changes and see our students grow their mathematical thinking of word problems." They have already and will continue to implement what they learned from the LS in their classrooms. Ms. Schultz said, "I feel far more capable in my abilities to design a lesson trajectory that would better serve my students' learning. I now find myself to be more intentional in observing and reflecting upon my students' thinking to more thoughtfully write and revise my lesson plans so that I may better support the success of their learning." They also indicated the opportunity for discourse on mathematical practices was beneficial to their professional growth.

Discussion and Conclusion

This study demonstrated how the Chinese LS approach inspired by LT and VP can help teachers improve a lesson that promotes students' conceptual understanding of comparison subtraction. Meanwhile, the teachers not only deepened their understanding of the concept of comparison subtraction, but also increased specific pedagogical skills such as selection of appropriate tools, sequencing of tasks, transfer from concrete to abstract, and analysis of student thinking, which has impacted their daily teaching. Each teacher acknowledged both the process (enactment and reflection) and product (final lesson) of the LS as being beneficial. This study suggests that part-part-whole should not be used for exploring comparison subtraction (Van de Walle et al., 2016). It also suggests that beginning with the physical action of matching sets in a one-to-one fashion, then moving to using linking cubes for matching, and finally using diagrams for matching can help students develop equations for comparison subtraction problems (Zhou & Lin, 2001). Since the highest level of variation problem regarding different unknowns was eliminated from the lesson after the first teaching, exploration regarding the scaffolding needed for students to successfully investigate this challenging but mathematically rich task is needed.

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TEACHERS' DEVELOPING QUESTIONING TO SUPPORT LINGUISTICALLY DIVERSE STUDENTS IN JUNIOR HIGH MATHEMATICS CLASSROOMS

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This study examines three junior high school teachers' questioning practices of English learners (ELs). We build on prior research demonstrating the utility of questioning for developing mathematical discourse and apply it to the novel endeavor of having teachers use questioning purposefully as a means to engage ELs in mathematics discourse. Using qualitative methods, this pilot study examines three teachers' questioning practices prior to participating in professional development, how the teachers' practices differed for ELs versus non-ELs, and how the teachers explored expanding their questioning practices during professional development as they prepared to teach a cognitively demanding mathematics lesson.

Keywords: Teacher Education-Inservice/Professional, Equity and Diversity, Middle School Education

Objectives

This study's objective is to examine three junior high mathematics teachers' questioning practices of English learners (ELs). This paper builds on prior research that demonstrates the utility of questioning for developing mathematical discourse and applies it to the novel endeavor of having teachers use questioning purposefully as a means to engage ELs in mathematics discourse, providing potential pathways for ELs to access and engage with mathematics content and language in meaningful ways. In this pilot study, we examine the following research questions: What does questioning look like in teachers' classrooms? How do the teachers' questioning practices differ for ELs versus non-ELs? How do teachers explore expanding questioning in professional development (PD) as they prepare to teach a cognitively demanding mathematics lesson?

Perspectives

We ground our work using four key principles for supporting ELs in mathematics classrooms: providing cognitively demanding mathematics tasks, building on and understanding students' resources, providing language rich opportunities, and understanding students' language demands (Roberts, Bianchini, Lee, Hough, & Carpenter, 2017). We see these as important for working with teachers and for teachers working with their ELs.

With these four key principles, this study focuses on questioning that develops, attends to, and elicits higher-order thinking. Such questioning has myriad functions. Teachers can use their questioning to access student thinking and to gain important knowledge about possible trouble spots or misconceptions their students have with regards to particular content (NCTM, 2014). Ideally, questioning allows teachers to support students to listen carefully and to develop a fruitful exchange of ideas with rich and purposeful mathematics discussions (Imm & Stylianou, 2012). We focus on using high cognitive demand questions to create an environment of inquiry for ELs.

Such questioning requires teachers to move beyond close-ended questioning, where teachers use questions for information gathering only or getting to a desired procedure or conclusion, which allows for little veering from a limited desired path (NCTM, 2014). Instead, this study seeks to press ELs to "communicate their thoughts clearly, and … to reflect on their thoughts and those of their classmates" (NCTM, 2014, p. 37) through teacher questioning.

Methods

This study took place in California at a junior high school with an enrollment of almost 450 students, with approximately 45% of these students classified as ELs, 34% as FEP, and 18% redesignated as FEP. The majority of ELs (>85%) at the school were Spanish-speakers. The teacher participants were three White teachers with 2.5-5 years of teaching experience and included a bilingual Spanish speaker and a bilingual Vietnamese speaker. The teachers reported that 20-60% of their classes were designated as ELs.

The teachers participated in initial hour-long interview that focused on the their pre-participation questioning practices. The research team also videotaped a single lesson of each teacher to capture how they questioned students, in particular ELs, prior to participating in PD. Finally, the research team video- and audio-recorded a two-hour professional development meeting focused on teacher questioning, and collecting written work and reflections.

During the PD, we used a cognitively demanding mathematics task, “Orange Fizz Experiment” (Georgia Department of Education, 2016), as a focal point for organizing teachers’ questioning of ELs in their mathematics classrooms. Following discussions of the four core principles for supporting ELs in mathematics (Roberts et al., 2017) and some examples of questioning from the literature, we provided teachers with an opportunity to solve this mathematics task. Because teachers previously rarely used such tasks and in depth questioning, we provided teachers with time not only to solve the task but also to develop questions to use with the task as they planned their lesson around the task.

We analyzed the teachers’ use of questioning across data sources using open coding (Strauss & Corbin, 1990). In order to identify consistencies and inconsistencies across the research questions, we generated themes within and across each question.

Results and Discussion

RQ#1: What does this questioning look like in teachers’ classrooms?

The first theme that we noted when we looked within each teacher’s classroom and across the teachers’ classrooms was that the teachers tended to use fairly close-ended questioning with their students. This is likely linked to the type of mathematical work that they usually used with their students. In their interviews and in the professional development, the teachers talked about the fact that they used mostly direct instruction with their students and they rarely used tasks. For example, Teacher 1 explained, “I use a mixture of drill and kill….and [I have students] look for a number answer.” Teacher 1 asked such questions as, “What are some quantities? What else are you seeing? Do we know how many some is?” Teacher 2 described using acronyms to help students remember particular mathematical processes, sharing, “I don’t, like, just to tell them. [I ask them,] What is the acronym?...How can we use this?” Teacher 2 focused on students memorizing particular processes, and her questioning attended to reminding students of these procedures. Teacher 3 also focused his questioning on procedural mathematics, telling us, “I will often ask what the next steps are. What do we need to be careful of?.... I ask a lot of times is this positive or negative.” These close-ended questioning strategies in many ways mirror what Cazden (2001) refers to as an *Initiate-Respond-Evaluate (IRE)* sequence, with a teacher providing a question such as, “Is this quantity negative?” A student would respond, “yes” or “no.” And then a teacher would then evaluate the correctness of that response.

Our classroom observations helped to triangulate the teachers’ self-reporting of these IRE sequences. While the teachers did not lecture the whole time, we had seen them do so in previous visits to the school. Teachers 1 and 2, on the day of our observations, used the same activity, a “speed-dating” activity in which students worked in pairs to solve one- and two-step equations.

When the teachers asked questions, the students had to describe the steps they took for solving the equations. Teacher 3 did a different activity in which students had to complete a worksheet where they had to identify how to write an equation. He asked such questions as, "How do we use these numbers to write an equation?" "Do we see $39 in the equation?" "How many friends did he take to the movies?" The teachers were focused on questions that generally elicited one-word answers and demanded lower-level cognitive demand. Teachers were using questioning that was close-ended, and we linked this questioning with the mathematics tasks the teachers were using, which required less in depth questioning.

RQ#2: How Do Teachers' Questioning Practices Differ for ELs Versus non-ELs?

Overall, the three participating teachers commented that their questioning practices were generally the same with their ELs and their non-ELs. Teacher 1 explained, "Yeah. It's not much different." However, there were some instructional practices that the teachers shared that they used when they were describing their questioning work with ELs. Teacher 2 shared in both her interview and the PD that she used sentence starters and often provided them for all students. Teacher 3 described, "I also code-switch and use some words in Spanish," explaining that he gave some mathematics vocabulary in Spanish and English and then used connecting words in Spanish to support students in understanding the new mathematics vocabulary in English. All three teachers used general support strategies for working with ELs and did not generally alter their questioning strategies when working with ELs versus non-ELs.

RQ#3: How Do Teachers Explore Expanding Questioning in Professional Development (PD) as They Prepare to Teach a Mathematics Lesson?

Our answers to the first two research questions helped us to recognize that we needed to provide some foundational experiences for the teachers in their first PD. We decided that providing cognitively demanding learning experiences for ELs would ground the work for our questioning work with the teachers. Without a good task, we found that the teachers had little to ground their questioning. We also knew from prior research that ELs often have less access to cognitively demanding work for myriad reasons. We had a cognitively demanding task serve as a *crossroad* for the teachers and the questioning work we could do in the professional development as they prepared for their future questioning. The teachers solved a task together that came from optional district curricular materials, the "Orange Fizz Experiment" (Georgia Department of Education, 2016). They planned a lesson and created supporting questions to enact during their upcoming unit on ratio and proportion. This problem has students use ratios to figure out which soda formula has the best tasting flavor, comparing three different formulas of orange concentrate and carbonated water.

The teachers all noted that this problem was different than the type of problems they usually used in their day-to-day instruction, because it was not a standard lecture-practice type of lesson. All three teachers noted that the Orange Fizz Experiment problem afforded them the opportunity to ask different types of questions than they might have asked typically. Teacher 3 shared, "I'm going to try to ask higher-level questions of my students." Teacher 2 also explained:

> I plan to work on my questioning in this lesson by creating more open-ended questions that require more depth of knowledge than procedural questions (i.e. asking the students to discuss what we were comparing vs. saying 'we are comparing part to whole, so what's the 'part' and what's the 'whole'?').

Teacher 1 had the biggest change in her planned questioning, because she anticipated having her students explain more of their thinking. Her questions included: "Which formula has the most 'orange' taste? How do you know?" However, she also included procedural questions like, "What rules do we know we can use to work through ratio tables?"

We pushed teachers to consider their questioning specific to ELs. Teacher 3 explained that questioning, “can support ELs by having them use more language that's specific.” Teacher 2 went into more depth and drew on her students’ prior knowledge: “Questioning can support ELs by giving them opportunities to voice their existing knowledge and enabling them to recognize that they have valuable prior knowledge that can be applied to the problem at hand.” Finally, Teacher 1 found questioning with ELs to be an instructional tool that created space for students to share their thinking: “Questioning can support ELs because it starts with the student's thinking. It provides space for students to reflect and connect knowledge through their past or others. Questioning also has students reflect and justify their thinking.” Even though the teachers acknowledged ELs in our prompts, we found that we need to engage them more thoroughly in thinking about how they can use questioning to engage ELs in mathematical discourse.

The teachers noted that they were looking forward to more opportunities to learn about and to develop their fluency with questioning. For example, Teacher 3 explained, “Today I learned what higher level questions are. I've been struggling with understanding what questioning…meant in a very practical way, and today helped me with that.” Teacher 2 was particularly interested in working toward specifically “preserv[ing] mathematical rigor while meeting the needs of our EL students” using questioning. Going forward, the professional development we provide is going to have to meet both teacher and student needs.

Conclusions

Cognitively demanding tasks provided a crossroads in this project on teacher questioning with ELs. Tasks provided an inroad for our work with teachers in helping them to understand and appreciate how to develop questioning practices that support ELs’ discourse practices. We were able to see marked differences in the types of questions teachers were asking and their approaches to questioning through the use of a different type of task. We believe that this work on questioning lays the foundation for future work on developing rich discourse opportunities for ELs in mathematics and creating and studying PD for teachers in executing such practices.

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THE INFLUENCE OF DAILY REFLECTION ON A MIDDLE SCHOOL TEACHER'S PRACTICE

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This report describes a framework used as a communication tool to reflect and adapt instruction to support student learning. Specifically, this framework was used as a bridge for collaboration between the classroom teacher and the researcher to reflect on daily lessons. The goal of this preliminary project was to adapt teaching and tasks to support seventh-grade students' learning of algebraic expressions and equations. The framework was used to document and summarize what occurred during the lesson on a daily basis and to identify learning goals for the subsequent lessons. The reflection process was helpful for the teacher to trace where students had been in their learning and to make decisions about how to foster learning in the following lessons. The framework served useful and made it easier for the teacher to adjust her instructional practices to support student learning in the classroom.

Keywords: Instructional Activities and Practices, Teacher Education-In-service/Professional Development, Middle School Education

Students' learning is influenced by how teachers design and implement lessons and assess student learning to make instructional decisions. A learning trajectories approach to teaching involves thinking about the goals and tasks, anticipating what students might do, and adjusting instruction and future tasks based on what students do and understand (Clements & Sarama, 2009). Research (Constantino & De Lorenzo, 2001; Danielson & McGreal, 2000; Glickman, 2002; Lambert, 2003) confirms the benefit of reflective practice in order to provide professional growth for teachers. Our goal was to understand how daily reflections on teaching algebra influenced the instructional practices of a middle school teacher. In doing so, we developed a reflection framework that documented what happened on a daily basis during the math lessons and what the teacher wanted to occur with respect to anticipated student learning in subsequent lessons. The framework served as an intersection point between the researcher and teacher to influence classroom teaching through a reflective process.

Theoretical Framework

The theoretical framework for this study was based on Danielson's (2009) four modes of educator thinking (Figure 1).

Mode of Educator Thinking	**Thinking based on:**
Formulaic:	prepackaged knowledge from external source
Situational:	decisions made on information gathered during a specific time in a specific context
Deliberate:	seeking more information than the immediate context provides
Dialectical:	deliberate thinking to gain understanding of a situation and generate solutions

Figure 1. A summary of the modes of educator thinking (adapted from Danielson, 2009).

An educator that thinks formulaically is grounded in general policies and rules that are part of the school culture and centers on standardized instructional decisions regarding curriculum. Situational thinking involves dealing with in-the-moment occurrences during teaching such as student dispositions. In deliberate thinking, an educator seeks to understand why or why not a method of instruction is successful. If the educator engages in finding a solution to the scenario during the deliberate thinking stage, then he/she is thinking dialectically.

The framework created and tested in this study was purposely designed for teacher reflection that was both deliberate and dialectical. Dialectical reflection in this case involved a logical discussion between the teacher and researcher; sharing ideas and opinions. One definition of dialectical thinking is to be concerned with or acting through opposing forces. The Reflection Framework sought to identify and understand the opposing forces within the classroom context during lessons over time. The daily reflection questions were developed to lead the educator through the trajectory of formulaic to dialectic reflection (see Figure 2). The framework began with mathematical meaning or the objective of the lesson. The teacher was asked to think about misconceptions generated and what may have caused this within the context of the lesson. Incorporating student comments from the lesson elicited deliberations regarding what it might be like to be a learner in the context. Finally, focusing on next steps provided a dialectical space to generate solutions.

Mathematical Meaning that you wanted to happen	Errors/Misconceptions that occurred for the students	What situation or activity led to this misconception?	Student comments/reflections	What did you change and why? What are you going to do for the next lesson?

Figure 2. The Reflection Framework that was used daily by the teacher.

Context and Methods

The teacher selected for this research was chosen because she had recently graduated with her doctorate in mathematics education, had 22 years of elementary teaching experience, and was in her first year of teaching seventh grade mathematics. The teacher taught at a public middle school in the western United States with an enrollment of 600 students, most from lower-to-middle-class families, and the majority of students were Latino/a or native. The teacher taught three seventh-grade mathematics classes with about 35 students in each class, ranging from 12 to 13 years old. The researcher in this study holds a doctorate in mathematics education, is a former middle school mathematics teacher, and met the teacher during her doctoral studies.

This research project evolved from a discussion between the researcher and the teacher regarding the teacher’s transition from elementary to middle school. The teacher identified that she was struggling with taking time to reflect on her teaching, the students’ understanding, and how to adjust her lessons based on student learning. After this discussion, the researcher and teacher created the Reflection Framework (Figure 2) on a shared Google Doc. Using the framework, the teacher reflected on teaching an algebra unit to her seventh-grade classes everyday for two weeks. Once the teacher wrote her reflection for each part of the framework, the researcher made comments and suggestions about how to adjust the lessons and address student misconceptions. At the end of the two weeks, the researcher and teacher individually answered the following questions prompted by their observations of using the framework:

1. Is the framework useful in general and why?
2. Why is this framework suitable for middle grades?
3. Did the framework help the teacher to shift her instructional practices? If so, in what ways?

To explore how the teacher reflected on her thinking using the Reflection Framework and how she adjusted her teaching based on these reflections, we read and examined the teacher's reflections, the researcher's comments and suggestions, and the answers to the questions for constructs, themes, and patterns. We then coded the themes using a constant comparative method (Strauss, 1987). Following the analysis, we were able to recognize three main themes regarding the benefits of the Reflection Framework.

Results

A lens for whole class learning of content

The process of reflecting on lessons everyday using the Reflection Framework allowed the teacher to focus on the mathematics content that the students' were learning. The teacher commented:

> In 7th grade, I teach roughly 120 students math each day. As a new teacher to this grade, it has been difficult to get a sense of what each student needs. This framework allowed me to analyze the lesson overall for the 7th graders. In accounting for their misconceptions and their understandings, I was able to address the learning of the group as a whole. This is very different from the elementary model where you focus on 30 students all day long and are quickly able to see the whole child as a learner in many different ways.

The teacher also revealed that feedback from a colleague gave her a new and different perspective on why the students might not understand the mathematics. For example, Figure 3 shows the teacher's reflection on students' misconceptions and suggestions and comments from the researcher.

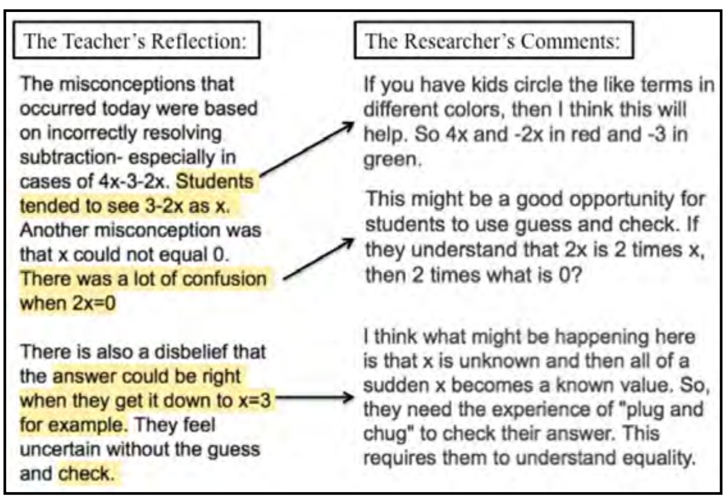


Figure 3. This is an example of the teacher's reflections on student learning and comments made by the researcher to offer suggestions on how the teacher could address these misconceptions.

Shifts in Instruction

The practice of reflecting using the framework made the teacher aware of what she needed to re-emphasize during the next class session. She wrote:

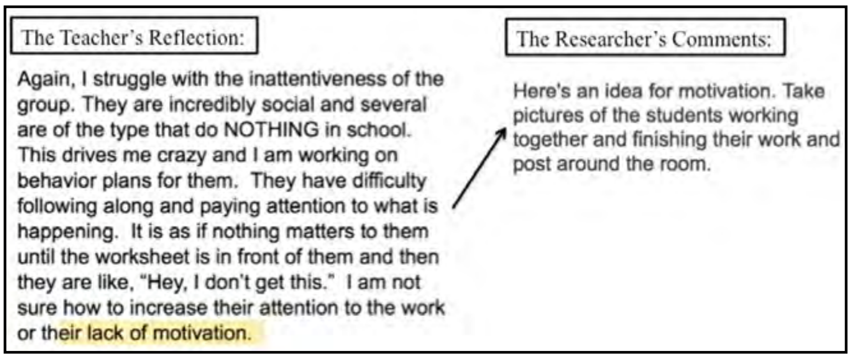
> In third period the algebra manipulatives were the most difficult for me to use. The students have poor self control and my initial perception was that this work was too easy. As I reflected on this each day, I realized that for most of this group, they did NOT understand what was happening.

> This shifted my approach as a teacher. Instead of expecting one assignment finished in one day, I allowed several days for completion continually providing those that were done with extensions.

By allowing extensions, several of the students took time to ask questions to help them make sense of the mathematics, rather than giving up. The teacher realized that she had to allow more time to scaffold her students' understanding.

Focus on Student Dispositions

As the teacher initially reflected using the framework, she focused on student dispositions in the class because she noted that their behaviors were inhibiting the whole class' learning of the mathematics (Figure 4).



The Teacher's Reflection:

Again, I struggle with the inattentiveness of the group. They are incredibly social and several are of the type that do NOTHING in school. This drives me crazy and I am working on behavior plans for them. They have difficulty following along and paying attention to what is happening. It is as if nothing matters to them until the worksheet is in front of them and then they are like, "Hey, I don't get this." I am not sure how to increase their attention to the work or their lack of motivation.

The Researcher's Comments:

Here's an idea for motivation. Take pictures of the students working together and finishing their work and post around the room.

Figure 4. This is an example of the teacher's frustrations with student dispositions and a suggestion made by the researcher on how to motivate the students.

The teacher recognized student dispositions and changed the lesson to make it less boring or more challenging to help her students focus. By taking student dispositions into account over two weeks, the teacher could see as a whole what had happened during the unit in terms of student learning and motivation.

Discussion

In our research, the Reflection Framework was used as a tool for keeping track of student learning and being able to identify the relationship between student learning and instruction to make more targeted teaching decisions. For the teacher and researcher, the framework served as the intersection for making sense of what happened in the classroom during a lesson and how to reflect on the teaching and learning that occurred. The framework should be explored further with inexperienced and experienced teachers to investigate if it supports them to make decisions when practicing a learning trajectories approach to teaching (Clements & Sarama, 2009) and changes their instructional practices to target students' learning needs.

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TRACING TEACHER RESEARCHERS' TALK ABOUT AND USE OF POSITIONING

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In this paper, we examine what teacher-researchers talk about and do as they engage with the idea of positioning in the context of study group discussions and cycles of action research. Using open coding, we analyze study group discussions and other artifacts across a five-year time span and examine how their talk and action changes over time. Broadly, we found that the teacher researchers were, at first surprised and unsure about how to positively influence positioning in their classrooms, then moved to a focus on the positioning of mathematics. As they adopted new curriculum materials, goals, and classroom norms, their talk and action shifted to focus on students' perspectives, voice, and issues of bias in their interactions with students. Such a longitudinal study can provide insights into how ideas like positioning might be used by teachers to work toward more equitable practices in mathematics classrooms.

Keywords: Classroom Discourse, Teacher Education-Inservice/Professional Development

We take it as central that the theoretical constructs we use matter most when we use them with teachers to see what is helpful to improving their practice toward their own ends of supporting student learning. In this way, we align ourselves with the views of crossroads as being an "intersection point" and see our collaborations with teachers as a "place of community" within which we (as mathematics education researchers and teacher educators) must learn. Here we examine the discussions and action research of a group of mathematics teachers the first author has collaborated with for five years to understand how the teacher researchers both talk about and use the idea of "positioning." Positioning refers to the "ways in which people use action and speech to arrange social structures" (Wagner & Herbel-Eisenmann, 2009, p. 2). In mathematics classrooms, words and actions carry implicit and explicit messages about who students are as learners, what they are capable of, and what it means to know/do mathematics. It has been shown that when particular positionings are repeated over and over, they can impact students' identity (Anderson, 2009) and disposition (Gresalfi, 2009) development. The results influence students' perceptions of themselves and others and are important to pay attention to, particularly in collaborative work with teachers. Thus, our goal is to answer the following question: When mathematics teachers talk about positioning across a five-year collaboration involving action research, what do they focus on and how do they report using it to improve their practice and student learning?

Positioning and Its Operationalization for Professional Development

Positioning theory is the "study of local moral orders" based on ongoing shifting patterns of "mutual and contestable rights and obligations of speaking and acting" (Harré & van Langenhove, 1999, p. 1). Important to issues of equity is that positioning theory does not assume that everyone in an interaction has equal access to rights and duties to perform any action (Harré, 2012). Although the theory focuses on local interactions (rather than the transcendental), it also shows the centrality of storylines and the communication acts that are employed in any interaction. Storylines are the ongoing repertoires that are already shared culturally or that can be invented as participants interact. We have described communication acts as the socially determined meaning taken from a communication action, which can be words, gestures, and physical positions and stances (Herbel-Eisenmann, et al., 2015). All three of these constructs--positionings, storylines, and communication acts-- mutually shape and constrain each other during an interaction. This theory has been increasingly used in the past decade of mathematics education research, with most of the articles appearing since 2009 (Herbel-Eisenmann, Meaney, Bishop Pierson, & Heyd-Metzuyanim, 2017). Very little of this work, however, actually involves collaborations with mathematics teachers to see what from the theory might be interesting and useful enough for them to change their practices. Our previous work that investigated theoretical constructs within the context of collaborations with teachers has illuminated how teachers make sense of the ideas and find them useful in their work but also has allowed us to reconceptualize the constructs in ways useful to practice (see Herbel-Eisenmann & Wagner, 2010; Wagner & Herbel-Eisenmann, 2014).

In the context of the collaborative work, we have used the *Mathematics Discourse in Secondary Classrooms* (Herbel-Eisenmann, et al., 2017) professional development (PD) materials to introduce the idea of positioning (and other constructs, which we do not focus on here) as a theoretical lens that can be used to interpret particular teacher discourse moves. In these PD materials, there are a series of "touchstone" readings that are used to formally introduce key concepts and tools of classroom discourse, one of which focuses on positioning. This touchstone document includes a focus on the positioning of people and the positioning of mathematics, which we describe very briefly here. In the positioning of people, teachers' attention is drawn to: (a) interactions between/among students and issues of status (Cohen, 1994), smartness (Featherstone, et al., 2011), and voice are highlighted and (b) interactions between the teacher and students, within which aspects of authority, agency, control, and competence are articulated. The positioning of mathematics highlights how the various activities, tasks, and words we use in relationship to the doing of mathematics shapes what students come to think it means to know/do mathematics. (We recognize that the positioning of mathematics is really about calling into question the storyline of typical school mathematics and not really about positioning. We decided to identify this as a type of positioning so that we did not have to bring in the additional idea of storyline.) Prior to reading the touchstone document, the teachers talked about ideas related to positioning by reflecting on videos, transcripts, and other practice-based artifacts. After they read the touchstone, the idea of positioning becomes a conceptual lens for considering how a range of specific discourse moves might be influencing students' opportunities to learn mathematics.

Context and Methods

The teacher research collaboration currently involves eight mathematics teachers who are working in a culturally, linguistically, and racially diverse school district and the first author of this paper, who works at a university near the district. The main school in which the majority of the teachers teach has about 800 students, across grades 6-8. Six of the eight teachers have been involved in the work for 4-5 years; two just joined the group when they were hired last year to teach 6th grade. Although the teachers have all taught at the middle school at some point in time, currently most of the teachers teach grades 6-8 mathematics and algebra, one teaches high school geometry, and one teaches multiple sections of 4th grade mathematics.

Our work is grounded in critical, sociocultural, and sociolinguistic perspectives, and as such, we see learning as related to how one participates in the discourse practices of a community. Our primary data source includes audio recordings of discussions from the study group meetings, which took place twice a month across the 5 years of the collaboration (approximately 16 sessions each year, one 4-hour meeting during the school day and another 1.5 hour meeting after school). We also examined the artifacts and information the teachers provided about their action research projects throughout the various cycles over the past four years. This included, for example, powerpoint presentations the teacher researchers did at a mathematics education conference, emails and journals they wrote about their action research projects, and a book chapter they co-authored with the first author of this paper that focused on how they use positioning in their teaching and action research.

We began by creating timelines of the work using agendas and field notes from the study group meetings. This information helped us reduce the amount of data by identifying where positioning-related discussions may have taken place. After narrowing the project discussion times, we used open coding (Esterberg, 2002) to code the nature of the focus of the discussions (e.g., whether they focused more on issues of authority or student status) as well as the various types of action the teachers reported taking related to positioning. We started with more recent meetings and worked backwards to see how the ideas appeared in previous years.

Preliminary Findings

Because we were still in the stage of open coding when this report was due, we share here broader scale findings about the changes in foci over time. We describe what they focused on about positioning but also some of the actions they described taking to focus on influencing student positioning. When we do our presentation, we will have more specific and finer grain sized findings to report. Generally, the year 1 discussions of positioning indicate that the teachers had not considered positioning in the ways described in the touchstone documents. They reported being aware of issues of social status related to things like popularity, but that they had not thought as much about this in relationship to *mathematics* learning. Their talk centered on their uncertainty about actions to take to counter positionings that they thought were not supporting student learning.

As they moved into their first cycles of action research in year 2, the talk about positioning focused on the positioning of mathematics. The teachers grounded these discussions about what kinds of tasks and activities they offered to students as well as what they expected students to do (e.g., how they would engage but also expectations for producing high quality explanations and justifications). Their action focused primarily on designing and finding high cognitive demand tasks to use with students. Toward the end of year 2, the talk about the positioning of mathematics shifted toward a slightly different kind of action: they identified the kinds of norms they could put in place, articulated a common set of goals they would work on, and planned for piloting new curriculum materials that would offer richer learning experiences for students (see Busby, et al. (2017) for more information). In year 3, all teachers used the same set of norms and goals and the teachers in grades 6 and 7 started to use the new curriculum materials. The shift in year 3, then, seemed to be away from the positioning of mathematics and more about issues of teacher authority (and the struggles associated with giving up control) and student agency (how they could get students to become more active participants).

By year 4 (2016-17), all of the teacher-researchers began to think about distributing authority more and focused on trying to get students to talk most of the time during whole group discussion. They continued to struggle, however, with the giving up of control and with some of the ways students seemed uncomfortable with being more active learners. Students' being uncomfortable with participating in more active ways, in fact, seemed to be especially acute in the 6^{th} grade where students had come from much more traditionally structured mathematics classes in the elementary schools. Three of the teacher-researchers began to develop instruments they could use to gather information from students about their experience, which included Likert scale items about how students felt about various learning activities and with the mathematics. The items also included information about students' developing agency for their mathematical thinking. Two teacher-researchers also began to give weekly reflections that required students to write about something they had learned from other students in the class discussions, which helped the teacher-researchers understand status in their classrooms. Although year 5 is still underway, the teacher-researchers continue to create structures to support students to have more space and voice in the classroom. One teacher-researcher has students in front of the room and co-facilitating parts of some of the activities. Three others have decided to focus centrally on how bias might be impacting their expectations of various groups of students along gender and racial lines. Thus, some of the shifts in attention to positioning in immediate classroom interactions have raised the prominence of student perspectives as well as how broader systems of privilege and oppression might be impacting the ways teacher-researchers interact with and support student learning.

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WORKING TOGETHER: USING CONSULTATIONS TO IMPROVE MATHEMATICS TEACHING FOR STUDENTS WITH SPECIAL EDUCATION NEEDS

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In this study, we investigate the impact of mathematics elementary school teachers' and special education specialists' participation in professional development regarding how to conduct mathematics-specific consultations to improve mathematics learning opportunities for students with special education needs (SEN). We found that engaging in this professional development increased the participants' confidence in their abilities to (a) engage in future mathematics-specific consultations, (b) engage in consultations in other content areas, and (c) differentiate mathematics instruction for students with SEN.

Keywords: Elementary School Education, Instructional Activities and Practices, Teacher Education-Inservice/Professional Development

Study Purpose and Research Questions

Concern for equitable learning opportunities for students with special education needs (SEN) has resulted in strong commitments for inclusive education from organizations worldwide (e.g., Australian Disability Discrimination Act, 1992; Individuals with Disability Education Act (IDEA), 2004; UNESCO, 2009). Explicating its vision of mathematics teaching and learning, the National Council of Teachers of Mathematics lists *Access and Equity* as the first essential element of excellence and states, "An excellent mathematics program requires that all students have access to a high-quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources need to maximize their learning potential." (2014, p. 59). Unfortunately, access to equitable mathematical learning opportunities is not available to many students with SEN. This lack of access, in turn, impacts their mathematical learning and performance as evidenced in their low performance scores compared to their peers without SEN on national representative assessments (e.g., NCES, 2013).

The need to provide access and equitable learning opportunities for students with SEN requires general education teachers to meet the learning needs of these students. However, it is unrealistic to expect teachers, who must become experts in content areas, to also know how to meet all the widely varying needs of students with one or more special education need. In order to help teachers better manage the demands of meeting needs of each of their students, we conducted a pilot study involving six elementary teachers of mathematics, two special education specialists, and an instructional coach at Hawthorn Elementary School (all names of participants and schools are pseudonyms). The teachers, specialists, and coach were asked to engage in mathematics-specific consultations (described below). The purpose of our study was to investigate the following research question: How does participation in professional development on mathematics-specific consultations impact elementary school teachers' and specialists' perceptions of their ability to meet the mathematics learning needs of students with SEN?

Theoretical Framework

Collaborations between mathematics teachers and special education specialists are one attempt to assist teachers in meeting the mathematics needs of students with SEN. These collaborations can be powerful as they bring together the specialized mathematical knowledge for teaching of the mathematics teacher with the expert knowledge of teaching students with SEN of the special education specialist (van Ingen, Eskelson, & Allsopp, 2016). Consultations between general education teachers and special education specialists are one form of teacher collaboration that can be used to support teachers in providing learning opportunities for students with SEN. Teacher consultations can take varying forms and can follow different models (e.g., Richards, Hunley, Weaver, & Landers, 2003; Truscott et al., 2012; Wesley & Buysse, 2004). While these models differ, they do share some common elements. Previously, we created a synthesized model of the common consultation elements (van Ingen, Eskelson, & Allsopp, 2016); this served as the consultation model for this study. This model consists of: (a) initiate rapport building, (b) negotiate consultation relationship, (c) identify the problem, (d) develop recommendations, (e) finalize recommendations and solidify plan, (f) implement the plan, (g) evaluate the plan, (h) learn from results, (i) check back, and (j) re-engage.

Methods

The participants choose to participate in this study in conjunction with their voluntary participation in professional development which we provided at the school. Our professional development team, composed of two mathematics teacher educators and one special education teacher educator, met with the teachers, special education specialists, and mathematics coach to provide professional development regarding how to conduct mathematics-specific consultations about the learning needs of students who struggle with mathematics. The professional development consisted of an introductory session, two weeks of implementation (practice) with authentic consultations, a second session, and then additional weeks of implementation.

During the first session, we introduced the concept of mathematics-specific consultations and modeled how inservice elementary teachers of mathematics and their special education counterparts might engage in consultations. We provided the participants with templates that prompted them to ask the types of questions and/or provide the types of responses that would allow for maximum sharing of their respective specialized knowledge bases. The participants were given time to work in pairs or trios to plan when and how they would engage in the consultations. During this first session, participants also completed a questionnaire in which they identified their confidence level in their own ability to engage in consultations related to mathematics instruction and consultations related to reading instruction. We asked about their experiences and perceptions related to reading consultations as multiple teachers had experience with these previously and we wanted to use these as a comparison to the mathematics consultations. The participants also described any prior experiences they had engaging in consultations (regardless of content area), as well as what they felt were key aspects of successful mathematics consultations. At the end of the first session, participants were asked to engage in authentic consultations during the following two weeks.

During the second professional development session, we asked the participants to talk about their experiences engaging in the mathematics consultations. We also discussed next steps they could take to continue to work together to meet the needs of the students with SEN they had identified and who were the focus of their consultations. During this session, the participants completed a second questionnaire similar to the first asking about their confidence in their abilities to engage in consultations related to reading and mathematics instruction. It also asked them to explain the extent to which this experience did or did not develop their thinking/skills related to teaching mathematics to students with special learning needs and consulting with a special education specialist or elementary teacher of mathematics. They were again asked to identify the key aspects of engaging in a consultation.

Data Analysis

Participants completed questionnaires related to their confidence in their abilities to engage in consultations using a Likert scale with responses of 1 = Strongly Disagree to 5 = Strongly Agree. Although there were not sufficient participants to run statistical analyses on these data, we did compare the means of the data from the first and second questionnaires to identify possible patterns. The remaining questions were open-ended. We looked for similarities and themes across participants and between the two surveys in these responses.

Results

As seen in Figure 1, the participants strongly agreed on both the pre- and post-professional development questionnaires that consultations between mathematics teachers and special education specialists play an important role in meeting the mathematics needs of students with SEN. Their confidence in their ability to engage in mathematics-specific consultations and in differentiating instruction to meet the mathematics needs of students with SEN increased after participating in the professional development. Interestingly, although the professional development focused solely on mathematics-specific consultations, participants' confidence in their ability to engage in consultations related to reading instruction also increased.

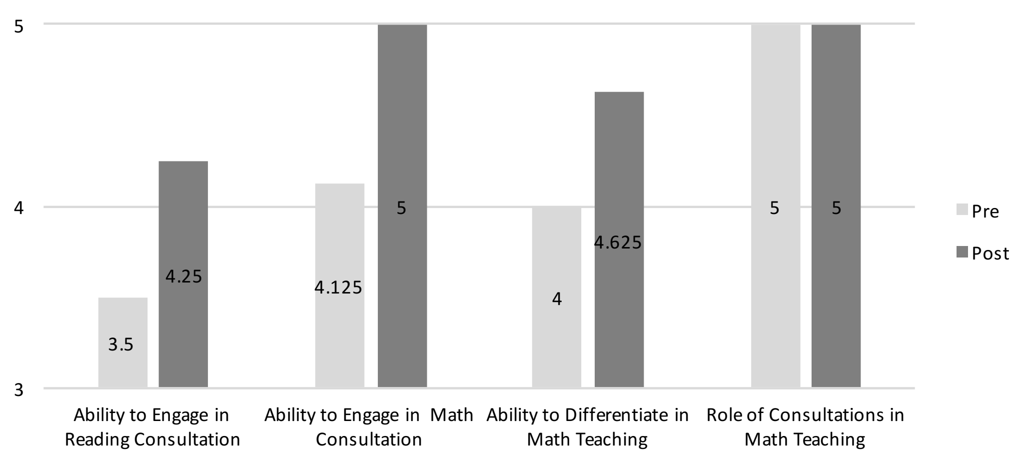

Figure 1. Mean scores of pre- and post-professional development self-reported data on perception of ability to meet the needs of students with SEN.

During the second professional development session, the participants reported on the impact of their participation in the mathematics consultations. They spoke positively about the experience and identified several ways in which they would use the same consultation process in the future and how the process helped them to meet the mathematics needs of students with SEN in their classes (see Table 1).

Table 1. Teachers Self-Report on the Impact of Participation in Mathematics Consultations

Effects on Ability to Engage in Productive Mathematics Consultations	Effects on Ability to Meet Math Learning Needs of Student with SEN
• A better understanding of questions to ask • A clear plan of how to prepare for and engage in consultation • Awareness of resources available • Awareness of the need to consider how a	• Greater understanding of how to support the mathematical practices when working with students with SEN • Thinking across grade levels and math content areas

student's overall learning issues may affect math learning in particular • Gained a deeper understanding of the specific needs of a student and how to set attainable goals	• Time to think about the specific needs of a student with SEN • An opportunity to create a specific action plan

Discussion and Significance

Meeting the mathematical learning needs of students with SEN can be extremely difficult for teachers of mathematics, yet they are under increasing pressure to do so. This study serves as a first step in exploring the impact of helping mathematics teachers to engage in mathematics-specific consultations with special education specialists. We found that that engaging in professional development that provides teachers and specialists guidance on how to participate in such consultations as well as the opportunity to do so was beneficial for both the mathematics teachers and special education specialists. As the teachers' and specialists' confidence in their own abilities to engage in these consultations and in providing differentiated instruction increase, we suggest that they will be more likely to have such discussions with one another and to focus on the mathematics needs of all their students, including those with SEN.

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A JOURNEY IN TEACHING MATH FOR SOCIAL JUSTICE WITH YOUNG CHILDREN

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As a developing early childhood mathematics teacher educator, the area of Teaching Math for Social Justice (TMfSJ) was relatively new to me. Given the diverse nature of children and families at our school and within the community, TMfSJ offered a space for children to become immersed in learning centered around both mathematics and social justice goals (Gutstein, 2006). With this goal, I engaged in an autoethnography which explored the question of: *What are the experiences of an early childhood educator working towards teaching mathematics for social justice?*

Drawing from critical praxis and sociopolitical norms for mathematics (Gutiérrez, 2013), I used an autoethnographic approach (Ellis, 2004) to challenge the idea that young children may not engage in this work. To do so, I taught four and five-year-old during a summer enrichment experience. As the school engages in Project Approach as a method of inquiry, lessons built off of topics the children were currently exploring, or demonstrated interest in the classroom through observation of informal conversations during elongated play. These included topics such as: access to public play spaces in our community, child hunger, and financial inequities. Data collected in a reflective journal, videotaped lesson implementation and children's work samples were examined to construct a narrative account of my experience. During narrative construction entries and videos were closely reexamined as notes were taken to weave into my narrative) and provide points for departure as I question my experience In this way, the narrative constructed represents a truthful and authentic account of the experience TMfSJ accounting for elements of story and attention to criteria for quality.

Based upon my experience, I offer points to consider when engaging in this work. During my lessons, issues of power and control emerged. These centered around management considerations, as I was focused initially on the children "behaving" and using tools appropriately, and questioning who had control over the lessons and thinking within the lesson. Having a focus on myself as the person constructing the lessons and sharing the knowledge, I fell into using deficit discourse to discuss the children and teachers at the school. While this occurrence was humbling, it led to my reexamination of the Funds of Knowledge children were bringing to the lesson as I reframed focus to highlight what children *could do*.

Furthermore, within the experience I began to reexamine my own beliefs about what young children were capable of mathematically. Children were successful in not only discussing social justice issues, but demonstrating connections between the social justice issues and mathematics topics often taught at upper elementary levels. These begins to challenge pre-existing ideas that mathematical learning at the early childhood level be solely organized by developmental age progression.

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COLLABORATING WITH TEACHERS: A DESIGN EXPERIMENT TO DEVELOP AMBITIOUS MATHEMATICS INSTRUCTION

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Keywords: Teacher Education/In-Service Professional Development, Classroom Discourse

Research (e.g., Fennema et al., 1993, Smith et al., 2001) has found that teachers who can observe student performance and respond by adjusting both content and methods empower diverse learners to succeed in doing mathematics. However, learning these kinds of ambitious instructional practices are challenging endeavors, but are manageable when collaboratively worked on with common materials (Lampert et al., 2011). Therefore, in this study, five third grade teachers and their students participated in a 20-week design experiment focusing on how to implement number talk instructional activities within their classrooms. The purpose of this design experiment was to implement a theory about how the pedagogies of enactment and investigation (e.g., McDonald et al., 2013) provide rich learning opportunities that would move teachers into roles where they prepared students to talk about mathematical ideas. To this end, this study aims to answer the following research question: how does the cycle of enactment and investigation support in-service teachers in learning about number talk instructional activities?

In order to answer the research question, video recordings of the teacher workgroup and enactments of number talks within teachers' classrooms were collected and analyzed. Transcripts of the teacher workgroups were coded for both the pedagogies of enactment and investigation and then for episodes of pedagogical reasoning (Horn, 2010) to gain understanding about how the teacher workgroup provided opportunities for teachers to make sense of both mathematics instruction and student learning. Additionally, classroom number talks were coded for levels of math talk learning communities (Hufferd-Ackels et al., 2004) in order to provide insight into how professional learning enables teachers to foster discourse within their classrooms.

Findings suggest that the pedagogies of enactment and investigation provide teachers with opportunities to interact with colleagues facilitating rich learning about how to promote mathematical talk with students. Moreover, analysis of classroom enactments suggest that the teachers' role in number talks shift from being the source of mathematical ideas to a role of facilitator, providing increasing opportunities for students to talk meaningfully about mathematics.

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AT THE CROSSROADS OF CONFIDENCE AND INSECURITY: A PHENOMENOLOGICAL STUDY OF MATHEMATICS TEACHERS

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Teaching is an affective practice, demonstrated by teachers' interactions with students, relationships with parents, and the shared communities of practice established with other educators (Hargreaves, 1998). This atmosphere affects teachers' feelings of confidence, which we define as "part of an individual teacher's ways of learning through experiencing, doing, being, and belonging" (Graven, 2004, p. 179). The crossroads between the affective and cognitive aspects of teaching are an under-explored and therefore invisible phenomenon that requires further research. Our study makes visible teachers' emotions of confidence and insecurity and explores: what are mathematics teachers' lived-experiences of confidence and insecurity in classroom teaching?

To explore the affective component of the mathematics teaching profession, we conducted phenomenological interviews (Moustakas, 1984) with seven practicing mathematics teachers, ranging from elementary through secondary level. This phenomenological study allowed for an in-depth exploration of the feelings of confidence or insecurity teachers encountered during mathematics instruction. Our interview sessions utilized Van Manen's (2015) approach, where we focused on teachers' innermost feelings and viewpoints regarding specific phenomenon.

Our teachers expressed feelings of confidence through a range of experiences: (i) student characteristics and interactions (e.g., knowledge of student backgrounds, students' abilities to help one another), (ii) knowledge and experience gained through teaching (e.g., curriculum familiarity, ability to engage students), and (iii) fostering mathematical connections with the real world and previous and future schoolwork. These teachers expressed feelings of insecurity along similar content lines including: (i) student characteristics and interactions (e.g., working with high-ability or poorly motivated students), (ii) interactions with authority (e.g., administration, policy), and (iii) negotiations of knowledge (e.g., teaching new subjects, lack of autonomy).

This study contributes to the body of research on teacher emotion in mathematics education, as well as implications for teacher professional development. It suggests an intersection point—a place where teachers' confidence and insecurity collide with their knowledge, experience, and authority—demonstrating the value of studying the affective realm of teachers' practice.

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DESIGN-BASED IMPLEMENTATION RESEARCH AS AN APPROACH TO DESIGNING VIRTUAL SPACES FOR MATHEMATICS TEACHER LEARNING

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Addressing problems of practice and implementation at scale is an issue that continues to challenge mathematics education researchers and practitioners (Cobb & Jackson, 2011). As part of a statewide research-practice partnership, we are currently using a design-based implementation research (Fishman et al., 2013) approach to organize the collaborative design of a multifaceted intervention for new state mathematics standards and the promotion of more equitable classroom learning opportunities for students and teachers. These efforts include common spaces and tools for teachers, leaders, and parents including: a closed webspace for teacher engagement, face to face professional development sessions, leader support tools, and parent outreach materials. The focus of this presentation in on the closed webspace for teachers seeking to promote the use of research on teaching and learning as teachers implement new state mathematics standards.

A review of the literature identified a number of characteristics of effective online learning environments that we drew upon to build our design principles for the webspace. Most importantly, opportunities for collaboration play a central role in both building community and the overall utility of online environments (Duncan-Howell, 2009). Therefore, a key design principle for the development of a common virtual space for teachers was the opportunity for teachers to collaborate with their colleagues around the new mathematics standards, their implications for mathematics teaching, and research on teaching and learning. In doing so, we conjecture that this space will foster collaboration and provide teachers with an asynchronous setting for learning and reflection at their own pace and tailored to their own needs (Clay, Silverman, & Fisher, 2012).

In this presentation, we will present aspects of our iterative design for creating a virtual space for teacher learning. We will describe our design conjectures for learning within this space and how they changed through findings from teachers' participation and feedback throughout the design process. We will outline aspects of the space and how we seek to create opportunities for teacher learning related to research on teaching and student learning. Finally, we will present preliminary results of research on teacher learning within the virtual space.

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DECISION-MAKING PROTOCOL FOR MATHEMATICS COACHING: CONNECTING RESEARCH TO PRACTICE

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The National Council of Teachers of Mathematics' (NCTM) *Principles to Actions: Ensuring Mathematical Success for All* [(*PtA*, NCTM, 2014)] acknowledges the critical role of mathematics coaches in enhancing teacher capacity and positively influencing teacher beliefs. While *PtA* calls on mathematics coaches to "take action," coaches need guidance to meet the challenges of enacting the eight research-based teaching practices in their unique school contexts. To lower potential barriers to implementation, we designed a Decision-Making Protocol for Mathematics Coaching (DMPMC) which recognizes ranges of coaching contexts, focuses on mathematics content, and empowers school communities to surmount obstacles as identified in *PtA*.

The DMPMC provides a reflective four-phase cycle of coaching actions to support classroom teachers in building more opportunities for student reasoning and sense-making. Guiding questions allow coaches to purposefully bridge content considerations, research-based teaching practices and high leverage coaching practices (Gibbons & Cobb, 2012). After evaluating their contexts and defining a content focus, coaches establish goals, select appropriate coaching and teaching practices, and engage in collaborative debriefs of classroom enactments to inform further actions.

In semi-structured interviews with two novice coaches and one veteran coach, we explored initial reactions and subsequent descriptions of enacting the DMPMC with selected teachers. Emergent themes were the potential of the DMPMC to formally structure interactions with teachers and to informally support in-the-moment decision making. The guiding questions of the protocol elicited the purposeful selection of coaching practices aligned with teaching practices prior to engaging in coaching situations. For one coach, the four DMPMC phases also became working knowledge that she drew upon during informal conversations with teachers.

While all participants situated their contexts within the DMPMC, the novice coaches frequently sought affirmation of their actions from the researchers. Participants shared the advancement of their coaching practice through consistent interplay between coaching and teaching practices, which enabled a simultaneous focus on relationship building and making instructional shifts. Future research includes focus groups for coaches to share their perspectives and experiences with the DMPMC. We seek to further our contextual understanding of how this protocol can build coaching effectiveness through structured consideration of content, practices, and relationship building in tandem.

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GRAPHING RULES OR CONVENTIONS? TEACHERS UNDERSTANDINGS

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Researchers have found that students and teachers often maintain insufficient meanings for function and their graphs (e.g., Montiel, Vidakovic, & Kabael, 2008; Moore, Silverman, Paoletti, & LaForest, 2014; Oehrtman, Carlson, & Thompson, 2008). Several of these researchers have noted that students maintain function meanings that are restricted to particular representations (e.g., the Cartesian coordinate system with the independent quantity represented by x on the horizontal axis). For example, Montiel et al. (2008) identified students applying the vertical line test, a common procedure in U.S. curricula, to determine if a graph represented in the polar coordinate system represented a function. As a second example, Moore et al. (2014) described pre-service teachers' responses to tasks in which hypothetical students provided mathematically correct work that did not follow conventions common to school mathematics (e.g., graphing the function $y = 3x$ with x on the vertical axis and y on the horizontal axis). Many of the pre-service teachers did not recognize that the hypothetical student's solution was correct, or identified that although the hypothetical student accurately represented the relationship defined by $y = 3x$, the student's solution was mathematically incorrect because he or she did not maintain conventions.

Extending Moore et al.'s (2014) work with pre-service teachers, we provided in-service teachers tasks similar to those by Moore and colleagues in pre and post course online surveys. The participants were enrolled in a fully online graduate mathematics course designed specifically for in-service teachers. The participants were geographically distributed across the U.S. and all had one to five years teaching experience. We coded their responses using a semi-open coding scheme, with the students' activities described by Moore et al. (2014) serving as the basis for an initial coding scheme. We adapted this initial scheme to capture the teachers' responses. We did this through an iterative process of analyzing teachers' responses, discussing commonalities across the responses, and adapting or creating new codes to satisfactorily capture the responses. In this poster, we present our coding of teachers' responses and compare our results with those reported by Moore and colleagues. Additionally, we compare differences in teachers' responses before and after a course designed to support their reasoning about relationships between quantities. Finally, we highlight differences between our results and those of Moore and colleagues that may stem from the methodological difference of collecting data from online surveys versus in-person interviews.

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EFFECT OF TEACHERS' PARTICIPATION IN A PROFESSIONAL DEVELOPMENT ON STUDENT ACHIEVEMENT: A LONGITUDINAL LARGE-SCALE STUDY

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Professional development (PD) programs aim to change teachers' current practices and positively impact student outcomes. In this poster, we present a preliminary model investigating the effect of teachers' participation in sustained PD on student achievement.

Background and Data

This poster reports on a study looking at the efficacy of a 3-year PD program for 3rd-5th grade teachers in a mid-sized, urban school district. The PD (Foreman, 2010) is focused on a set of core mathematical habits for students and teachers that promote high-level reasoning and productive discourse (e.g. Stein, Engle, Smith, & Hughes, 2008) in mathematics classrooms. Student achievement was measured by mathematics scores on standardized assessments at the end of each grade. Over four years, 239 teachers spanning 34 schools participated in the study for a total of 10,076 students and 14,750 student assessment scores. Each school was either part of a sustained year long PD (Studio), or part of the summer sessions only (BP-only).

Model

To investigate the relationship between student achievement and teachers' participation in the PD, we developed a hierarchical linear model (HLM) with three levels to address students, teachers, and schools. Since over the years, students are not nested within teachers, we used a cross-classified random effects model with levels: (1) time series data of student scores (normalized), (2) row-factor: students; column-factor: teachers, and (3) schools as clusters.

Results and Discussion

Our preliminary HLM model suggests that our PD program had a positive impact on student outcomes as measured by end of the year standardized assessments. We found that by the end of the second year of being in a studio school, students scored .08 standard deviations higher ($p = 0.019$) than their counterparts. This result is reflecting a positive impact of a multi-year participation to the PD. This impact is robust remaining significant and positive when other variables such as free and reduced lunch at school and student level, student ethnicity, teachers' MQI scores, and hours of participation in the PD are added to the model.

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MATHEMATICAL MODELING A CROSSROAD PROVIDING ACCESS FOR ELLS TO HIGHER LEVEL MATHEMATICS

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Keywords: Modeling, Mathematical Knowledge for Teaching, Equity and Diversity

Mathematical modeling is especially well poised to be successful in closing the opportunity gap for ELL students since it offers the opportunity to tackle "big, messy, realistic problems, helps students connect mathematics to life and empowers them to use their mathematics to solve relevant problems" (GAIMME Report, 2016, p. 23). High concentrations of ELL students, when immersed in classrooms where they are engaged in small group work involving authentic problem solving, showed significant gains on mathematics achievement, not only in computation, but also problem solving and language acquisition, two important components of mathematical modeling. However, many teachers have no formal preparation in teaching mathematical modeling.

The Study

The research featured in this poster is from a small, funded professional development project for teachers in rural western United States in which science and mathematics teachers were engaged in utilizing and creating mathematical modeling problems and enacting them in all classes through highly interactive classroom environments. It consisted of a two-part summer institute augmented with three additional workshops throughout the year. The emphasis of this professional development was, in part, on fostering language and content development of ELLs through mathematical modeling and also cultural responsive pedagogy as means of closing the opportunity gap for ELLs.

This study sought to address two research questions: (1) *How does mathematics and science teachers' conception of teaching mathematical modeling in ways that foster content learning and language development for ELLs emerge; and (2) what challenges do mathematics and science teachers face, as they experience year-long professional development focused on mathematical modeling.* This study used a mixed-methods design with focus group interviews, classroom observations, surveys, and lesson plans. Data collection is currently ongoing but preliminary data suggests that although students seem to benefit from well-designed mathematical modeling tasks, teachers' beliefs about pedagogical practices are very hard to change. Also, teaching mathematical modeling in ways that foster language development for ELLs can be challenging to implement due to time constraints since teachers are on a prescribed curriculum that they have to follow. This is significant because as we connect research to practice consideration should be made in terms of revising the curriculum to allow opportunities for teachers to implement pedagogical practices such as mathematical modeling, that help their diverse students access higher level mathematics.

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EXPERIENCED AND NOVICE GRADUATE STUDENTS NAVIGATING MATHEMATICS INSTRUCTION TOGETHER

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Mathematics graduate student instructors (GSIs) teach hundreds of thousands of undergraduate mathematics students each semester, yet typically lack guidance and support to teach undergraduate students effectively (Rogers & Steele, 2016; Speer & Murphy, 2009). GSIs' initial teaching experiences represent a crossroad between how they teach in the short term in graduate school and in the long term as potential future faculty members (Lortie, 1975). To address this concern, we developed and implemented a peer-mentoring program at two universities in the US whereby experienced GSIs mentor first- and second-year GSIs (novices).

After a semester of being trained in ways to effectively serve as a peer-mentor, nine experienced GSIs mentored three or four novices on teaching similar content. Mentors facilitated bi-weekly small group meetings with novices as part of an NSF-funded peer-mentorship grant (IUSE #1544342 & 1544346). During these discursive meetings, mentors provided context-specific support, resources, and guidance to novices using practices that augment productive discourse (Smith & Stein, 2011). Within small groups, novices determined specific teaching and learning topics they wanted to discuss. Topics also included concerns mentors and novices raised later in the semester, or adapted from ideas other small groups discussed. We examined: *What topics from small-group peer-mentoring meetings did novices value?*

To answer our research question, we applied a social constructivist lens within each small group to identify how the small group topics were valued. Participants included 30 novices and nine mentors from two universities. At the end of one mentoring semester, novices rated how (a) valuable they found each topic discussed during their small group meetings and (b) interested they would have been in discussing topics from other small groups. Discussion topics were objective (e.g., Grading Practice and Determining Assessment Questions) and subjective (e.g., Work-Life Balance and Qualities of a Good Teacher). We first qualitatively coded data as either within a group or from other groups. We then quantitatively analyzed each novice's ratings of their own small-group meetings. From this analysis we identified topics that novice GSIs value and topics that their peer-mentors may struggle to facilitate well. Topics valued by novices that were facilitated well by mentors included (a) Managing Time During Class, (b) Presenting Mock Lessons, and (c) Incorporating Formative Assessments. Topics valued by novices that were not facilitated well by mentors included (a) Writing a "Good" Exam, (b) Dealing with Overbearing Students, and (c) Incentivizing Group Work. Moreover, these results offer insight and synergy between educating GSIs and improving undergraduate mathematics teacher pedagogy.

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OPPORTUNITIES FOR ACTIVE LEARNING IN A MOOC DESIGNED FOR EFFECTIVE TEACHERS PROFESSIONAL DEVELOPMENT

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Keywords: Teacher Education-Inservice/Professional Development, Technology

Over the last 17 years researchers have been investigating the features of effective professional development (PD) that affect teacher's knowledge and skills, and lead them to changes in classroom practice (e.g., Garet et al., 2001). Based on results from face-to-face PD, Garet and colleagues (2001) proposed six characteristics of high-quality and effective PD, in which opportunities for active learning is highlighted as a core feature of effective PD. *Active learning* denotes the extent to which PD affords opportunities for teachers to become engaged in meaningful discussion, collaborative work, planning, and reflection regarding their practice (Garet et al., 2001; Borko, 2010). Massive Open Online Courses (MOOCs) have generated notable disruption in higher education, and MOOCs as venues for teachers' PD are a relatively new field of study. There is a need to thoroughly understand which opportunities for active learning are present in MOOCs designed for effective teachers PD.

Context, Data and Methods

The context of this study is a MOOC offered by a large American university that has been designed for statistics teachers. This study aims to build answers to the research question: *What are the opportunities for active learning depicted in MOOCs designed for effective professional development of statistics teachers?* Qualitative research methods of content analysis were used to identify and describe *opportunities for active learning* as enacted in a MOOC designed for PD in statistics teaching. Data comprised 2370 forum posts distributed across 546 discussion threads.

Results and Implications

Results showed that by engaging in forums, teachers had opportunities to (a) interact with each other regarding the MOOC content, (b) share their experiences regarding statistics content and their practice of teaching, and (c) reflect on their practice. For example, teachers experienced the process of evaluating different statistics tasks according to a framework for statistical investigations, and subsequently engaged in discussion forums, sharing their perspective about how those tasks would promote productive statistical habits of mind and how these tasks could foster students' learning. The process of analyzing, reflecting, and suggesting improvements in the tasks produced meaningful opportunities for participants' active learning as stated in the literature. Implications from this study show the relevance of the connections and the network established by teachers in MOOCs. It indicates that future MOOCs for teachers' PD should be designed to nurture participants' connections and to help them in establishing virtual communities for shared practices.

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INTEGRATING FACE-TO-FACE PROFESSIONAL DEVELOPMENT AND A MOOC-ED TO DEVELOP TEACHERS' STATISTICAL KNOWLEDGE FOR TEACHING

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Keywords: Data Analysis and Statistics, Teacher Education-Inservice/Professional Development, Design Experiments

In our poster we will report on our effort to support middle grades teachers to develop the content knowledge, pedagogy, and self-efficacy necessary to support students to learn about statistical inquiry through authentic data investigations. There is a growing consensus that "every high school graduate should be able to use sound statistical reasoning to intelligently cope with the requirements of citizenship, employment, and family and to be prepared for a healthy, happy, and productive life" (Franklin et al., 2007, p. 1). This acknowledgement is represented in various content standards for mathematics at the K-12 level (CCSSM, 2010; NCTM, 2000). However, many practicing teachers have not experienced sufficient preparation to facilitate students' development of statistical literacy, and research has shown that most teachers do not have a deep understanding of the foundational concepts related to statistical inference (Franklin et al., 2015).

Thus, we have designed an innovative approach to supporting teachers through the integration of 1) a summer institute, 2) professional learning communities (PLCs), and 3) a Massive Open Online Course for Educators (MOOC-Ed). During the summer institute, teachers will meet together for a week-long professional development. The summer institute will focus on developing deep understandings of statistical concepts and exploring the ways students might engage with these ideas. We will use tasks from the research-based Data Modeling curriculum during the institute (Lehrer, Kim, & Schauble, 2007) to expose teachers to the need for statistical inference by exploring questions in the midst of widespread variability in data and explore different sources of variability and will discuss the conceptual principles of scale, interval, grouping, and order in visual displays of data. The MOOC-Ed, Teaching Statistics Through Data Investigations (Friday.institute/tsdi), was designed by Hollylynne Lee to develop teachers' knowledge of teaching statistics through the statistical investigative cycle (Franklin et al., 2007).

Our poster will report on the design framework for integrating Data Modeling resources and the MOOC-Ed to support sustained teacher support and to build collaborative communities. The poster will also report on early data about how new concepts and teaching practices travel across these different settings.

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RESEARCH PRACTICE PARTNERSHIPS: DESIGN-BASED IMPLEMENTATION RESEARCH EFFORTS ON A STATEWIDE SCALE

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In this poster, we share details of a statewide Design-Based Implementation Research project's efforts supporting teachers' implementation of new state secondary mathematics standards. This includes design of project spaces and tools and preliminary results related to teacher interactions and learning.

Keywords: Teacher Education-Inservice/Professional Development, Design Experiments, Standards

Mathematics education researchers have built a strong body of research that continues to move the fields of both research and teaching forward; however, a divide continues to exist between research and practice and issues of implementation at scale (Cobb & Jackson, 2011). Responding to this challenge, researchers and funding agencies are encouraging research-practice partnerships where researchers and practitioners work together to iteratively design and research problems of practice (Penuel & Farrell, in press).

As part of a statewide research-practice partnership between a state education agency, district leaders, teachers, and mathematics education researchers at several institutions, our work draws upon Design-Based Implementation Research (DBIR) (Fishman et al., 2013) as an approach to facilitate the collaborative design of an intervention related to new state mathematics content standards and efforts to promote more equitable classroom learning opportunities for students. These efforts include common spaces and tools for teachers, leaders, and parents including: a closed webspace for teacher engagement, face to face professional development sessions, leader support tools, and parent outreach materials.

In this poster presentation, we will present DBIR as an approach to this work and share the ways in which we draw upon both communities of practice and boundaries to conceptualizing teacher learning (Wenger, 1998). We will describe the design process for both boundary encounters and boundary objects for which communities of high school mathematics teachers, teacher leaders, and parents/community members can collaborate and engage around research on teaching and student learning of mathematics. Finally, we will present preliminary findings on research related to interactions and learning within the spaces.

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ONLINE FACULTY COLLABORATION TO SUPPORT INSTRUCTIONAL CHANGE

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Keywords: Post-Secondary Education, Mathematical Knowledge for Teaching

Research has shown that mathematicians may struggle implementing a new curriculum without support (Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). Further, research has shown how summer workshops and online forums aid mathematicians in sustaining instructional change (Hayward, Kogan, & Laursen, 2015). However, new technology exists in which faculty collaboration can take place (e.g., online video calls). Research is underway that explores diverse ways to engage mathematicians and support them in reaching their goal for instructional change. This work is part of an ongoing project to support mathematicians' instructional change. This study focuses on one participant, Dr. J., and seeks to find links between his experiences in the online faculty collaboration and his instructional practice. The poster will address the following research question: How does one mathematician's instructional practice develop while participating in a faculty collaboration for inquiry-oriented differential equations?

Methods

Data for this case study (Yin, 2013) comes from observations of the faculty collaboration online meetings, Dr. J's classroom observations, and audio recordings of three interviews with Dr. J. Analysis of the classroom instruction uses the inquiry oriented instructional framework (Kuster, Johnson, Andrews-Larson, & Keene, n.d.). Analysis of the transcripts of the faculty collaboration uses a priori coding to explore Dr. J.'s changes in participation. Lastly, analysis of interview transcripts will be open in nature and serve as triangulation of data.

Preliminary Results

Analysis is ongoing but preliminary results seem to indicate that Dr. J.'s implementation of the IODE materials was influenced by his participation in the OWG. Although the interviews indicate that his desire to make changes in his instruction began prior to participation in the online working group, he uses the language and ideas of the inquiry-oriented instructional components presented in the project as a way to focus his teaching, offer feedback to other participants, and otherwise participate in the discussions. Further work will describe the connections and offer ideas to other facilitators of faculty online instructional support groups.

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SECONDARY TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' PROPORTIONAL REASONING

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Keywords: Teacher Education-Inservice/Professional Development

Professional noticing of children's mathematical thinking is a sense-making activity specific to the profession of teaching (Jacobs, Lamb, and Philipp, 2010). It distinguishes itself from other noticing conceptualizations in two ways: by including teachers' in-the-moment decision-making, and by focusing on children's mathematical thinking. Additionally, it is an essential part of instruction that builds on and responds to students' ideas. Herein we consider the professional noticing of students' mathematical thinking of 18 practicing secondary mathematics teachers in the domain of proportional reasoning (Lobato & Ellis, 2010).

Participants were a part of a larger professional development[1] aimed at supporting secondary mathematics and science teachers to improve in their practice and become leaders of their teaching communities. At the time of data collection, 13 participants had received 2.5 years of sustained professional development around students' mathematical thinking, and 5 had just begun. Each participant considered three samples of student work about proportional reasoning, and responded to the three professional noticing prompts (Jacobs et al., 2010): (1) Describe in detail what each student did. (Attending); (2) What did you learn about the students' mathematical understandings? (Interpreting); and (3) Pretend you are the teacher of these students. What problem might you pose next, and why? (Deciding how to respond).

The authors then coded participants' responses according to the amount of evidence demonstrated of considering the students' mathematical thinking: robust, limited, or a lack of evidence (Jacobs et al., 2010). Percentages of scores can be seen in Table 1. In sum, we find more evidence that improvement in professional noticing of students' mathematical thinking requires sustained professional development around students' mathematical thinking

Table 1: Percentages of Initial and Advancing Participants' Codes Per Component-Skill

	Attending		Interpreting		Deciding how to Respond	
	Initial	Advancing	Initial	Advancing	Initial	Advancing
Robust	40%	38%	0%	31%	0%	31%
Limited	40%	62%	80%	69%	20%	31%
Lack	20%	0%	20%	0%	80%	38%

Endnotes

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RE-STORYING THE RITUAL OF CRITICAL INCIDENTS TO EMBRACE VULNERABILITY AND EMOTIONS AS EPISTEMOLOGY

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Keywords: Teacher Education, Research Methods

Critical incidents (Goodell, 2006) in mathematics teacher education have historically focused on teaching as a practice of problem solving. This poster describes findings from a collaborative self-study on the impact of this “problems of practice” perspective on ourselves as knowers in mathematics teacher education. Instead of continuing to ritually make use of critical incidents to explore such “problems of practice,” we sought to explore whether critical incidents could function in more positive and sustaining ways.

Early, reflective analyses of our conversational transcripts revealed that our initial sharing of critical incidents provoked largely negative activating and deactivating emotions. Our discussions of the critical incidents as “problems of practice” and how to solve these problems left us feeling mostly defeated. Previous mathematics education scholarship (Confrey, 1995; Forgasz & Clemans, 2014) has discussed the marginalization of emotion as a separate and inferior form of sense-making than cognitive perspectives of knowledge. We sought to challenge this historic tendency in our field and looked for ways to discuss practice that provoked positive activating emotions. We began crafting and discussing critical incidents that were at the crossroads of cognition and positive activating emotion.

Thematic analysis of 6 narratives of critical incidents and transcripts of 11 of our 30- 90 minute conversations over 5 months evoked by the incidents, resulted in several findings. For example, an analysis of one of the transcripts revealed that expressions of vulnerability that were brought to the group (such as: “If I can be really vulnerable…” and, “someone else positions us in such a way that we’re blindsided into being vulnerable”) suggest that vulnerability may be an important characteristic for mathematics teacher education development that allows the sharing of critical incidents to become productive. Interpretation of additional transcripts indicates that this discussion of vulnerability and our own willingness to open ourselves to be vulnerable with each other contributed to a re-storying of our work and ourselves in more positive ways (Brown, 2006). Further analysis of transcripts revealed ways in which the emotional terrain constructed during the experience of articulating critical incidents and re-storying served an epistemic role. Emotions allowed us to make sense of the incidents in new ways. We assert that a critical friends group and the sharing of positive incidents has the potential to provoke and sustain explorations of mathematics teacher educator practice without concomitant negative and deactivating emotions that had been provoked by explorations through a “problems of practice” lens.

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THE CROSSROADS OF STAKEHOLDERS' VIEWS OF CCSSM IMPLEMENTATION

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Keywords: Policy Matters, Standards (broadly defined), Teacher Education-Inservice/Professional Development, Elementary School Education

The current effort to implement the *Common Core State Standards for Mathematics* (*CCSSM*) is the latest in a series of mathematics standards implementation efforts in the United States over the last half century. When implemented, previous standards efforts have either failed or been less successful than anticipated for a variety of reasons. Two oft-cited reasons are a lack of a shared understanding about what the standards are and how to incorporate them effectively at various levels of an existing education system, and perceived and/or real flaws in the standards themselves.

With this past in mind, as well as requests for studies of this type from the mathematics education research community (Heck, Weiss, & Pasley, 2011), this study sought to document whether and to what extent these problems exist within Michigan's education system as the state implements the *CCSSM*. More specifically, this study sought answers to two research questions: To what degree is there alignment between Michigan Department of Education (SDE) officials', regional professional development providers', and elementary teachers' views of the goals of *CCSSM* implementation? Also, do those outside MDE charged with the implementation feel adequately supported in effecting their part of the transition to the *CCSSM*?

MDE officials, regional professional development providers, and elementary teachers were surveyed and interviewed as part of this mixed methods study in order to gather their thoughts on what they believe the goals of the *CCSSM* to be, what they believe their roles in the implementation effort are, and how they are supported in that effort. Responses were analyzed for commonalities and differences in the perceptions of individuals at the varying levels of the state's education system.

While elementary teachers were confident in their abilities to implement the *CCSSM* effectively, they still desired more professional resources related to the CCSSM and were generally unfamiliar with several resources others in Michigan's education system were promoting. Furthermore, as the *CCSSM* became a political issue that was widely discussed outside the education community, each of the three stakeholder groups that participated in this study were affected by and dealt with that development in different ways. These responses can be compared and contrasted with the findings of McDonnell and Weatherford (2016) as well as Polikoff, Hardaway, Marsh, and Plank (2016).

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SYNERGY ACROSS UNIVERSITIES: EXAMINING THE EFFICACY OF STATEWIDE PROFESSIONAL DEVELOPMENT

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Borko (2004) outlined four elements of a professional development (PD) experience (i.e., the PD program, teachers, facilitators, and context), and suggested three phases of research to investigate the interactions between these elements: (1) focusing on an individual PD experience at a single site, (2) expanding the study to a single PD experience enacted by more than one facilitator at more than one site, and (3) comparing multiple PDs at multiple sites. Garet et al.'s (2001) survey of teachers highlighted desirable core characteristics of PD experiences (i.e., focus on content knowledge, opportunities for active learning, coherence with other learning activities) and structural features (i.e., form of activity, collective participation of teachers from the same school, grade, or subject; duration of the activity).

In this poster, we report on a three-year phase 2 PD experience across four universities involving 60 middle school and secondary mathematics teachers. The project aims to enrich teachers' knowledge and skills for teaching algebra. Here, we investigate the ways in which PD for middle school and secondary mathematics teachers may be evaluated in order to give a holistic view of the PD experience. Eight objectives (e.g., collaborate to locate and develop algebra activities, engage students in solving rich algebra tasks) were set at the beginning of the program with six planned instruments to measure progress towards achieving those objectives. The following data sources were used to measure the progress of the project: a knowledge for teaching algebra assessment, lesson plans, PD and critical friend reflection forms, action research projects, and students' state standardized test scores.

Most compelling of our preliminary results has been the benefits of multiple data sources to provide a broad picture of teachers' development. Giving a pre- and post-test in knowledge for teaching algebra provides one lens for evaluating the PD, while the lesson plans and reflection forms give insight to how the teachers implemented PD foci (e.g., mathematics teaching practices) and illustrate teachers' creativity. Critical friend reflection forms show growth in teachers' ability to provide feedback to peers in ways that support algebraic thinking, and PD survey data indicate that teachers find value in the PD work itself and the ability to collaborate with their colleagues. Action research projects are individualized or collaborative and allow us to gain insights into the teachers' needs and how they perceive their progress. Collectively, these items give us a holistic view of the PD experience.

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Chapter 6

Mathematical Knowledge for Teaching

Research Reports

Brief Research Reports

Posters

A STUDY OF PRE-SERVICE TEACHERS USE OF REPRESENTATIONS IN THEIR PROPORTIONAL REASONING

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Proportional reasoning is important to the field of mathematics education because it lies at the crossroads of additive reasoning in the elementary school and multiplicative reasoning needed for more advanced mathematics. This research reports on the representations used by pre-service teachers (PSTs) as they responded to tasks involving proportional reasoning. The findings highlight three common difficulties that were prevalent among participants' responses. An analysis of the representations used by participants revealed that the representations that PSTs created in their effort to solve the problems often enabled them to overcome these difficulties. Prior research is used to hypothesize explanations of the extent to which different forms of representations were useful and productive for the participants. Implications include ways that use of these multiple representations could aid in the teaching of proportional reasoning.

Keywords: Mathematical Knowledge for Teaching, Rational Numbers, Modeling

Proportional reasoning is important to the field of mathematics education because it lies at the crossroads of transitioning from additive reasoning in the elementary school to multiplicative reasoning necessary for proportional reasoning and more advanced mathematics. Lesh, Post, and Behr (1988) describe the importance of proportional reasoning, saying that it is "widely recognized as a capability which ushers in a significant conceptual shift from concrete operational levels of thought to formal operational levels of thought" (p. 101). This shift in understanding can lead to advanced mathematical thinking and is paramount in achieving success in higher level mathematics courses.

Pre-service teachers (PSTs) enter college with prior assumptions about mathematics and mathematical concepts. Often PSTs have many deep-rooted misconceptions about the multiplicative relationships involved in proportional reasoning and struggle with solving tasks that involve these concepts (Simon & Blume, 1994; Smith, Silver, Leinhardt, & Hillen, 2003; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). The question becomes: What mathematical knowledge do PSTs have in relation to proportional reasoning? Understanding this knowledge is important in helping them develop the specialized content knowledge necessary for teaching. And how do PSTs use representations to deepen their understanding of proportional relationships? This report focuses on particular tasks that were used to elicit proportional reasoning of PSTs, the misconceptions that surfaced and how PSTs used representations in their problem solving to overcome these obstacles. These findings can help improve mathematics teacher education, as we can gain a better understanding of how PSTs think about proportional reasoning.

Conceptual Framework

While the definition of representation in mathematics education vary, most researchers differentiate between external and internal representations where external representations are embodiments of ideas or concepts such as charts, tables, graphs, diagrams, etc., and internal representations are cognitive models that a person has (e.g., Janvier, Girardon, & Morand, 1993). In this study, representations are external mathematical embodiments of ideas and concepts that provide the same information in a drawing, picture, table or graph.

According to Dufour-lanvier et al. (1987) the role of representations in mathematics education has several characteristics: (1) Representations are an inherent part of mathematics, (2) Representations provide a concrete example of a concept, (3) Representations are used locally to mitigate certain difficulties and (4) representations are intended to make mathematics more interesting. It is this third role that this report will illustrate in terms of PSTs proportional reasoning. In particular this study focuses on the use of representations to overcome difficulties and misconceptions.

The use of multiple representations is advocated by many mathematics educators and supported by the National Council of Teachers of Mathematics (NCTM) Standards (NCTM, 2000). It is suggested that multiple representations provide an environment for students to abstract and understand major mathematical concepts. Constructivist theory also suggests that we need to understand students' thinking processes in order to facilitate their learning in more empowering ways (Steffe, 1991). Therefore, it is necessary for mathematics teacher educators to understand how PSTs use representations, not only to understand their thinking, but to develop a repertoire of useful representations for teaching and discussing proportional reasoning. The results of this report will provide mathematics teacher educators with multiple representations that are productive in the teaching and learning of ratio and proportion.

Methodology

Twenty-five elementary and secondary math education PSTs were selected for this study at the beginning of their first mathematics methods courses at a large research university. A nine-problem questionnaire was developed and used to ascertain each PST's current level of understanding about proportional reasoning. (See Johnson, 2013 for more details on questionnaire). The responses were coded and participants were divided into four groups based on the analysis of their responses. Group 1 was distinguished by having a high level of proportional reasoning while Groups 2 and 3 showed moderate levels of proportional reasoning and group 4 gave evidence of little to no proportional reasoning. Eleven individualized interview schedules were created in order to challenge the PST understandings and misconceptions about proportional reasoning that surfaced from the questionnaires; the interviews were implemented, videotaped, transcribed and annotated. Individual interview data was coded and analyzed to create descriptions of the nature of the participants' understanding of proportional reasoning. A group of trained graduate students also coded the data and these codes were then discussed and revised to provide a higher degree of validity and reliability (Johnson, 2013). Another pass through the analysis showed patterns that emerged within each of the four groups in terms of their use of representations and it was noted that there were stark differences between those students in Group 1 and the students in the other groups. This report discusses and illustrates how PSTs in Group 1 utilized representations in solving these proportional problems during the interview and why these representations were important for them in overcoming certain challenges. Additionally, I will contrast these representations with those created by participants who were less successful in reasoning proportionally about these problems.

How Did Pre-Service Teachers Use Representations When Given Tasks Focused on Proportional Reasoning?

For this study tasks were designed to address distinct aspects of PSTs' difficulties with proportional reasoning that surfaced from the questionnaire. Three of these misunderstandings were: (1) The epistemological obstacle of linearity (Brousseau, 1997), (2) confusion between ratio and fraction (Karplus, Pulos, & Stage, 1983), and (3) inability to reason quantitatively (Thompson, 1994). Below, I illustrate how these particular tasks were designed to challenge PSTs' prior assumptions and how PSTs used representations to reason proportionally.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Difficulty #1: Epistemological Obstacle of Linearity.

Modestou and Gagatsis (2007) studied students' improper proportional reasoning as an epistemological obstacle of linearity. An epistemological obstacle is NOT one in which there is a lack of knowledge, but one in which a piece of knowledge is appropriate only within particular contexts. The epistemological obstacle often generates false responses outside that context (Brousseau, 1997). These responses are recurrent, universal, and resistant to a variety of forms of support aimed at overcoming the problem (De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003). For example, problems involving proportionality are often characterized as an epistemological obstacle in linearity. Missing value problems often include the basic structure of four quantities (*a*, *b*, *c* and *d*) of which, in many cases, three are known and one is unknown. Additionally, many proportional problems involve the context of speed. The Bike problem is this type of scenario; it involves students riding their bikes to school at the same speed. It provides PSTs with three numbers and asks them to find the fourth (see Figure 1).

Ben and John both ride their bikes to school at the same rate. Ben leaves his house first and meets John after riding 10 blocks at which time John has only rode 7 blocks. If John's trip to school is 14 blocks, how many blocks does Ben ride his bike to school? (Adapted from Cramer, Post and Currier [1993, p. 159])

Figure 1. Bike problem designed to elicit the obstacle of linearity.

Despite the structure of a missing value problem and context of speed, this problem does not involve a proportional relationship between the quantities. Research has found that this structure and context evoke a strong tendency of students to use direct proportions even if it does not fit the problem (DeBock, et al., 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). 64% of the PSTs answered the bike problem incorrectly which demonstrates that students are drawn to the illusion of linearity in this problem and desire to solve it by setting up a proportion and cross multiplying even though there is not a proportional relationship. Verschaffel, Greer and DeCorte (2000) claim it takes a radical conceptual shift to move from the uncritical application of this simple neat mathematical formula to the modeling perspective that takes into account the reality of the situation being described. It is not surprising that the 36% of the PSTs interviewed who correctly used additive reasoning to solve this problem all created a diagram as part of their reasoning. The diagrams all illustrated the context of the problem (see figure 2).

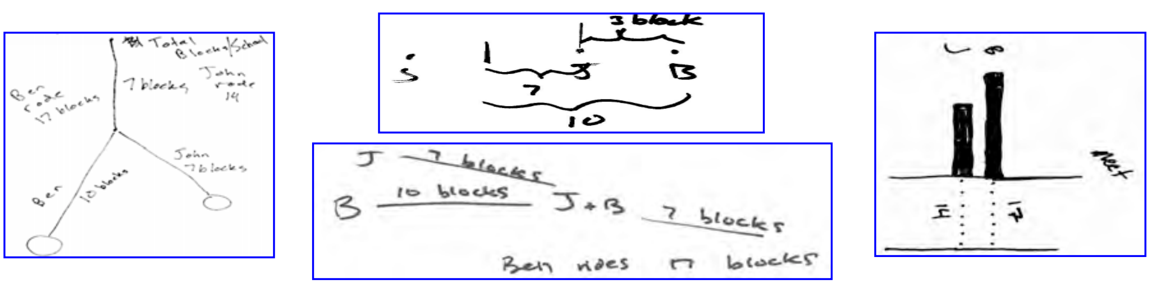


Figure 2. Representations created by PST to solve the Bike problem and overcome the obstacle of linearity.

It was the use of the diagram that helped them to situate and solve the problem as well as overcome the obstacle of linearity. Their representations modeled the additive reasoning necessary to solve the task despite its context and structure that led most PSTs to overextend the concept of proportions. These PSTs used the representation to illustrate that the two boys ride at the same speed

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

then meet and travel together meaning that Ben rides his bike 3 more blocks than John. It is through modeling the context of the problem that meaning is achieved.

Difficulty #2: Confusion between Ratio and Fraction.

A link between fractions and ratio is often not made explicit in mathematics textbooks or classrooms. The difficulties surface if ratio and fraction are understood as equivalent mathematical terms when they are fundamentally different. Even though there are similarities in representations of ratio and fraction (i.e., fraction 2/3; ratio 2/3), the interpretations of those representations differ in important ways. In the case of ratio both the numerator and the denominator can represent parts (i.e., 2 parts to 3 parts); this is not the case with fractions. The Dog/Cat problem (see figure 3) was meant to elicit this type of difficulty in reasoning by PSTs. The correct interpretation of the situation is a part-to-part relationship. The numbers were chosen so that regardless of whether the participant interprets the ratio as a part-whole relationship or a part-part relationship, the solution will be an integer.

When animal lover Mr. Henry died he left 240 thousand dollars to be divided amongst two animal shelters using a 2:3 ratio between the amounts that Cat Best Home and Dog Lover Home gets. How much money should each shelter get? (Adapted from Peled [2007, p. 21])

Figure 3. Dog/Cat problem to elicit understanding of part to part ratios.

This problem was given on the initial questionnaire and a third of the PSTs interpreted 2:3 as 2/3rds and arrived at an incorrect solution of $80,000 to the Cat home and $160,000 to the Dog home. When interviewed these students who were asked to explain their reasoning and some were able to recognize that 2:3 is a part to part relationship, not a part whole relationship. In order to explain this relationship, these PSTs utilized representations to find the solution, either in a table or a model. For example, Eve started by doing an easier problem of 100 thousand dollars and created a pie chart to show the distribution of money. She then used a similar pie chart to determine the distribution for the 240 thousand dollars in the problem (see figure 4).

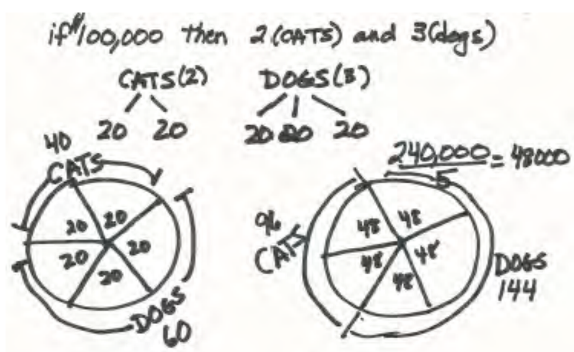


Figure 4. Eve's representation of her solution to the Dog/Cat problem.

Her representations illustrate a deep understanding of the part-to-part relationship presented in this problem. However, many of the PSTs who did answer this problem correctly on the questionnaire were unable to explain the procedure they used to find the solution. When asked how they solved the problem they would re-iterate the steps in the procedure (i.e. you add the two numbers given in the ratio, then you create two fractions 2/5 and 3/5 and multiply by them by the 240) but when asked to explain why it makes sense, replied, "I don't know, this is what I was taught to do when solving this type of problem, I don't know what it means or why it works." In order for these students to develop the specialized content knowledge needed for teaching proportional reasoning a discussion of illustrations such as Eve's would be beneficial. The pie chart can provide an understanding of what part-to-part relationships represent and why their procedure finds the solution. The slices of the pie show the number of pieces in the whole created by adding the two parts together and provide a visual representation as to why 2/5 times 240 (i.e. it is 2/5ths of the whole) results in the amount of money the Dog home receives.

Difficulty #3: Inability to Reason Quantitatively

PSTs in this study struggled with proportional reasoning situations that involved the distinction between quantitative reasoning and computation. Quantitative reasoning is making sense of relationships among measureable attributes of objects in a situation (Thompson, 1994) while computation is the result of an arithmetic operation to evaluate quantities. In general, reasoning about quantitative situations involves conceiving of circumstances in terms of quantities by constructing networks of quantitative relationships. For example, PSTs often set up proportions but do not understand what the ratios represent in the context of the situation.

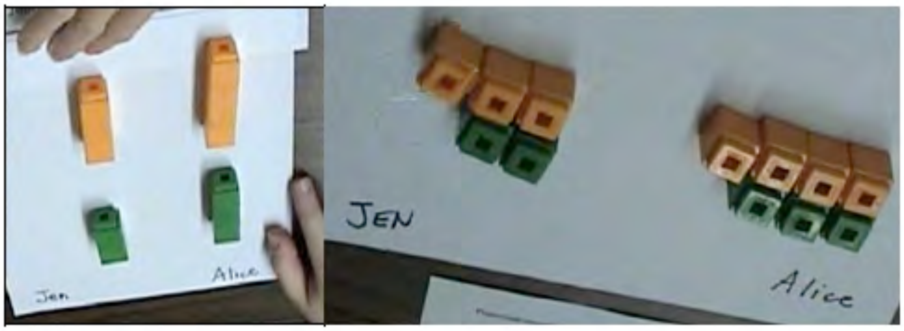
The Lemon/Lime task was used to challenge the PSTs' misconception about quantitative reasoning and computation (see figure 5). In this task, participants were asked to compare two different lemon/lime mixtures (3 lemon:2 lime and 4 lemon:3 lime) to determine which was more lemony, **without doing ANY calculations** but by representing the mixtures with green and yellow unifix cubes. The request to not use calculations posed a high degree of difficulty for most of the PSTs interviewed, because it forced them to reason quantitatively and conceptually rather than computationally.

Jen and Alice are making lemonade. Jen mixes 3 cups of lemon juice with 2 cups of lime juice. Alice mixes 4 cups of lemon juice with 3 cups of lime juice. Whose mixture is more lemonier? Justify your solution without using calculations. You may rearrange the cubes in such a way that they demonstrate which one's more lemony tasting or that they are the same. (Heinz [2000, p. 70])

Figure 5. Lemon/Lime Problem.

60% of the PSTs interviewed used an additive relationship when they reasoned without calculations, claiming that there was "one more cup of lemon in each mixture so the mixtures were the same." However, when allowed to utilize calculations, these same PSTs created a multiplicative relationship (i.e. 3/2 = 1.5 and 4/3 = 1.333) by dividing the quantities in order to compare the decimal representations of the mixtures. Their calculation of the relationship caused them to reevaluate the original statement that the mixtures were the same and state that the 3 lemon:2 lime mixture had more lemon taste than the 4 lemon:3 lime mixture. But why did the PSTs not recognize the multiplicative relationship when reasoning quantitatively (without calculations)? What is surprising is how the PSTs used the cubes when they initially reasoned about the situation. The 40% of the PSTs who utilized multiplicative reasoning ALL created models where the green and yellow cubes were separated (see figure 6), while the 60% who reasoned additively all created models with the green and yellow cubes attached (see figure 7).

Separating the lemon from the lime allowed the PSTs to recognize the multiplicative relationship between the two quantities and not focus on the fact that there is one-cup difference between the two mixtures. In contrast, those representations created by the PSTs where the lemon and lime remained attached seemed to force the PSTs' focus on the fact that there was one more cup in each mixture. The reason may be because the attached cubes resemble lines that would have length. Kaput and West (1994) found that there was a strong tendency to adopt additive reasoning when problems involved linear measurements. So by, separating the cubes by color the PSTs were able to attend to both quantities multiplicatively because their prior assumptions about length would not have been brought forward as strongly by the representation. Knowing this distinction we can create powerful discussions about the quantitative reasoning necessary for solving proportional problems.



(a) (b)

Figure 6. Examples of PSTs representation of the Lemon/Lime problem that utilized multiplicative reasoning.

(a) (b)

Figure 7. Examples of PSTs representatoin of the Lemon/Lime problem that utilized additive reasoning.

Conclusion

In all of the cases presented in this report, PSTs used representations to clarify and explain their proportional reasoning. Whether they used tables, drawings, pie charts, or unifix cubes, the models that represented the context and particular situation of the tasks led to reasoning that had deep meaning. Lo (2004) suggests that providing pre-service teachers with mathematical tasks that are rich in context and encouraging them to develop drawings and representations that convey the meaning of their solution methods to other students deepens their mathematical reasoning. The use of representations when teaching proportional reasoning can provide opportunities to distinguish between proportional and non-proportional situations, explain part to part relationships involved in ratios, and support students' multiplicative reasoning necessary for the development of deep proportional reasoning. Lobato and Ellis (2010) discuss the use of representations in many of their proposed essential understandings of ratio and proportion; however, the role of representation in developing and modeling reasoning is not given the priority it warrants.

This study suggests that greater importance should be given to representations that students produce while solving proportional problems and that the use of multiple representations while

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teaching about ratio will lead to deeper understanding of the concept of proportion. Representations allow individuals to attend to important aspects of their reasoning, including the two quantities involved in the ratios, the context of the problem, and the multiplicative relationship needed in proportional reasoning. Yetkiner and Capraro's (2009) research summary for National Middle School Association stated that until teachers can develop specialized content knowledge in multiplicative and proportional reasoning, they would struggle to provide students with multiple representations that can address the different learning styles found in their classroom. As mathematics teacher educators we should begin to address the difficulties PSTs have with proportional reasoning by providing multiple representations in our own classrooms and discussing their benefits. This report illustrates several representations of proportional reasoning that proved to be useful.

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CONNECTION BETWEEN TOPIC-SPECIFIC TEACHER KNOWLEDGE AND STUDENT PERFORMANCE IN LOWER SECONDARY SCHOOL MATHEMATICS

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The interpretive cross-case study focused on the examination of connections between teacher and student topic-specific knowledge of lower secondary mathematics. Two teachers were selected for the study using non-probability purposive sampling technique. Teachers completed the Teacher Content Knowledge Survey before teaching a topic on division of fractions. The survey consisted of multiple-choice items measuring teachers' knowledge of facts and procedures, knowledge of concepts and connections, and knowledge of models and generalizations. Teachers were also interviewed on the topic of fraction division using questions addressing their content and pedagogical content knowledge. After teaching the topic on division of fractions, two groups of 6th grade students of the participating teachers were tested using similar items measuring students' topic-specific knowledge at the level of procedures, concepts, and generalizations. The cross-case examination using meaning coding and linguistic analysis revealed topic-specific connections between teacher and student knowledge of fraction division. Results of the study suggest that student knowledge could be reflective of teacher knowledge in the context of topic-specific teaching and learning of mathematics at the lower secondary school.

Keywords: Teacher Knowledge, Rational Numbers, Mathematical Knowledge for Teaching

Purpose of the Study

In the last several decades, research on teacher knowledge initiated by work of Shulman (1986) has focused on teacher knowledge as a major predictor of student learning and achievement. Since then the field benefited from numerous studies that substantially advanced the conceptualization of teacher knowledge. Scholars (e.g., Chapman, 2013; Izsak, Jacobson, & de Araujo, 2012) examined different facets of teacher knowledge without explicitly emphasizing its connection to student learning. Studies also stressed the importance of the kind of knowledge a teacher possesses because it impacts his/her teaching (Steinberg, Haymore, and Marks, 1985). Another line of research (e.g., Hill, Rowan, & Ball, 2005; Baumert et al, 2010; Author, 2011) specifically targeted the effects of different types of teachers' knowledge on student achievement.

Recently, scholars have advanced the field by examining teacher knowledge in variety of domains including number sense (Ball, 1990), algebra (McCrory et al., 2012); geometry and measurement (Nason, Chalmers, & Yeh, 2012), and statistics (Groth & Bergner, 2006). However, the field lacks research that provides an in-depth analysis of the various facets of teacher knowledge and its connection to student knowledge at *a topic-specific level*. To know what kind of teacher knowledge impacts student learning in the topic-focused context is an important issue worth of studying. Considering the importance of topic-specific knowledge, this study was guided by the following *research questions*: (1) Does what a teacher knows matter in regard to her students' topic-specific knowledge and performance? (2) What is the nature of topic-specific connections between teacher and student knowledge?

Conceptual Frame: Topic-Specific Content Knowledge

Division of fractions is one of the topics in lower secondary school mathematics curriculum for grade 6th in Russia (Ministry of Education and Science of Russian Federation, 2004) where the study was conducted. Scholars (Ball, 1990; Ma, 1999) found that the teachers have limited topic-specific content knowledge and they lack conceptual understanding of the topic. One of the main reasons is that the topic of division of fractions is traditionally taught by using the "flip and multiply" or "cross-multiply" procedure (e.g., invert-and-multiply algorithm) without helping learners to understand why it works (Siebert, 2002).

Although some teachers consider the traditional algorithm as one of the efficient procedures to divide fractions, they lack understanding of its connection to the inverse nature of division and multiplication (Flores, 2002; Lamon, 1999). Moreover, profound understanding of division of fractions requires connections to other topics such as measurement and sharing/ partitioning interpretations of division (Ball, 1990; Flores, 2002). Other important meanings of fraction division are "finding a whole given a part", "missing factor problem" interpretation of fraction division (Flores, 2002), and "the common-denominator algorithm" (Sharp and Adams, 2002).

In order to develop students' knowledge and comprehension of fraction division teachers themselves need to understand underlying meanings of the algorithms and procedures (Ball, 1990) to make their mathematical knowledge connected and conceptual (Ma, 1999). To be connected topic-specific teacher knowledge should address different *cognitive types*: knowledge of facts and procedures, knowledge of concepts and connections, and knowledge of models and generalizations (Author, 2011).

Analyzing cognitive types of teacher knowledge and its connection to student knowledge within a topic-specific context will contribute to the field of mathematics education and provide tools to enhance teacher education and professional development in order to improve student learning.

Methodology

The interpretive cross-case study (Merriam, 1998) focused on the topic-specific connections between teacher and student knowledge of lower secondary mathematics. Two teachers were selected for the study. Teachers completed the Teacher Content Knowledge Survey (TCKS) before teaching a topic on division of fractions. The TCKS consisted of 33 items measuring teachers' knowledge of facts and procedures, knowledge of concepts and connections, and knowledge of models and generalizations. Teachers were also interviewed on the topic of fraction division using questions addressing their content and pedagogical content knowledge. After teaching the topic on fraction division, students of the participating teachers were tested using similar items measuring students' knowledge of procedures, concepts, and generalizations. The cross-case examination was performed using meaning coding and linguistic analysis techniques (Kvale & Brinkmann, 2009) to report connections between teacher and student knowledge of fraction division.

Participants

The study participants were selected using non-probability purposive sampling technique based on the following set of criteria: 1) selected teachers should represent upper and lower quartiles of the total scores on the TCKS; 2) selected teachers should have similar teaching experience; 3) selected teachers should have similar teaching assignments; 4) selected teachers should teach at similar school settings.

The TCKS was administered to the initial sample of lower secondary (grades 5-9) mathematics teachers (N=90) in Russia (Author, 2015) and then the sample was subdivided by quartiles using teachers' overall TCKS scores. The maximum teacher score on the TCKS was 27 (out of 33) and the minimum score was 13. With regard to the first criteria, the overall sample of teachers was reduced to 43 teachers: 17 of which represented the upper quartile with the range of total TCKS scores 24-27 and 26 teachers represented the lower quartile with the range of total TCKS scores 13-18. After applying the remaining set of criteria 2)-4) we identified two teachers who met the requirements of the purposive sampling (names of the teachers are changed to keep the data anonymous) - Irina (with a total score of 25 on the TCKS) and Marina (with a total score of 17 on the TCKS). Both selected subjects are experienced lower secondary mathematics teachers and both of them are females of the same ethnic origin. Irina has 33 years of teaching experience and Marina - 21 years of teaching experience. Participants have similar teaching assignments - 5-8 grade mathematics with content addressing the following main objectives: Arithmetic, Algebra and Functions, Probability and Statistics, and Geometry. They both teach at urban public schools with similar student population concerning its' ethnic distribution and SES level.

We purposefully selected two contrasting cases with regard to teachers' mean scores on different cognitive types of content knowledge to closely examine the impact of teacher topic-specific knowledge on student performance while solving a set of problems related to division of fractions. The cluster sample of N=55 6th grade students of participating teachers (29 students in Irina's group and 26 students – in Marina's group) was used for collecting student level data after they studied a topic on division of fractions. The topic was a part of the chapter on operations with rational numbers placed in the 6th grade mathematics curriculum at the beginning of the fall quarter (Ministry of Education and Science of Russian Federation, 2004). Additionally, both Irina and Marina were teaching mathematics to the participating cohorts of students for the second consecutive year starting at 5th grade. Therefore, one may say that they established a certain teaching and learning "history" with these students.

Data Sources

The study used the following data sources: 1) TCKS to collect data on cognitive types of teacher knowledge; 2) structured teacher interviews on the topic of division of fractions; and 3) student data on solving three tasks related to the topic of division of fractions.

TCKS is the instrument that was designed to assess teacher content knowledge based on the cognitive types identified above. The survey consisted of multiple choice-items addressing main topics of lower secondary mathematics: Arithmetic, Algebra and Functions, Probability and Statistics, Geometry and Measurement. Specification table along with item analysis was performed to ensure content and construct validity of the TCKS along with its' reliability measured by the Cronbach alpha coefficient at *.839* (Author, 2011).

Teachers were interviewed using two sets of questions related to the topic of fraction division. First set of questions was aimed at tapping into teachers' pedagogical content knowledge whereas the second set was focused on different cognitive types of teacher content knowledge.

Students' written work on solving three tasks related to similar questions on division of fractions was collected and analyzed to examine connections to teacher knowledge. We purposefully used similar questions for teachers and students in order to trace linguistic, procedural, and conceptual traits in their reasoning as well as to analyze non-parametric quantitative trends in student topic-specific knowledge.

Data Analysis

For the qualitative phase of analysis, the teacher interviews were audio recorded and transcribed. Student written work was collected after the completion of the unit on division of fractions. In order to respond to the research questions we conducted meaning coding and linguistic analysis (Kvale & Brinkmann, 2009) of teacher narratives as a primary method of analysis. The data-driven meaning coding technique was used for the purpose of "breaking down, examining, comparing, conceptualizing and categorizing data" (Strauss & Corbin, 1990, p. 61). The linguistic analysis technique addressed "the characteristic uses of language, … the use of grammar and linguistic forms" (Kvale & Brinkmann, 2009, p. 219) by participating teachers and students within the topic-specific domain of lower secondary mathematics. The meaning coding and linguistic analysis was performed and cross-checked by two of the co-authors of this paper.

Considering ranked nature of the quantitative data collected in the study, we employed non-parametric technique (Chi-square test of goodness of fit) to detect group differences using frequency data in student responses.

Results

In this section, we will present major findings of the study starting with teacher responses on pedagogical content knowledge questions. Then we will report data on questions and tasks focused on teacher content knowledge and student knowledge of fraction division. Finally, we will present results of the quantitative analysis of students' performance on selected tasks.

In the qualitative phase of the study, we conducted structured interviews with two of the study participants – Irina and Marina. Irina's mean scores on the TCKS items are as following: score on items measuring knowledge of facts and procedures – 80%, items measuring knowledge of concepts and connections – 46%, and items measuring knowledge of models and generalizations - 30%. Irina's total TCKS score is 51%. Marina's mean scores on the TCKS are as following: knowledge of facts and procedures – 90%, knowledge of concepts and connections – 69%, knowledge of models and generalizations – 70%, and total score – 75%.

The qualitative phase of the study included two stages: (1) teacher interview and (2) student problem solving. The teacher interview consisted of the following two sets of questions: a) the subset of questions 1)-2) tapping into teachers' pedagogical content knowledge and aiming at teachers' understanding of learning objectives for the topic of fraction division; and b) the subset of questions 3)-6) focusing on teachers' possession of cognitive types of content knowledge. The first subset included the following questions: 1) When you teach fraction division, what are important procedures and concepts your students should learn? 2) What is the meaning(s) of division of fractions?

The second subset consisted of the following questions: 3) What is the fraction division rule? 4) Divide two given fractions 1 3/4 and 1/2. 5) Construct a word problem for the fraction division from the previous question. 6) Is the following statement $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$ *(a, b, c,* and *d* are positive integers) ever true?

Responses were audio recorded and teachers were provided with a scratch paper. We used open coding followed by axial coding technique (Strauss & Corbin, 1998) applied to the transcribed narratives to analyze meaning expressed and language used in teachers' responses. Below we present teachers' narratives to the first two pedagogical content knowledge questions.

Teachers' Responses to Pedagogical Content Knowledge Questions

Participants' responses to the question 1 is transcribed below. Based on Irina's response to the first question, it is evident that she capitalizes on her procedural knowledge with little or no attention to concept development. There is a slight indication of applying the rule in "standard situations" (lines 8-9 of Irina's interview excerpt for the question 1) with no further clarification on the nature of this application. Whereas Marina extends the application of the fraction division rule to the "non-routine problem solving situations" (lines 6-7 of Marina's interview excerpt for the question 1). Also, we thought that Irina's reference to "factorization of polynomials" in teaching fraction division was not further elaborated by her and, therefore, was confusing.

IRINA

Before introducing the fraction division, I would like my students to recall the topic on factoring a polynomial, recall the rule of fraction multiplication, and recall reciprocals. After the lesson on fraction division I expect my students to know fraction multiplication and division rules, acquire skills to use these rules *in standard situations* [emphasis added], as well as apply factorization of polynomials.

MARINA

When I teach fraction division, first of all, I expect students to learn fraction division rule as it applies to the case of common fractions. Then, I expect them to know how to apply the rule to mixed fractions. Further, students need to understand how to use the fraction division *in routine and non-routine problem solving situations* [emphasis added]. Pedagogy wise, I always support students' motivation through engaging students in small group work and classroom discussion.

Responding to the question 2, Irina used the part-to-whole interpretation of fraction division whereas Marina offered two different but somehow related interpretations of the meaning for division of fractions.

IRINA

Well… there are two main problems in school arithmetic: finding a part of a whole and finding a whole knowing its' part. Said that, the meaning of *fraction division is finding a whole knowing its' part* [emphasis added].

MARINA

From my perspective… There are at least two meanings for division of fractions. First meaning is based on the interpretation of *division as operation opposite to multiplication* [emphasis added]. In other words… to divide a fraction A by a fraction B means to find a fraction C such as A=B x C. For example, 5/6 divided by 1/6 means that there is a fraction C such as 5/6 = 1/6 x C. Or 5/6 ÷ 1/6 =5. On the other hand, *division is kind of "sorting"* [emphasis added]. For instance,1/2 = 1/4 + 1/4 = 1/4 x 2 meaning that 1/4 goes 2 times into 1/2 whereas 1/2 goes 1/2 times into 1/4. Hope it makes sense… [smiles]

Teachers' Responses to Content Knowledge Questions

Irina's response to the question 3 further confirmed that she has well-established procedural knowledge of the fraction division rule.

IRINA: The rule of fraction division is reduced to the rule of fraction multiplication. Therefore, you need to multiply the *first fraction* [emphasis added] by the reciprocal of *the second one* [emphasis added].

INT: What do you mean by reduced to fraction multiplication?

IRINA: As students say, *cross multiply* [emphasis added] fractions.

Marina's response to the question 3 is depicted below. Surprisingly, Marina used a similar conclusion connecting fraction division to multiplication as Irina did in her response to the same question.

MARINA: In order to divide fractions, you need to multiply *dividend* [emphasis added] by the reciprocal of the *divider* [emphasis added]. For example, $\frac{15}{4} \div \frac{3}{10} = \frac{15}{4} \times \frac{10}{3} = \frac{25}{2}$ (writes on a scratch paper). Generally speaking, fraction division "boils down" to multiplication.

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Irina's response to the question 4 consisted of the solution only (she wrote it on a scratch paper) without any commentary: $1\frac{3}{4} \div \frac{1}{2} = \frac{7}{4} \times \frac{2}{1} = \frac{7}{2} = 3.5$.

Unlike Irina, Marina supported her response to the question 4 with step-by-step comments.

MARINA: First, we convert given mixed fraction 1 3/4 to common one 7/4. Notice, here the numerator is larger than denominator. Then, we replace division by multiplication *reversing the divider* [emphasis added]. Hence, $1\frac{3}{4} \div \frac{1}{2} = \frac{7}{4} \times \frac{2}{1} = \frac{7}{2}$ (writes on a scratch paper).

With regard to the question 5, it took a while for Irina to think about the problem. Then Irina clarified whether she can write down the word problem she came up with on a scratch paper.

IRINA: May I write down the problem on the paper?
INT: Yes, of course.
IRINA (writes on a scratch paper): Area of a rectangle is equal to 1 3/4 cm^2, its length is equal to 1/2 cm. Find width of the rectangle.

Irina was consistent in applying the part-to-whole interpretation in her response, more specifically – using "the missing factor problem" as a meaning for division of fractions (Flores, 2002).

After some thinking, Marina offered the following word problem in her response to the question 5.

MARINA: Here is my word problem: an automated machine packs butter in 1/2 kg bricks. How many bricks one can make out of 1 3/4 kg of butter?
May I draw a picture!?
INT: Sure.
MARINA: (draws a picture on a scratch paper)

We noticed that Marina herself offered drawing a picture to illustrate her word problem.

In Irina's answer to the question 6, she basically repeated her response to the question 2.

IRINA: The given statement is not correct. In order to divide fractions you need to multiply the *first one* [emphasis added] by a reciprocal of the *second one* [emphasis added].

Question 6 was the most challenging to Marina. Nonetheless, she confessed that she liked it.

MARINA: I like this question. It makes me think.
INT: Good.
MARINA: Alright, notice that in order to solve this problem *ac/bd* should be equal to *ad/bc*. Right?
INT: So…
MARINA: Therefore, $c/d = d/c$. This is possible only if $c = d$.

Students' Responses to Fraction Division Questions

At the stage of student problem solving, we asked groups of 6th grade students of participating teachers (Irina's group had n=29 students and Marina's group n=26) to solve a subset of questions corresponding to different cognitive types of content knowledge similar to those presented to teachers: 1) Divide two given fractions. 2) Construct a word problem for fraction division from the previous question. 3) Is the following statement $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$ *(a, b, c,* and *d* are positive integers) ever true?

Questions were presented to students after they completed a chapter on the basic operations with fractions. Student wrote down their responses on a paper and student work was collected for further analysis. The number of correct students' responses on questions 1-3 along with the chi-square statistic comparing student performance between groups on each question is presented in the table 1 below.

Table 1. Chi-Square Analysis' Results of Student Responses on the Fraction Division Questions

	Question 1	Question 2	Question 3
Irina's Group (n=29)	28	12	1
Marina's Group (n=26)	25	15	10
Chi-square and *p*-value (df=1)	χ^2=.41 *p*=.522	χ^2=.88 *p*=.348	χ^2=8.43 *p*=.0037

As we mentioned before, questions selected for the student problem solving session reflected different cognitive types of content knowledge on the topic of division of fractions.

Discussion: Does What a Teacher Knows Matter?

The most important finding of the study was the evidence collected and analyzed in support of the first research question: what a teacher knows matters in regard to his/her students' topic-specific knowledge. As we expected based on similar teachers' scores on cognitive type 1 items (measuring knowledge of facts and procedures), there was no difference observed between student performances in two groups on the procedural question 1. There was some difference, not significant though, observed on the question 2 measuring knowledge of concepts and connections (in favor of students in Marina's group). The most evident difference between student performance in two groups was observed on the question 3 (measuring knowledge of models and generalizations) was statistically significant (χ^2*=8.43, p=.0037).* We were surprised by the partially-correct student's response from Irina's group considering the fact that Irina herself was not able to correctly solve the question. Overall students' responses were reflective of their teachers' knowledge: student performance in Marina's group was stronger than in Irina's group, particularly in solving questions 2 and 3 with difference in responses to question 3 being statistically significant.

The data collected and analyzed to respond to the second research question - What is the nature of topic-specific connections between teacher and student knowledge? – revealed that teacher's mastery of cognitive types of content knowledge is associated with the students' topic-specific knowledge. Thus, findings of this study contribute to the body of research claiming that teacher content knowledge is critical for student learning (Hill, Rowan, & Ball, 2005; Baumert et al., 2010). Teacher interviews and students' problem solving helped us to closely look at the nature of the relationship between teacher knowledge and student performance.

We are cognizant that the study had its limitations such as teacher sample size, multiple-choice format of the teacher content knowledge survey, to name a few. Following on the discussion about complexities of assessing teacher knowledge (Schoenfeld, 2007), we are aware of the limitations of the multiple-choice format in test construction and assessment of teacher knowledge (p. 201). Therefore, we included the qualitative phase of the study to zoom further into teacher knowledge and understanding. We are also cognizant that classroom observations could be another source of data in this study. However, to explicitly address the research questions we purposefully focused the study on the link "teacher knowledge-student performance" in the topic-specific context. Considering these limitations, we are sensitive enough to not overgeneralize the results obtained in the study. The major findings of this study open an opportunity to discuss the importance of different cognitive types of the topic-specific teacher knowledge and its potential impact on student knowledge and learning.

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ESTRATEGIAS DIDÁCTICAS Y CONOCIMIENTO ESPECIALIZADO DE PROFESORES DE MATEMÁTICAS. UN CASO EN ÁLGEBRA ESCOLAR

TEACHING STRATEGIES AND MATHEMATICS TEACHER'S SPECIALIZED KNOWLEDGE. A CASE IN SCHOOL ALGEBRA

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Presentamos los resultados del análisis de los conocimientos puestos en acción por una maestra de matemáticas de secundaria en su práctica en el aula. Éstos toman forma y se despliegan como estrategias didácticas específicas para gestionar su clase cuando incorpora Computer Algebra Systems. A partir de observaciones no participantes de clases cotidianas y entrevistas, encontramos que los conocimientos de esta maestra (matemáticos, didácticos y tecnológicos) son movilizados en varios momentos y a través de diversas estrategias didácticas. Ello depende de una amplia diversidad de factores entre los que destacan el objetivo de la planeación, el momento específico de la clase, las participaciones de los estudiantes y el uso de la herramienta tecnológica. En esta complejidad la maestra aplica sus estrategias didácticas de manera flexible y logra controlar e incluso modificar la gestión de su clase.

Palabras Clave: Conocimiento Matemático para la Enseñanza, Álgebra y Pensamiento Algebraico, Educación Secundaria, Technology

Introducción

Para acercarse al conocimiento profesional del profesor de matemáticas y sus prácticas han surgido diferentes marcos de análisis, metodologías y propósitos que aún siguen en refinamiento y en discusión al interior de la propia comunidad (Ball, Thames y Phelps, 2008; Ponte y Chapman, 2006; Gaeber y Tirosh, 2008). Ponte y Chapman (2006) sugieren tomar en cuenta la estrecha relación de este tipo de conocimiento y la práctica, las condiciones de trabajo y los objetivos explícitos e implícitos de dicha labor. Por su parte, Davis (2014) señala que el conocimiento necesario para un profesor de matemáticas es una red compleja en la que interactúan "una mezcla de varias asociaciones/instanciaciones de conceptos matemáticos y una conciencia de procesos complejos en los cuales se producen las matemáticas" (p. 155). De hecho, el conocimiento del profesor es "enactuado (puesto en acción/*To be enacted*) e implícito" (p. 155).

La investigación sobre el uso de Tecnologías Digitales (TD) para la enseñanza de las matemáticas es amplia. Se ha encontrado que la incorporación de Computer Algebra Systems (CAS) a los salones de clases generan cambios en las prácticas matemáticas (Pierce y Stacey, 2004) y que el profesor es central para proveer condiciones que ayuden a los estudiantes a su comprensión matemática, con uso de estas herramientas (McFarlane, Williams y Bonnett, 2000).

Elegimos el tema de la enseñanza del álgebra con CAS debido a que una de las ventajas de esta herramienta es que puede usarse para proveer de significado a las transformaciones y expresiones algebraicas. Numerosas investigaciones han mostrado evidencia sólida de que la tecnología puede ser un elemento activo en la construcción de significados de los conocimientos algebraicos. (Puig & Rojano, 2004; Hitt y Kieran, 2009; Kieran y Drijvers, 2006; o bien por Solares y Kieran, 2013.) Nuestro estudio se ubica en esta línea, centrándose en el estudio de los conocimientos y las prácticas docentes en clases cotidianas de álgebra con CAS. Buscamos profundizar en la comprensión de cómo los profesores movilizan estos conocimientos (matemáticos, didácticos y tecnológicos) y les dan sentido en términos de sus prácticas (Ponte y Chapman, 2006). Específicamente buscamos responder: ¿Cómo los conocimientos didácticos y matemáticos se ponen en acción durante una clase en la que se usa CAS? y ¿cómo estos conocimientos se manifiestan por medio de las estrategias didácticas?

Perspectiva Teórica: Conocimiento Especializado del Profesor de Matemáticas

Nos interesa identificar los conocimientos puestos en acción por un profesor cuando enseña álgebra con el uso de CAS a partir del análisis de sus estrategias didácticas. Para considerar la especificidad de este conocimiento respecto a la enseñanza del álgebra escolar y para efectos de análisis partimos del modelo del *conocimiento especializado del profesor de matemáticas* (MTSK por sus siglas en inglés) (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013).

Este modelo se refiere al conocimiento específico (en su conjunto) del profesor de matemáticas el cual se compone de dos dominios: Conocimiento Matemático (MK) y Conocimiento Didáctico del Contenido (PCK). El primer dominio está compuesto de tres subdominios: Conocimiento de los Temas, de la Estructura de la Matemática y de la Práctica Matemática. El segundo, el Conocimiento Didáctico del Contenido, está compuesto por: Conocimiento de la Enseñanza de las Matemáticas, de las Características del Aprendizaje de las Matemáticas y de los Estándares de Aprendizaje de las Matemáticas.

Por las características de nuestro objeto de investigación, nos centramos en el segundo dominio (PCK). A continuación, describimos brevemente en qué consiste cada uno de los tres subdominios que lo componen. El *conocimiento de la enseñanza de las matemáticas* (KMT) está vinculado con estrategias y teorías de enseñanza, materiales y recursos vinculados con el contenido a enseñar; el *conocimiento de las características del aprendizaje de las matemáticas* (KFLM) incluye asuntos sobre cómo se aprenden ciertos conceptos, intuiciones, errores y formas como los alumnos interactúan con cierto contenido matemático y, finalmente, el *conocimiento de los estándares de aprendizaje de las matemáticas* (KMLS) se refiere a lo qué es esperable (en términos del propio currículo) en un nivel escolar determinado incluyendo también las indicaciones y formas de aprender dicho contenido desde un "referente estandarizado" (p. 598).

Lo anterior da cuenta de la complejidad en la que está inmersa la tarea de enseñanza de un profesor al momento de impartir su clase. Sin embargo, no todos los conocimientos se adquieren en instancias institucionales, muchos de ellos se construyen en la práctica de aula, en el intercambio con otros colegas y en el propio contexto donde se realiza la tarea de enseñar. Desde nuestra perspectiva los conocimientos de los profesores para la enseñanza de las matemáticas se ponen en acción y toman forma específica como estrategias didácticas que ellos modifican al momento de gestionar su clase.

Metodología

Los tres investigadores participamos en el análisis de los videos de las clases de una maestra de matemáticas de secundaria en México (voluntaria) con experiencia en el uso de tecnología y con disponibilidad para ser grabada. Esta maestra participó previamente, junto con otros profesores, en un taller de 20 horas sobre CAS con la calculadora (TI-82). De manera general, el taller trató sobre cómo resolver problemas algebraicos usando las funciones *factorizar, desarrollar, resolver* y *evaluar*. La discusión en el taller estuvo centrada en las diferentes maneras en las que se puede resolver un problema algebraico con calculadora, así como cuándo es adecuado usar CAS en una clase, cómo y para qué. Decidimos observar y grabar dos clases regulares de la maestra, sin que los investigadores determináramos el tema y tratamiento didáctico al mismo. La maestra durante el taller mostró tener conocimiento de las reglas del CAS y su uso con la calculadora, así como, habilidades para explorar nuevas situaciones relacionadas con las reglas propias de esta herramienta y proponer nuevos usos a sus alumnos. Cabe señalar que se han reportado resultados de otro estudio

(García-Campos & Rojano, 2008) en el que se menciona que para que el uso de CAS logre impactar en la práctica docente es necesario brindar mayor acompañamiento a los profesores, pues a corto plazo éstos no logran incorporar CAS de manera cotidiana en el aula.

En este reporte nos centraremos en los resultados del análisis de la práctica de la maestra Clementina, de quien sabemos que en el momento de la investigación tenía más de 20 años de experiencia dando clases en secundaria, dominaba los contenidos a enseñar y conocía las componentes de la calculadora que permiten tener, de manera simultánea, distintas representaciones para los objetos algebraicos (tablas, gráficas y expresiones algebraicas), y las instrucciones para resolver ecuaciones.

Para el análisis de los datos se hizo una transcripción de los episodios que los tres investigadores seleccionaron al haber identificado conocimientos didácticos del contenido expresados a través de las siguientes estrategias didácticas.

Estrategias Didácticas como un Acercamiento a Aspectos del Conocimiento Didáctico del Contenido Matemático

Los autores proponemos estas tres estrategias didácticas como resultado de un estudio más amplio en el que se ha estado trabajando mediante el análisis de videos de clases de más profesores de matemáticas de secundaria (incluso telesecundaria). En dichas clases hemos identificado estas estrategias, las cuales dan cuenta además de algunos de los conocimientos especializados que los profesores generan y movilizan al momento de impartir sus clases (*To be enacted*) como lo señala Davis (2014). Los resultados de tal investigación serán comunicados ampliamente en un futuro.

En el análisis de los videos de las clases de Clementina identificamos que su conocimiento especializado se manifestó en tres estrategias didácticas:

- Mantenimiento de la planeación de la clase. Consiste en la toma de decisiones para mantener el desarrollo de la clase de tal manera que se cumplan los propósitos de aprendizaje que planteó el profesor considerando el contenido del currículo a enseñar y los recursos a su disposición. Estas decisiones son, por ejemplo, elección de soluciones, procedimientos y errores para discutir o mostrar con el grupo completo, recapitulaciones, balances, formalizaciones, etc. En esta estrategia se hace énfasis en los conocimientos del profesor en los subdominios KMT y KMLS.
- Rol otorgado a los estudiantes. Un mismo profesor puede promover distintas formas de participación en sus estudiantes en una misma clase (KMT). Por ejemplo, que exploren sus soluciones y las expongan, que pasen al pizarrón o usen CAS (con el TI presenter) para explicar sus procedimientos, soluciones o hipótesis, o que simplemente sigan las instrucciones.
- Usos de las herramientas tecnológicas. Los profesores proponen el uso de los recursos tecnológicos disponibles para verificar resultados, explorar procedimientos y soluciones, aplicar técnicas, entre otras. (KMT y KFLM)

En este reporte nos centraremos en los resultados del análisis de la práctica de una maestra que llamaremos Clementina. De la entrevista, sabemos que en el momento de la investigación Clementina tenía más de 20 años de experiencia dando clases en secundaria, dominaba los contenidos a enseñar y conocía las componentes de la calculadora que permiten tener, de manera simultánea, distintas representaciones para los objetos algebraicos (tablas, gráficas y expresiones algebraicas), y las instrucciones para resolver ecuaciones.

Análisis y Discusión. El Caso de Clementina

A continuación, presentamos resultados del uso de las estrategias para dar cuenta del tipo de conocimiento que Clementina moviliza en la clase. Este análisis nos permite describir la dinámica de cómo las componentes del MTSK se entretejen. La maestra tiene una *planeación* detallada y precisa de su clase usando CAS. Ha preparado un problema específico. Al parecer, ella misma lo redactó o hizo la adaptación correspondiente. Tanto en la planeación como en el desarrollo de la clase se evidencian dominio y experiencia en el manejo de las ecuaciones de primer grado (KMLS), cómo enseñarlas (KMT) y las diferentes posibilidades de sus alumnos para abordar la tarea propuesta (KFLM).

Breve Descripción de las Acciones de la Maestra en la Clase

Al inicio de la sesión, la maestra dicta el problema, el objetivo y da instrucciones de cómo trabajar en la clase, como se muestra en el siguiente fragmento.

> Maestra: Tema: Ecuaciones lineales. Propósito general: Despejar literales en diferentes tipos de fórmulas. Relacionar una fórmula con la tabla de datos que genera y con su gráfica.
> Problema: En una tienda se compran 9 paquetes de libros, el cual tiene un precio adicional de $75 [sic]. ¿Cuánto cuesta cada paquete si en total se pagan $1400?
> Maestra: Leer el problema. Ya que lo tengas planteado, puedes usar la calculadora para resolverlo. Lo pueden resolver entre tres o dos o uno, los que sean en cada mesa.
> [Fragmento 1, sesión 1]

A pesar de que sus clases son muy estructuradas y se desarrollan de acuerdo a la planeación definida, se observa que en su práctica siempre da un momento de reflexión y lectura del problema, seguido de uno de exploración con los recursos y estrategias que los alumnos decidan. Los alumnos usan la calculadora libremente y levantan la mano para preguntar dudas a la maestra. Ella recorre el salón, pasando entre los equipos y aclarando las dudas que surgen. Consideramos que ésta es una manifestación de su conocimiento sobre cómo se aprende este contenido (KFLM).

Un Problema Didáctico no Previsto en la Planeación

Un problema didáctico que emerge en la clase está vinculado con la palabra "adicional" usada en la redacción del enunciado del problema. Los estudiantes tienen problemas para interpretar a qué se refiere y, espontáneamente, comienzan a explorar con valores numéricos específicos. Entonces la maestra interviene de la siguiente manera.

> Maestra: Tu compañera dice que cuando se emplea el término *adicional* se está refiriendo …
> Alumna: A que le tienes que aumentar.
> Maestra: En este caso, ¿cuánto le tienes que aumentar?
> Alumna: 75
> Maestra: ¿Cómo quedaría tu ecuación? […] ¿En qué consiste el problema? Tú vas a elaborar una tabla [se dirige a una estudiante] ¿En base a qué vas a elaborar la tabla?
> [Fragmento 2, sesión 1]

En el fragmento 2 podemos identificar dos intervenciones de la maestra. La primera es para ayudar en la traducción del problema al lenguaje algebraico. La palabra *adicional* se traduce como *aumentar* o *sumar*. La segunda ayuda consiste en sugerir que se enfoquen primero en encontrar una ecuación que modele el problema. Además, les dice a sus alumnos que partir de la ecuación podrán hacer cuentas, encontrar valores y construir la tabla de variación correspondiente, al revés de lo que ellos están haciendo: explorar con valores numéricos para establecer la ecuación. Cabe señalar que en México es una práctica recurrente obtener primero la expresión algebraica y después la tabla o la gráfica. Esto se propicia desde el mismo programa de estudios (SEP, 2011).

Desde nuestra perspectiva, lo anterior es una manifestación de que la maestra aplica la primera estrategia: *mantiene la planeación de la clase*. Aunque los estudiantes exploran con ejemplos numéricos con objeto de establecer las relaciones dadas en el problema, Clementina interviene para orientar esos procedimientos y hallar la ecuación que modele el problema (KFLM y KMT). De esta manera, resuelve el problema didáctico que se le presentó y mantiene su planeación.

Pero las dudas y confusión continúan. Entonces, la maestra hace una intervención grupal. Toma las dudas de uno de los estudiantes para discutirlas y aclararlas grupalmente. El estudiante ha estado probando con varios posibles valores para el costo de cada paquete. Al parecer, ha recurrido al *ensayo y error* (aproximaciones numéricas sucesivas) para ajustar los valores.

Maestra: ¿Qué es lo que quiere saber tu problema? [inaudible] Quiere saber cuánto cuesta un paquete, ¿cuántos paquetes compró?

Alumna 2: Nueve

Maestra: Ese nueve qué le harías. Dice tu compañera que tiene que elaborar una tabla. Pero, ¿cómo lo establecerías? […]

Alumno 3: Mil cuatrocientos treinta y nueve (1 439)

Maestra: Y entre qué lo vas a dividir, ¿y los adicionales? [Siguen discutiendo]

Maestra: Otra vez, un paquete no te puede costar $75. Es adicional. Vamos a considerarlo como un aumento […] ¿Cómo te quedaría tu ecuación? Nuestro tema, ¿qué dice? [los estudiantes leen lo que la maestra dictó al inicio de la clase] por lógica tenemos que sacar una ecuación [...] Muchos ya se fueron a tabular, ¿qué vas a tabular? Yo quiero ver nada más el planteamiento.

[Fragmento 3, sesión 1]

Las intervenciones de la maestra [fragmentos 2 y 3] orientan la clase hacia identificar la incógnita, las relaciones entre los datos y la incógnita, y la igualdad. Nuevamente, mantiene su objetivo principal para esta sesión: obtener una ecuación que modele el problema (KMLS). En este momento ella se enfrenta a otro problema didáctico, pues los estudiantes siguen obteniendo distintos resultados numéricos sin encontrar ninguna ecuación.

Una de las alumnas encuentra una ecuación y la maestra le pide que pase al pizarrón. La alumna escribe "$9x + 75 = 1400$". La maestra elige socializar en el grupo esta solución y, al mismo tiempo, deja sin discutir soluciones diferentes, como las numéricas obtenidas mediante ensayo y error. Así, la maestra privilegia la ecuación resultante de la traducción del problema (y su solución numérica), sin importar las técnicas que se puedan usar para resolver la ecuación, con lápiz y papel o con CAS (presencia de conocimientos KMT y KFLM).

Después de planteada esta ecuación (objetivo principal de la clase), la maestra pide a los estudiantes que la resuelvan, dándoles libertad de tiempo y de elegir usar calculadora o lápiz y papel. Para Clementina, el uso de CAS es opcional. Se trata de la estrategia que define el *uso que se da a la tecnología*.

Una vez resuelta la ecuación pasa a los equipos para presentar sus soluciones, en su mayoría son numéricas. Entonces elige a una alumna que resolvió la ecuación algebraicamente, paso a paso y con lápiz y papel. Esta solución se muestra en la Figura 2(a).

Al percatarse de que la mayoría de los alumnos no están de acuerdo con el valor $x = 147.22$, pregunta qué soluciones encontraron. Aquí la maestra pone en marcha otra de sus estrategias: usa las producciones de sus estudiantes (*rol de los estudiantes)* para contrastarlas, confrontar errores y procedimientos (los últimos en este caso) y hacer aclaraciones (presencia de KFLM y KMT).

9x + 75 – 75 = 1400 – 75 9x = 1325 9x/9 = 1325/9 **x = 147.22** 147.22 x 9 + 75 = 1400 (a)	75 1400.00 725 /9 = 80.55 (residuo 1) x 9 - 675.00 675 0725.00 **R = 80.55** (b)

Figura 2. Soluciones propuestas (a) algebraica y (b) numérica

Ante las dudas que siguen manifestándose, la maestra promueve que los alumnos expongan sus soluciones. Los estudiantes tienen claro que ella había pedido que plantearan una ecuación que modelara el problema, por eso preguntan "¿pero con ecuación?". Entonces, la maestra elige a una alumna que obtuvo una solución numérica, la de la Figura 2(b). Al parecer, el procedimiento de la alumna es aritmético: deshace las operaciones. La maestra aprovecha esta solución para indagar, preguntando: "¿cada paquete dices que cuesta?" Y La alumna responde "$80.55". Posiblemente, la maestra identifica que la diferencia entre los dos procedimientos está determinada por el lugar en que se pone el paréntesis. Es decir, las ecuaciones correspondientes a los procedimientos de la figura 2 serían: $9x + 75 = 1400$, la primera; y $9(x + 75) = 1400$, la segunda. En esta intervención la maestra da muestra de su conocimiento de cómo los alumnos están interactuando con la ecuación y las ideas matemáticas (jerarquía de operaciones, asociatividad) que están detrás de estas propuestas (KLM). Quizás por ello, después de la explicación, pasa a la alumna que resolvió algebraicamente la ecuación para corregirla. Sin embargo, esta alumna agrega paréntesis a la comprobación: (147.22 x 9) + 75 = 1400, para obtener la respuesta correcta. La maestra dice que no se trata de ver quién está bien, sino de quién tiene el procedimiento válido. En sus palabras: "¿Cuál de los dos procedimientos creen ustedes que conviene? Repito: no es por mayoría."

En las intervenciones de la maestra, se evidencia que está enfocada en la estrategia didáctica que le permite enfatizar *el rol de los alumnos*, promoviendo la comparación de las distintas interpretaciones de las relaciones entre los datos y la incógnita del problema que los estudiantes han hecho. Los alumnos se enganchan en esta comparación, dan argumentos tanto a favor como en contra. En nuestro análisis identificamos que la maestra hace explícitas y pone a discusión dos interpretaciones de la palabra "adicional": 1) *adicional por paquete*, esto es, $9(x + 75)$; y 2) *adicional por los nueve paquetes*, $9x + 75$.

En términos de la riqueza de la discusión matemática que los alumnos desarrollan en esta parte de la clase, el problema ha sido fructífero: hay argumentaciones, comprobaciones, se usan los distintos medios tecnológicos y las distintas técnicas (CAS y lápiz y papel). Pero, en este momento, el tiempo de la clase prácticamente se ha agotado; así que el objetivo central de la clase ha quedado incompleto en lo que se refiere a "Relacionar una fórmula con la tabla de datos que genera y con su gráfica". La maestra intenta hacer un cierre de la discusión preguntando lo siguiente:

Maestra: ¿Cuál de los dos planteamientos o soluciones (procedimientos) consideran ustedes que es el correcto, de los problemas?
Alumnos: Los dos.
Maestra: Esto quiere decir que alguno de los dos está fallando [...] No necesariamente tienen que ser exactitos sin sobrar [...] Vamos a redondear [...], a uno le faltan dos centavos y al otro, cinco centavos […] Puedes perder décimas.

[Fragmento 6, sesión 1]

Clementina explica que los dos procedimientos tendrían que dar el mismo resultado y que no es el caso. En su argumentación apela a la unicidad de la solución de una ecuación, tema que al parecer no ha sido abordado en la clase. Tampoco es algo que haya sido considerado en su planeación (posible deficiencia en KMLS).

Hasta antes de este momento, la maestra había entretejido las estrategias de *mantener la planeación* y la del *rol de los alumnos* (manifestaciones de KMT y KFLM). Sin embargo, en este momento toma la decisión de modificar la planeación de su clase adaptándola a las necesidades de sus alumnos, a sus interpretaciones generadas tanto con lápiz y papel como con CAS. En los minutos restantes permite que los alumnos usen estrategias de solución en las que se sienten más cómodos y confiados, no necesariamente los típicos algebraicos (como el despeje de la incógnita). Abre espacios para que la alumna que planteó la ecuación y la alumna que propuso la interpretación aritmética expongan nuevamente al grupo sus interpretaciones y cómo obtener la solución. La maestra le pide a la alumna del procedimiento "aritmético" (Figura 2 (b)) que explique, en el pizarrón. Esta alumna dice que con otra compañera llegaron a la conclusión de que su procedimiento es correcto.

Alumna: Es que ahí en el problema no se comunicaba si eran 75 por cada uno o por todos. Y aquí ya comprobamos que si eran 75 por todos. Porque no puede ser por cada uno porque no queda, serían 1447.95

[La maestra pide a varios equipos que le lean el problema a la alumna para aclarar si los 75 eran por todos por cada uno.]

Maestra: ¿Habla [se refiere al enunciado] del paquete o de todo?

Alumna: Al enunciado le hizo falta una palabra.

Maestra: ¿Cuál sería?

Alumna: Que te dijeran que son 75 pesos adicionales por cada paquete [...]

Maestra: Para eso sirven los problemas, para ver cuando no están bien planteados, cada alumno lo va a resolver de diferente forma… Aquí de lo que se trata, hay que leerlo…

Alumna: si yo fuera maestra este problema tendría dos objetivos… No sólo es leer sino establecer las relaciones entre los datos y la solución.

[Fragmento 6, sesión 1]

Grupalmente, los niños llegan al acuerdo que los 75 adicionales son por los nueve paquetes, y que entonces la ecuación correcta es: $9x + 75 = 1400$. La maestra despliega un conocimiento de la enseñanza que le resulta efectivo en este momento: "cede" el control de la clase a sus estudiantes.

Reflexiones Finales

En el desarrollo de su clase, Clementina se enfrentó a un problema didáctico no previsto en su planeación debido a la redacción ambigua de la situación problemática planteada. Es por ello que hizo adaptaciones al objetivo de la clase, a su planeación y a la participación y los roles que tenía contemplados para sus estudiantes. Además, en este proceso, Clementina adaptó su propio rol, modificó sus expectativas, sus intervenciones e incluso cedió temporalmente el rol de "maestra hipotética" a una de sus estudiantes. Posibilitó el consenso entre el colectivo, cediendo el control de la clase a los estudiantes, y logrando establecer el significado de adicional y su implicación en la ecuación ($9x + 75 = 1400$). En el cierre de la clase retomó el objetivo planteado inicialmente, ubicándolo en el trabajo realizado y sin dejarlo a la deriva. Este tipo de adaptaciones de la planeación, del problema, de los objetivos, de las formas de participación de los estudiantes, pero también de las intervenciones y del rol mismo del profesor requieren de gran flexibilidad y adaptabilidad en los conocimientos especializados de los profesores. Consideramos que esta flexibilidad y adaptabilidad no son previsibles ni tematizables como contenidos a desarrollarse en programas de estudios de formación inicial. Para adquirirlos es necesario construirlos en la práctica. Indagar a mayor profundidad sobre estas estrategias y la interacción entre distintos tipos de conocimientos, puede tener implicaciones fructíferas para la formación de profesores, así como para la propia investigación en el área.

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We present results of the analysis of knowledge used by a secondary school mathematics teacher in her classroom practice. This knowledge takes shape and is displayed as specific teaching strategies in the management of her class when she incorporates Computer Algebra Systems. Based on observations of regular classes, we find that her knowledge (mathematical, pedagogical and technological) is put into action at various moments during the lesson and through a variety of teaching strategies. These strategies depend on many factors, such as the planned objective of the lesson, the specific moment in the class, student participation, and the use of technological tools. Within such complexity, the teacher applies her teaching strategies in a flexible way, and manages to control and even modify the course of her class.

Keywords: Mathematical Knowledge for Teaching, Algebra and Algebraic Thinking, Middle School Education, Technology

Introduction

To approach the professional knowledge and practices of teachers of mathematics, several methodologies and frameworks for analysis have been suggested (Ball, Thames and Phelps, 2008; Ponte and Chapman, 2006; Gaeber and Tirosh, 2008). Ponte and Chapman (2006) propose the importance of considering the complexity of this sort of knowledge and its close relation with practice, working conditions and explicit and implicit objectives. Davis (2014) notes that the knowledge required by a mathematics teacher is a complex network where there is interaction between “a sophisticated and largely enactive mix of various associations /instantiations of mathematical concepts and an awareness of complex processes through which mathematics is produced” (P. 155). In fact, “the most important knowledge for teaching tends to be enacted and tacit” (p. 155).

With respect to the research on the use of digital technology (DT) in the teaching of mathematics, incorporating Computer Algebra Systems (CAS) into classrooms has been discovered to generate change in mathematical practice (Pierce and Stacey, 2004), and the teacher has a key role in providing conditions that help students understand mathematics with these tools (McFarlane, Williams and Bonnett, 2000).

We chose to study the teaching of algebra with CAS because it can be used to give meaning to algebraic transformations and expressions. Many investigations have provided solid evidence that technology can be an active element in building algebraic knowledge (Puig & Rojano, 2004; Hitt and Kieran, 2009; Kieran and Drijvers, 2006; or by Solares and Kieran, 2013.) We seek to understand how teachers mobilize this knowledge (mathematical, pedagogical and technological) and how they make sense of it in terms of their practice (Ponte and Chapman, 2006). We specifically want to know how pedagogical and mathematical knowledge is put into action during a lesson where CAS is used; and how this knowledge is revealed through teaching strategies.

Theoretical Perspective: Mathematics Teacher’s Specialized Knowledge

We have taken into account the *Mathematics Teacher’s Specialized Knowledge* (MTSK) model (Carrillo, Climent, Contreras & Muñoz-Catalán, 2013) as a starting point for analysis and in our consideration of the specificity of the knowledge that teachers put into action when they teach school algebra.

This model refers to the mathematics teacher's specific knowledge, which consists of two domains: Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). The first domain is composed of three subdomains: Knowledge of Topics, of the Structure of mathematics and of the Practice of Mathematics. The second, Pedagogical Content Knowledge is composed of: Knowledge of Mathematics Teaching, of Features for Learning Mathematics and of Mathematics Learning Standards.

We concentrated on the second domain (PCK) due to the characteristics of our research objective. Its three subdomains are briefly described as follows: *Knowledge of Mathematics Teaching* (KMT) is the integration of mathematics and teaching. "It is the kind of knowledge of resources from the point of view of their mathematical content or the knowledge of approaching a structured series of examples to help pupils understand the meaning of a mathematical item" (Carrillo et al, 2013, p. 2991); *Knowledge of Features for Learning Mathematics* (KFLM) include theories and models of how student learn mathematics, that is to said, how certain concepts are learned, intuition, mistakes, and the way students interact with specific mathematics content; and finally, *Knowledge of Mathematics Learning Standards* (KMLS) refers of "curricular specifications, the progression from one year to the next, conventionalized materials for support, minimum standards and forms of evaluation […] We include objectives and measures of performance developed by external bodies such as examining boards, professional associations and researchers" (Carrillo et al, 2013, p. 2991).

In our research, we focus on the knowledge that teachers display within classroom practice and that acquires a specific form as teaching strategies that they modify as they manage their lessons.

Methodology

Three researchers participated in the analysis of videos of a voluntary secondary school mathematics teacher in Mexico. The teacher had already participated in a 20-hour workshop on CAS with calculators (TI-82). In general, the workshop was about solving algebra problems using functions such as *factorize, develop, solve* and *evaluate*. The workshop discussion was centered on the various ways in which an algebra problem can be solved with a calculator, as well as when it is adequate to use CAS in the classroom, how and why. We decided to observe and make video recordings of two of the teacher's regular classes, without a prior decision by the researchers about the topic and how it would be treated from a pedagogical perspective. During the workshop, it was clear that the teacher knew the rules for CAS, as well as the skills to explore new situations related to these rules and propose new uses to students. It should be mentioned that a previous study (García-Campos & Rojano, 2008) states that in order for CAS to impact teaching practices, it is necessary to provide more support for teachers, because in the short term, they do not incorporate CAS on a daily basis in the classroom.

In this report, we focus on the results of the analysis of the practice of Clementina, a teacher who, at the moment of our research, had more than 20 years of experience teaching at secondary school level, had good proficiency of the mathematical topics to be covered, and was familiar with the elements of a calculator that allow a variety of simultaneous representations of algebraic objects (tables, graphs, expressions), and the instructions to solve equations.

Data analysis was developed using a transcription of the episodes selected by the three researchers once they had identified the pedagogical content knowledge expressed by means of the following teaching strategies.

Teaching Strategies as an Approach to Aspects of Pedagogical Content Knowledge (PCK)

We propose the following three teaching strategies that result from a broader study in which we have been working by analyzing videos of secondary school mathematics teachers. The strategies have been identified in these classes and account for some of the specialized knowledge that teachers generate and enact when they are teaching. The results of this investigation shall be published in detail in the future.

In the analysis of the videos of Clementina's lessons, we identified that her specialized knowledge was clear in three teaching strategies:

- Maintaining the lesson plan. This involves making decisions to keep the lesson going in such a way that the learning objectives determined by the teacher according to curriculum content and available resources are achieved. Examples of these decisions include: choice of solutions, procedures and mistakes to discuss or show to the entire group, summaries, assessments, formalizations, etc. The emphasis is on the teacher's knowledge of the KMT and KMLS subdominions.
- The role given to students. Teacher can promote various forms of participation in the same class (KMT). For example, students can be invited to explore and share solutions, to come to the board or use CAS (with the TI presenter), to explain procedures, solutions or hypothesis, or simply to follow instructions.
- The use of technological tools. Teacher proposes the use of technological resources available to verify results, explore procedures and solutions, apply techniques, etc. (KMT and KFLM).

Analysis and Discussion. The Case of Clementina

The results of the use of teaching strategies to account for the type of knowledge promoted in class by Clementina, are presented below. Our analysis allows us to describe how the components of MTSK come together. The teacher has a detailed and precise plan for the use of CAS in her class. She has prepared a specific problem, which she apparently wrote herself. Both planning and progress of the class show proficiency and experience in the solution of linear equations (KMLS), how to teach them (KMT), and the various ways her students will approach the task proposed (KFLM).

Brief Description of the Teacher's Actions in The Lesson

At the start of the session, the teacher dictates the problem and the objective of her lesson plan, and provides the instructions regarding how to work in the lesson, as shown in the following fragment.

> Teacher: Linear equations. General aim: Solving literal equations. Relating a formula to a table of data and its graph.
> Problem: We buy 9 packs of books that have an additional cost of $75 at a store [sic]. How much does each pack cost if the total price is $1400?
> Teacher: Read the problem. When you have posed it, you can use the calculator to find the solution. You can work on your own, or in groups of two, three or as many as there are at each table.
>
> [Fragment 1, Session 1]

Although her classes are very well structured and they develop according to a well-defined plan, we observe that Clementina always provides their students a moment to read and reflect on the problem, followed by time to explore resources and strategies. Students use the calculator freely, and raise their hands when they have questions. The teacher goes around the classroom from one team to another, answering questions as they arise. We consider that this demonstrates her knowledge regarding how to learn this subject matter (KFLM).

A Teaching Issue not Foreseen in Planning

A teaching issue related to the use of the word "additional" in the sentence that formulates the problem arises in the class. Students do not know how to interpret what the word refers to, and spontaneously begin to explore with specific numerical values. Then the teacher takes part in the following way:

> Teacher: Your classmate says that when you use the word *additional,* it means…
> Student: That you have to increase.
> Teacher: In this case, how much do you have to increase?
> Student: 75
> Teacher: What would your equation be like? […] What is your problem about? You are going to make a table [referring to a student] What is your table going to be based on?
> [Fragment 2, Session 1]

In Fragment 2, there are two moments at which the teacher intervenes. The first is to help translate the problem into algebraic terms; the word "additional" means to *increase* or *add* as she helped the student to realize. In the second instance, she suggests that the students should focus first on finding an equation that models the problem. In addition, she tells her students that with the equation they will be able to do the numbers, find values and build a corresponding table, which is the opposite of what they were actually doing: exploring numerical values to establish the equation. It is worth noting that in Mexico it is often observed that students first get the algebraic expression and then the table or graph. This practice is promoted by the national curricular plans (SEP; 2011).

From our perspective, the above shows that the teacher applies the first strategy: *maintaining the lesson plan.* Although the students explore the problem with numerical examples to establish the relationships posed by the problem, Clementina orients these procedures and to find the equation that models the problem (KFLM and KMT). In this way, she solves the teaching issue that arose and goes on with her lesson plan.

However, the questions and confusion continue. The teacher then invites the students to a group discussion. She picks up on the questions of one of the students to discuss and clarify them together in a group. The student has been trying several possible values for the costs of the packs of books. Apparently, he is applying a *trial and error* strategy (successive numerical approximations) to adjust the values.

> Teacher: What does your problem want to find out? [inaudible] It wants to know how much a pack costs, how many packets did it buy?
> Student 2: Nine
> Teacher: What would you do to that nine? Your classmate says she has to make a table. But how would you do that? […]
> Student 3: One thousand, four hundred and thirty-nine (1439)
> Teacher: What will you divide it by? And the additional? [Discussion continues]
> Teacher: Once again, the pack can't cost $75. That's additional. Let's consider it as an increase […] What would your equation be like? What does the general aim of the lesson tell you? [Students read what the teacher dictated at the start of the lesson], we logically have to obtain an equation [...] Lots of you have started to make tables, what are you going to tabulate?
> [Fragment 3, Session 1]

The teacher's participations (Fragments 2 and 3) orient the class to identifying the unknown quantity, the relation between the data and the unknown quantity, and the equation. Again, she maintains the objective of her lesson: obtaining an equation that algebraically models the problem (KMLS). Now she faces yet another teaching issue as her students begin to obtain different numerical results and still haven't found an equation.

In spite of the students difficulties, one of them finds an equation and the teacher asks her to come to the board. She writes "*9x + 75 = 1400*". The teacher chooses to discuss this solution with the group while she ignores other solutions, such as those obtained by trial and error. Thus, the teacher favors the equation that results from translating the problem and its numerical solution, no matter the techniques that can be used to solve it, either paper and pencil or CAS (KMT and KFLM knowledge).

Once the equation has been proposed (which was the main objective of the lesson), the teacher asks the students to solve it, giving them the time and freedom to choose a calculator or paper and pencil for this purpose. For Clementina, the use of CAS is optional. This is the strategy that defines *the use given to technology* for this lesson.

When the equation has been solved, the teacher asks the teams to present their solutions, which are mostly numerical. She then chooses a student who solved the equation algebraically, step by step, with paper and pencil. The solution is shown in Figure 2(a).

9x + 75 – 75 = 1400 – 75 9x = 1325 9x/9 = 1325/9 **x = 147.22** 147.22 x 9 + 75 = 1400 (a)	75 1400.00 725 /9 = 80.55 (residuo 1) x 9 - 675.00 675 0725.00 **R = 80.55** (b)

Figure 2. Proposed solutions (a) algebraic and (b) numerical.

When the teacher realizes that most students do not agree with $x = 147.22$, she asks for the solutions they found. Here she puts into action another of her strategies: she uses what the students have produced (role of the students) to contrast, compare mistakes and procedures, and to clarify (presence of KFLM and KMT).

As questions continue to arise, the teacher invites students to present their solutions. They understand that she had asked them to propose an equation that could model the problem, so they ask, "With an equation?". Then the teacher selects a student that obtained a numerical solution, shown in Figure 2 (b). Apparently, the student's procedure is arithmetical: she did the operations separately, written both operations and results. The teacher uses this solution to ask "What is the cost of each pack?" And the student answers "$80.55". It may be that the teacher identifies in this moment that the difference between the two procedures is the placing of brackets. In other words, the equation that corresponds to the procedure in Figure 2 (a) would be: $9x + 75 = 1400$, while for Figure 2 (b) it would be $9(x + 75) = 1400$. Now the teacher shows she knows how the students are interacting with the equations and mathematical ideas (order of operations, associativity) that are behind these proposals (KMT). Possibly due to this, after explaining, she asks the student who came up with the algebraic solution to go the board and correct the equation. However, in order to obtain the correct answer, she adds brackets to the verification: (147.22 x 9) + 75 = 1400. The teacher adds that the point is not to see who is right, but who has come up with the valid process. In her own words: "Which of the two processes do you think is convenient? Let me be clear, it's not a majority vote."

The teacher's participation shows that she is focused on the teaching strategy that allows emphasis on the *role of the students*, causing them to compare the various interpretations they have made of the relations between the data and the unknown value of the problem. The students get caught up in this comparison, arguing for and against. In our analysis we identify how the teacher makes explicit and promotes the discussion of two different interpretations of the word "additional": 1) *additional per pack,* i.e., $9(x + 75)$; and 2) *additional per nine packs*, $9x + 75$.

In terms of the richness of the mathematical discussion carried out by the students, the problem has given fruits: there has been reasoning, verification, use of different technological media and a variety of techniques (CAS, paper and pencil). Nevertheless, at this moment class time is almost over and the main objective of the lesson is not complete with regards to "relating a formula with the data table it produces and with its graph". The teacher tries to close the discussion with the following question:

Teacher: Which of the two approaches or solutions to the problem do you consider to be correct?
Students: Both.
Teacher: This means that one of the two is not right [...] They don't necessarily have to be exactly the same [...] Let's round off [...], one is missing two cents, and the other five cents […] You might lose a few decimals.

[Fragment 6, Session 1]

Clementina explains that both procedures should come up with the same result, and that this is not the case. She argues on the unicity of the solution of an equation, a topic that apparently has not been dealt with in class; it hasn't been considered in the lesson plan either (a possible deficiency in KMLS).

Until now, the teacher has intertwined the strategies for *maintaining the lesson plan* and the *role of the students* (manifestations of KMT and KFLM). However, she now makes the decision to modify her lesson plan and adapt it to her students' needs and to the interpretations they have come up. In the remaining minutes, she allows students to use the solution strategies they feel most comfortable and confident with, not necessarily those typically algebraic. She allows the student who proposed the equation and the one who proposed an arithmetic procedure to show their interpretations to the class on how to obtain the solution to the problem once more. The teacher asks the student with the "arithmetic" procedure (Figure 2 (b)) to explain on the board. This student says that she and one of her classmates have concluded that her procedure is right.

Student: The thing is that the problem didn't state clearly if it was 75 for each one or for all of them. And here we proved that it was 75 for all of them. Because it can't be for each one because it doesn't work, it would be 1447.95
[The teacher asks several teams to read the problem to the student to clarify if the 75 referred to all or to each one of the packs.]
Teacher: Is the sentence talking about the pack or all of them?
Student: The sentence is missing a word.
Teacher: Which word?
Student: It should tell us if it's 75 additional pesos for each pack [...]
Teacher: This is what problems are for, to see that if they are not stated well, each student will solve them in a different way… What this is about is, you have to read it …
Student: If I was the teacher, this problem would have two objectives… It's not just about reading, it's about determining the relation between the data and the solution.

[Fragment 6, Session 1]

The children of the class agree that the additional 75 pesos are for the nine packets, so the correct equation would be: $9x + 75 = 1400$. The teacher exhibits pedagogical knowledge that is effective at this moment: she "gives" control of the lesson to her students.

Final Remarks

During her class, Clementina faced an unforeseen teaching issue due to the ambiguous description of the problem situation. Consequently, she modified the class objectives, her lesson plan, as well as the roles and participation she had planned for her students. In the process, she also adapted her own role, modified her expectations and participation, and even gave up her role as “teacher” to one of the students. She allowed the group to reach a consensus, giving control to her students and establishing the meaning of the word “additional“ and its implication in (9x + 75 = 1400). At the end of the lesson, she went back to the initial objective, related it to the work done in class and gave it meaning. Such adaptations to the lesson plan, the problem, objectives, forms of participations of the students and even the role of the teacher require great flexibility and adaptability from teachers and their specialized knowledge. We believe flexibility and adaptability like this cannot be anticipated nor is it possible to include as a topic within a teacher training program; they can only be acquired through classroom practice. Inquiring more deeply into these strategies and the interaction between various types of teaching knowledge may have productive implications in the formation of teachers, as well as on research in the area.

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EXTENDING APPROPRIATENESS: FURTHER EXPLORATION OF TEACHERS' KNOWLEDGE RESOURCES FOR PROPORTIONAL REASONING

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In this study we extend our prior exploration focused on the extent to which middle school teachers appropriately identified proportional situations and whether there were relationships between attributes of the teachers and their ability to identify proportional situations. For this study, we analyzed both a larger dataset (n=32) and two dynamic scenarios in which participants were asked to consider aspects of the relationship shown in the diagrams. We found teachers who were correctly able to discern that a situation was not proportional were more likely to use important knowledge resources to evaluate the tasks.

Keywords: Teacher Knowledge, Rational Numbers, Mathematical Knowledge for Teaching

Purpose and Background

Proportional reasoning is an important mathematical concept in middle school mathematics. Despite its prominence in both the mathematics (National Governors Association & Council of Chief State School Officers, 2010) and science standards (NGSS Lead States, 2013), proportional reasoning has not enjoyed a rich history of research relative to its importance (e.g., Lamon, 2007). Research available on teachers' understanding is sparse, but indicates that, like students, teachers struggle with proportions (e.g., Akar, 2010; Harel & Behr, 1995; Izsák & Jacobson, 2017; Orrill, Izsák, Cohen, Templin, & Lobato, 2010; Post, Harel, Behr, & Lesh, 1988; Riley, 2010).

One necessary element of a robust understanding of proportions for teachers is the ability to distinguish those situations that are proportional from those that are not. Orrill et al. (2010) observed that the middle school teachers in their studies had trouble identifying situations as appropriate or inappropriate for using proportional reasoning. For example, when teachers were given a problem with three values and asked to find a missing fourth value, teachers tended to treat those situations as directly proportional even if the actual relationship was inversely proportional or linear. Teachers also struggled to apply proportional reasoning in a qualitative task (e.g., one that does not rely on manipulating numbers) that asked them to compare one pile of blocks to another pile, similar to those tasks used by Harel, Behr, Post, and Lesh (1992), instead they relied on additive reasoning.

Such findings led us to wonder how pervasive these issues were, what kinds of situations might confuse teachers, and what knowledge teachers rely on to determine whether a situation is proportional. In this paper, we extend our earlier findings (Nagar, Weiland, Brown, Orrill, & Burke, 2016) related to this topic by looking at data from more teachers and by expanding our task set to include a dynamic task that appropriately modeled a proportional relationship with a "thermometer" representation (see Figures 1 & 2). Specifically, we consider which knowledge resources were most frequent and what trends emerged among teachers who were able to differentiate proportional from non-proportional relationships versus those who struggled to do this. This work is at the crossroads because it brings together theory and practice in a way that is expressly aimed at impacting practice. By understanding how teachers think about proportional situations, we are better able to create teacher professional development experiences that meet the teachers where they are, thus maximizing the potential for impacting students' experiences with mathematics.

Theoretical Framework

We work from the knowledge in pieces perspective (diSessa 1988, 2006), which asserts that individuals develop understandings of various grain sizes that are used as knowledge resources in a given situation. These resources are connected, over time, through learning opportunities that lead to the refinement of the resources and the development of rich connections. More rich connections between knowledge resources allow them to be available in more situations. This is parallel to the research on expertise that has shown experts have both more knowledge and a different organization of knowledge than novices in their domain (e.g., Bédard & Chi, 1992). It is also aligned with Ma's (1999) interpretation the need for teachers to have profound understandings of fundamental mathematics. By having a robust set of knowledge resources that are coherently connected, we posit teachers will be more able to access their understandings to apply them to a wider range of mathematics and teaching situations than others whose knowledge resources are less coherently connected. We refer to this richly connected collection of knowledge resources as being *coherent* and assert that more coherent teachers will be better able to support student learning (e.g., Thompson, Carlson, & Silverman, 2007). This approach differs from much research on teacher knowledge in that we are not trying to identify deficiencies in teachers' understanding of mathematics, rather, we are trying to understand how teachers understand the mathematics they teach and how different knowledge resources are drawn upon for solving problems and teaching.

Methods

This study is part of a larger project investigating teachers' knowledge of proportional reasoning for teaching. The participants included a convenience sample of 32 in-service, grade 5-8 mathematics teachers, whose teaching experiences ranged from one to 26 years. The participants were from four states. They taught at a variety of schools (public, private, and charter). Twenty-four of the teachers identified as female and eight identified as male. Six of the teachers identified as a race other than white.

The data analyzed for this study were collected through a task-based clincial interview that was videotaped using two cameras trained on the participant's hands to ensure we captured anything the participant wrote or pointed to in the interview. Each interview lasted about 90 minutes. Additional data were collected in the form of a written assessment of proportional reasoning that included the LMT Proportional reasoning instrument (e.g., Hill, 2008) augmented by additional questions focused specifically on whether participants could discern proportional situations from non-proportional situations.

The qualitative analysis of the participant's clinical interview responses was carried out by coding the participants' utterances using a coding scheme that was developed using emergent coding focused on the knowledge resources participants used to reason about a variety of situations. This coding scheme, which included of 23 codes, relied on codes from the literature (e.g., Lobato & Ellis, 2010) as well as from open coding (Corbin & Strauss, 2007). This approach of relying on both literature and emergent codes is consistent with certain grounded theory approaches (e.g., Charmaz, 2014). To create the coding scheme, we coded several interviews, with 2-5 members of the team coding each interview until we were certain that the coding scheme included all the relevant resources we were observing. The full coding scheme included knowledge resources related to reasoning about ratios and proportions, the relationship between fractions and ratios, the relationship between similarity and proportions, the use of representations to reason about proportions, and a few pedagogically-related code, such as one to capture those instances in which a teacher indicated she would ask the student for additional information. For the purposes of this study, we present only those codes that appeared across both studies (shown in Table 1) specifically relevant to proportions (e.g., excluding those for representations and pedagogy). Our coding relied on a binary approach in which each utterance was coded as a 1 or a 0 based on whether a particular knowledge resource was observed. Once the coding scheme was stable, each interview was coded by at least two researchers and 100% agreement was reached on all coding.

Table 1: Codes of Knowledge Resources Used in Thermometers Task

Code	Description
Comparison of Quantities	States that ratio as a comparison of two quantities.
Multiplicative Comparison	Participant sees that there is a way of describing the relationship of the quantities in the ratio that is multiplicative
Covariance	Recognizes that as one quantity varies in rational number the other quantity must covary to maintain a constant relationship.
Unit Rate	Uses the relationship between the two quantities to develop sharing-like relationships such as amount-per-one or amount-per-x.
Equivalence	Describes proportion as a relationship of equality between ratios or fractions.
Constant Ratio	Recognizing the invariant multiplicative relationship between two quantities.
Scaling Up/Down	Uses multiplication to scale both quantities to get from one ratio in an equivalence class to another.
Horizon knowledge	Demonstrates knowledge that extends into mathematics beyond proportions
Rule	Shares a verbal or written rule (e.g., Red = Blue - 2) stated in a way that conveys a generalizable relationship.

For this analysis, we revisited the Thermometers task from our earlier study (Nagar et al, 2016). The thermometers task relied on a dynamic sketch presented to participants with two thermometers, one red and one blue, whose lengths could be varied by dragging a point on a number line (as shown in Figure 1 and Figure 2). Two scenarios were shown to participants (one at a time) and with each scenario participants were asked: (a) whether there was a relationship between the thermometers; (b) whether the relationship was proportional; (c) whether they could provide a rule and a story problem or real-world situation for that relationship; and (d) whether they see a scale factor involved in the situation. For the original study, we analyzed 13 participants' responses to the first scenario in which the thermometers were designed to maintain a constant difference of two units in length of the lines as the point on the slider is dragged from left to right (Figure 1). This situation represents a non-proportional linear relationship between the two thermometers. Our earlier findings showed that five of the 13 teachers initially misidentified the situation as proportional. In that analysis, we also found that two teachers (Group 3) remained convinced that the situation shown in Figure 1 was proportional, whereas the other three teachers (Group 2) started out thinking it was proportional, but then changed their mind. The eight teachers in Group 1 started out, and remained, convinced that the situation was not proportional. We then analyzed which knowledge resources the teachers relied on to determine that the situation was not proportional. Our analysis showed that teachers in Groups 1 and 2 used *Rules*, *Scaling Up/Down*, and *Equivalence* to appropriately identify this Thermometers task as non-proportional. We also found that across all three groups, teachers used language that sounded very additive rather than relying on multiplicative reasoning language.

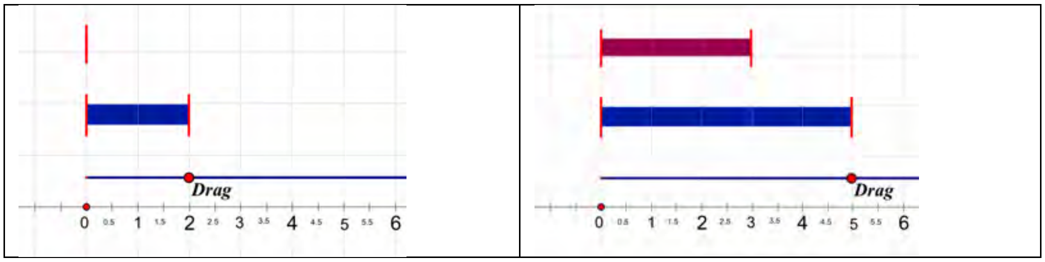

Figure 1. Screenshot of thermometers scenario 1 task.

In the study reported here, we extend the earlier work in two ways. First, we now have our entire dataset analyzed for the Thermometers Scenario 1 task (Figure 1), therefore, we consider 32 teachers' responses to that item (this includes the 13 teachers in the original study plus 19 additional teachers). Second, we analyzed Scenario 2 in the Thermometers task, a situation in which the dynamic environment models a proportional relationship (see Figure 2).

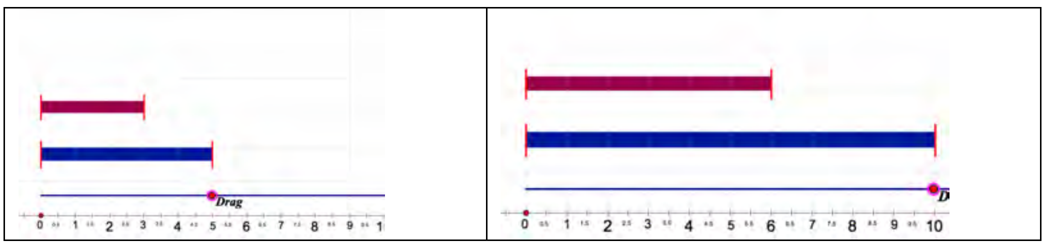

Figure 2. Screenshot of thermometers scenario 2 task.

Results

Our driving research question for this study was: what knowledge resources do teachers seem to rely upon in determining whether a situation is proportional or not proportional? We will first consider this question for Scenario 1 (the non-proportional situation), then for Scenario 2 (the proportional situation). In both scenarios, we focus on trends in the groups. All names reported in this section are pseudonyms.

Scenario 1: Linear Relationship

We began by separating the participants into groups the same way we had in the earlier study. The analysis of 32 teachers in the non-proportional Scenario 1 task showed that 19 teachers (59%) correctly identified the situation as non-proportional (Group 1). Seven participants (22%) first identified the situation as proportional but changed their mind during the interview to identify the situation as non-proportional (Group 2). And, six participants (19%) identified the situation as proportional (Group 3).

As in our earlier study the subset of codes shown in Table 1 were used in making sense of the situation. The most notable trend in the dataset was that the Group 3 teachers relied very little on these knowledge resources to make their determination. In fact, only three members of Group 3 (50%) used any of these resources. Peter used both *Unit Rate* and *Equivalence* while David used only *Equivalence* and Bridgette used *Horizon Knowledge*. In contrast, in Group 1, only four teachers (21%) did not use the knowledge resources included in this analysis. And, across the teachers there was much more variation with at least one person using each of the listed knowledge resources at least one time. For Group 2, two of the teachers (29%) did not use any of the resources.

Table 2: Number of Occurrences of Each Code by Group for Each Scenario

	Scenario 1				Scenario 2			
Code	Group 1 (*n*=19)	Group 2 (*n*=7)	Group 3 (*n*=6)	*Total (n=32)*	Group 1 (*n*=19)	Group 2 (*n*=7)	Group 3 (*n*=6)	Total (*n*=32)
Comparison of Quantities	1	0	0	1	3	1	0	4
Multiplicative Comparison	0	4	0	4	13	2	4	19
Covariance	3	1	0	4	5	1	1	7
Unit Rate	2	0	1	3	5	3	0	8
Equivalence	5	2	2	9	3	2	1	6
Constant Ratio	4	2	0	6	11	1	3	15
Scaling Up/Down	6	2	0	8	12	4	1	17
Horizon knowledge	5	3	2	10	5	1	1	7
Rule	21	6	9	36	12	4	1	18

Consistent with our earlier study, *Scaling Up/Down, Equivalence,* and *Rule* were some of the most used knowledge resources on this task. We note that *Scaling Up/Scaling Down* was not used at all by Group 3. It was used somewhat consistently in Group 1 with five of 19 teachers (26%) using it a total of six times. In Group 2, only one teacher out of seven (14%) used *Scaling Up/Down* in her reasoning twice. *Equivalence* was used nine times across all the teachers for Scenario 1, making it the third most commonly used code. In Group 1, three (Diana, Greg, Larissa) of the teachers used *Equivalence* a total of five times. In Group 2, two teachers each used it one time, and in Group 3, two teachers used is one time each.

Interestingly, the two most commonly used codes for this larger dataset on Scenario 1 were *Rule* and *Horizon Knowledge*. An example of *Horizon Knowledge* in this context would be recognizing that Scenario 1 is not a proportional relationship because the *y*-intercepts for the blue and red bars differ, as Alan did:

> Oh, as an eighth grade math teacher you'd say they have the same slope but a different Y intercept. Yeah, I know it's probably not what you're thinking about but, yeah when you go all the way back here it's, this is always going to be two ahead. So that starts a zero, this starts two, but then they grow at the same rate, so it's always two ahead.

The commonality of the *Horizon Knowledge* code is interesting as it suggests that having more formal understandings of mathematical structures to be able to generalize might matter in Scenario 1. It also suggests that understanding related mathematical topics may support teachers in invoking knowledge resources to better understand a given situation.

The *Rule* code was the most frequently observed in this coding scheme for Scenario 1. In Group 1, 14 of the teachers (74%) were able to generate a rule describing the relationship. For example, Diana said, "Red plus 2 would be blue." In Group 2, this dropped to three of seven teachers (43%). However, Group 3 had four of six teachers (67%) able to generate a rule relate to the situation. This suggests that the teachers in Group 3 may not be connecting their knowledge resources for determining whether something is a proportion to their generalization of a mathematical situation. For example, Brianna (in Group 3) clearly stated that the relationship was, "Whatever red is plus 2 would equal blue" but, she maintained that the relationship was proportional. This suggests that explicitly understanding the mathematical structure of the problem may not be necessary to generate a rule about the relationship presented.

Scenario 2: Proportional Relationship

In Scenario 2 (Figure 2), we asked the same questions of our participants about a similar dynamic representation that showed a proportional relationship. Many more of the teachers got this task correct. In fact, only eight teachers (25%) gave wrong responses and two of those changed their response to be correct during the course of the interview (one in Group 2 and one in Group 3). Of the teachers who answered incorrectly and did not change to a correct interpretation, three were in Group 1, one was in Group 2, and two were in Group 3. This is consistent with our earlier finding that teachers have an easier time recognizing situations that are proportional than those that are not proportional (Nagar et al., 2016).

While *Scaling Up/Down* and *Rule* continued to be important in Scenario 2, *Equivalence* became less important and two new codes became more important: *Multiplicative Comparison* and *Constant Ratio*. *Multiplicative Comparison* was used only when an utterance demonstrated the participant understood a relationship between the quantities of the ratio as multiplicative. For example, understanding the blue thermometer is 5/3 as long as the red thermometer. While only one person (Kanita in Group 2) used *Multiplicative Comparison* as a resource for Scenario 1, in Scenario 2, 12 participants (38%) used it one or more times. In Group 1, nine participants used this knowledge resource 13 times for Scenario 2. In Group 2, two participants used *Multiplicative Comparison* two times in Scenario 2. Only one member of Group 3 (Patricia) used *Multiplicative Comparison* four times.

Constant Ratio was coded when participants indicated there was a fixed relationship between the two numbers in a ratio. It was not as precise as *Multiplicative Comparison* in that participants needed only to note the relationship existed without specifying the nature of that relationship (i.e. that it is multiplicative). In Scenario 1, six participants (19%) used this knowledge resource whereas 15 participants (47%) used it in Scenario 2. In Group 1, this was used eleven times across nine participants (47%) in Scenario 2. For Group 2, it was used just one time, and in Group 3, it was used three times by one participant (Patricia). This trend in *Constant Ratio* and *Multiplicative Comparison* codes suggests that there is something different about the way many members of Group 1 use their knowledge resources than the members of Groups 2 and 3. We note that Patricia in Group 3 appears to be an outlier in terms of her use of knowledge resources.

Generating rules was harder for teachers in Scenario 2 than in Scenario 1, but was still an important code with 18 instances across all three groups. For Scenario 2, six Group 1 teachers (32%) generated 12 rules, four Group 2 teachers (67%) generated four rules, and one Group 3 teacher (Patricia) generated one rule (17%). An example of one teacher's rule was Ella's, "So if I say the red bar is 3/5 of the distance to the blue bar, so the blue… so the blue bar… let me see if five… I don't want to like change this up. Five equals… so that would be like B=5/3R." The relative struggle the participants experienced in identifying a rule is interesting given that teachers were more successful identifying the situation as being proportional and reinforces our assertion that teacher knowledge is shaped by the specific context.

Conclusions

Consistent with our earlier findings, this study showed teachers are better at determining whether a situation is proportional if it is actually proportional. In the current study, 13 of the 32 teachers started out believing Task 1 was a proportion and only seven changed their thinking to recognize that the situation was not proportional. In contrast, only eight teachers were unable to initially identify a proportional situation as such, with two of those eight figuring out the situation was proportional as they worked. This is consistent with research on students that also shows problems discerning proportional relationships from non-proportional ones (De Bock, Van Dooren, Janssens, & Verschaffel, 2002).

When this finding is combined with the particularly sparse use of relevant knowledge resources by teachers in Group 3, one reasonable assertion would be that teachers need more opportunities to apply their understandings and make connections between those understandings. For example, we are confident that middle school teachers understand ratios must be equivalent for a proportion to exist. However, few teachers applied this understanding, which could have helped provide evidence that Scenario 1 was not a proportion.

We also note that there may be some need for additional development of knowledge resources. For example, *Multiplicative Comparison*, which is a critical understanding for reasoning about the relationships within a proportion, was seen in a relatively small number of utterances. In Group 1, almost 2/3 of the teachers used it, but in Groups 2 and 3 combined, the resource was used by only three teachers. This suggests that having the *Multiplicative Comparison* resource available may lead teachers to be more accurate in their ability to discern proportional relationships. It also suggests that several teachers are lacking, or failing to activate, a critical understanding of proportional reasoning.

We assert that the lack of presence of *Multiplicative Comparison* may also explain the limited presence of the *Rule* code in Scenario 2, while it was very prevalent in Scenario 1. It may be that in teacher preparation and professional development teacher educators are over-emphasizing linear situations rather than multiplicative ones. It is also possible that the lack of comfort with the multiplicative relationship between quantities, implied by the limited use of the *Multiplicative Comparison* code, prevents teachers from seeing applications of proportional relationships in the real world. Perhaps focusing more on problem generation and the multiplicative nature of the relationship between quantities in a proportion would strengthen teachers' abilities to recognize proportional situations.

Combined, the findings of this study intersect theory with research to inform practice. By looking at teachers' actual use of knowledge through the knowledge in pieces lens, we are able to suggest that professional development be sensitive to both knowledge resources development and the development of connections between and among those resources. Failure to address both of these approaches creates a situation in which teachers are unable to capitalize on the knowledge they have to support their students.

Acknowledgment

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TEACHERS' INTERPRETATIONS OF STUDENT STATEMENTS ABOUT SLOPE

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This paper describes seven in-service teachers' interpretations of student statements about slope. The teachers interpreted sample student work, conjectured about student contributions, assessed the students' understanding, and positioned the students' statements in the mathematics curriculum. The teachers' responses provide insight into their Knowledge of Content and Students (KCS) and Knowledge of Content and Curriculum (KCC). Results suggest these teachers value academic terminology related to slope, have limited perspectives on slope in real world contexts, and struggle to describe the extension of slope to precalculus.

Keywords: Algebra and Algebraic Thinking, Mathematical Knowledge for Teaching, Teacher Beliefs, Teacher Knowledge

Introduction

Ball, Thames, and Phelps (2008) introduced the Mathematical Knowledge for Teaching (MKT) Model based on Shulman's (1986) work as a means to consider the multifaceted knowledge that teachers need for their craft. MKT consists of two different types of knowledge: pedagogical content knowledge and subject matter knowledge. Pedagogical content knowledge has been outlined in terms of three domains: Knowledge of Content and Curriculum (KCC), Knowledge of Content and Students (KCS), and Knowledge of Content and Teaching (KCT). In this study, we focus on KCS, how students learn mathematics, and on KCC, where the mathematical topics students are learning fit in the curriculum. Subject matter knowledge also has been portioned into three domains. We will focus on Horizon Content Knowledge (HCK), which refers to understanding future mathematical topics and how the math at hand provides a foundation for those topics. We investigate these areas of MKT as related to the concept of slope, a key topic in the middle school mathematics curriculum upon which advanced mathematical (Moore-Russo, Connor, & Rugg, 2011) and statistical (Casey & Nagle, 2016; Nagle, Casey, & Moore-Russo, 2017) ideas are built. In addition to coverage across the curriculum, the multitude of ways in which students can reason about slope make it well suited for this study.

Slope Network

Nagle and Moore-Russo (2013a) proposed a network of five slope components, each with visual and non-visual as well as procedural and conceptual subcomponents (see Table 1).

The slope network outlines the multi-faceted nature of slope, but research has not described how these subcomponents may interrelate and be leveraged to help students develop a connected understanding of slope. In this study, we consider teachers' interpretations of student statements related to the various slope components (KCS) and their accounts for how they fit together across the secondary mathematics curriculum (KCC). Using this lens, we consider teachers' perspectives on the relative sequencing of these components and gain insight into their valuation of the components. In particular, we investigate the following research questions:

1. How do teachers interpret common student statements about slope? What notions of slope do teachers value in student thinking?

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2. What are teachers' perceptions of how the notion of slope is developed across the secondary curriculum?

Table 1: Slope Network (adapted from Nagle & Moore-Russo, 2013a)

Slope Component	**Description**	**Subcomponents (shown as subscripts)** v = visual, n = nonvisual, p = procedural, c = conceptual	
Ratio	Slope viewed as a ratio; extends to explain why linear behavior results in a constant ratio.	$R_{v,p}$: rise/run or vertical change over horizontal change	$R_{v,c}$: similarity of slope triangles yields a constant ratio of rise/run regardless of the position on the graph
		$R_{n,p}$: change in y over change in x; $(y_2\text{-}y_1/x_2\text{-}x_1)$	$R_{n,c}$: constant rate of change between two covarying quantities; equivalence class of ratios thus a function
Behavior Indicator	Relates slope to the increasing or decreasing behavior of a linear function or graph; links sign of the quantity m with the function's or graph's behavior.	$B_{v,p}$: increasing (or decreasing) lines have positive (or negative) slope	$B_{v,c}$: positive (or negative) rise corresponds to positive (or negative) run for an increasing (or decreasing) line, yielding a positive slope
		$B_{n,p}$: value of m in the equation for a linear function indicates whether f is an increasing (m>0) or decreasing (m<0) function.	$B_{n,c}$: If function f is increasing then $f(x_1)<f(x_2)$ for every $x_1<x_2$, so $[f(x_2)\text{-}f(x_1)]/(x_2\text{-}x_1)>0$; similar generalization for decreasing function
Trig. Conception	Describes slope in terms of the angle of inclination of a line with a horizontal; extends to relate steepness to the tangent of the angle of inclination.	$T_{v,p}$: steepness of a line; slope as the angle of inclination of the line with a horizontal	$T_{v,c}$: the angle of inclination determines the rise/run; a steeper line has a greater rise per given run than a less steep line
		$T_{n,p}$: slope is calculated as $\tan\theta$ where θ is the angle of inclination	$T_{n,c}$: the angle of inclination determines the ratio of $(y_2\text{-}y_1/x_2\text{-}x_1)$, which is equivalent to $\tan\theta$.
Determining Property	Property that determines if lines are parallel or perpendicular; property can determine a line if a point on the line is also given.	$D_{v,p}$: parallel (perpendicular) lines have the same (negative reciprocal) slope; slope and point determine unique line	$D_{v,c}$: parallel lines have the same vertical change for a set horizontal change; may be seen in terms of congruent slope triangles
		$D_{n,p}$: $y_2\text{-}y_1/x_2\text{-}x_1$ is equal for parallel lines and results in negative reciprocals for perpendicular lines; slope and a point determine a unique linear equation	$D_{n,c}$: parallel lines have equivalent differences in y values for a set difference in x values, yielding equivalent slope ratios
Calculus Conception	Limit; derivative; a measure of instantaneous rate of change for any (even nonlinear) function; tangent line to a curve at a point	$C_{v,p}$: slope of a curve at a point is the slope of the tangent line to the curve at a given point	$C_{v,c}$: visual interpretation of secant lines approaching tangent line
		$C_{n,p}$: derivative f' is used to calculate slope of function f at a particular point	$C_{n,c}$: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ as the average rate of change over increasingly small intervals

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Methodology

Participants

Participants included seven secondary mathematics teachers who elected to participate in a funded, year-long professional development cohort focused on promoting conceptual understanding. Of the seven participants, two had fewer than 5 years teaching experience, four teachers had between 5 and 10 years, and one had over 10 years of experience. All teachers reported experience teaching introductory algebra, including the concept of slope.

The Tasks

Prior to the first professional development meeting, each teacher submitted responses to a series of tasks related to slope PCK. The task analyzed in this study is provided below. All teachers in the cohort were emailed the task and given three weeks to complete it. The student statements below were generated by the researchers as typical responses noted in past research.

As an instructor, you asked each of the students in your class to make a statement about slope. For each student response [in Table 2]*, please answer all of the following.*

a. Provide a visual representation (a graph, an equation, etc.) that you would expect each student could easily have created to accompany her statement about slope.

b. If each student had been asked to contribute a problem to a study sheet on slope, provide an example of a problem that each would have been most likely to submit.

c. Using the scale [in Table 3]*, rate (and justify) each student's understanding of slope.*

d. By which level of schooling would you expect each student's response? Explain.

Table 2: Student Statements Regarding Slope and Associated Slope Components

Slope Component	Student Statements Given to Teachers
Ratio	A: Slope is rise divided by run of a graph. B: Slope is found by taking the change in *y* values divided by the change in *x* values. D: Slope tells the rate of change between two variables, *x* and *y*. K: The slope of a line is constant regardless of which two points on the graph are chosen to calculate the value.
Behavior Indicator	J: Slope indicates if a line is increasing, decreasing, or constant.
Determining Property	I: Slope can be used to determine if lines are parallel or perpendicular.
Trigonometric	C: Slope describes the steepness of a line. F: Slope is related to a line's angle of inclination with respect to a horizontal line.
Calculus	G: The derivative function tells the slope of a function at a particular time.
Open – No specific component intended	E: Slope is represented by *m* in equations and formulas. H: Slope can be used in real world situations. L: Slope refers to the straightness of a line; the fact a line doesn't curve.

Table 3: Scale for Rating Each Student's Understanding of Slope

	1 - Strictly Procedural	*2- Procedural with Limited Conceptual*	*3- Emerging Conceptual*	*4- Robust Conceptual Understanding*
Description	*Demonstrates a strictly procedural focus on how to calculate slope through rote manipulation without any interpretation of the meaning of the concept*	*Demonstrates a primarily procedural focus on how to calculate slope with very limited attention to interpreting the meaning of the concept*	*Demonstrates an understanding of the meaning of slope in a particular situation or context*	*Demonstrates a flexible, deep understanding of slope that allows for understanding in multiple situations or contexts*

Data Analysis

Using the slope network (from Table 1), two researchers coded the teachers' responses to parts *a* and *b* of the task for the slope components and the visual or non-visual subcomponent evidenced. A number of responses did not provide enough detail to code the conceptual versus procedural subcomponents, so this coding was omitted. The researchers also recorded each teacher's rating of student understanding and recorded the level of schooling at which the teacher expected such a response. The schooling responses were categorized into PreAlgebra, AlgebraI/II, Geometry/Trig/Precalculus, and Calculus categories. The researchers completed all coding independently before meeting to compare codes. When discrepancies were found, a third researcher was brought in to discuss the coding until a consensus was reached. When all the data had been coded, all three researchers looked for trends within and across teachers' responses.

Results and Discussion

The teachers' responses to the student statements are summarized in Table 4. For each student statement (A-L), the first column indicates the slope component(s) and subcomponent(s) illustrated in the teachers' responses to parts *a* and *b* of the task. The data were combined for these parts of the task. Thus, only one slope component is recorded when the teacher used the same component for both the representation and example. When two slope components are listed, that means that the teacher included both slope components in both parts of the tasks or that the teacher included one component in part *a* and the other in part *b*.

Table 4: Teacher Responses to Students' Statements about Slope

Tchr	Hypothetical Student Statements																							
	A		B		C		D		E		F		G		H		I		J		K		L	
1	R_v	1 P	R_n	1 P	T_v	2 P	R_v	2 P	B_n	1 P	T_v	4 G	C_n	4 C	R_v	3 P	D_n	4 A	R_v	1 P	R_n	4 P	D_v R_n	2 A
2	R_n R_v	1 P	R_n R_v	2 P	T_v	2 P	R_v	3 A	R_n	1 P	T_v	3 ^	C_n	3 ^	R_n R_v	3 vr	D_n D_v	2 A	B_v	2 P	R_v	3 A	R_v	2 P
3	R_v	1 P	R_n	2 P	T_v	3 P	R_v	4 -	^	1 P	T_n T_v	3 G	C_n C_v	4 C	^	1 P	D_n D_v	2 A	B_n	3 P	R_v*	2 P	-	3 C
4	B_v R_v	1 P	R_n	1 P	T_v	1 P	R_n	1 P	R_n	1 A	T_v	3 A	C_n	1 C	^	1 A	D_n D_v	3 A	B_v	1 P	R_v	2 P	^	1 -
5	R_v	1 P	R_n	1 A	T_v	3 P	B_v* R_n	3 A	^	1 P	Tv	3 A	C_n	4 C	R_n	3 P	D_n	2 A	B_n	2 P	R_v	1 -	-	- -
6	R_v	1 P	R_n	2 A	T_v	1 P	R_n	3 A	^	1 P	R_v	3 A	C_n	1 C	R_n	3 A	D_n	2 A	B_n	1 P	R_n*	3 A	R_v	1 P
7	R_v	1 P	R_n	2 P	T_v	3 P	R_n	4 P	R_n	1 P	^	3 ^	Cn	? C	^	? G	D_n D_v	2 A	B_v	2 P	R_v	2 P	^	? A

Key: P=PreAlgebra, A=AlgI/II, G=Geometry/Trigonometry/Precalculus, C=Calculus, vr=varies

In a few instances, the researchers thought the response showed strong promise of indicating the conceptual subcomponent according to the slope network. In those cases, an asterisk is marked in the table. The second column reports teachers' responses to part *c* regarding the procedural versus conceptual rating (on the 1 to 4 scale). We distinguish between responses that did not align with any slope component (^), were left blank (-), or acknowledged uncertainty of how to interpret the given statement (?). Part *d* responses are below those to part *c* in Table 2. Consider the row 1 and column A intersection in the table. It reveals Teacher 1 responded to student statement A by providing a representation and example problem aligned with the *Ratio (visual)* component of slope, rated the statement as *1 (strictly procedural)* and placed the statement in PreAlgebra.

In the following sections, we report on teacher responses to the specific student statements in light of the anticipated slope components (from Table 2).

Results for Ratio Components

Statements A, B, D (*Ratio* component). All seven teachers' responses to these statements included the *Ratio* component. Furthermore, all teachers included a visual interpretation of statement A and a non-visual interpretation of statement B, as expected. Interesting trends emerge across the various *Ratio* components. Although statements A, B, and D all express that slope is a ratio, statement A describes it visually, statement B does so non-visually, and statement D describes it as a rate of change. Despite the statements' similarity, teachers interpreted them quite differently. All teachers rated statement A as strictly procedural and at the PreAlgebra level. Five of the seven teachers rated statement B (that had received R_n codes) as either more advanced in grade level or more conceptual (or both) than statement A (that had received R_v codes). Teachers 1 and 4 rated both statements as strictly procedural and at the PreAlgebra level. For the rest of these teachers, the visual approach seemed to be de-valued, as was apparent in many teachers' written explanations. Teacher 2 wrote, "B understands the idea of the slope as the change in the values, instead of just rise over run," and Teacher 7 wrote, "B is using academic vocabulary that suggests that she has a basic understanding of slope." Furthermore, six teachers reported that statement D was more conceptual or more advanced (i.e., grade level) than both the other ratio statements. Several teachers related the "rate of change" language of statement D to using slope in real world situations. Teacher 2 explained, "D has a firm grasp on how slope is applied in real life scenarios," and Teacher 3 justified her rating of this statement as robust conceptual understanding by stating, "the student understands the concept and can relate it to everyday solutions." These teachers are equating the phrase "rate of change" with slope applied to real world situations and conceptual understandings of slope.

Statement K (*Ratio* component). Responses to this statement varied from strictly procedural to robust conceptual. Teachers tended to put it at the PreAlgebra or Algebra level. Teachers 3 and 6 provided sample problems that the researchers felt showed promise of relating to the conceptual *Ratio* subcomponent, with Teacher 3 doing so with a visual emphasis while Teacher 6 did so in a strictly non-visual manner.

Results for Other Components

Statements C, F (*Trigonometric* component). Teachers' responses to statement C were quite consistent. Despite the researchers' interpretation of this statement as being open to visual and non-visual sub-components, every teacher's response emphasized a visual interpretation. These often included graphs of several lines with varying slopes, indicating that the line got steeper as the absolute value of slope increased. Interestingly, four teachers provided a real world context comparing two or more roads or roofs and making reference to steepness. Despite the potential to link steepness to the slope in these contexts, none of these teachers did so in a meaningful way that involved reference to the angle of inclination nor described steepness in terms of a ratio. All seven teachers saw statement C as a PreAlgebra interpretation of slope, with some variation in whether it was more procedural or more conceptual. Statement F was seen as emerging or conceptually robust by all teachers, and was categorized at the Algebra I/II level or later. Most teachers' responses emphasized a visual interpretation. Teachers' 6 and 7 responses suggested their inabilities to interpret this statement or place it in the curriculum.

Statement J (*Behavior Indicator* component). All teachers saw this as a PreAlgebra interpretation of slope, and six of the seven teachers' responses illustrated the *Behavior Indicator* component split equally between visual and non-visual interpretations. Visual interpretations tended to show graphs of increasing, decreasing, and horizontal lines with positive, negative, and zero slopes labeled accordingly. Non-visual representations tended to give the equation of a linear relationship and described the relationship in terms of the parameter *m* in the equation. Teacher 1 provided a graph of a line and asked whether it was increasing, decreasing, or constant but never linked this with slope. Thus, this response could not be linked with any slope component. Teacher 1's sample problem presented the graph of a horizontal line and asked what the graph represents. The research team interpreted this as asking for the equation of the line—which would not require use of the *Behavior Indicator* component. All but one teacher saw this statement as more procedural than conceptual (1 and 2 ratings).

Statement I (*Determining Property* component). Teachers' responses consistently evidenced the intended component. Non-visual representations generally presented two linear equations and asked whether the lines were parallel, perpendicular, or neither. Four teachers incorporated both visual and non-visual representations in their responses. Responses incorporating both representations included equations and graphs of the lines—showing how the relationship between the slopes was displayed graphically via lines that never intersected, intersected in right angles, or intersected in some other way. There was very little variation in the example problems and representations presented. All seven teachers agreed this notion of slope would appear in Algebra I/II, and most teachers rated this as a 2 (mostly procedural understanding), with one 3 and one 4 rating. Overall, the teachers were in agreement with where this fits in the curriculum.

Statement G (*Calculus* component). It is interesting that with this open statement, only one teacher linked this to a visual representation of a function's graph with tangent lines drawn at various points. Most teachers included $f'(x)$ notation and provided an example involving finding the derivative of a polynomial. Six of the seven teachers unsurprisingly placed this conception as occurring in Calculus. There was, however, great variation in whether teachers viewed this as procedural or conceptual in nature. Two teachers rated this as strictly procedural and three teachers rated it as robust conceptual understanding, highlighting a very distinct mismatch. Teacher 7 indicated that she was not sure how to rate this problem.

Statement E (*open* - no component). Three teachers' responses to statement E did not link any understanding of slope to the statement. Each gave a problem or representation that provided an equation in slope-intercept form and then labeled *m* in the equation as the slope with no indication of what *m* meant for the equation or its graph. Three of the remaining teachers linked this statement with R_n, acknowledging *m* in the equation $y=mx+b$ and writing $m=(y_2-y_1)/(x_2-x_1)$. It is interesting that these teachers viewed these algebraic representations as related, especially since none showed how one formula could be manipulated to achieve the other. All teachers viewed this understanding as strictly procedural, and all but one placed it in PreAlgebra.

Statement H (*open* - no component). The researchers expected this statement to elicit a variety of slope components in teachers' responses, but the teachers' responses were relatively uniform. Four of the teachers linked this statement to the *Ratio* component of slope, with two teachers focusing on non-visual aspects, one on visual aspects, and one on both. The link with the *Ratio* component was made via an equation or graph labeled with real world variables and a description of the slope in terms of the problem context. The final three teachers provided responses that could not be coded as indicating any slope understanding. For instance, Teacher 3 sketched a picture of a car driving up what appeared to be a hill with no indication of how slope was demonstrated. The others acknowledged that the statement itself did not indicate much about the student's understanding.

Teacher 4 wrote: “H does not show much with this statement. Sure it can be used in real world situations but how? If she knows how, then we are getting somewhere.” Thus, this code does not mean that this teacher misinterpreted this student’s understanding, but acknowledged the lack of clarity in the statement itself. In terms of responses, the most interesting result may be the absence of the *Trigonometric* component. One of the fundamental uses of slope in real world situations is to consider steepness of physical objects (e.g., ramps). In the one instance where such a connection was hinted at, the connection stopped short of showing how slope was demonstrated. The ratings and grade levels for this statement varied greatly. Interestingly, the three teachers who did not attach this statement to any particular conception of slope ranked it as strictly procedural. The remaining four teachers, who had interpreted this statement as being linked with the *Ratio* component, all rated the statement as mostly conceptual. For those teachers who linked this to a *Ratio* component, they seemed to value the use of *Ratio* in a real world context as indicating a more robust understanding of slope.

Statement L (*open* - no component). Statement L proved to be surprisingly difficult for teachers to interpret. Only three teachers provided codable responses, with two stating that they did not understand L’s statement and the remaining two providing vague responses that couldn’t be coded (e.g., a graph of a line and the graph of a curve with no mention or indication of slope on the graph). Of the three who did provide codable responses, two interpreted it using visual aspects by providing the graph of a line and describing in words or denoting on the graph that every time “you move right one unit on the graph, the corresponding vertical change on the graph is constant.” This was accompanied by statements such as “therefore the function will be a line.” Teacher 1 linked this statement with the *Determining Property* by asking how many lines can be drawn through a given point with a specified slope. She also asked whether three points lie on the same line, linking to R_n. Teachers’ responses regarding grade-level and knowledge rankings varied greatly, adding to evidence of their overall uncertainty about this statement.

Implications

The results reveal important insight into the teachers’ PCK in terms of their KCC and KCS. In particular, teachers’ responses revealed (1) their valuation of academic language, (2) the nature of real world problems for slope and (3) their views of slope beyond the algebra curriculum.

Value of Academic Language

The responses to statements A, B, and D suggest that teachers value student use of academic terminology. Although attending to precision and using correct mathematical terminology is a key part of the mathematics curriculum (NGA & CCSSO, 2010), these results raise a red flag that teachers may equate academic terminology with conceptual understanding. Teachers’ responses to statements A and B suggest that the teachers may devalue visual thinking by equating it with non-mathematical terminology. Likewise, responses to statement D suggest teachers valued the academic language of “rate of change” even though that expression could be used as a mnemonic just as “rise over run” often is. Together, these results highlight two important aspects of teachers’ KCS: (1) distinguishing between students’ use of terminology and their understanding of the terminology and (2) encouraging students to connect multiple representations to integrate academic terminology with visual reasoning.

Rate of Change and Real World Situations

Teachers’ responses also revealed some interesting trends related to the role of real world situations in students’ learning about slope. The real world situations provided by teachers either demonstrated the *Ratio* component within the context of a functional situation (e.g., time worked versus dollars earned) or the *Trigonometric* component within the context of physical situations (e.g., steepness of roof). Furthermore, when physical situations were mentioned, they were done so trivially without explicit attention to how slope was related to steepness. These results suggest teachers may miss valuable opportunities to help students connect the *Ratio* and *Trigonometric* components of slope through real world situations (KCS). As a result, their students may fail to connect the ideas of slope and steepness (Nagle & Moore-Russo, 2013b).

Role of Slope in Advanced Mathematics

The results raise questions about how the teachers view slope as informing students' work with non-linear functions. Nagle and Moore-Russo (2014) describe the CCSSM's high school focus on extending the notion of a constant rate of change of linear functions to interpret and understand non-linear functional relationships. Recall that teachers generally were not sure how to interpret statement L that "slope refers to the straightness factor of line," a statement that links naturally to CCSSM's focus on moving from linear to non-linear relationships by understanding variable rate of change. Furthermore, other than the *Calculus* component of slope, the teachers tended to provide algebraic interpretations of slope, even when statements were open to more trigonometric or geometric interpretations. Even teachers who do not teach beyond the Algebra I/II curriculum, should have sufficient knowledge of the curriculum (KCC) and how slope is foundational to more advanced concepts, such as the derivative (HCK), to include more advanced interpretations of slope.

Future Work

By analyzing teachers' interpretations of student statements, we have described the teachers' apparent values related to student thinking about slope. We have not investigated how these values are carried out through teachers' intended or enacted instruction on slope. Future work should investigate to what extent the tendencies described for teachers do or do not play out in their intended and enacted lessons on slope. Doing so will allow for confirmation of these valuations and for exploration of the manner and extent to which teachers' valuations of understanding inform their written and enacted lessons (KCT).

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TEACHERS' QUANTITATIVE UNDERSTANDING OF ALGEBRAIC SYMBOLS: ASSOCIATED CONCEPTUAL CHALLENGES AND POSSIBLE RESOLUTIONS

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While the mathematics education community encourages teachers to support students in developing a more meaningful contextual understanding of algebraic symbols, very little is known about teachers' quantitative understandings of algebraic symbols themselves. The goal of this study was to fill this gap and examine secondary teachers' ability to contextualize algebraic symbols, in particular notation that results from algebraic generalization. The results led to the identification of various conceptual hurdles that teachers encountered as they endeavored to articulate the underlying quantities as well as various conceptualizations they invoked, both productive and unproductive, in their attempt to overcome these challenges.

Keywords: Mathematical Knowledge for Teaching, Algebra and Algebraic Thinking

Introduction

Traditionally, algebra instruction in the United States has focused on symbol manipulation. Teachers tend to emphasize formal methods, involving abstract mathematical symbols, over other approaches that involve representations that are more closely grounded in context such as diagrams, tables, and graphs (Kieran, 2007; Smith & Thompson, 2007; Yerushalmy & Chazan, 2002). Unfortunately, such an approach has failed to meet the needs of many, if not most, students. Struggling to cope with abstract notation, abruptly introduced and presented as detached from a coherent system of referents, students often fail to develop meaningful interpretations of algebraic symbols and the associated operations (Kieran, 2007; Harel (2007); Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Sfard & Linchevski, 1994).

The ability for students to not only manipulate symbols, but interpret the contextual quantities that expressions represent has been emphasized as a core component to algebraic thinking. The Common Core State Standards (2010) underscores this understanding, including it as one of the eight practice standards (SMP 2: Reason abstractly and quantitatively) as well as a high school algebra content standard (HSA.SSE.A.1). Likewise, many scholars have articulated the significance of this understanding. Kaput and colleagues (2008) noted that without such an understanding, students' actions are guided strictly by the rules of the notational system without support from the previously learned structure of the reference field. As such, knowledge is more fragile with students tending to overgeneralize symbolic rules such as $(a + b)^2 = a^2 + b^2$.

To support students in developing a contextual understanding of symbols, researchers have advocated for the introduction of algebra through inquiry-based activities grounded in more concrete representations such as tables, situations, and words (Koedinger & Nathan, 2004; Nathan, 2012). One example of such an approach is through figural pattern generalization. These are tasks in which students are provided drawings of sequential stages and asked to find subsequent stages and eventually write an expression to model their understanding of a general stage. Exploring these patterns affords students the opportunity to convey their generalizations through a variety of increasingly abstract representations, leading to a more meaningful interpretation of the eventual symbolic forms.

In order to support students in developing a quantitative understanding of the notation, teachers must possess specialized content knowledge that goes beyond simply the ability to write expressions (Ball, Thames, & Phelps, 2008). They must understand how to relate, with precision, the various mathematical representations to the contextual quantities they represent. Although several researchers have investigated students' understanding of representations in algebra (e.g., Knuth, 2000; Nathan & Kim, 2007), less attention has been given to examining teachers' understandings of algebraic notation and their ability to draw connections to the context. Stylianou (2010) studied middle school teachers' beliefs about the instructional use of multiple representations, but not their knowledge. Harel, Fuller, and Rabin (2008) documented ways in which teachers failed to support students to develop meaningful interpretations of symbols, but without exploring teachers' symbolic reasoning or other potential causes for the failure.

While the field has emphasized the need for students to develop a contextual understanding of symbolic representations, we know very little about teachers' understanding in this area. Having a better image of the specific challenges teachers face and how to overcome these challenges will inform teacher educators how to better support teachers in working with their students to develop this ability. Therefore, the goal of this study was to examine secondary teachers' understandings of the quantitative meanings of algebraic symbols, in particular notation that results from algebraic generalization. The results led to the identification of various conceptual hurdles that teachers encountered as they attempted to make sense of and connect the underlying quantities and quantitative relationships as well as various conceptualizations they invoke, both productive and unproductive, in their attempt to overcome these challenges.

Theoretical Perspective

Although there is a lack of empirical studies addressing teachers' understandings of mathematical representations, considerable thought has been devoted to establishing the importance and role of representations theoretically. Multiple scholars have developed theoretical rationales to explain why the ability of expressing the meaning of numeric and algebraic figures is foundational to the understanding of mathematical notation.

Quantitative Reasoning

In order for students to be able to contextualize algebraic notation, they must possess a strong understand of the quantities the symbols represent. Therefore, a key component to possessing a deep understanding of algebraic symbols is quantitative reasoning. According to Thompson (1994), "quantitative reasoning is not reasoning about numbers, but reasoning about objects and their measurements (i.e., quantities) and relationships among quantities" (p. 8). As such, problem solving is not about determining the sequence of operations that will result in the correct answer, but about developing a conceptual understanding of how the quantities in a given problem are interrelated and how they combine to create new quantities. By focusing on the relationship between quantities, students develop a deeper understanding of the problem situation. Smith and Thompson (2007) argue that students must possess a sophisticated enough understanding of the structure of the problem to warrant the use of algebraic tools. Without a grasp of the quantities that shape the problem situation, students are unable to see algebraic notation as a representation that communicates quantitative relationships and consequently are left interpreting symbolic expressions as simply a tool that serves to calculate numerical values.

Symbolization

While understanding the contextual situation is foundational for developing meaning of algebraic expressions, for such an understanding to become embedded in abstract symbolic forms and for students to see notation as communicating the quantitative structure, various cognitive developments must take place. Kaput, Blanton, and Moreno-Armella (2008) described a process they refer to as *symbolization,* in which through one's experience in working with mathematical ideas, their related understandings become infused in the mathematical objects used to represent the phenomenon. They noted that over time and with multiple iterations of reflection, students' understanding of the context becomes instilled in more and more densely compressed forms of symbolization. Initially, students use more contextually connected representations such as oral, written, and drawn descriptions to express their experiences. They then use these representations to reflect on this same experience. This process leads to a newly mediated conceptualization of the mathematical phenomenon and possibly to new representations. Each interaction with the mathematical phenomenon, whether individual and or socially mediated, results in a new conceptualization. Eventually these conceptualizations converge into a conventional and compact symbolic form, establishing a rich, densely packed interpretation of the mathematical phenomenon. Kaput and colleagues noted that in the end, instead of the symbols' representing the referent as a separate entity, the two become interpreted as one. Actions applied to the symbols are construed as actions on the referent itself. At this stage, a student does not look *at* symbols, but *through* them, seeing the mathematical phenomenon and the notation as one.

Connections Between Representations

The role of multiple representations in mathematics and the importance of teachers to engage students in making connections among mathematical representations has been recognized by many scholars (National Council of Teachers of Mathematics [NCTM], 2014). Several studies have demonstrated the ability to translate between representations as a characteristic of more robust and flexible knowledge (e.g. Pape & Tchoshanov, 2001; Stylianou & Silver, 2004). In particular, Lesh, Landau, and Hamilton (as cited by Lesh, Post, & Behr, 1987) observed that students working through mathematics problems seldom came to the solutions successfully using a single representational mode. Explaining this phenomenon, Tripathi (2008) noted that using these "different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective makes the picture richer and deeper" (p. 439). Extending this idea, Dreyfus and Eisenberg (1996) argued that representations differ not only in the way information is expressed, but also in terms of the information itself. They maintained that "any representation will express some, but not all of the information, stress some aspects and hide others" (pp. 267). Subsequently, mathematical ideas are not embodied by a single representation but rather lie, at the intersection of these representations. Finally, Lesh et al (1987) asserted that establishing a relationship from one representational system to another supports students in developing a stronger understanding of the various properties within the situation as they are encouraged to focus on what structural characteristics are preserved in the mapping.

Methods

To investigate the various conceptual hurdles associated with teachers' quantitative understanding of the algebraic notation used to describe figural generalizations, I engaged four 8^{th} grade teachers each in a 1.5-hour individual semi-structured clinical interview (Ginsburg, 1997). Wanting to identify particular challenges associated with connecting algebraic notation to quantities as well as productive conceptualizations teachers formulate to overcome these difficulties, I chose teachers with significant experience with algebraic generalization. The teachers selected all had previously participated in multiple days of professional development focused on algebraic generalization as well as significant experience teaching and interviewing students in this area. Although a study of a more representative group of teachers might provide more generalizable information, choosing more knowledgeable participants allowed me to investigate in detail the subtleties of contextualizing algebraic notation.

During the interview, the teachers were presented two different figural generalizing tasks (see Figure 1). These afforded many different decompositions including interpretations of groups of varying sizes and overlapping groups. Also, the two patterns picked differed in that one was more conducive to being construed as consisting of a constant number of groups of increasing size, while the other could be more readily understood as comprising of an increasing number of groups of constant size. During the interview the participants were asked to provide numerical expressions for specific stages and a general algebraic expression for the n^{th} stage. After each expression they formulated, I asked them to explain what each symbol represented. In addition, I asked the participants to analyze the quantitative meanings of students' work to examine their understanding of decompositions that might differ from their own. Throughout the interview, questions focused on the teachers' understandings of individual symbols and collections of symbols. In addition, I asked participants to comment on their interpretations of various initial symbolic rules as well as on intermediate expressions that arise through syntactical manipulation.

Each interview was videotaped and transcribed. Teachers' responses were reviewed using a grounded theory approach (Strauss & Corbin, 1994) in which I used open coding and the constant comparison method to analyze their responses. I began by identifying particular areas of difficulty across the four teachers. I then compared the actions and comments in these areas among the participants as well as among similar items on different problems.

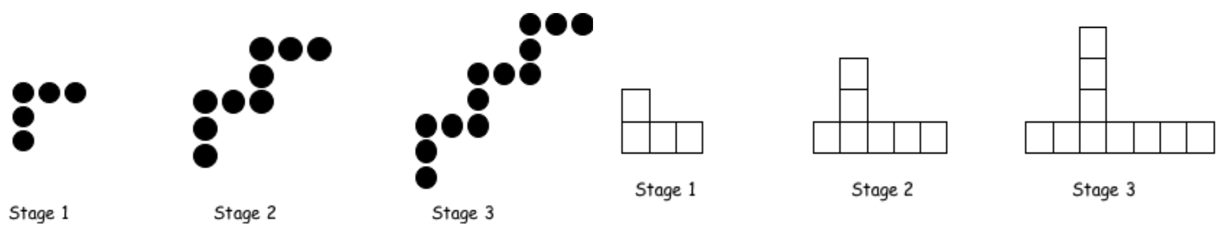

Figure 1. Figural Generalizing Tasks

Results

All four teachers approached the generalizing tasks quantitatively. That is, rather than using a procedure based on numerical values to arrive at a correct linear expression, they began by decomposing the figures into various quantities and then formulating expressions to express their understanding of the quantities they saw. In addition, all of the participants were successful in writing different expressions that corresponded to distinct decompositions of the pattern when asked to analyze the pattern differently and were able to explain possible interpretations of the pattern when exposed to students' expressions that differed from their own. That being said, while the teachers were able to connect the expressions to the quantities in the pattern in general, they struggled articulating the precise contextual quantities that symbols represented. In the end two different challenges emerged along with 3 different conceptualizations to overcome each challenge, one unproductive and two productive.

Challenge #1. Interpreting the Coefficient of *x*

The first conceptual difficulty centered on the participants' understandings of the coefficient of x and its relationship to the variable. Initially, all four participants described the coefficient and the variable together as representing groups of a particular size (i.e. $5x$ represents the number of groups of 5), but struggled to disentangle the two and articulate the specific meanings of the symbols independently.

Unproductive conceptualization: Detaching meaning of the symbols from details of context. To overcome this challenge one participant, Denise, reconceptualized the coefficient as the constant difference between stages even when such a construal was inappropriate for the context. To illustrate this type of thinking, I describe Denise's explanation of the expression $3x+1$ for the second task.

Initially, she stated that the *3* and the *x* both represented number of groups of 3. When asked to clarify, she began vacillating between various interpretations (the number of groups of three, three tiles, the number 3, the three added on) before ultimately concluding that it represented the three tiles that were being added at each stage. When asked to highlight "the three added on" in the figure, she seemed to imposed her notion of adding three on the diagram, selecting tiles that did not correspond to how the pattern changes between stages. Initially she said it did not matter which ones, circling what seemed like arbitrary groups of 3 dots (see left image of Figure 2), before eventually deciding the three additional tiles each stage were the one far left tile and the two far right tiles (see right image of Figure 2).

Denise's vague and even problematic explanations of the various symbols' referents are evidence that she was not using symbols to communicate her interpretation of the figure. Instead, she seemed to reinterpret the coefficient as representing a decontextualized growth factor and attempted to improvise a quantitative interpretation on the figure.

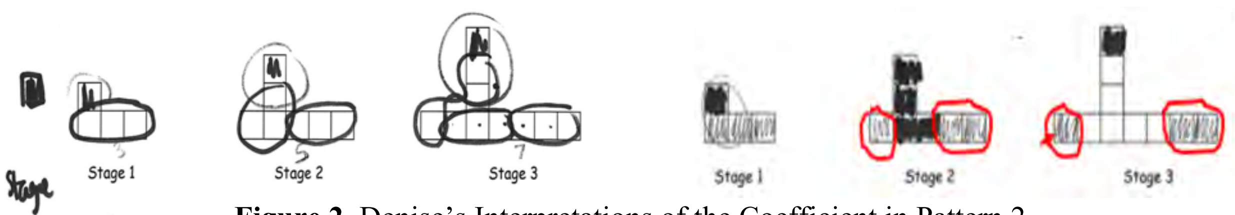

Figure 2. Denise's Interpretations of the Coefficient in Pattern 2.

Productive conceptualizations: Interpreting the variable as the number of groups and coefficient as a ratio of tiles per group or vice versa. While the other three participants also struggled identifying the meaning of the coefficient and variable separately, they eventually disentangled their meanings, describing the variable as the number of groups and coefficient as a ratio of dots or tiles per group for the first task. Notable was their explanation of the symbols in the expression $3(x - 1) + 4$ for the second task. While two of the participants switched their interpretation of the symbols relative to the first task, with the coefficient now representing the 3 constant groups and the variable corresponding to their varying size, the third teacher did not. To make sense of the 3, he imagined orbits of 3 tiles being added to each stage. Such a conceptualization matched his previous interpretation of the variable as the number of groups with the coefficient representing its size. While all three participants had quantitative interpretations of the symbols, the first two flexibly adapted their interpretations of the symbols to accommodate their quantitative understandings of the figure, while the third had a more fixed view of the symbols, reconceptualizing the quantities in the figure to match his previously formulated understanding of the symbols.

Challenge #2. Interpreting Expressions Where the Variable Appears More Than Once

The second conceptual challenge that emerged for the teachers was negotiating the meanings of variables that appeared more than once in a single expression or between expressions after algebraic manipulation. Such a situation exists in the first pattern when decomposing the figure into overlapping groups of 5, first with the expression $5x - (x - 1)$ and then in the subsequent simplified expression $4x + 1$. Initially all four participants interpreted the various xs in these expressions as representing different quantities in the figure. They understood the x in $5x$ as the number of groups of 5, the x in $x - 1$ as the number of overlapping dots, and the final x in 4x as the number of groups of 4. Expecting a single variable to a have a consistent meaning, they struggled to explain this apparent conflict.

Unproductive conceptualization: Imposing the interpretation of one variable onto another.

Three of the participants, in an effort to coordinate the symbols' referents, initially imposed an interpretation of the *number of groups of five* on the *x* in the expression *x – 1*. In doing so they then incorrectly reinterpreted the minus *1* as accounting for the difference in sizes of the groups of five and the groups of four (i.e., the difference in the number of dots) rather than the difference between the total number of groups of five and the number of overlapping dots. While all three teachers devoted at least 5 minutes to this incorrect construal, eventually they all noticed their inappropriate interpretation. Of these three teachers, two then formulated productive conceptualizations of the variables to overcome this problem, while the third participant did not. Instead, in an attempt to resolve this inconsistency, she oversimplified the symbols' referents, arriving at a final interpretation of all the *xs* as simply a dot. Accordingly, she construed *5x* to mean 5 dots and *4x* to mean 4 dots, but was unable to indicate which exact dots in the figure. Such a conceptualization of *x* as a dot essentially treats the variable as a label and the coefficients as decontextualized numbers, removing any quantitative meaning of the symbols and failing to explain any quantitative relationships between the symbols.

Productive conceptualizations: Coordinating numerical values and reinterpreting symbols to align quantitative meanings. Two of the participants were able to articulate viable, yet different solutions to reconcile the diverging meanings of the symbols. Although both teachers initially tried to make sense of the *x*s by using a literal translation of the words like the participant described previously, they eventually formulated productive conceptualizations.

The first participant did so by a**ssociating the quantities numerically.** By evaluating the expressions multiple times and stating the quantities and their numerical relationships, she was able to see that the number groups, initially of 5 dots, is always equal to the number of overlapping dots to be removed, which is equal to the number of groups of 4 dots. In the end, although she continued to interpret the same variable as different referents, she realized that the numerical value of each of these quantities is always equal.

The second productive conceptualization resulted through a reinterpretation of the quantitative meanings of the variables. Similar to the previous participant, the third teacher verbalized his understanding of the various variables in the expression *5x–(x+1)* and *5x–x+1*, while carefully examining their values. This process helped him to not only coordinate the values of the two quantities (groups of 5 and overlapping dots), but also to connect them physically, noticing that the overlapping dots were members of the same groups of 5. In addition, he had added, apparently somewhat serendipitously, a coefficient of *1* in front of the second expression (resulting in *5x–1x–1)*. Together, these various semiotic acts supported him in reconceptualizing the overlapping dots as group of size 1. This conceptualization allowed him to reinterpret *x* as the purely the number of groups, without attaching a size, and the varying coefficients as the size. In the end, he was able to formulate an understanding of the variable so that it retained consistent quantitative meaning throughout as well as explained the varying coefficients.

Discussion

As this analysis reveals, contextualizing algebraic notation is challenging, even for experienced teachers. There are many nuances that experts overlook when they use algebraic expressions to solve problems and communicate their generalizations. In this final section I will revisit some of these challenges and discuss implications that I see stemming from these results.

Conceptual Complexities of Interpreting Algebraic Expressions

To highlight the complexity of contextualizing algebraic notation, I want to revisit a particular conceptualization that emerged. This example serves to not only illustrate the sophisticated understanding necessary, but also emphasizes that the quantities the teachers came to see in the notation were not intrinsic properties in the figure but rather, mental constructs that they themselves created.

One challenge identified in this study was articulating, with precision, separate meanings for the variable and coefficient that explained the relationship between the two. To overcome this difficulty, participants not only interpreted *x* as the number of groups and the coefficient as the size of each group, but also reversed this mapping and conceptualized the coefficient as the number of groups and the variable as the group size. To see both ways requires an abstract and flexible interpretation of the symbols. To illustrate the abstraction of perceiving both ways, I will use an alternative figure (see Figure 3) in which the transition between these two views requires only a subtle, cognitive shift in defining the group. As I point out, while this pattern can be modeled by the expression of *3n*, depending on your perspective, the 3 and then *n* can take on different meanings. In interpretation 1, the *n* indicates the number of groups of size 3 and in interpretation 2, the *n* represents the number of dots in the constant 3 groups.

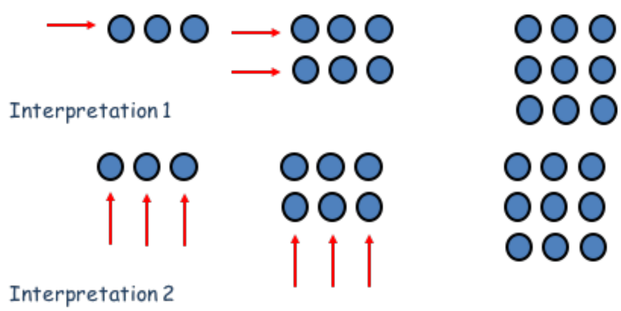

Figure 3. Flexible Conceptualization of Variable and Coefficient.

As this example illustrates the quantitative structure of the pattern is not an inherent characteristic of the figure or of the corresponding notation used to communicate it. The capacity to interpret symbols in multiple ways is an understanding that must be explicitly developed.

Implications

While I see several implications that stem from this study, I will highlight two which are interrelated. As noted in the introduction, algebra classrooms are dominated by a symbolic focus without attention to meaning. While only a few studies have specifically tackled this issue from the teacher's perspective, the consensus seems to be that the primary cause is teachers' orientations. The results of this study indicate that the challenge to transform the current symbolic focus in algebra classrooms is not simply an issue of beliefs. By detailing teachers' struggles with the complexity of this topic, this study demonstrates that, at least in part, the difficulties teachers experience in shifting their instruction is connected to their knowledge bases. Consequently, a second, related implication is the need for teacher preparation programs to explicitly develop this understanding. While definitely a daunting task, the results of this study contribute to this endeavor by identifying both conceptual hurdles and conceptual resources on which to focus instructional attention to support teachers in developing this knowledge and ultimately helping their students foster a deeper, quantitative understanding of the notation.

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CONCEPTUALIZING MEASURES OF MATHEMATICAL KNOWLEDGE FOR TEACHING IN TERMS OF UNDERLYING COMPONENTS

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Most existing measures of mathematical knowledge for teaching (MKT) assume a unidimensional structure and report scores that reflect the "amount" of MKT possessed by each examinee. These measures aggregate scores on MKT items that each tap different kinds of teacher work (e.g., explaining concepts, using representations) and different curricular topics. Although this approach is useful for some purposes, the author argues that a new way of conceptualizing measures of MKT will be more useful for gathering data to address open questions about MKT. A review of existing measures is used to argue that conceptualizing measures of MKT in terms of underlying components has promise for building MKT theory.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Teacher Education-Preservice

Only in the last decade has teachers' mathematical knowledge been reliably linked with student achievement. The breakthrough was writing job-embedded items that require teachers to coordinate specific kinds of teacher work with specific curricular topics in order to assess mathematical knowledge for teaching (MKT; Ball et al., 2008; Hill, Ball, Schilling, 2008). In existing MKT measures, however, teachers' responses on work- and topic-specific items are aggregated. The resulting scores subsume many different kinds of teacher work and span large content areas such as all of elementary or secondary mathematics (e.g., Baumert, et al., 2010) or content covered over one or more years such as algebra (McCrory et al., 2012), geometry (Herbst & Kosko, 2014) or number and operation (e.g., Hill, Rowan, & Ball, 2005).

Efforts to measure MKT have come to an important crossroads. General MKT scores provided by existing measures are important for answering some questions but are too coarse to reveal how underlying components of MKT (i.e., the kinds of teacher work and the specific curricular topics) are related to instructional quality, student learning, or teacher development. These questions are critical for improving MKT theory, and in this paper, we propose a new way of conceptualizing how MKT is measured in terms of its underlying components, that—if realized—promises useful data for building theory.

Existing Measures of MKT

Three large-scale studies have found correlations between MKT and student achievement (Baumert et al., 2010; Hill et al., 2005; Tchoshanov, 2010). Ball, Thames, and Phelps (2008) proposed a framework for MKT that can be used to classify the different approaches used by these studies. The framework extends the notion of *pedagogical content knowledge* (PCK) introduced by Shulman (1986) by grouping PCK with content knowledge and identifying strands within both categories of teacher knowledge. The term PCK captured Shulman's seminal idea that teachers' knowledge of the content they teach is transformed by and merged with their knowledge about how to teach it to form a new domain of knowledge. Within the MKT framework, one kind of PCK is *knowledge of content and students* (KCS), a category that includes, "familiarity with common errors and deciding which of several errors students are most likely to make" (Ball et al., p. 401). Within the MKT framework, two kinds of content knowledge are *common content knowledge* (CCK), the knowledge expected of educated adults including knowledge of the content that students will learn, and *specialized content knowledge* (SCK), knowledge that teachers but few other adults or professionals would possess, such as knowing how tasks and representations correspond with mathematical ideas and evaluating non-standard student reasoning.

Each of the research teams on the three large-scale studies of MKT took its own approach. Tchoshanov (2010) designed the Teacher Content Knowledge Survey (TCKS) measure with three subcontracts, all "closely aligned with … standardized tests for students" (p. 148); thus, it measured CCK. The Learning Mathematics for Teaching (LMT, Hill et al., 2005) measure included CCK items in addition to SCK items involving job-embedded mathematical tasks, for example, to determine the mathematical validity of student strategies. The CCK and SCK items were modeled on the same scale. Items were also developed to measure KCS, but factor analysis revealed "significant problems with multidimensionality" (Hill, Schilling, & Ball, 2004, p. 26).

The COACTIV project (Baumert et al., 2010) created an MKT measure that had two distinct dimensions: content knowledge and what they called PCK. The content knowledge dimension was similar to the instrument used by Tchoshanov (2010) and focused on teachers' understanding of mathematical topics in the curriculum. The PCK dimension included identifying multiple solutions for tasks (PCK-Tasks), understanding students' thinking (PCK-Students), and selecting representations and explanations for instruction (PCK-Instruction). Thus, it overlapped with all three categories of CCK, KCS, and SCK in Ball et al.'s (2008) framework. Baumert et al. (2010) argued the LMT instrument assessed (their) PCK: "mathematical knowledge related to the instructional process" (p. 141).

Categories of Teacher Work

As the COACTV case suggests, the terminology and definitions of the Ball et al. (2008) framework have been influential but have not proved canonical. Several other research groups have defined and validated large-scale MKT measures after delimiting and subdividing the MKT domain in other ways. Table 1 shows three crosscutting categories of teacher work emphasized in the constructs and sub-constructs of published MKT measures. First, teachers use mathematical knowledge to (1) understand what they teach. The second and third categories of teacher work reflect the components of PCK specified by Shulman (1986, p. 9). Teachers use mathematical knowledge to (2) appraise students' conceptions and reasoning and to (3) select and use instructional representations. Moreover, Table 1 illustrates how terms one might expect to have common meanings (e.g., PCK) do not agree across projects.

Table 1: Three Categories of Teacher Work Intersect Subconstructs of MKT Measures

	Categories of Teacher Work		
Project or Instrument	Understanding the content to be taught (deeply)	Appraising students' conceptions and reasoning	Selecting and using instructional representations
COACTIV	• PCK-Tasks	• PCK-Students	• PCK-Instruction
DTAMS	• Type 1, 2 & 3	• Type 4 (PCK)	• Type 4 (PCK)
LMT	• CCK items	• KCS & SCK items	• SCK items
SimCalc	*none*	• MKT Items	*none*
TCKS	• Type 1, 2 & 3	*none*	*none*
TEDS-M	• MCK	• MPCK	• MPCK

Curricular Topics

Projects also differ in how they divide and delimit MKT relative to mathematical topics. Often, the mathematical demarcations are explicitly related to curriculum. For example, two of LMT instruments measure “Number and operation (K-6)” and “Geometry (3-8)” (Learning Mathematics for Teaching Project, n.d.). The curricular focus of MKT measures can be quite broad; both the TEDS-M and COACTIV projects measured MKT for entire mathematics curricula across multiple grades. Focal content can also be quite narrow; many small studies have used interviews and observation to investigate MKT over a range of specific curricular topics (see review, Depaepe et al., 2013). Rather than being empirically grounded, these divisions of MKT reflect divergent conceptualizations.

As is clear from this review of measures, MKT does not have a simple structure. The LMT excluded KCS pilot items because they were multidimensional. In spite of this complexity and lack of agreement between projects, all these projects except DTMR used unidimensional psychometric models to construct measures, and—differences in how the domain of MKT is defined and divided notwithstanding—all reported unidimensional measures that had sound psychometric characteristics. It may be that the item selection process used to create these measures reduced the diversity of the retained items in non-obvious ways, with each project converging on a different dimension within a multidimensional space.

Conceptualizing A Novel Way to Measure MKT

The inconsistent results and framings of MKT across the field stem in part from scholars who disagree on the paradigmatic foundations of MKT. Two such concerns are whether MKT can be distinguished from mathematical knowledge and whether MKT is situated knowing-to-act that depends on classroom context or if it is cognitive knowledge-about-action that can be validly assessed with survey questions (Depaepe, Vershaffel, & Kelchtermans, 2013; Graeber & Tirosh, 2008; Mason, 2008; Petrou & Goulding, 2011). Different kinds of data than are now available are required to address these gaps and open questions in MKT theory.

At the elemental level, I conceptualize MKT as the mastery of a specific kind of teacher work for a specific curricular topic in mathematics. Rather than assuming a single general structure (i.e., a unidimensional latent trait), I hypothesize that MKT may be multidimensional and structured by the kind of work and the curricular topics involved. I propose developing new MKT measures that systematically vary the kind of work and topic. By leveraging recent advances in psychometrics (Rupp, Templin, & Henson, 2010), a 16-item test (4 kinds of work crossed by 4 topics) could provide a score on each of the 8 components for every examinee. Data from such an assessment bear on several pressing open questions about MKT.

One question concerns how the components of MKT shape and constrain the knowledge teachers develop. I hypothesize that knowledge for a certain kind of teacher work learned in relation to a specific curricular topic may not readily transfer to different topics even if the teacher work is similar. In a study of 40 experienced first-grade teachers, Carpenter, Fennema, Peterson, and Carey (1988) found they could identify which whole number addition and subtraction problems would be most difficult for their students but could not explain why. It is unlikely that teachers could apply tacit professional knowledge such as this to different curricular topics such as fraction addition or subtraction. Comparing relevant MKT component scores could confirm this hypothesis.

A second question concerns how different kinds of teacher work and different curricular topics moderate the relationship between teacher knowledge and student learning. Sadler and colleagues (2013) found that middle grade science teachers’ knowledge of student misconceptions for an individual student item predicted large gains for that teachers’ students on the same item at the end of the year; however, teachers’ average knowledge of misconceptions across all items was only weakly related to increases in their students’ overall scores. When it comes to the teacher work of identifying common student errors, MKT may affect student learning very narrowly, topic by topic. Correlation analysis between student achievement for specific topics and teacher MKT for corresponding topics would shed light on this question.

Conclusion

Existing MKT items are written to combine specific kinds of teacher work with specific curricular topics, yet existing measures are not sensitive to these components. New kinds of MKT measures should be developed that can resolve MKT in terms of the specific kinds of teacher work and curricular topics assessed. Such measures coupled with fine-grained, mixed-methods, and longitudinal analyses, have promise for addressing the critical gaps in MKT theory.

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HIGH SCHOOL TEACHERS' PEDAGOGICAL CONCEPTIONS THAT SUPPORT TEACHING THROUGH PROBLEM SOLVING

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This study investigated the pedagogical conceptions held by three exemplary high school mathematics teachers for teaching about and through problem solving and how the conceptions were held that made them effective in supporting students' engagement with problem solving. It was framed in a perspective of mathematics problem-solving knowledge for teaching [MPSKT] consisting of five components of knowledge. Analysis of data, consisting of interviews and class-room observations, indicated that there were important relationships in how the teachers held their conceptions, with one component of their MPSKT forming the core knowledge that gave meaning to the other four components. A core knowledge of "knowledge of students" was a critical factor in supporting their problem-solving-based classrooms.

Keywords: Problem Solving, Teacher Knowledge

Problem solving [PS] is a central aspect of learning and doing school mathematics as well as teaching it. Teaching through PS is promoted as a way to foster students' development of mathematical thinking and autonomy to solve challenging, meaningful problems (e.g., National Council of Teachers of Mathematics [NCTM], 2000). However, this learner/learning-focused approach to teaching is likely to be a challenge for teachers depending on their understanding of PS in mathematics. Similar to the perspective of Ball, Thames, and Phelps (2008) that suggests general mathematical ability does not fully account for the knowledge and skills needed for effective mathematics teaching, the knowledge teachers need to effectively engage students through PS should be more than general PS ability (Chapman, 2015; Schoenfeld, 1985). Understanding this knowledge from a practice-based perspective can provide insights to support teachers' learning and use of it. This paper contributes to this understanding based on a study that investigated the conceptions held by three exemplary high school mathematics teachers regarding teaching about and through PS and how the conceptions were held that made them effective in supporting students' engagement with PS.

Related Literature and Theoretical Perspective

In recent years, considerable attention has been given to mathematics knowledge for teaching (MKT) (e.g., Ball et al, 2008; Rowland & Ruthven, 2011) as a basis for understanding and improving the teaching of mathematics. *Mathematical problem solving knowledge for teaching* (MPSKT), an important aspect of MKT, requires specific attention to address how to support teachers in creating PS-based classrooms. PS is considered here as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 52). In PS-based classrooms, students are engaged in PS as a way of thinking mathematically and doing and learning math.

Several studies have highlighted issues with teachers' PS ability and knowledge of PS. For example, they tend to lack flexibility in choice of PS approaches (van Dooren, Verschaffel, & Onghena, 2003), apply a stereotypical solution to a problem (Leikin, 2003), have a lack of strategies for interpreting the information given to them in word problems and to recognize the appropriate procedure to use (Taplin, 1998) and make sense of PS as a linear process (Chapman, 2005). However, less attention has been given to teachers who hold knowledge that supports PS-based classrooms, which is the focus of the study being reported in this paper.

The study is framed in a theoretical perspective of MPSKT suggested by Chapman (2015). Based on a structure similar to Ball et al.'s (2008) perspective of MKT, two categories of MPSKT are adopted in this study. Category 1 (PS content knowledge) has three components: knowledge of problems, PS, and problem posing. Description of these components includes understanding of meaningful problems, of mathematical PS as a way of thinking, of how to interpret students' unusual solutions and implications of students' different approaches, and of problem posing before, during and after PS. Category 2 (Pedagogical PS knowledge) has two components: knowledge of students as problem solvers and instructional practices for PS. This includes understanding what a student knows, can do, and is disposed to do and understanding how and what it means to help students to become better problem solvers.

The participants' knowledge is based on their conceptions of each component connected to their teaching of and through PS.

Research Process

This study is part of a larger project that investigated prospective and practicing elementary and secondary school mathematics teachers' thinking and teaching of PS. It focuses on three of the high school teachers who consistently taught through PS. They were experienced grades 10 to 12 mathematics teachers with 16 to 20 years of teaching experience. They were from different local public schools and received teaching awards as excellent mathematics teachers.

Main sources of data were open-ended interviews, PS tasks, classroom observations, role play, teaching/learning artifacts, and students' work. The interviews explored the participant's conceptions of and experiences with PS. This included questions/scenarios on: PS ability; nature of tasks, PS, and learning; instructional approaches; contexts; planning; and intentions for PS in their teaching. Participants also commented on the nature of five different types of relevant school mathematics problems and on when and how they may or may not use them in their teaching. Interviews were audio recorded and transcribed. Classroom observations and field notes focused on the teachers' actual instructional behaviors and students' engagement during the lessons. Ten lessons (60 to 85 minutes each) involving PS or when a new concept was being introduced and developed were observed and audio-recorded.

Data analysis involved the researcher and a trained research assistant working independently to thoroughly review and code the data and identify themes, which were validated through an iterative process of identification and constant comparison. Coding was guided by the five com-ponents of MPSKT. This included highlighting participants' conceptions and what they valued regarding each component. For example, for PS, their conceptions included: "It's like anything else that you don't know what the outcome will be and you're kind of game for anything, so you just take your chances and you try and use the tools that are available to you, see what happens." "It is a process, it is not a solution. It is whatever takes you to get to that solution. So, thinking, trying out things, writing, using whatever tools you have to find a solution." The data were also examined for connections that were consistently expressed in their conceptions as a basis for understanding how they held their MPSKT. For example, their conceptions of problems were always expressed in relation to the problem solver/learner/student regardless of the context involved; e.g., "Using problems in class, the big thing is that they have to be interesting to some-one, to those kids." "All word problems are real problems if students have not encountered them before." "Students don't have a predetermined solution process." "It's a problem you want to have the answer to, … something that is needed, is practical, is worthwhile, that has some kind of relevance to the student." In general, for all participants, explicit connections to *students* was dominant for each component of MPSKT, which was consistent with their observed classroom actions. Thus, *knowledge of student* emerged as central to how they held and used their know-ledge. Further analysis focused on unpacking and confirming the relationships with *students.*

Finding

While there were differences in the teachers' conceptions and teaching approaches, only the common features that were central to how they engaged students in PS are presented here. The teachers held conceptions that showed positive disposition to PS. They viewed PS not as separate topic but as a central aspect of learning mathematics that should be integrated throughout a course. Their PS-based classrooms included engaging students in solving challenging problems, exploring patterns, designing tasks, formulating and checking conjectures, reasoning and generalizing, and communicating/discussing ideas and outcomes. New concepts were introduced through PS. For example, Teacher 1 started her Grade 11 unit on functions by giving students a set of graphs and asking them "to put them together in some form, like group them or look at them in some way that they are mathematically connected or similar or not." This was followed by further exploration. Teacher 2 started her unit on relations and functions with the following scenario for which students had to determine independent and dependent variables with justification, represent the scenario in three different modes, explain the type of graph and reason for it, explain the meaning of the steepness and intercepts.

> You are running in the Terry Fox Run at school. One of the ways to earn money in support of the Terry Fox Run is to obtain sponsorship. The school has decided the sponsorship rate will be $1/km run in the designated time slot.

Table 1 summarizes central aspects of the teachers' conceptions of the components of MPSKT that defined their teaching.

Table 1: Teachers' MPSKT

Components of PSKT	Teachers' Conceptions
Students	Students are designers, interpreters, evaluators, inquirers and agents of their learning and doing of PS.
Problem	A problem emerges based on how the student experiences the task. The task provides possibilities, while the student provides the interpretation that gives meaning to the problem, e.g., as interesting, relatable, and challenging or not.
Problem posing	Problem posing involves students creating tasks, modifying/redesigning tasks contextually and extending a problem. It allows them to realize their interests and be creative.
PS	PS involves what students are able to do that makes sense to arrive at a solution.
Instructional strategies	A key strategy is not teaching heuristics or concepts in an explicit way, but allowing them to emerge out of students' experiences in trying to solve a problem and reflecting on the process and concept.

In addition to the nature of the teachers' conceptions, an important finding is that the components of MPSKT were not held by the teachers in isolation of each other but were connected through one component that formed the core of a network consisting of the five components. For these teachers, the core knowledge was *knowledge of students*. Students were viewed in terms of their ways of learning/knowing and affective factors/beliefs impacting them as problem solvers/ learners. Knowledge of each of the other four components was held in relation to knowledge of student. For example, *problems* were viewed as relationship between student and task. *Problem* *posing* was viewed in relation to students as task designers. For example, after mentioning the Goldilocks story as an example, Teacher 3 assigned the following task to her grade 12 students:

> … rewrite a children's story to solve permutations or combinations. The math should be an integral part of the story line, perhaps illustrating difficult decision making or large numbers of choices to be made. …This is a children's story, so make the story line and illustrations simple, although the problems encountered should be relatively complex in nature.

PS was viewed in relation to the students' thinking/inquiry process and instructional strategies were viewed in relation to supporting students' agency and autonomy in engaging in PS. As Teacher 3 explained:

> I go around and listen to the groups. …I can sit next to any group and they talk, and I ask them questions if they're stuck but that's about it. I simply watch how the groups are working together and if I see a group is stuck, I try to come up with a question that will allow them to continue, but I will not give anybody the answer at any time …they can always ask a question, but if they want to know how to do it, or are they right, they may not talk to me.

Conclusion

This study contributes to our understanding of the nature of and relationship between exemplary teachers' MPSKT and practice. It indicates the nature of the teachers' MPSKT and that they held their conceptions of the five components of MPSKT, not independent of each other but, as a network of interdependent knowledge with one component forming the core or anchoring component that gave meaning to the other four components. The findings suggest that the nature of this core component is a critical factor in supporting PS-based classrooms. In particular, they suggest that a teacher's MPSKT with a core component of *knowledge of students* as genuine problem solvers is important to enable the teacher to support PS-based classrooms with "effective teaching that engages students in meaningful learning … to make sense of mathematical ideas and reason mathematically" (NCTM, 2014, p.7). Ongoing investigation in the larger project explores this relationship between MPSKT, core component, and teaching to empower students in learning about and through PS and the implication for teacher education.

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KNOWLEDGE FOR TEACHING INTEGERS: ATTENDING TO REALISM AND CONSISTENCY IN A TEMPERATURE CONTEXT

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Research has illustrated that prospective teachers struggle to construct word problems for integer operations. This paper extends the current research by examining the ways in which four classes of prospective teachers responded to a child's temperature story for an integer addition number sentence (i.e., -9 + -6 = □). One hundred prospective teachers responded to whether they believed that the story matched the number sentence and provided justification for their reasoning. The child's story, an actual story posed by a Grade 5 student, did not utilize temperature realistically (realism), nor was it consistent with the given number sentence (consistency). The results indicated that when prospective teachers were asked to evaluate the child's story, they tended to either focus on realism or consistency, but not both. Benefits of using children's thinking during instruction as well as implications for research are discussed.

Keywords: Number Concepts and Operations, Teacher Education-Preservice, Mathematical Knowledge for Teaching, Teacher Knowledge

Background

Research has shown that prospective teachers (PTs) struggle to think conceptually about integer addition and subtraction, often focusing on procedures (e.g., Bofferding & Richardson, 2013). Additionally, both children and PTs have difficulties when posing stories for integer number sentences (Kilhamn, 2009; Roswell & Norwood, 1999). Since PTs will need to extend their own knowledge about integer operations (Chrysostomou & Mousoulides, 2010) to make sense of children's reasoning with contexts for integer addition and subtraction, we developed a study focusing on the ways in which PTs attended to a child's posed temperature story for an integer addition number sentence.

Utilizing Contexts with Integer Addition and Subtraction

Research has illustrated that both children and PTs have varying degrees of success when generating stories for integer operations (Kilhamn, 2009; Wessman-Enzinger & Tobias, 2015). Difficulties include posing unrealistic stories, changing the number sentence to a different number sentence, and using contexts that do not support opposites (Kilhamn, 2009; Wessman-Enzinger & Mooney, 2014). Wessman-Enzinger & Tobias (2015) found that although some PTs could successfully pose temperature stories, many either posed unrealistic stories or did not use temperature as a context. Likewise, Kilhamn (2009) asked PTs to solve and describe their thinking for integer number sentences (e.g., -8 – -3= □). Kilhamn found that only a small amount of PTs utilized a model or context to explain the mathematics. Interestingly, those who did either used number lines or temperature to explain their reasoning.

Though the research described above has given us some insight about how children and PTs struggle with posing stories for integer operations, little is known about how PTs reason about children's thinking. This study seeks to extend current research by examining the multiple ways PTs responded to an integer temperature story posed by a fifth-grade student and what their understandings of the child's story means for their own content knowledge for teaching. Consequently, the guiding research question for this study was: *What do PTs attend to when they evaluate a child's temperature story related to integer addition?*

Methodology

Background

One hundred and three elementary and middle school PTs in four different classes participated in a study on integer addition and subtraction. The PTs were enrolled in an introductory mathematics content course focused on number concepts and operations. Two of the three authors were professors for this course and all three of the authors analyzed the written tasks from the PTs. The mathematics content course was designed to promote conceptually oriented discourse. PTs were first given problem situations. Then they worked on the problems individually and/or in small groups. This was followed by whole-class discussions where the instructor acted as a facilitator. Throughout the course PTs were asked to solve problems in multiple ways, present their own solution strategies, make sense of the reasoning of others, and ask questions when clarification was needed.

Data Collection and Analysis

Data were collected across two academic semesters, Fall 2013 and Spring 2014. The PTs were given four integer addition and subtraction number sentences with stories that Grade 5 children posed for those number sentences. The PTs were asked to provide a response for whether the child's story made sense and justify their responses. PTs' responses to one of these stories, The Sabrina Task (see Table 1), is the focus of this report.

Table 1: The Sabrina Task

Student	Number Sentence	Story
Sabrina	-6 + -9 = □	It is -6 degrees 2 days ago. It was -9 yesterday. Now it is -15 degrees.

The data for this study were comprised of the PTs' written responses (i.e., responding yes or no for matching and their explanations). Of the 103 written responses, only 100 explanations were readable and consequently analyzed for the study.

When coding the responses, we analyzed them to determine themes for coding, such as attention to language, attention to the solution, attention to realism, and attention to consistency. We then went through an initial pass of coding, checking to see if these themes were evident in each PTs' response. As we discussed themes like language and attention to solution, we noticed that our discussions for this problem centered on realism and consistency. Realism focused on PTs' attention to the fact that adding temperatures from previous days does not determine the temperature the following day as well as everything within PTs' responses being realistic. Consistency included PTs' attention to the story not being consistent with addition, and PTs' explanations themselves being consistent. After each round of coding, the authors met, compared codes, and negotiated any differences. We then obtained counts and percentages of these codes to make sense of the qualitative data (Miles & Huberman, 1994).

Results

Of the 103 PTs responses, 17 thought that Sabrina's story did match the number sentence. Eighty-four PTs thought that Sabrina's story did not match, and 2 PTs thought that Sabrina's story did and did not match at the same time. The responses from PTs contained various levels of attention to consistency and realism.

Attention to Consistency

When attending to consistency we found that fifty-three of the 100 PTs' written responses mentioned that the story was not consistent with the addition sentence. 47 did not mention a consistency issue, including students who claimed that the story was consistent with a subtraction problem instead of an addition problem. Additionally, 28 of the PTs themselves suggested an inconsistent story when trying to fix Sabrina's problem.

When PTs recognized that there was not an addition operation present in the story they expressed this in various ways – utilizing language such as "no addition," "no operation," "no relationship," or "no math." An example of this is PT B20's response: *Sabrina's sentence did not make sense because in her sentence she just stated different degrees on different days. There was no mention of any addition or subtraction.* This PT noted that there was no mention of addition or subtraction and that the story just lists the temperatures from different days.

Attention to Realism

When coding for realism we found thirty-eight of the 100 PTs' responses mentioned that it is not possible to sum previous days' temperatures to obtain the next day's temperature. Sixty-six of the 100 responses were entirely realistic.

We found PTs whose entire justification included the idea that combining the previous two days' temperatures do not give you the temperature for the third day. For example, PT B4 wrote:

The separate days' temperatures don't affect today's temperature. So, the problem doesn't make sense. Others, such as PT B21, also stated that Sabrina's story was not realistic. However, PT B21 offered a suggestion that was coded as unrealistic, because it involved adding the two temperatures together.

> … her story does not make sense. The temperature from two days ago and one day ago do not necessarily dictate what the temperature will be today, there is no relationship there. It would have been fine if she asked, "what are these temperatures together?"

Though PT B21 noted that the temperatures from the previous two days do not determine the temperature for the third day, she stated that Sabrina should have asked what the two temperatures are together, which is unrealistic in the context of temperature.

Discussion and Implications

Our results indicated that PTs used varying levels of justification when discussing Sabrina's story and were generally successful in determining that Sabrina's story did not match the number sentence (81 out of 100 PTs). Of these, only 13 attended to the fact that Sabrina's story was both unrealistic and inconsistent with the number sentence. The majority (64 of 81) focused on only one aspect, either realism or consistency, and four did not attend to either. These results provide insight into PTs' mathematical knowledge for teaching as well as extend previous research focusing on PTs' understanding of integer operations.

Providing PTs with a task that included examining a child's story that was both unrealistic and inconsistent may have helped them to focus on realism and consistency more so than if they had just been asked to pose a story themselves. We found that the majority (76 PTs) did this automatically by discussing realism, consistency, or both even when instructed to just explain whether Sabrina's story matched the number sentence. Giving PTs this story may have given them an avenue to reflect on what makes a story realistic or consistent as previous research has suggested that PTs do not necessarily attend to these same aspects when posing their own stories (Wessman-Enzinger & Tobias, 2015).

The results also indicate that PTs used a variety of justifications when analyzing an integer addition story for a given number sentence. Focusing discussions around these justifications can be a productive way for PTs to develop an understanding of the limitations of certain contexts that support integer operations as well as what it means to operate with integers. For example, the Sabrina Task included a temperature situation that was not possible (adding two temperatures together). By exploring realism, PTs can develop an understanding of why contexts, such as temperature, are not possible for integer addition when the second addend is negative. These conversations can then lead PTs to develop similar understandings for subtracting with a negative as it is also not realistic for a temperature to drop by a negative amount.

Conclusion and Final Remarks

PTs' responses to the Sabrina Task highlight the need for content and methods courses in mathematics education to provide PTs with a variety of classroom experiences, including but not limited to posing problems, examining stories in multiple contexts, and analyzing children's thinking in the domain of integers. The results extend previous research by providing insight into the ways in which PTs coordinate various contextual nuances with temperature and integers, such as realism and consistency in comparison to a given number sentence. We found that despite having experiences with temperature and integers, PTs did not readily extend this knowledge to children's thinking. In addition, although many of the PTs attended to realism and/or consistency in their responses, some stated other things that were unrealistic or inconsistent. Thus, as mathematics educators, we have an indispensable responsibility to provide PTs with opportunities where they can make sense of children's thinking to deepen their own content knowledge as well as prepare them for what they will be required to do in their own classrooms.

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MEASURING MATHEMATICAL KNOWLEDGE FOR TEACHING: THE EFFECT OF THE "I'M NOT SURE" DISTRACTOR

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This paper presents a quantitative analysis of data from an MKT instrument administered on a sample of 284 practicing teachers in Norway and Slovakia. It confirms concerns raised by several qualitative studies: Some teachers choose the "I'm not sure" distractor despite having solid knowledge. On the other hand, we found that the statistical impact of these cases is limited and a qualified use of the MKT measures is likely to provide valid and useful information.

Keywords: Mathematical Knowledge for Teaching, Measurement, Teacher Knowledge

Discussions of the knowledge needed to teach mathematics and issues with the related measures have received notable coverage in research literature. One of the most cited models of teachers' professional mathematical knowledge is the Mathematical Knowledge for Teaching (MKT) that emerged from the research at the University of Michigan (Ball, Thames, & Phelps, 2008). These researchers also developed measures of MKT (the MKT measures, sometimes referred to as the LMT measures) that served many important purposes. For example, they were used to statistically validate the existence of various domains of MKT (Hill, Ball, & Schilling, 2008) and demonstrate the positive effect of teacher's MKT on students' achievement (Hill, Rowan, & Ball, 2005). Due to the unique properties of the MKT measures, several international researchers (see for example Blömeke & Delaney, 2012) adopted them and studied their validity in new cultural contexts.

Our study builds on the cross-national MKT measures adaptation research and involves data from Norway and Slovakia collected by using the same MKT instrument (MSP_04_formA containing 61 multiple-choice items with known IRT parameters for the U.S. population). This report focuses on a specific result that our data set revealed: How the "I'm not sure" distractor affects the response patterns and potentially the overall validity of the MKT instruments.

Theoretical Background

The MKT measures were developed as multiple-choice questions to allow a robust quantitative analysis involving several thousands of respondents. To discourage guessing, which may occasionally result in a correct answer without having the knowledge, the authors of the measures included the distractor "I'm not sure" (coded as incorrect, cf. Hill, 2007). However, the format of the MKT measures was criticized by some researchers and multiple issues were raised. For example, a multiple-choice approach to measuring teacher knowledge ignores its complexities (Beswick, Callingham, & Watson, 2012), and as such might be measuring constructs different from the intended one (Schoenfeld, 2007). Other researchers employed qualitative methods to study how answering MKT measures relates to the actual teacher knowledge or their performance in a classroom. For example, Hill and colleagues describe a case of a teacher who scored significantly high on the MKT instrument (89^{th} percentile) and yet her quality of instruction was only average with undue emphasis on routines and mnemonics (Hill, Umland, Litke, & Kapitula, 2012). Fauskanger (2015) combined MKT measures with constructed-response questions to reveal the cognitive types of knowledge teachers use when answering MKT measures. She found that one teacher (out of 28 in the study) provided three incorrect responses on multiple-choice items and yet demonstrated connected conceptual knowledge in related constructed-response questions. On the other hand, 13 teachers chose correct multiple-choice answers and yet their constructed responses indicate limited, instrumental understanding. In another qualitative study, Fauskanger and Mosvold (2015) looked at the reasons for choosing the "I'm not sure" option and found out, that out of 15 teachers in the study, six choose this distractor because they lacked the knowledge, five had limited, instrumental understanding of the content and four chose "I'm not sure" even though they possessed connected, conceptual (relational) understanding.

Research Questions

Research suggests that at least in some cases, there is a mismatch between teacher's MKT scores and their actual knowledge for teaching mathematics and that teachers choose the "I'm not sure" option even if they possess deep, conceptual understanding of the topic. This is problematic as the "I'm not sure" option is coded as incorrect and thus affects calculations of the test parameters and consequently teachers' abilities. If the number of knowledgeable teachers who chose the "I'm not sure" distractor is relatively high (relative to the number of teachers who chose it for the lack of knowledge) then item difficulty estimates can be inaccurate. Similarly, the ability estimates can be inaccurate for teachers who chose this option repeatedly.

We therefore decided to investigate if the quantitative patterns in our data can reveal how the overall knowledgeability of respondents is related to their decision to choose the "I'm not sure" option. Particularly, we were interested in answering the following questions:

3. Does our quantitative data support the findings of other qualitative studies? Identification of respondents who choose the "I'm not sure" option despite their high overall level of knowledge will provide such support.
4. If so, what is the statistical importance of such cases? Specifically, we will try to assess the extent to which these cases affect the item parameters (and consequently teachers' abilities).

Methods and Results

Our instrument contains 61 MKT items and we administered it in Slovakia and Norway to collect data from 284 practicing elementary teachers. The sample size allows us to use a two-parameter Item Response Theory (IRT) model. IRT scales the person's MKT (ability θ) and item difficulty (b) on the same continuum: A person with the ability (MKT) of θ has a 50% chance to answer an item with the difficulty $b = \theta$. The parameters are scaled so that the average θ for the whole population is 0 and the population standard deviation is 1.

We performed several data simulations to explore various assumptions. We recoded all "I'm not sure" answers the same way as "No Answer" is coded to be ignored in the calculations. Our new estimates of item difficulties (denoted b') and teacher abilities (θ') thus operate on an assumption that teachers were choosing these two options interchangeably and their inclination towards the "I'm not sure" option does not indicate lacking knowledge.

For each item, we then recorded the ability estimates (θ') of all teachers who chose the "I'm not sure" option for that item. In other words, we created ability profiles of teachers who choose the "I'm not sure" option across all items. We did the same analysis for missing answers. Only the items with more than twenty "I'm not sure" or skipped answers were included in the analysis.

Table 1 summarizes important parameters and shows that the mean ability of teachers who chose "I'm not sure" is lower than the mean ability of teachers who chose not to answer items. The difference is statistically significant ($t(40) = -2.85$, $p < 0.005$). This result indicates that statistically, the distractor "I'm not sure" works as intended: teachers who choose it are more likely to be "less knowledgeable" than the ones who simply skip the answer.

Table 1: Ability estimates θ' of teachers who chose "I'm not sure" or skipped answers

	Number of items with frequency >20	Ability (θ') mean	Std. deviation
"I'm not sure"	24	-0.495	0.229
No answer	32	-0.335	0.170

The analysis of individual items provides further insight. Points in the scatter plot in Figure 1 represent the average abilities θ' (y-axis) of those teachers who selected "I'm not sure" for a given item (x-axis). Vertical segments show the standard errors of the θ' estimation. The plot provides visual information on the relationship between the average ability of teachers who selected "I'm not sure" option for a particular item and the population mean (bold horizontal line θ' = 0).

We can see that even if the teachers who choose "I'm not sure" are overall likely to select it for the lack of their knowledge to choose another option, concerns raised by Fauskanger and Mosvold (2015) are valid. For example, the group of teachers who chose "I'm not sure" for the item 16 is on average significantly more knowledgeable than the whole population. As many such teachers are knowledgeable overall and some of them are very knowledgeable (six teachers ranked above the 87th percentile), it is likely that their choice of "I'm not sure" was not caused by the lack of their knowledge. Similarly, the teachers who chose this option for the item 43 or 37 are on average not significantly less knowledgeable than the population average. Further insight into the possible reasons for choosing "I'm not sure" can be gained from a closer analysis of these items and the national samples; Such analysis, however, is beyond the scope of this report.

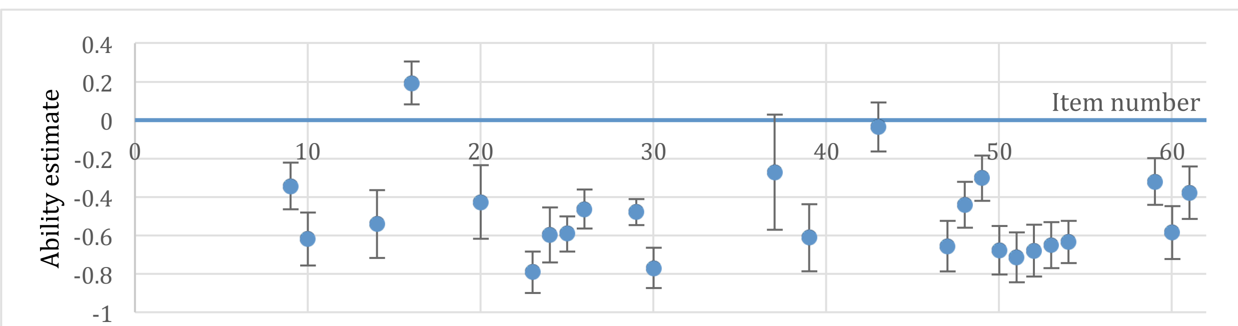

Figure 1. Average abilities of respondents who chose the "I'm not sure" distractor.

When assessing the statistical importance of this group of knowledgeable teachers, we selected all teachers (26) who chose "I'm not sure" in either of the questions 16, 37 and 43, and whose ability estimates were above the population mean (taking the standard error into account). We assumed that these knowledgeable teachers likely chose "I'm not sure" for reasons other than the lack of knowledge. We removed these respondents and recalculated the test parameters.

Due to a limited scope of this paper, we only present a partial analysis of the item difficulties calculated from the new sample (b'') and the original difficulties (b). We can see from the Table 2 that the new difficulties b'' of the items 16, 37 and 43 are within the standard errors of the original difficulty estimations b. The extension of this analysis to all items yields similar results: The group of knowledgeable teachers, who choose the "I'm not sure" option for reasons likely unrelated to the lack of their knowledge does not significantly affect the test parameters.

Table 2: Original (b) and Simulated (b'') Difficulties of the Items 16, 37 and 43

	Item 16	Item 37	Item 43
original item difficulty b and the standard error	0.51 ± 0.22	-0.19 ± 0.44	-1.36 ± 0.3
new item difficulty b"	0.39	-0.22	-1.2

Conclusion

Our quantitative results confirm that concerns raised by the qualitative analyses of Fauskanger and Mosvold (2015) are valid. The analysis of the items 16, 37 and 43 revealed knowledgeable teachers who occasionally choose the "I'm not sure" option for reasons likely not related to the lack of their knowledge. On the other hand, we saw that the group of teachers who chose "I'm not sure" is on average less knowledgeable than those who skipped the answers. Moreover, the statistical impact of the group of teachers who choose the "I'm not sure" option despite their sound knowledge was not significant in our data. We view these results as a reminder of the importance of a qualified use of the MKT measures. If an MKT instrument, such as ours, is used to assess the ability of an individual teacher (e.g. for certification purposes), it is possible that the resulting MKT score will not correctly capture their actual knowledge. If, however, it is used for the purpose, for which it was designed (for example to measure knowledge growth of a group of teachers participating in a professional development), it is likely to provide valid and useful results.

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MIDDLE SCHOOL TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING PROPORTIONAL REASONING

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Vast amounts of money are spent on professional development for teachers, but little is known about its quality or efficacy. IMPRINT, an 18-month long professional development project, was aimed at improving the teaching of proportional reasoning using mathematical tasks set in authentic contexts. IMPRINT framed teachers' analyses of student work using particular domains of MKT. Outcomes indicate that IMPRINT positively influenced participants' MKT in ways that fostered more student-centered instruction, with students working on authentic application tasks set in engaging and motivating contexts. Compared to a control group, students of IMPRINT teachers showed greater knowledge and understanding of ratio and proportion.

Keywords: Teacher Education-Inservice/Professional Development, Mathematical Knowledge for Teaching, Middle School Education; Teacher Knowledge

Introduction

Proportional relationships are foundational to much of the mathematics curriculum, and have been called the capstone of elementary concepts in mathematics (Lesh, Post, & Behr, 1988). However, evidence exists that teachers' knowledge of proportional reasoning, and how to teach it, is not robust (Izsak, Jacobson, & de Araujo, 2012). Although professional development is conducted in virtually every district across the United States, empirical evidence regarding the relationships among professional development, teacher growth, and student learning is scant (Desimone, 2009). Few studies demonstrate a link between professional development and measures of teacher learning, instructional performance, or student learning outcomes at scale. Gersten, Taylor, Keys, Rolfhus, and Newman-Gonchar (2014) found that only two forms of professional development had positive effects on students' achievement in mathematics: intensive courses in mathematics content with follow-up workshops, and lesson study.

Cognizant of these challenges, we designed the **I**mproving **P**roportional **R**easoning **I**nstruction through e**N**gineering **T**asks (IMPRINT) professional development to enhance teachers' mathematical knowledge and pedagogical skill in developing strong understandings of ratio and proportion. IMPRINT was designed around the idea that improvement of mathematics teaching requires expansions of teachers' mathematical knowledge for teaching (MKT) (Ball, Thames & Phelps, 2008). Devising teacher learning environments situated in the demands of instruction (Ball & Cohen, 1999) guided our work. To develop teachers' MKT, we designed mathematically rich experiences anchored in what teachers see as their real work. Mathematical tasks played a large role because teachers are engaged with mathematical tasks in their daily work. The research described in this paper sought to understand whether and how teachers' MKT (Ball, Thames & Phelps, 2008) in proportional reasoning was enhanced through their participation in IMPRINT's professional development program of intensive mathematics content with follow-up workshops and lesson study over an 18-month period, and how this affected student learning in turn.

Methods

Participants in this study were the 16 elementary and middle school teachers who completed the IMPRINT program, which, included working on mathematical tasks that feature proportional relationships in authentic STEM contexts, evaluating word problems in school curricular materials, studying student thinking including known misconceptions in proportional reasoning, revising and creating mathematical tasks for proportional reasoning, and analyzing student work collaboratively. The Survey of Enacted Curriculum (CCSSO SEC, 2005) was administered to teacher participants as both a pre- and a post-test to measure any changes in instructional practice. Improvement in participants' MKT in proportional reasoning was assessed with the Learning Mathematics for Teaching (LMT) 4-8 Proportional Reasoning Assessment (Hill, Schilling, & Ball, 2004) used as a pre- and post-test for project teachers and a control group. Students of IMPRINT and control group teachers were assessed for their knowledge of proportional reasoning using a project-developed 11-item assessment at the beginning and end of the project.

Teachers' analysis of student work provided qualitative data. Teachers administered a proportional reasoning task twice during the project. Each time, teachers used the same task, analyzing it collaboratively. Their analyses focused on the varied strategies students used. In another analysis cycle, a task was chosen by each teacher and administered twice, focusing their analysis on their own students and the specific problem they wanted to do with their students.

LMT data were analyzed using non-parametric tests to detect any statistically significant differences between IMPRINT and control group teachers, and SEC data were analyzed using two proportion *z*-tests to detect any changes in the IMPRINT teachers during the project. Teachers' analyses of student work were categorized, coded, and examined for emerging patterns and themes (Miles & Huberman, 1994). Triangulation of data ensured the reliability of the study.

Results

Teachers reported in surveys that their understanding and teaching of proportional reasoning improved during IMPRINT, and analyses of the quantitative and qualitative data bear this out.

> Through the time that I have been a part of IMPRINT I have definitely strengthened and expanded my knowledge of ratios, rates, and proportional reasoning. Before beginning IMPRINT, I thought I had a solid understanding of these concepts and that I was doing a good job teaching these concepts to my students, however there were so many new and different ways of thinking, teaching, and looking at the concept of proportional reasoning, rates, and ratios.

As this teacher noted, we saw instances of deeper thinking about the content as well as how to teach the content. Below, we provide further evidence bolstering self-reports of improved MKT.

Quantitative Evidence

Due to small sample sizes, the LMT pre-test results were analyzed with a Mann-Whitney U test. The project and control groups were not significantly different (p=.0.582). When the differences from pre- to post-test for each group were tested for significance using the Wilcoxon signed rank test, the control group did not show any significant differences from pre-test to post-test (p=0.330), while the project group showed a significant positive difference (p=0.017).

Post-test results of the student assessment were analyzed using ANCOVA, controlling for pre-test scores. At α=0.10, a significant difference in favor of the project group was found (p=0.087). At α=0.05, ANCOVA also revealed significant differences in favor of the IMPRINT group on two of the individual items (p=0.02 and p=0.04).

We used sections of SEC pertaining to instructional and assessment practices and teacher opinions/beliefs. When the SEC items were analyzed, statistically significant changes from pre- to post-test were detected in the proportions of participants who answered items in ways consistent with the goals of IMPRINT.

Taken together, the LMT results show that IMPRINT teachers' MKT in proportional reasoning increased over the life of the project, while the SEC results demonstrate that IMPRINT teachers were changing instruction in ways that aligned with the project's objectives. Moreover, students of the project teachers outperformed those of a control group of teachers on the 11-item proportional reasoning assessment.

Fostering Teachers' MKT With Student Work

Teachers analyzed their students' work on two different proportional reasoning tasks assigned at two different times. The goal was to develop teachers' MKT by having them appraise students' responses and strategies. The assignments encouraged teachers to track different solution strategies, as well as examine their own and students' habits in solving proportional reasoning problems. This supported development of a wide variety of solution strategies.

Building on this work, teachers chose and implemented a proportional reasoning task. Next, they responded to questions about the tasks they chose, explained why they selected them, and shared student work. They also commented on one another's selections. Analysis of tasks selected and reasons for selecting them shows IMPRINT teachers most valued tasks relevant to students' lives. They also preferred open-ended proportional reasoning tasks leading to multiple solution pathways or leading to fruitful discussions. These discussions often touched on the MKT domain of Knowledge of Content and Students (Ball, Thames, & Phelps, 2008).

We also found teachers slightly more concerned with what students did not understand than what they did understand. We surmise that concern with lack of understanding stems from their efforts to improve the task and students' understanding of it, hence their focus on what students did not understand. There was little mention of specific errors or explanations for errors in the student work, possibly due to the nature of questions teachers were asked as they analyzed their results. Regardless, there is strong evidence teachers were focusing on students' proportional reasoning, some teachers more so than others.

After analyzing what teachers noticed in students' work, we looked at their task revisions. We categorized the revisions as consistent, somewhat consistent, or inconsistent with what they noticed. For example, one teacher used a problem which asked students to find ratios of different colored LEGO pieces. She noticed that students struggled with comparing multiple parts to a whole, as well as filling in a ratio table. She revised the task, but only addressed the ratio table where students lacked understanding of equivalent fractions. We coded this as a "somewhat consistent" revision with a shift in the content of the lesson. The revision was coded "somewhat consistent" because only one part of what was noticed was used in revising the task. Shifts in the content of the lesson occurred when teachers decided to teach a different concept based on observations made from teaching an earlier concept. In the LEGO example, the teacher went from comparing parts to parts and parts to wholes to focusing solely on equivalent fractions, because it is "foundational to proportional reasoning." Changes in the direction of instruction occurred when teachers reflected on students' learning trajectories as a result of analysis. They shifted to more challenging content or reviewed content they deemed their students lacked. In the LEGO example, after implementing a revised task on equivalent fractions, the teacher decided to also teach addition and subtraction of fractions. Finally, we also observed teachers' deliberations on the revision cycle based on analysis of student work. For example, one teacher wrote:

> In this problem, students had to calculate the hourly pay rate and then use that to determine how much money they would earn for a given number of hours worked. It was very straightforward, however, and didn't lend itself to much discussion. There was only one right answer although students could come up with the answer in different ways. The new task I created is much more engaging and has multiple answers.

For this teacher, the correct calculation of a proportion was insufficient. She sought discussion, engagement, and creativity in her students' work. We see this as evidence of improved MKT.

Discussion and Implications

In sum, the IMPRINT experience contributed to teachers' improved MKT and fostered more student-centered teaching practices aligned with content standards in mathematics. We also observed teachers' growing interest in using tasks that featured authentic applications of mathematics. Participating teachers commented that interesting and motivating contexts were reasons for their choice of tasks. Finally, our interpretation of the quantitative data analysis suggests that the students of participating teachers had increased knowledge and understanding of proportional relationships by the end of IMPRINT, in comparison with the control group. This suggests that extended attention to authentic tasks, immersion in rich and focused mathematics content, and the fine-grained collaborative analysis of student work contribute in positive ways to the teaching of proportional reasoning, and ultimately, to student learning.

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PRESERVICE MATHEMATICS TEACHERS UNDERSTANDING OF MODE

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This paper explores two preservice mathematics teachers' understanding of mode. Participants' initial understanding and understanding following use of an interactive virtual manipulative is examined. Findings suggest that participants initially operated with less-effective definitions of mode. Preservice teachers' developed understanding following a learning trajectory is discussed.

Keywords: Data Analysis and Statistics, Middle School Education

Purpose

Mode is a statistical representation, often in the form of a single value or classification, for the distribution of a data set. Mode is an important concept that is typically introduced to students as early as middle school. Unfortunately, researchers have found that a significant portion of students at various levels (i.e., middle-grade, high school, and undergraduate) have difficulty understanding and explaining the concept mode (Groth & Bergner, 2006; Jacobbe & Carvalho, 2011). Given that teachers' mathematical conceptions can influence those of their students (Ball, Thames & Phelps, 2008; Jacobbe & Carvalho, 2011), the present study focuses on the nature of teachers' understanding of mode. Thus, it is the purpose of the present study to explore preservice mathematics teachers' (PSMTs) conceptions of mode.

Perspective

Relatively few studies have focused specifically on conceptions of mode of either teachers or students. Those that have generally done so in the context of or in relation to statistical mean. Groth and Bergner (2006) interviewed PMSTs to investigate their understanding of mean, median, and mode. Their research study suggested that most PSMTs (34 out of 45) believed that a data set may have only one mode, as they defined mode as 'most frequent number.' Only a few participants (3 out of 45) believed that a data set may have more than one mode or no mode at all (Groth & Bergner, 2006). Jacobbe and Carvalho (2011) reported that often preservice and inservice mathematics teachers confuse the concept of the mean with the concept of mode. In fact, many mathematic teachers (over 30%) defined mode incorrectly (Jacobbe & Carvalho, 2011). Barr's (1980) observation of college students revealed that 68% of the participants considered the frequency count, and not the classification (in this case, a number) as the mode. Thus, even at the collegiate level where students learn to become teachers, many consider mode in a manner that does not meet the mathematical definition.

While previous studies reported lack of PSMTs' knowledge about the mode, they only reported PSMTs' amount of understanding, but did not provide reasons as why that is the case. Furthermore, prior study has not examined how conceptions of mode develop or evolve. By contrast, the present research study sought to explore PSMTs' understanding of mode and provide descriptions of learning trajectories regarding conceptions of the mode.

Method

An individual teaching experiment was used to guide this study. In an *individual teaching experiment*, a researcher (a.k.a. teacher-researcher) interviews individual students on one-on-one basis with the intent to model their conceptual understanding of a particular topic (Steffe & Thompson, 2000). Also, the researcher observes and identifies individuals' actions so that the researcher can model learning processes, conceptions, and less-effective understandings. Because of its flexibility, an individual teaching experiment allows a researcher to focus on one student or participant at a time.

The present study consists of nine teaching episodes across a period of nine weeks. Consistent with teaching experiment design guidelines, an observer-researcher was present in all episodes to improve reliability of researcher-constructed models of PSMTs' conceptions. For each episode, tasks were prepared ahead of time. While main tasks were designed to focus on a specific topic or concept (e.g., definition of mode, finding mode of a given data set), probing questions were used during each episode) to clarify participants' answers and press them to explain or explore aspects of the task.

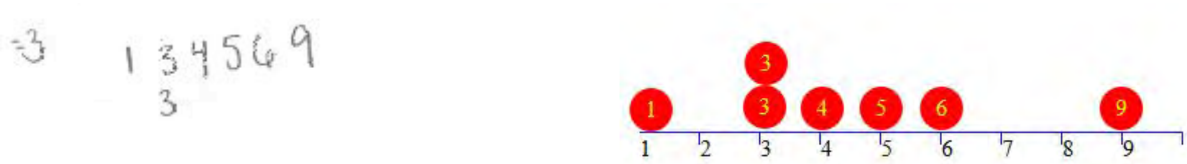
In order to facilitate PSMTs' engagement in tasks related to the mode, a computer-based virtual manipulative in the form of a number line was used. Number lines are widely considered as an appropriate tool for investigating various measures of central tendency (Amiruzzaman & Kosko, 2016; Gravemeijer, 2004). Thus, the Interactive Statistical Number Line (ISNL) was incorporated within specific tasks (see Figures 1 and 2 for screenshots of ISNL). ISNL allowed PSMTs to manipulate the location of data as discrete elements within a dataset. For mode-based tasks, this generally resembled a line plot. Although visually similar to a paper-based line plot, use of ISNL was hypothesized to engage participants in considering how all elements are represented aspects of the data set, given the virtual manipulation of each element in the data set.

Analysis and Findings

Two Preservice Mathematics Teachers participated in this study: Alex and Bob. Considering the length of this paper only Alex's answers are presented here. Data were collected from a larger study focusing on various aspects of measures of central tendency (mean, mode, & median). However, only data focusing on PSMTs' interaction with mode-based tasks is reported. Considering the length of this paper, we limit our description of the analysis and findings of Alex's actions within episode 4.

Episode 4 was intended to explore Alex and Bob's understanding of the mode. Given the initial task to find the mode of a given data set, Alex sorted the data set as, 1, 3, 3, 4, 5, 6, 9, and then told me that mode is 3 (see Figure 2). Following this, he used the ISNL to develop a mathematical model for the data set and indicated 3 as a mode of the data set. To explain his work, he said that he needed to sort the data set, so that he could see the repeated numbers. In the example, there are two 3s, Alex saw them together once he sorted the numbers. He said, "[after sorting] …now I see two 3s…so 3 is mode." Alex knew that the mode is a repeated number and his algorithmic scheme helped him to find the mode (see Figure 1). We refer to this scheme as *algorithmic* because Alex followed a step-by-step procedure to find the mode.

To confirm the model and his initial understanding, Alex was asked to define the mode. Alex answered, "What occurs the most is mode, in the example, 3 occurred the most, so 3 is mode." So, Alex was asked to find mode from, 1, 2, 3, 4. He responded that all of them are mode as all of them occurred most. Thus, Alex's algorithmic scheme aligns with a definition of mode as "the most."

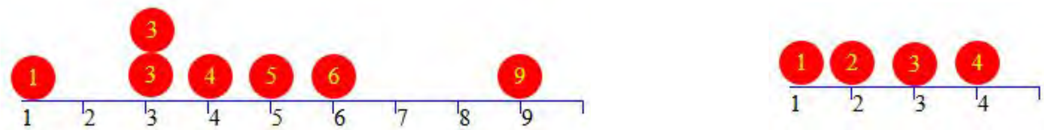


(a) (b)

Figure 2. Alex's approach to find mode from a data set for Task 1 (a) and Task 2 (b).

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

To press Alex's use of this definition, the teacher-researcher asked him to compare the first data set (1, 3, 3, 4, 5, 6, 9) with the second data set (1, 2, 3, 4). Alex used the ISNL to model both data sets (see Figure 2). While he was comparing models of both data sets, he identified some differences, "I see that … data sets are not same….in the first data set, there was a repeated number which was the mode… in the second data set there were no repeated numbers…" After analyzing both data sets via the ISNL, he decided to revise his definition of mode (see Figure 2).



(a) (b)

Figure 3. Alex's models for two data set. One with repeated number and another without repeated number.

Alex modeled additional datasets on the ISNL to find a mode before revising his definition. He said, "The mode is the value that appears most often in a set of data. A data element will be considered to be a mode if it has a frequency of at least two and there is no other data element that has a higher frequency than that. A data set may have no mode, or one mode, or more than one mode." After revising the definition of mode, Alex went back to find the mode of the data set consists of 1, 2, 3, 4. He analyzed the data set once again and developed a mathematical model using ISNL, and he claimed that the data set 1, 2, 3, 4 has 'no mode'. He further added, "…If there are no data elements that occur most frequently, then that situation indicates that the data set has no mode…"

Alex's initial understanding confirms the finding of Groth and Bergner (2006). Knowing that mode is the most frequent number is not enough. In fact, in some cases 'most frequent number' will help to find a mode of one data set, but not necessarily another (i.e., the first data set versus second). By comparing the two sets on the ISNL, Alex identified that his definition of the mode was incomplete and was not suitable for all scenarios or data sets. We conjecture that by manipulating the individual elements of data along a number line, with the purpose of finding the mode (to represent the set of data). Note that, Alex was confronted with scenarios that were at odds with his prior definitions. The comparison task, in particular, perturbed his acceptance of both his prior definition and operational scheme. Specifically, Alex initially considered his algorithm before considering the nature of the data. By the end of the episode, Alex demonstrated attention to the elements of the data as part of a dataset. Although seemingly a subtle distinction, this shift in how Alex operated on the data to find mode allowed for more flexible considerations, and more correct identifications of mode.

Conclusions

This paper described one PSMT's (Alex's) points of cognitive dissonance related to the concept of mode, and provides an initial learning trajectory. A key facilitator of Alex's developing conception of mode appeared to be his use of visualization of mathematical models. Such models may be helpful for both PSMTs and K-12 students to develop more sophisticated schemes in operating with data.

It is a common assumption in mathematics education that merely memorizing definitions and working with a few examples is not sufficient to develop a deeper understanding of concepts. Yet mode is often considered a simple concept, and this study provides evidence that many PSMTs (and potential their future students) may not know that they have understood less useful definition of mode (or even one considered incorrect by the discipline). Alex considered his definition of mode as insufficient only after he developed mathematical models and visualized both models side-by-side. However, some individuals may have needed additional models that can further press their definition of mode. Specifically, Alex visualized a comparison of multiple datasets, and manipulated elements within those datasets through construction of mathematical models. Such engagement pressed Alex to consider a different way of determining mode, and allowed a transition from an algorithmic scheme to a whole set unit scheme. In the *whole set unit* scheme, Alex considered the whole data set as a unit before finding the mode. A similar transition may be possible with other PSMTs or with middle school students. However, future study is needed to both confirm and extend the findings presented here.

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TEACHER KNOWLEDGE RESOURCES FOR PROPORTIONAL REASONING

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Taking the perspective that a robust understanding is necessary for middle grades mathematics teachers, this study explores what particular knowledge resources teachers make use of in their own problem solving. We analyzed the knowledge resources used by 17 middle grades teachers as they solved proportional reasoning tasks in two protocols designed for this purpose. We report on how the occurrences of these knowledge resources were related to teacher performance on proportional reasoning LMT items.

Keywords: Rational Numbers, Mathematical Knowledge for Teaching

Purpose and Background

Proportional reasoning is an important component of the mathematics curriculum (Lamon, 2007; Lobato & Ellis, 2010) that has only increased in prominence and emphasis by being considered its own content domain in the *Common Core Standards for Mathematics* (National Governor's Association & Council of Chief State Schools Organization, 2010). While middle school mathematics teachers are required to guide students in their efforts to develop this complex set of understandings, research suggests that, like students, teachers struggle with proportional reasoning (e.g., Harel & Behr, 1995).

While research on teacher knowledge of proportional reasoning is limited (e.g., Lamon, 2007), there is a growing consensus about the kinds of understandings that may be important for teachers to have (e.g., Lobato & Ellis, 2010). For example, teachers need to understand that a ratio is a comparison of two quantities, where quantity is defined as "a measurable quality of an object- whether that quality is actually quantified or not" (Lamon, 2007, p. 630). A teachers' understanding of ratio should go beyond ways to express it, to include the understanding that a ratio is a multiplicative comparison and not an additive comparison (Lamon, 2007; Lobato & Ellis, 2010; Sowder et al., 1998). This is a critical understanding of the concept of ratio and is considered crucial for the transition from additive to multiplicative reasoning, and teachers need to be able to discern whether students are using additive or multiplicative reasoning (Sowder et al., 1998). Furthermore, a coherent and robust understanding of ratios for middle school teachers must go beyond that of their students (Clark et al., 2003; Lobato & Ellis, 2010). In order to have a robust understanding for teaching, teachers' understanding must also include specialized content knowledge and pedagogical knowledge resources (Ball, Thames & Phelps, 2008).

This study investigates teachers' knowledge resources used in solving proportion tasks. We aim to contribute to research on teachers' understanding of proportional reasoning. Specifically, we examine how the frequency of 17 middle school math teachers' use of knowledge resources, related to teaching proportional reasoning, was correlated with their scores achieved on the proportional reasoning *Learning for Mathematics Teaching* (LMT, 2007) assessment.

Theoretical Framework

We rely on the Knowledge in Pieces theory (KiP; diSessa, 2006) for this study. KiP asserts that individuals hold understandings of various grain sizes that are used as knowledge resources in any given situation (Orrill & Burke; 2013). For novices, these knowledge resources are not likely to be well-connected to each other. As expertise develops, interconnections among the knowledge resources allow them to be invoked in appropriate situations. KiP offers a unique lens for exploring the development of expertise, which is dependent not only on the amount of knowledge resources but also the extent of the *coherency of knowledge* (Orrill & Burke; 2013), which we define as multiple knowledge resources that are connected in robust ways allowing for *in situ* access to the resources. Coherence, combined with a robust set of knowledge resources, allows teachers to deal with complex situations in more efficient ways. This is consistent with previous research on expertise (e.g., Bédard & Chi, 1992) and Ma's (1999) concept of profound understandings of mathematics. We hypothesize that as a teacher develops coherence among knowledge resources, the teacher will be more fluent at teaching and doing mathematics.

KiP represents a departure from the deficiency model traditionally used in the study of teachers' knowledge. Much prior research has focused on what knowledge teachers do not "have" and the misconceptions that they display. In contrast, our application of KiP assumes teachers have a wide variety of knowledge resources available to them, but that different situations invoke different knowledge resources. Teachers develop these resources and make connections between resources depending on the situation or task they are facing. Thus, in KiP, the focus is on what knowledge resources are elicited and used by teachers and how those resources are connected. In this study, our focus is specifically on what knowledge resources elicited in novel proportional reasoning tasks were highly correlated to traditional measures (e. g. the LMT assessment) of teacher's mathematical knowledge for teaching proportional reasoning.

Methods

Data Sources

In this analysis, we used a convenience sample of 17 middle grades mathematics teachers from four states. Participants completed two task-based interviews as well as the LMT proportional reasoning assessment. One paper-based interview was mailed to participants and completed using a LiveScribe pen, which recorded their marks on the paper protocol as well as their spoken words. Participants were asked to speak aloud about their reasoning as they solved the tasks. A second interview was a 90-minute clinical interview conducted in person and recorded on video. Tasks included in the interviews were situated in the work of teaching in that some asked the participants to solve a novel proportional situation, while other tasks asked the participants to make sense of student work or respond to an inquiry made by another teacher.

Data Analysis

Interviews were transcribed verbatim and then analyzed by at least two members of the research team for reliability. Each utterance was coded for the presence of knowledge resources in participants' reasoning based on a set of 23 codes for teaching proportional reasoning grounded in literature as well as from the data (Weiland, Orrill, Brown, Nagar, & Burke, 2016). A correlational analysis was performed using Kendal's Tau correlation coefficients on the frequency of participant's use of the knowledge resources in the two interviews.

Results

Five knowledge resources were found to be significantly correlated with the participants' standardized theta scores on the proportional reasoning LMT (see Table 1). The codes that were significantly correlated with participant performance on the LMT pertained to understanding the structure of ratios or proportional reasoning, which are listed in Table 1. For example, recognizing the relationship between quantities is multiplicative or identifying a scenario as involving proportional reasoning.

Table 1: Significant Kendal's Tau Correlations between Participant's Frequency of Use of Knowledge Resources their Standardized Theta Scores on the Full 73 Item Proportional Reasoning LMT

Knowledge Resource	LMT item Theta	Description
Multiplicative comparison	.413* (.032)	Shares description of the relationship of the quantities that is multiplicative.
Covariance	.447* (.016)	Recognizes that as one quantity varies in rational number the other quantity must covary to maintain a constant relationship.
Ratio as measure	.391* (.036)	Identifies an abstractable quantity created from the combination of the two quantities (e.g., flavor or speed) or discusses the effect of changing one attribute in terms of its effect on the ratio.
Constant ratio	.538** (.004)	Recognizes the invariant multiplicative relationship between two quantities
Proportion situation	.421* (.020)	Recognizes that a situation involves proportional reasoning.

Discussion

Based on the results presented in Table 1, there is a significant relationship between the uses of the following knowledge resources: multiplicative comparison, covariance, ratio as measure, constant ratio, and proportional situation, and how well a participant did on the LMT assessment. Note that all relationships are positive so as the use of these knowledge resources increases, the success on LMT increases. The results also suggest knowledge of the structure of ratios is an important subset of knowledge resources teachers rely on to reason proportionally. In particular, recognizing the invariant multiplicative relationship between quantities in a ratio and recognizing that the quantities in a ratio must covary in a particular way to maintain their relationship are a particularly interesting finding. They suggest the presence and importance of dynamic understandings that would not necessarily be evident in familiar missing value proportion problems that teachers frequently and easily solve with procedures such as cross multiplication. The ability to recognize when reasoning proportionally about a situation was or was not appropriate is also shown to be a knowledge resource correlated with success on the proportional LMT. Previous research found teachers' identification of a situation as appropriately proportional was not correlated with their history of taking content or methods courses (Nagar, Weiland, Brown, Orrill, & Burke, 2016).

Implications

That there is a set of understandings that correlate to high performance on a measure of teacher knowledge can provide guidance for the design of learning opportunities for in-service and pre-service teachers. Opportunities to consider both proportional and non-proportional situations together could address the recognition of proportion. Ratio concepts that involve the idea of variation and invariance may benefit from tasks that are not static instances of equating two ratios, but rather involve some way of experiencing the invariance and covariation that simultaneously exists between quantities in a proportion. Resources higher-scoring teachers use would be powerful tools in the hands of all middle school mathematics teachers.

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CONOCIMIENTO COLECTIVO DE PROFESORES: UNA APROXIMACIÓN BASADA EN ESTUDIO DE CONCEPTOS Y ANÁLISIS DE REDES
TEACHERS' COLLECTIVE KNOWLEDGE: AN APPROACH BASED ON CONCEPT STUDY AND NETWORK ANALYSIS

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Keywords: Conocimiento del Profesor, Conocimiento Matemático para la Enseñanza

En este estudio piloto exploramos una aproximación metodológica para investigar el conocimiento colectivo para la enseñanza de las matemáticas en grupos de profesores basada, por un lado, en el *análisis de redes sociales* (Scott & Carrington, 2011) que tiene fundamentos en la teoría de gráficas, y por el otro en el *estudio de conceptos* matemáticos (Davis y Renert, 2014; Preciado-Babb, Solares-Rojas, Davis y Padilla-Carrillo, 2015) como modelo de formación profesional en el que grupos de educadores trabajan juntos para interrogar y discutir las asociaciones conceptuales de conceptos matemáticos a distintos niveles escolares.

Si bien el conocimiento para la enseñanza de las matemáticas es un tema prominente en la educación matemática, todavía existen preguntas sobre cómo se puede evaluar este conocimiento. Por ejemplo, Davis y Renert (2014) han enfatizado la necesidad de estudiar formas de evaluar el conocimiento colectivo de profesores de matemáticas. En este estudio analizamos las conexiones entre conceptos identificados de forma colectiva en distintos grupos con diferentes niveles de experiencia en matemática educativa. Proponemos que el estudio de estas conexiones puede generar información relevante para evaluar el conocimiento colectivo para la enseñanza de las matemáticas en grupos de profesores.

Uno de los autores impartió tres talleres basados en el *estudio de conceptos* compuestos de dos sesiones cada uno. Dos talleres tuvieron lugar en la Ciudad de México y el tercero en la Ciudad de Colima, México. El primer taller estuvo conformado por estudiantes de grado y de posgrado en educación matemática (n=14). El segundo taller estuvo conformado por académicos y estudiantes de posgrado en educación en matemáticas (n=15). El tercer taller estuvo conformado por estudiantes de licenciatura en educación (n=28). Los datos colectados incluyeron mapas mentales desarrollados como parte de las actividades del taller y una encuesta de salida. La encuesta incluyó preguntas sobre los conceptos clave discutidos durante taller, percepciones sobre el tema del taller (estudio de conceptos matemáticos), así como el impacto del taller en futuras prácticas docentes.

Para estudiar la relación entre los conceptos matemáticos identificados por los participantes del taller utilizamos medidas de centralidad de análisis de redes. Usamos el programa Gephi 0.9.1 para elaborar gráficas correspondientes a las *relaciones de centralidad*. Contrastamos la gráficas considerando los distintos niveles de experiencia en cada grupo. Encontramos que las diferencias entre las gráficas se pueden relacionar con los niveles de experiencia y conocimiento especializado de los grupos. Esto sugiere que esta aproximación metodológica permite evaluar el conocimiento colectivo para la enseñanza de las matemáticas en cada grupo.

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Keywords: Teacher Knowledge, Mathematical Knowledge for Teaching

In this pilot study we explore a methodological approach to study collective knowledge for teaching mathematics in groups of teachers based, on the one hand, on *social network analysis* (Scott & Carrington, 2011), that has its foundations on graph theory, and, on the other hand, on mathematics *concept study* (Davis & Renert, 2014; Preciado-Babb, Solares-Rojas, Davis & Padilla-Carrillo, 2015) as a model for professional development in which groups of educators work together to interrogate and discuss conceptual associations of mathematical concepts at different school levels.

Although teacher's knowledge for teaching mathematics is a prominent topic in mathematics education, there still exist questions about how this knowledge can be assessed. For instance, Davis and Renert (2014) have stressed the need to study ways to assess mathematics teachers' collective knowledge. In this study we analyzed the connections collectively identified in different groups with different levels of expertise. We contend that studying these connections can generate relevant information to assess collective knowledge for teaching mathematics in a group of teachers.

One of the authors delivered three workshops based on *concept study*, with two sessions each. Two workshops took place in Mexico City and the third in Colima City, in Mexico. The first workshop included undergraduate and graduate students in mathematics education (n=14). The second workshop included academic staff and graduate students in mathematics education. The third workshop included undergraduate students in education (n=28). Collected data included mental maps created as a part of the workshops and an exit survey. The survey included questions about the key concepts discussed during the workshop, participants' perceptions on the workshop themes (mathematics concept study), as well as the impact of the workshop on future teaching practices.

We used network analysis measure of centrality to study the relationship between identified mathematical concepts during the workshop. We used Gephi 0.9.1 software to generate graphs corresponding *centrality relationships*. We contrasted the graphs against each other considering the different levels of expertise in each group. We found that the differences among the graphs can be related to the levels of expertise and specialized knowledge in each group. This finding suggests that this methodological approach allows us to assess collective knowledge for teaching mathematics in each group.

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BUILDING MATHEMATICAL KNOWLEDGE FOR TEACHING IN A GEOMETRY COURSE FOR PRESERVICE TEACHERS

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Keywords: Geometry and Geometrical and Spatial Thinking, Mathematical Knowledge for Teaching

Mathematics teachers utilize mathematical knowledge for teaching (MKT) during instruction. MKT includes not only common knowledge of content but also specialized content knowledge usually needed only by teachers (Ball, Thames, & Phelps, 2008). Prospective secondary mathematics teachers (PSMTs) often learn advanced content knowledge and pedagogical knowledge in separate courses. Grossman, Hammerness, and McDonald (2009) argued that this separation is problematic for PSMTs because it fosters a disconnect between content knowledge and the work of teaching in a classroom. Furthermore, they argued the separation of content and pedagogy can result in PSMTs inability to apply specific content knowledge when making pedagogical decisions in teaching.

As part of a national initiative, this project includes developing, piloting, and studying the effectiveness of modules for use in a College Geometry course that interweaves common and specialized content knowledge elements into a rigorous geometry content course. PSMTs engage in mathematical practices while developing a deep understanding of advanced content throughout three modules (Axiomatic Systems, Transformational Geometry, and Similarity). PSMTs also engaged in completing simulations of teaching practice activities that required them to draw on MKT in their responses. We posed the following question for this study: How does PSMTs' MKT change as a result of interacting with the Geometry Modules?

Data for this study included pre- and post-assessments that included open response items for which PSMTs responded to student thinking during a simulation of teaching practice. Other data sources included open response pre- and post-assessments of the geometric knowledge PSMTs believed necessary for teaching at the secondary level. Qualitative data were analyzed using Silverman and Thompson's (2008) framework which focuses attention on evidence of understanding student thinking, developing key developmental understandings, and decentering. In addition, PSMTs' MKT was measured in a pre- and post-assessment format using the Geometry Assessment for Secondary Teaching (GAST; Mohr-Schroeder, Ronau, Peters, Lee, & Bush, accepted). Preliminary findings indicate that PSMTs increased their MKT as measured by the GAST and developed key aspects of MKT. In particular, PSMTs moved from generalized discussion of student thinking to more specific responses that indicated application of MKT.

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FAMILIAR CONTENT IN UNFAMILIAR CONTEXTS: TWO CASES FOR THE MATHEMATICAL DEVELOPMENT OF NOVICE ELEMENTARY TEACHERS

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Keywords: Teacher Education-Preservice, Teacher Education- Inservice/Professional Development, Mathematical Knowledge for Teaching

In an elementary methods course, the first author references Gladwell's (2011) description of struggles to learn the counting number sequence in English. After observations, novice teachers asked the first author for support in teaching money concepts to students. Novice teachers cannot remember how they learned about numbers and coins. We, a mathematics education professor and two undergraduate mathematics students, want to re-create struggles to learn these concepts.

Conceptual Perspective

Ma (1999) wrote that American teachers struggle to explain subtraction with re-grouping well. The work in performing multi-digit subtraction is as an algorithm, without connection to place value. Rowland and colleagues (2005) described one component of their Knowledge Quartet as "structural connections within mathematics itself [and] awareness of the relative cognitive demands of different topics" (p. 263). In this study, participants encounter place value through tasks: familiar and unfamiliar coins.

Research Question and Design

Two questions guide this study. The first is in what ways do novice teachers' mathematical knowledge support answering word problems in unusual contexts? The second is in what ways do the same teachers' knowledge inhibit answering word problems? We will use task-based interviews to answer these questions. Novice teachers refer both to pre-service teachers and teachers in their first one or two years of teaching.

Data Collection Techniques

For place value concepts, participants work individually to count, add, and subtract using digital base-five blocks. Data include videos of work and participants' reflections. For money concepts, participants in small groups will be asked several questions using Euro coins correctly.
Data include scans of groups' written work to word problems and transcripts of discussions.

Preliminary Findings

One pre-service teacher attempted to coordinate the base-five system with the decimal system simultaneously. Another participant made the transition to the new base quickly. We conjecture the use of words in the American currency that convey limited or no decimal place value will create a struggle for these teachers to explain the Euro coins.

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DEVELOPING A FRAMEWORK FOR MATHEMATICAL KNOWLEDGE FOR IMPROVING THE CONTENT PREPARATION OF ELEMENTARY TEACHERS

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Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice

The intent of this poster is to work towards building an overarching framework for the *mathematical knowledge for teaching teachers* (MKTT) needed for the content preparation of prospective elementary (K-8) teachers (PTs). Often the endeavor of elementary teacher preparation falls to those without explicit training in elementary mathematics education (Masingila, Olanoff & Kwaka, 2012). Given the diversity of backgrounds that Mathematics Teacher Educators (MTEs) bring to their practice, it is imperative to establish a framework for the knowledge needed to develop elementary PTs' Mathematical Knowledge for Teaching (MKT; Ball, Thames, & Phelps, 2008). We aim to build a foundation for such a framework by analyzing and synthesizing the existing literature on the work of MTEs through the lens of elementary mathematics content course development.

Much of the extant research conceptualizes MTE knowledge as an extension of teacher knowledge (Jaworksi & Huang, 2014). While such conceptualizations make explicit the theoretical domains that are critical to MTEs' work, such broad conceptualizations fail to capture the tacit knowledge embedded in MTEs' daily teaching practice (particularly in regards to teaching content courses for elementary PTs). Building off the work of Chavout (2009), we will consider and synthesize recently emerging theoretical mappings of MTE knowledge, while comparing and contrasting them with the well-established domains of MKT. Our goal is to provide insight as to how domains of MKTT are embedded in the work of developing elementary PTs' MKT.

The exploration of MKTT is a relatively new avenue for research in mathematics education; it resides at the crossroads of mathematics teacher knowledge and the development of MTEs. As the field grows, it becomes imperative to attend to how MTEs can be better supported for the demands of preparing elementary PTs. We believe our work in this area will have implications for improving both the preparation and professional development of MTEs.

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PROSPECTIVE TEACHERS' USE OF CONCEPTS AND MODELS FOR EXPLAINING FRACTION ADDITION

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Keywords: Mathematical Knowledge for Teaching, Number Concepts and Operations, Teacher Education-Preservice

This study investigated specific aspects of prospective elementary teachers' (PSTs') knowledge for teaching fraction addition. We examined changes in their use of the subconcepts underlying fraction addition and their use of models in their explanations. This is an important area of study because it can be difficult for PSTs to effectively utilize their specialized content knowledge for teaching (Morris, Hiebert, & Spitzer; 2009). Our research questions were the following: What underlying concepts do PSTs mention when describing how they would help a child understand fraction addition? When a drawn model is provided as a representation, how do PSTs use the model?

The setting for this study was a midwestern university where elementary education majors take two sequential mathematics content courses followed by a mathematics methods course. All three courses are taught in the mathematics department. The study occurred during a semester when seven different instructors each taught one section of the first mathematics content course. A pre-test, which including one item addressing fractions, was administered at the beginning of the semester in all sections of both mathematics content courses. A related item on the final exam also assessed similar concepts. At the beginning of the following semester, the pre-test was again administered in the second mathematics content course. This study compared PSTs' responses on the pre-test item with their responses on the same item administered in the second content course the following semester. The study also examined PSTs' responses on the final-exam item. The responses to the pre-test item and the final-exam item were analyzed using the six subconcepts underlying fraction addition identified by Morris, Hiebert, and Spitzer (2009). The responses to the final-exam item were also analyzed for PSTs' use of models, using the four pedagogical purposes for using drawn models identified by Izsak (2008).

Our results showed moderate changes, following completion of the first mathematics content course, in PSTs' use of underlying concepts in their written descriptions of how they would help a child understand fraction addition. Our results also revealed wide variations in PSTs' use of drawn models to add fractions, with some PSTs demonstrating no ability to use a drawn model, most using the model to *illustrate* the solution, and a few using the model to *deduce* aspects of the solution. Finally, our results indicated providing the drawn model could both promote and hinder PSTs' attempts to make the underlying concepts explicit.

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"THAT'S NOT A MODEL! YOU DON'T HAVE ANY NUMBERS!"

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Keywords: Teacher Beliefs, Modeling, Teacher Education-Inservice/Professional Development

Teachers' perspectives of mathematical modeling affect how modeling tasks are designed and implemented (Lesh & Lehrer, 2003). As reported in previous studies, middle grades (5-9) teachers were affected by their educational backgrounds and interpreted the term "mathematical model" in many different ways (Bautista, Wilkerson-Jerde, Tobin, & Brizuela, 2014).

This study analyzed the mathematical modeling views of four middle school teachers using interviews with the teachers, observations of teachers' task solutions and lesson development during a year-long professional development workshop, and student work samples provided by the teachers. This poster focuses on the manifestation of the teachers' views of mathematical modeling in their teaching and their implementation of tasks.

The theoretical framework of this study was created to connect the idea of *mathematical thinking styles* (Borromeo Ferri, 2006) to Kuhs and Ball's (1986) framework for *mathematics teaching styles* and Thompson's (1992) notion of *conceptions* regarding conceptual and calculational orientations. The three previous frameworks are related in their objective to describe teachers' views and the role of these views in their teaching practices. Borromeo Ferri (2006) found that teachers are likely to emphasize different aspects of the modeling process or even avoid certain modeling elements depending on their preferences toward a certain mathematical thinking style. This study extends her results by closely examining each teacher's view of what counts as a mathematical modeling task.

This framework allowed for a three-fold analysis. First, teachers' perceptions of mathematical modeling aligned with three common views in the literature. Second, teachers' views of mathematical modeling reflected specific elements of the modeling process that they emphasized during their teaching. All four teachers listed different criteria for the elements that constitute the successful completion of a modeling task. Third, teachers' perceptions of their role during the planning and implementation of mathematical modeling tasks aligned with their teaching views of modeling. These findings suggest that teachers should consider their own perceptions of mathematical modeling prior to implementing tasks in their classrooms.

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GENERATING A FOLLOW-UP PROBLEM TO CONFIRM STUDENT THINKING AND UNDERSTANDING: WHAT CAN PRESERVICE TEACHERS DO?

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Keywords: Mathematical Knowledge for Teaching; Teacher Education - Preservice

Posing an additional problem to confirm or learn more about student thinking is a strategic move that teachers can make when eliciting student thinking. However, the selection of such a problem is not straight-forward. Teachers need to attend to critical features in tasks to strategically choose what to maintain and what to vary. This work draws upon mathematical knowledge for teaching (MKT) (Ball, Hoover, & Phelps, 2008). This exploratory study examines the preservice teachers' (PSTs') skills at generating a follow-up problem and articulating the rationale for using the problem to confirm a particular student's thinking.

We made use of a teaching simulation in which each PST engaged with a "simulated" student to elicit the student's process and understanding. Then, PSTs were asked to generate a follow-up problem that could be used to confirm their interpretations of the student's thinking (Shaughnessy & Boerst, in press). The student task is to find the total number of squares needed to cover a rectangle which has some squares drawn in. The "simulated" student uses the marked squares to determine the number of squares in a row and then iterates the row count (Battista et al., 1998). Our sample is 39 PSTs in three cohorts in a two-year elementary teacher education program: pre-admission (Pre-admits), beginning of year 1 (Yr1), and beginning of year 2 (Yr2).

PSTs from all cohorts generated follow-up problems for finding the total number of squares needed to cover a rectangle, but we found differences across cohorts in the features that they attended to or changed. For example, more Pre-admits and Yr2 PSTs created problems which had dimensions which were similar to the original problem than Yr1 PSTs. Examining PSTs' rationales for generating these problems, we found that more than two-thirds of the PSTs focused on confirming one core step of the student's process: counting by rows. They did so by either maintaining or changing some features of the problem: some generated similar problems that maintained all of the core features of the problem but others increased or decreased the difficulty.

These findings point to the need for further investigation of PSTs' capabilities in generating follow-up problems. Future studies might examine skill in generating follow-up problem for a range of content and instructional experiences that support the development of such capabilities.

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PRESERVICE TEACHERS' GENERALIZATIONS ABOUT AN AREA STRATEGY

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Skillful mathematics teaching involves being able to size up children's strategies and determine whether a strategy would work in general (Ball, Thames, & Phelps, 2008). This application of mathematical knowledge for teaching may be particularly challenging for novice teachers who have less familiarity with alternative or nonstandard approaches. Yet, valuing and leveraging student thinking is imperative for teachers to teach mathematics equitably.

This study focuses on preservice teachers' (PSTs') generalizations about the validity of a student's strategy for finding the area of a rectangle. The student's strategy is to use marked squares to find the number of squares in one row and then to skip count by that number (Battista et al., 1998). PSTs analyze a student work sample, interact with a "simulated" student, then engage in a structured interview about their interpretation of the student's process and understanding (Shaughnessy & Boerst, in press). We investigated: *What do PSTs attend to when generalizing about the validity of this strategy?* Our sample consists of 38 PSTs at three stages in a two-year elementary teacher education program: pre-admission, beginning of Year 1, and beginning of Year 2.

Through qualitative analysis of interview responses, we found that PSTs often attended to problem features such as having whole unit squares and no partial squares, but some did not articulate the core idea that each row needs to contain the same number of unit squares. Variation in generalizations was most evident when PSTs explained why the strategy would *not* work for a shape. All 38 PSTs accurately identified a shape for which the strategy would not work, but only 20 PSTs (54%) explained that the strategy would not work because such shapes lack the same number of squares in each row. PSTs who were further along in the program were more likely to articulate this reason. In addition, we noticed considerable variation in PSTs' attention to the properties of shapes and use of precise mathematical language.

Further investigation of these ideas could allow us to explore the relationship between program learning experiences and differences in mathematical knowledge for teaching. More broadly, use of generalization questions with simulations could allow researchers to distinguish between cases of superficial and deep understanding.

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Chapter 7

Mathematical Processes

Research Reports

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Brief Research Reports

Posters

AN ANALYSIS OF STUDENTS' MISTAKES ON ROUTINE SLOPE TASKS

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This study extends past research on students' understanding of slope by analyzing college students' mistakes on routine tasks involving slope. We conduct quantitative and qualitative analysis of students' mistakes to extract information regarding slope conceptualizations described in prior research. Results delineate procedural proficiencies and conceptual underpinnings related to various slope conceptualizations that can help both teachers and researchers pinpoint students' understanding and make appropriate instructional decisions to help students advance their understanding.

Keywords: Algebra and Algebraic Thinking

Functions play a crucial role throughout the mathematics curriculum. The concept of slope is critical to the study of linear functions in beginning algebra and extends to describe non-linear functions in advanced algebra (Nagle & Moore-Russo, 2014), the line of best fit in statistics (Casey & Nagle, 2016), and the concept of a derivative in calculus (Stanton & Moore-Russo, 2012). Research has documented students' difficulties with interpreting slope in both functional and physical situations (Simon & Blume, 1994) and with transferring knowledge of slope between problem types (Planinic, Milin-Sipus, Kati, Susac & Ivanjek, 2012). Moore-Russo and her colleagues (Moore-Russo, Conner & Rugg, 2011) have refined and extended the conceptualizations of slope Stump (2001) offered, resulting in 11 conceptualizations which have been documented among secondary and post-secondary students and instructors (Nagle & Moore-Russo, 2013; Nagle, Moore-Russo, Viglietti & Martin, 2013). Procedural knowledge of slope is also important; students need a comprehensive knowledge of a procedure, along with an ability to make critical judgments about which procedure is appropriate for use in a particular situation (National Research Council, 2001).

In the case of slope, procedural knowledge includes familiarity with the symbols typically used in relation to it and the rules used to calculate it (e.g., m, $\frac{rise}{run}$, $\frac{y_2-y_1}{x_2-x_1}$) (Nagle & Moore-Russo, 2013). Conceptual knowledge enables students to make connections between the various notions of slope and to explain why particular procedures for calculating slope work. In a recent study of eleventh grade students' interconnected use of conceptual knowledge and procedural skills in algebra, Egodawatte and Stoilescu (2015) used error analysis to show how prevalent procedural errors sometimes indicated weak conceptual understanding. As described earlier, research has documented students' weak conceptual understanding of slope. However, findings that many students confuse *rise over run* and *run over rise* in the formula for slope and are unsure of the procedure to find a perpendicular line's slope also suggest that students may lack procedural knowledge of slope as well (Stump, 1999).

Since slope is the constant rate of change of two linearly related variables, it is important to consider how students apply covariational reasoning as they conceptualize slope. Described as the "mental coordination of two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen & Hsu, 2002, p. 354), covariational reasoning has been identified as a key prerequisite for advanced mathematical thinking (Carlson, Oehrtman & Engelke, 2010). Carlson and colleagues (2002) describe five developmental stages of covariational reasoning. The first three stages, namely L1 Coordination, L2 Direction, and L3 Quantitative Coordination, are foundational for students' thinking about slope (Casey & Nagle, 2016).

The Present Study

Past research on slope has described the multitude of ways which students might conceptualize it and described students' limited proficiency. However, these areas of research have not been merged. In particular, past research has not engaged in error analysis of students' solutions on common slope tasks to extract information regarding students' procedural and conceptual knowledge of the various slope conceptualizations. We conduct quantitative analysis of students' solutions to routine slope tasks in order to delineate procedural proficiencies and conceptual underpinnings that can be attributed to those mistakes. We link these to the previously identified slope conceptualizations to provide insight into the procedural and conceptual knowledge underlying each notion of slope. The research questions are:

1. What mistakes did students make when solving the various slope tasks?
2. Which tasks did students have the most trouble with and what mistakes were most prevalent?
3. What do students' mistakes reveal about procedural proficiencies and conceptual understanding of different slope conceptualizations?

Methods

Participants and Assessment

Participants in this study were primarily college freshmen and sophomores at a single four-year college in the Northeastern region of the United States. Seven mathematics instructors representing 13 sections of Quantitative Reasoning (Elementary Algebra), Algebraic Problem Solving (College Algebra or Intermediate Algebra), and Precalculus agreed to administer the slope assessment to their students during class time. The assessment was administered during the second half of the semester, after slope was taught. In all, 256 students completed the assessment with fairly even distribution among the three courses: Quantitative Reasoning (QR, n = 79), Algebraic Problem Solving (APS, n = 94), and Precalculus (Precalc, n = 83). The researchers developed a 15-question assessment containing standard slope questions similar to those that students solved on homework and exams. The 15 questions belonged to six broad categories: (1) write an equation of a line given particular information, (2) write the equation of a line given its graph, (3) write the equation of a line given its graph and interpret in terms of a real problem situation, (4) use a table of values to write a linear equation, (5) determine whether graphs of two equations are parallel, perpendicular, or neither, and (6) sketch a line given particular information. One sample problem from each category, with an actual student response, is provided in Figure 1. The fifteen-item assessment included questions that called on nine of the eleven slope conceptualizations described by Moore-Russo, Connor, and Rugg (2011) as shown in Figure 1. Only the Trigonometric and Calculus conceptions of slope (Moore-Russo et al., 2011) were not reflected in the items included on the assignment.

Data Analysis

Coding began with one researcher grading all responses using a four-point scale: 4 points for a completely correct answer, 3 points for a mostly correct answer, 2 points for a half correct answer, 1 point for a partially (less than half) correct answer, and 0 points for a blank or nonsense answer. Next, the researchers used grounded theory (Glaser & Strauss, 1967) to code students' solutions for mistakes. For every answer that did not receive a perfect score, the researchers analyzed the students' solution to determine what mistake(s) were made. We define a mistake as a wrong action or inaccuracy or lack of action that was demonstrated in the problem solution. We recognize that the same mistake may stem from different sources of misunderstanding and we do not distinguish between these when coding for mistakes. Based on the students' solutions, we generated a list of possible mistakes. When a new solution suggested the need for an additional mistake category, the code was added to the list and all responses were revisited in light of the revised list. After generating a list of possible codes, one researcher revisited all student work and completed the coding according to the list of mistakes.

Results

Classifying Mistakes

In total, 18 mistake categories emerged from the grounded theory approach to coding students' solutions on the slope tasks. Table 1 provides a description of all such mistakes and indicates the assessment question(s) on which the mistake was made as well as the frequency of the mistake across all students and questions (n = 3840). Figure 1 illustrates sample responses with the corresponding mistake codes and overall item score (out of 4 points) assigned to the response.

Table 1: Mistake Codes, Related Questions, and Frequency

Code #	Abbreviation code	Description of Mistake	Related Questions	Frequency
1	*NoResponse*	No response or nonsense answer	All questions	496
2	*Arithmetic*	Any type of addition, subtraction, multiplication, or division mistake	All except 14	310
3	*SimpleFraction**	Not changing a fraction to the simplest form	All except 1, 3, 13, 14	128
4	*NoXvariable*	Don't put the *x* variable after the slope in the equation	All except 6, 11, 12, 13, 14	54
5	*SlopeRunRise*	Calculating a slope as run/rise instead of rise/run	2, 5, 6, 7, 8, 9, 10	57
6	*CoordiPoints*	Calculating $\frac{y_2-y_1}{x_1-x_2}$, hence getting the opposite of the actual slope.	2, 5, 6, 7, 8, 9, 10	17
7	*SubtractCoord*	Calculating $\frac{y_2-x_2}{y_1-x_1}$	2, 8, 9, 10	8
8	*OppSignSlope*	Putting a negative sign for an increasing line's slope or vice versa	5, 6, 7, 8	95
9	*BlockSlope*	Using blocks instead of axis' units to calculate a slope	5, 6	94
10	*MentalAction1*	Does not coordinate the *value* of one variable with changes in the other variable	7, 8	32
11	*MentalAction2*	Does not coordinate the *direction* of change in one variable with changes in the other variable	7, 8	30
12	*MentalAction3*	Does not coordinate the *amount* of change in one variable with changes in the other variable	7, 8	118
13	*CalcYintercept*	Don't know how to calculate the *y*-intercept with many non-routine points	9, 10	101
14	*NoSlopeInter*	Not revising a standard form to a slope-intercept form when using the coefficient of *x* as the slope	11, 12	55
15	*GraphOpposite*	Graphing opposite direction with a given slope	13, 14, 15	73
16	*PlotXYchange*	Plotting a point using *x*-coordinate value as a *y*-coordinate and vice versa	13, 14, 15	29
17	*NoOppPerp*	Using reciprocal but not opposite slope to apply to the perpendicular line's slope	4	32
18	*NoRecPerp*	Using same slope to apply to the perpendicular line's slope or just put opposite sign	4	29

*We recorded this as a "mistake" to track the frequency of its occurrence, but students were not penalized when a fraction was not written in simplest form.

Overall Performance on Slope Problems

The mean percentage on the assessment for all 256 students was 65.66%, with APS students scoring highest (66.76%), Precalc students scoring in the middle (65.13%), and QR students scoring lowest (64.92%). A single factor ANOVA showed no significant difference on overall percentage based on the students' course of enrollment [F (2, 253) = 0.15, p = 0.86 >> 0.05]. It is interesting that not only did Precalc students not score significantly higher than students in the more basic Algebraic Problem Solving and QR courses, but they actually scored lower in overall percentage (albeit not statistically significant) compared with the APS students.

Questions with Lowest Average Percentage Scores

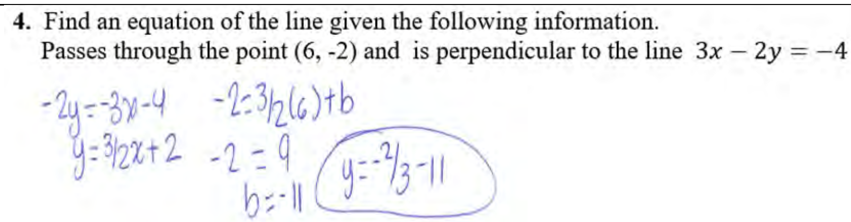
Across the 15 questions, the four lowest average percentage scores were Questions 10 (45.4%), 4 (54%), 7 (55%), and 8 (55.5%). Figure 1 illustrates sample responses highlighting typical mistakes for these four questions. Despite being a standard task, students scored the lowest on Question 10. Many students made a mistake when coordinating points in the slope formula, resulting in a positive slope instead of a negative slope. Question 4 had the next lowest average score. The sample response to Question 4 (see Figure 1) illustrates the common mistake of calculating the y-intercept before finding the perpendicular line's slope. Although this solution uses the negative reciprocal slope of -2/3 in the final slope-intercept form of the equation, notice that the original slope of 3/2 was used when calculating the slope-intercept of the perpendicular line. The variable x is also omitted from the slope-intercept form of the equation. Questions 7 and 8 both required students to write an equation (given a graph) and interpret the equation in light of the real world context that was provided. These items, and their common responses, are discussed in the next section.

Covariational Reasoning and Overall Performance

Students' challenges on Questions 7 and 8 generally related to interpreting the equation in terms of the problem situation. The codes *MentalAction1*, *MentalAction2*, and *MentalAction3* emerged from students' difficulties interpreting the slope of this linear equation in context. A code of *MentalAction1* indicated that a student did not demonstrate knowledge of the two covarying quantities (L1 Coordination). This was often seen in responses that considered only a single variable changing. A code of *MentalAction2* indicates that a student *did* demonstrate L1 covariational reasoning but either did not attempt or made errors in L2 Direction covariational reasoning. This generally appeared when students described the direction of change incorrectly (e.g., "the value of the HDTV increases as the number of month increases"). The *MentalAction3* code indicates that a student demonstrated both L1 and L2 covariational reasoning but either did not attempt or made an error when reasoning using L3 Quantitative Coordination covariational reasoning. Generally, this code indicated that a student did not attend to the amount of change (e.g., "the value of the HDTV decreases over time") or did not correctly interpret the slope as a ratio of change in y variable over unit change in x variable. We conducted additional analysis of how students' covariational reasoning levels were related to their overall performance on the slope tasks. Students who exhibited higher levels of covariational reasoning scored higher on the slope assessment as a whole. Demonstrating fluency with L3 covariational reasoning on both Question 7 and 8 was correlated with a higher overall score on the slope assessment (r = 0.294). Fluency with L2 reasoning was also positively correlated with overall score (r = 0.203).

Category 1: Write an equation of a line given particular information.

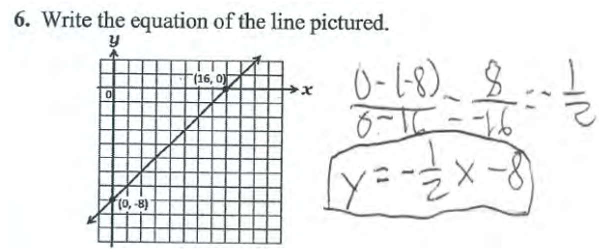
Question 4. (Slope Conceptualizations: Parametric Coefficient, Determining Property)



Response Coding: *NoRecPerp, NoXvariable* (Score 1)

Category 2: Write the equation of a line given its graph.

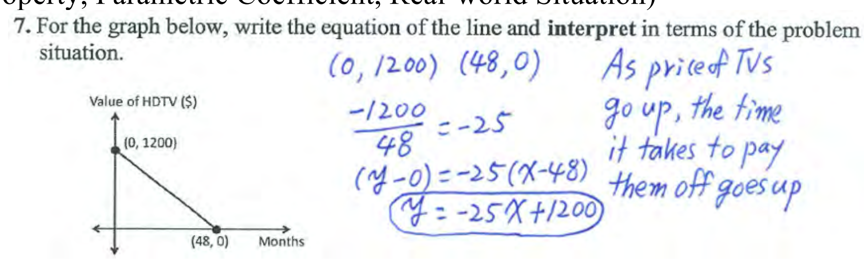
Question 6. (Slope Conceptualizations: Algebraic Ratio, Geometric Ratio, Parametric Coefficient)



Response Coding: *Arithmetic, CoordiPoints, OppSignSlope* (Score 2)

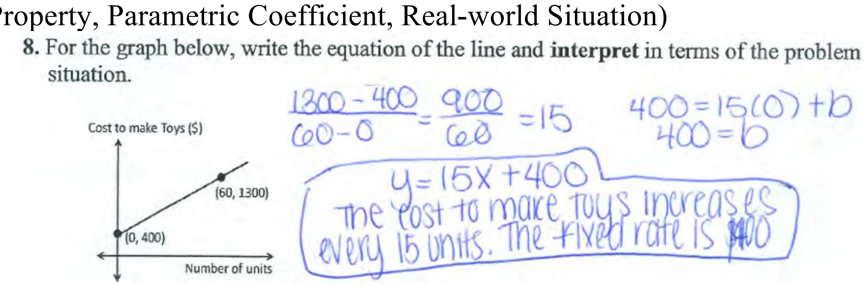
Category 3: Write the equation of a line given its graph *and* interpret it in the problem situation.

Question 7. (Slope Conceptualizations: Algebraic Ratio, Physical Property, Functional Property, Parametric Coefficient, Real-world Situation)



Response Coding: *MentalAction2* (Score 2)

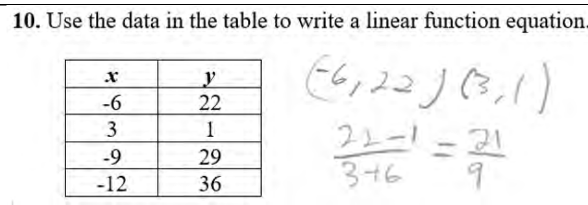
Question 8. (Slope Conceptualizations: Algebraic Ratio, Physical Property, Functional Property, Parametric Coefficient, Real-world Situation)



Response Coding: *MentalAction3* (Score 3)

Category 4: Use a table of values to write a linear equation.

Question 10. (Slope Conceptualizations: Algebraic Ratio, Parametric Coefficient, Linear Constant)


10. Use the data in the table to write a linear function equation.

x	y
-6	22
3	1
-9	29
-12	36

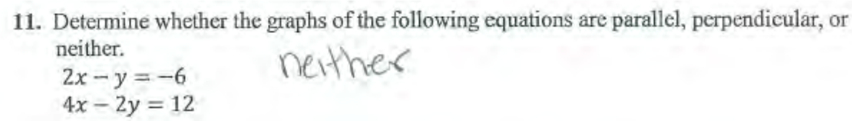
(-6, 22) (3, 1)

$\frac{22-1}{3+6} = \frac{21}{9}$

Response Coding: *CoordiPoints, SimpleFraction* (Score 1)

Category 5: Determine whether graphs of two equations are parallel, perpendicular, or neither.

Question 11. (Slope Conceptualizations: Parametric Coefficient, Determining Property)


11. Determine whether the graphs of the following equations are parallel, perpendicular, or neither.

$2x - y = -6$

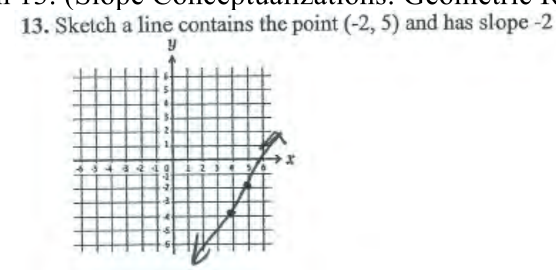
$4x - 2y = 12$

neither

Response Coding: *NoSlopeInter* (Score 0)

Category 6: Sketch a line given information.

Question 13. (Slope Conceptualizations: Geometric Ratio, Behavior Indicator)


13. Sketch a line contains the point (-2, 5) and has slope -2

Response Coding: *GraphOpposite*, *PlotXYchange* (Score 0)

Figure 1. Sample assessment items with anticipated slope conceptualizations, actual student response, and resulting codes.

Discussion

Our study of students' mistakes on routine slope tasks has built on previous literature by analyzing particular mistakes that may hinder students' abilities to reason successfully with the various slope conceptualizations. A total of 18 mistake categories emerged from the grounded theory approach to coding students' solutions. The mistakes indicate that there are many procedural proficiencies required for students to work with the various slope conceptualizations. Arithmetic mistakes were the most widespread mistakes regardless of a student's class of enrollment. These errors carried over into algebraic manipulation with many students making mistakes when adding or subtracting a variable term to the other side of the equation or dividing by the coefficient of the x-term when converting from standard to slope-intercept form. This is a reminder that even when a student has a strong conceptual grasp, a lack of procedural proficiency may hinder his or her ability to reason successfully on slope tasks.

Procedural Proficiencies and Conceptual Underpinnings of Slope Conceptualizations

Past research has focused on describing the many different conceptions of slope. Our analysis in this paper does not attempt to distinguish between a student's procedural and conceptual understanding of slope. However, by analyzing the mistakes students made on problems related to each slope conceptualization, we were able to develop a preliminary list of the underlying procedural proficiencies and conceptual underpinnings that may have been linked with the mistakes we saw on the assessment. Next, by linking the mistakes with the slope conceptualizations each problem illustrated, we were able to make a preliminary list of the procedural proficiencies and conceptual underpinnings which may be linked to the various slope conceptualizations (see Table 2). This is an important step which allows teachers and researchers to begin to break down the underlying ideas and practices that are necessary for a student to work fluidly with a particular notion of slope.

Table 2: Procedural Proficiencies and Conceptual Underpinnings for Each Category

Category	Procedural Proficiencies	Conceptual Underpinnings
Geometric ratio	Count "units" for vertical change. Count "units" for horizontal change. Attach a sign to indicate direction (up or right is positive, down or left is negative).	Rise and run are oriented (signed). Units are determined by graph increments (not blocks). The "rise over run" ratio and "run over rise" ratio are reciprocals.
Algebraic ratio	Subtract y-coordinates for change in y. Subtract x-coordinates for change in x.	"Change" is oriented (signed). The "change in y over change in x" and "change in x over change in y" ratios are reciprocals.
Functional property	Interchange the word slope with the phrase "rate of change".	Slope describes the coordinated change of two covarying quantities.
Parametric coefficient	Algebraically manipulate an equation into slope-intercept form or point-slope form. Identify the coefficient m of x.	The coefficient of x reveals different information depending on the form of the linear equation.
Real-world situation	Identify the real-world quantity associated with the input and output variable (using any type of representation).	Interpret change as it relates to a real-world variable (i.e., a decrease in price shows depreciation over time).
Determining property	Calculate the negative reciprocal. Recognize that equal slopes indicate two lines are parallel. Recognize that negative reciprocal slopes indicate two lines are perpendicular.	Slope indicates the number of points shared by two linear relationships and how they intersect (if at all).
Behavior indicator	Visually determine if a line increases/ decreases.	An increasing (decreasing) relationship is one in which the variables change in the same (opposite) direction. MA2: A positive rate of change indicates two variables change in the same direction.
Linear constant	Choose any two points on a graph/in a table when given multiple points.	Slope is independent of the points chosen since the ratio of change between the dependent and independent variables is constant.
Physical property	Visually recognize a line's "steepness".	MA3: The rate of change indicates the amount of change in the dependent variable per unit change in the independent variable.

Future research should analyze the pattern of student mistakes to better understand whether procedural proficiency or conceptual grounding may be the root of the mistake. In particular, a simple isolated incident may mean a student made a procedural slip while repetition of a mistake across problem types and representations may indicate deep-rooted conceptual misunderstandings (Egodawatte & Stoilescu, 2015).

Slope Questions for Instruction

The questions on which students had the most difficulty can also provide important insight for teachers. Results suggest that teachers should consider including tables with x-values that have varying increments and which are non-monotonic. This is supported by students' difficulties with Question 10, a seemingly standard question other than the lack of a pattern in the x-coordinates provided in the table. Students' difficulties with Questions 7 and 8 highlight the need for teachers to link the *Algebraic* and *Geometric Ratio* conceptualizations with the *Functional Property* idea of slope as a rate of change of two covarying quantities. Many students struggled on these examples because although they were able to explain that the two variables changed together, many even describing the corresponding directions of change in the variables, they struggled to interpret the slope as the amount of change in the dependent variable per a unit change in the independent variable. Thus, our results remind teachers that L3 covariational reasoning is a conceptual underpinning that helps to link the *Functional Property* conception of slope as the rate of change of two variables with *Behavior Indicator* and *Physical Property* conceptions of slope that focus on the direction and magnitude of change, respectively.

Future Study

Our work has described procedural fluencies and conceptual underpinnings related to nine slope conceptualizations. Future work should repeat error analysis with more diverse pool of students to see whether other mistakes emerge. Future studies could also investigate the patterns of student mistakes over multiple items to analyze whether they indicate procedural errors or more foundational conceptual misunderstandings using the framework we have described.

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COLLABORATIVE GESTURES WHEN PROVING GEOMETRIC CONJECTURES

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Research in mathematics education has established that gestures – spontaneous movements of the hand that accompany speech – are important for learning. In the present study, we examine how students use gestures to communicate with each other while proving geometric conjectures, arguing that this communication represents an example of extended cognition. We identify three kinds of "collaborative gestures" – gestures that are physically distributed over multiple learners. Learners make echoing gestures by copying another learner's hand gestures, mirroring gestures by gesturing identically and simultaneously with another learner, and joint gestures where multiple learners collectively make a single gesture of a mathematical object using more than one set of hands. The identification and description of these kinds of collaborative gestures offers insight into how learners build distributed mathematical understanding.

Keywords: Reasoning and Proof, Geometry and Geometrical and Spatial Thinking, Cognition

Introduction

Theories of embodied cognition posit that learners understand ideas, even abstract mathematical ideas, through their bodies and senses (e.g., Lakoff & Núñez, 2000). One important form of embodiment is *gesture* – physical hand movements that people spontaneously formulate to accompany speech. Hostetter and Alibali (2008) argue that gestures are an outgrowth of mental simulations of actions enacted by learners as they think and reason. Considerable research has suggested that gesture production predicts students' learning and performance across a variety of content areas, including mathematics (Goldin-Meadow, 2005; Valenzeno, Alibali, & Klatzky, 2003; Cook, Mitchell, & Goldin-Meadow, 2008).

While the importance of gestures to student learning has been established in a variety of studies, less work has been done detailing how gestures allow for cognition to be physically distributed over multiple learners. Here we focus on how multiple learners use gestures in their interactions with each other, during mathematics classroom learning activities. We argue that these gestures exemplify *extended cognition* (Clark & Chalmers, 1998), the idea that cognitive processes themselves include physical resources beyond the skull. We show evidence of extended mathematical cognition by documenting *collaborative gestures* – gestures made collectively by multiple students as they work together to make sense of mathematical ideas. We discuss the emergence of collaborative gestures in the context of proving geometry conjectures.

Literature Review

Justification and Proof

Justification and proof are central activities in mathematics education (National Council of Teachers of Mathematics, 2000; Yackel & Hanna, 2003). In fact, "proof and proving are fundamental to doing and knowing mathematics; they are the basis of mathematical understanding and essential in developing, establishing, and communicating mathematical knowledge" (Stylianides, 2007, p. 289). Research on mathematicians' proving practices has suggested that proof "is a richly embodied practice that involves inscribing and manipulating notations, interacting with those notations through speech and gesture, and using the body to enact the meanings of mathematical ideas" (Marghetis,

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Edwards, & Núñez, 2014, p. 243). The multimodal nature of proof is also evident for novice students in classroom settings, as students' proofs often take on spontaneous, verbal forms, as opposed to formal, written ones (Healy & Hoyles, 2000), and both teachers and students use gestures as a way to track the development of key ideas when exploring mathematical conjectures (Nathan et al., 2017). Thus, gestures serve as crucial embodied grounding mechanisms for proof-related reasoning in geometry classrooms.

Dynamic Gestures and Dynamic Geometry Systems

One type of gesture identified in prior research as being particularly important is *dynamic gestures* (Göksun, Goldin-Meadow, Newcombe, & Shipley, 2013; Uttal et al., 2012). These are gestures where learners use their bodies, usually their hands and fingers, to physically formulate and then manipulate mathematical entities (see Walkington et al., 2014). For example, when proving that the sum of any two sides of a triangle must be greater than the remaining side, a learner might physically formulate two sides of the triangle with straight hands, and then "collapse" these two sides to show that if the two sides were not larger, the triangle would not be able to close. The presence of dynamic gestures has been associated with more accurate proofs of geometric conjectures, with a medium effect size (Walkington et al., 2014; Nathan & Walkington, in press).

Dynamic gestures allow students to formulate shapes and lines with their bodies in a manner that can be similar to using dynamic geometry software (DGS). DGS allows users to construct, measure, and manipulate objects by dragging and connecting defined objects on a computer screen (Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2004). The direct manipulation of DGS allows users to experiment freely and to have instantaneous interactions with geometric objects and their spatial relations (Marrades & Gutierrez, 2000). Dynamic gestures are limited compared to DGSs in that there is no feedback on whether manipulations are mathematically possible, nor is there exact measurement of geometric objects. However, gestures are highly portable and meaningful to the learner, and are part of the natural way in which human beings communicate, making them a powerful tool for mathematical reasoning. Research has shown that when gesture is facilitated or directed, reasoning is improved (Goldin-Meadow, Cook, & Mitchell, 2009), and when it is inhibited, reasoning is impaired (Hostetter, Alibali, & Kita, 2007). Walkington et al. (2014) found that for geometry conjectures specifically, even the inhibition of sitting in a chair and having a pencil in-hand and paper available reduced the incidence of dynamic gestures, and caused students to formulate correct proofs less often.

Distributed and Extended Cognition

Work in professions involves the coordination of many different inscriptions and representational technologies by differently-positioned actors whose actions occur across a range of social and physical spaces (Goodwin, 1995; Hutchins, 1995). Through joint, coordinated activity, cognition becomes distributed over a patchwork of discontinuous spaces and representational media. In this conceptualization of *distributed cognition*, the environment is used to offload cognitive demands. Theories of *extended cognition* go even further to argue that the social and physical environment of learners is actually constituent of their cognitive system (Clark & Chalmers, 1998). The implication is that cognition, rather than existing in the head of an individual, is distributed over the bodies of multiple learners and the environment around them as they interact. One way in which cognition can be extended across learners is through the use of gestures that extend over multiple persons.

Prior research on students learning origami from instructors has identified *collaborative gestures* as gestures through which a learner interacts with the gestures of a communicative partner (Funiyama, 2000). In the context of this past research, these gestures often involved a learner pointing to or manipulating a teacher's gestures about origami folds. Here we reimagine the idea of collaborative gestures to be relevant to learner-learner interactions around mathematical sense-making, and take such gestures to be a case of extended cognition.

Research Purpose

In the present study, we address the following research question: *What are the ways that team members use collaborative gestures when proving geometric conjectures?* We focus specifically on cases where the physical, gestural activity is distributed over multiple learners, rather than cases of a single student gesturing and another student interpreting that gesture.

Method

Setting and Sample

Eleven undergraduate students enrolled in a teacher education program (ten female and one male) aged 20-22 years voluntarily participated in this 75-minute study. The undergraduates were enrolled in the elementary mathematics method course from a private university situated within a large city in the southwestern United States. Informed consent was obtained from all participants. Sixty-four percent of the participants identified as Caucasian, 18% identified as Asian, and the remaining 18% identified as Latino/a. The undergraduates had already declared a non-education major, but were simultaneously enrolled in a 33-credit hour undergraduate major in education preparing them to pursue teaching careers, work in the social sciences, or informal education paths in non-profit organizations. Students were divided into two groups around two separate gaming systems with each group being video recorded while playing the video game. We focus our analyses on one of the two groups of students, with four females and one male.

Procedure and Measures

The focus of the study was the playing of an educational video game about learning geometry (see Nathan & Walkington, 2017 for more information about the game). Specifically, through the Kinect video game platform, students were prompted to perform specific arm motions and then prove geometry conjectures that were related to those arm motions. While only one participant (the gamer) of each group controlled the Kinect with their body movements, the remaining participants in each group worked collaboratively with the gamer to mathematically prove or disprove the conjectures. The role of the gamer rotated throughout the group so that each participant had the opportunity to perform the directed arm motions and also to take the lead in communicating the proof. In this study, rather than focusing on the directed arm motions that the game directed learners to perform before proving the conjecture, we focus on the hand gestures they spontaneously made while formulating their proofs.

Before playing the video game, students were given a pre-test measuring their knowledge of geometry (basic properties of triangles, circles, and quadrilaterals) and their attitudes towards geometry (items drawn from Linnenbrink-Garcia et al., 2010). Although a detailed analysis of these pre-measures is beyond the scope of this paper, results suggest that the students had neutral or slightly negative attitudes about geometry, rating items like “I enjoy doing geometry and “Geometry is exciting to me” on average between 2 and 3 on a 5-point scale (SDs ≈ 1.0). In addition, results suggest that students had somewhat strong knowledge of basic geometric properties (pre-test items included statements like “the angles of a triangle add up to 180 degrees”), scoring an average of 80% (SD = 13%) on the pre-assessment.

Analysis Techniques

The video captured while the participants played the game was transcribed using Transana (Woods & Fassnacht, 2012) in order to integrate text and video data into the analysis. These transcripts and videos were then analyzed to find where the students performed collaborative gestures – gestures that were distributed in some way over multiple individuals. Transcripts from the group formulating proofs for their six conjectures were analyzed using multi-modal analysis (McNeil, 1992) of gestures. Multimodal analysis involves analyzing, interpreting, and reporting the use of gestures in conjunction with speech transcripts, in order to provide the fullest possible picture of learner reasoning. Here we employ a multiple case studies approach (Yin, 1994), since our research goal is to describe phenomena of potential theoretical importance, rather than the manipulation of a relevant behavior. Case study research recognizes that the rich context in which the interactions occur contain many variables interacting simultaneously.

Results

Through a multi-modal analysis of the focal group proving six conjectures, we discovered three types of collaborative gestures. Although we present a single group's activities, these gesture types were also present and important in subsequent work that examined 4 additional classes of students. We give a case for each gesture type. All student names are pseudonyms.

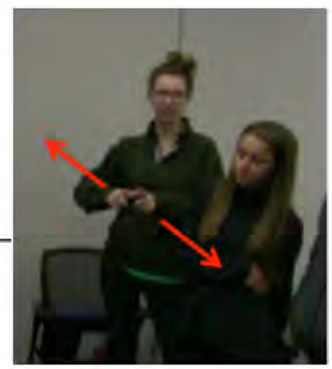
Echoing Gestures

Our first case is taken from the group proving the conjecture, "If you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements." Tanya (bottom image, Figure 1) was in front of the game, with the other students, including Karen (top image, left, Figure 1), assisting her in formulating a proof.

1 Tanya: Okay, given that you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements.

2 Karen: Because you can always…you have the…you can always scale it once you have those three lengths, like the three angles you scale it.

((Karen makes angles with her index fingers and thumbs and moves them in and out.))



3 Tanya: Okay, so I say false because you have certain angles, but you can bring them in and out to make the side lengths bigger or smaller.

((Tanya moves her hands up and down to show how to make the sides of the triangle bigger or smaller.))

Figure 1. Transcript of echoing gestures.

Once Tanya reads the conjecture (Line 1), Karen explains why the conjecture must be false, and uses a dynamic gesture where she formulates a triangle with her thumb and index fingers, making it grow and shrink (Line 2). Tanya seems to immediately understand and take up this gesture, repeating the gesture herself, and putting Karen's explanation into her own words (Line 3). Tanya and Karen performed echoing gestures, where one person made a dynamic gesture, and then a second person repeated that gesture while making the accompanying verbal reasoning her own. Other literature has identified *gestural catchments* as repeated similar or identical hand gestures used by a single gesturer

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

(usually an instructor) to convey similarity of or highlight important conceptual connections (McNeill & Duncan, 2000). Next, we describe a related use of gesture where one learner echoes and repeats the gestures of another learner.

Mirroring Gestures

Our second case is taken from the group proving the conjecture, “If one angle of a triangle is larger than a second angle, then the side opposite the first angle is longer than the side opposite the second.” In this sequence (Figure 2), Haley, shown in the left of the images, works to formulate a proof using gestures. She first draws two angles of a triangle in the air with her fingers, and then points to the angles of the triangle (Line 4). At the same time, Karen (shown on the right, partially cut off) represents a side of the triangle with her arm, interweaving her reasoning (“and the side opposite the first…”; Line 5) into Haley’s narrative proof. Haley and Karen perform identical gestures where they form equilateral-like triangles with their thumbs and forefingers (Line 6). Haley then performs a dynamic gesture where she collapses one side of this equilateral-like triangle inwards in order to vary the angle measurements and check how this impacts the side lengths (Line 7). After the transcript ends, they come to a consensus that the conjecture is true, both repeating their prior gestures as they clarify their reasoning.

4 Haley: Okay, so if one angle is larger than the second angle...
((Haley points to a second angle of a triangle that she drew in the air with her finger while Karen connects the first angle with second angle with her forearm.))
5 Karen: ...and the side opposite of the first...
6 Haley: ...and the side opposite of the first angle...
Okay, so one angle of a triangle...
((Haley and Karen both represent a triangle with both index fingers and thumbs.))
7 I am just trying to picture a triangle ...is larger than the second angle.
((Haley swings the left side of the triangle to make a right triangle on the upper right side of the triangle.))
8 So it's like that then the side opposite of the first angle is longer than the side. Yeah, I would say it's true because if you're saying if this angle is here then there's more space here then this angle makes a smaller line.
((Haley points to where there is more space in her triangle.))

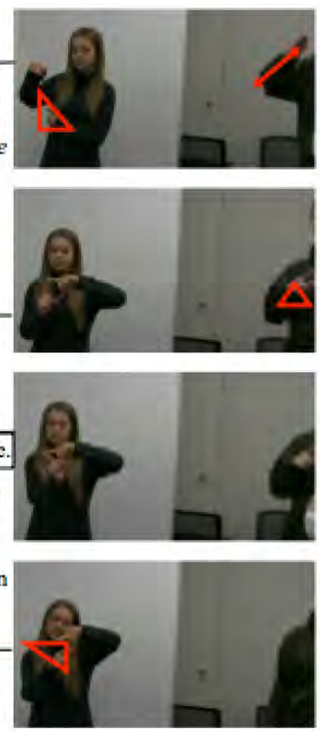


Figure 2. Transcript of mirroring gestures.

Karen and Haley performed *mirroring gestures* as they were gesturing at the same time in response to the same line of reasoning and jointly formulating a mathematical argument. In addition, at times their gestures were structurally identical. Mirroring gestures differ from echoing gestures in that they occur simultaneously – learners are using their bodies in conjunction with each other as they reason together in-the-moment. Echoing gestures, on the other hand, may capture instances where one learner’s reasoning is later taken up by another learner, *after* the initial string of reasoning has been communicated and interpreted.

Haley's and Karen's gestures are representing two distinct geometric shapes, with one shape being imagined in the air in front of each of them. In our final case, we observe gestures where two learners operate on a single imagined geometric object using gestures.

Joint Gestures

Our third case is taken from the group proving the conjecture, "The measure of any central angle of a circle is twice the measure of an inscribed angle intersecting the same two endpoints on the circumference." In this sequence (Figure 3), Karen begins by trying to represent both the circle and the angles using her gestures, but struggles to properly represent the conjecture (Line 9). Stephanie misunderstands the reasoning she is communicating using this gesture (Line 10), so Karen seeks a different approach to make her thinking clear to her group. She calls upon Haley to use her hands to make the circle (Line 15), and then Karen layers her hands over Haley's circle to formulate a central angle and then an inscribed angle.

Stephanie, who is controlling the game for this conjecture, then mimics their gesture (Line 20) and agrees with their conclusion that the central angle would be smaller (Line 22). Haley questions their reasoning at two points during the discussion (Lines 19 and 23), but ultimately the group concludes that the central angle is smaller than the inscribed angle (Lines 26-27). This is a common misconception – the central angle is the large angle since it sweeps out more space.

Discussion and Implications

Situated cognition holds that cognitive behavior is embodied, embedded, and extended. An embodied cognition perspective (e.g., Lakoff & Núñez, 2000) focuses on ways body states and body-based resources shape behavior. Embedded and distributed cognition holds that cognition is mediated by the physical and social environment and the environment is used to off-load operations that could otherwise be performed mentally (Hutchins, 1995). Extended cognition takes this further, positing that social actors and the physical environment, in concert with the mind of the one doing the reasoning, constitute the cognitive system (Clark & Chalmers, 1998).

Here we identified three novel ways in which students socially coordinate hand gestures and speech that exemplify extended mathematical cognition. In echoing another's gestures, one learner makes a hand gesture representing a mathematical object, and then another learner repeats it, often making the reasoning it illustrates personally meaningful. In mirroring gestures, two learners simultaneously make the same or similar gestures with each of their set of hands, as a way of following each other's' reasoning in real time. This strategy goes beyond simply observing another's gestures – by making the same gesture, learners may better understand a collaborator's reasoning. Finally, joint gestures illustrate how multiple learners collaboratively build and manipulate mathematical objects that are too complex for one set of hands. Taken together, these findings suggest collaborative gestures have the potential to provide learners with additional tools that facilitate mathematical communication and proof...."

An interesting question for future research is how collaborative gestures influence student learning – our third case shows an ultimately unsuccessful use of collaborative gestures – and whether collaborative gestures are more effective than other tools of extended cognition (e.g., manipulatives, pencil and paper, DGS). We are of the view that there is not one *optimal* tool for learning about geometric properties and conjectures; rather that students need a variety of experiences exploring geometric ideas with different tools for cognition and collaboration. In the present paper, we argue that collaborative gesture should be one element of students' toolboxes as they learn proof in geometry. In this way, we seek to answer the question "***How can we lay the groundwork for future crossroads between theory, research, and practice?***" We use educational research to lay the groundwork for the potential importance of collaborative gestures, connecting our research to theories relating to gesture and extended cognition. By studying these gestures within classrooms where students are engaged in mathematical reasoning, we begin to consider how this research might inform practice.

9 Karen: Well, I think of a circle and it's like the inside angle versus the outside angle and I feel like the outside angle is bigger because the inside angle would be like this whereas the outside angle is like this.

((Karen uses her fingers to draw an inside angle.))

10 Stephanie: Yeah, it would make sense. I would guess that it is true.

11 Karen: Oh no, I am saying the opposite.

12 Stephanie: Ok.

13 Karen: Right?

14 Stephanie: Oh no!

15 Karen: So like the top of the circle and like...can you hold your hands up for me? You have like this angle that you are drawing from the center, right?

((Karen has Haley make a circle with her hands and then uses the circle to draw an angle from the center.))

16 Stephanie: Mm mm.

17 Karen: That's going to be smaller than like any angle out here, isn't it?

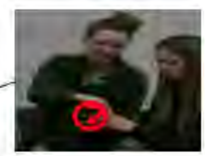
((Karen shows an angle on the outside of the circle.))

18 Stephanie: Sure.

19 Haley: What do you mean?

20 Karen: Yea, like if we have a circle and we draw like this, that point would be…

((Stephanie is tracing the central angle of her circle, while Karen is tracing the central angle of the circle Haley created with her hands.))



21 Karen: What is it called? Then central angle…

22 Stephanie: The central angle would be smaller.

23 Haley: But it is bigger than what angle?

24 Karen: Then the inscribed angle. And the inscribed angle is what the vertex is on the circumference.

((Karen is pointing to the vertex of an inscribed angle that is on the circumference of Haley's circle.))

25 Stephanie: So the central angle is smaller than the outside angle?

26 Karen: That would make sense to me but…

27 Stephanie: I would think too.

Figure 3. Transcript of joint gestures.

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CONTEXTUALIZED MATHEMATICS PROBLEMS AND TRANSFER OF KNOWLEDGE: ESTABLISHING PROBLEM SPACES AND BOUNDARIES

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In this study, we examine how inservice secondary mathematics teachers working together on a contextualized problem negotiate issues arising from the ill-structured nature of the problem such as what assumptions one may make, what real-world considerations should be taken into account, and what constitutes a satisfactory solution. We conceptualize the process of negotiating these questions as the construction of a "problem space," characterized by the boundary between considerations deemed relevant or essential to the problem and ones thought to be beyond the scope of the problem. We use data from group discussions of the problem to consider ways in which problem spaces are co-constructed by learners, instructors, and problem authors and how these problem spaces evolve over time. We conclude by discussing implications of these findings for the design and implementation of contextualized mathematics problems.

Keywords: Problem Solving, Modeling, Classroom Discourse, Affect, Emotion, Beliefs, and Attitudes

Background

The focus of this paper is on teacher work on one particular mathematics problem. The problem is contextualized and open-ended and involves aspects of both problem-solving and mathematical modeling. Lesh and Zawojewski (2007) define *problem solving* as

> the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing and revising mathematical interpretations – and sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics. (p. 782)

The problem we discuss in this paper requires integration of multiple mathematical concepts, as well as interpretation, modification, and revision of ideas within and outside of mathematics. With respect to mathematical modeling, when learners work on a problem involving a real-world context, part of the problem solving process may involve the construction of *mathematical models*, or systems of objects, relationships, and rules that can explain or predict the behavior of other systems (Doerr & English, 2003). Although we do not claim that the problem discussed in this paper is a modeling problem *per se*, participants engage in aspects of the modeling process (*e.g.*, developing a model and interpreting solutions) as they solve the problem. The problem used in this study is contextualized and ill-structured, and requires that the learner find and use information from the real world.

Our focus in this paper is on the negotiation of problem spaces. We defined a "problem space" as the collection of mathematical ideas and classroom and real-world issues and resources that learners take up and use as part of their solution process. These ideas, issues, and resources become visible as *the boundaries of the problem spaces* are constructed and explicitly negotiated. For example, while working on the problem of designing an enclosure with the greatest possible area given a fixed perimeter, a learner may decide (by themselves or by asking a teacher) that they only need to consider rectangular shapes. This decision about the problem boundary leads to a problem space that includes rectangles but not other shapes. By investigating the development of problem boundaries, we hope to better understand the ways in which problem spaces are created and how they evolve, as well as surface implicit assumptions about problem spaces and boundaries that we may take into our own mathematics teaching.

In our research, we assume that learners' problem-solving work is situated within particular mathematics classroom contexts, with associated norms and expectations that will influence the negotiation of the problem space, as will learners' previous experiences in mathematics classrooms. We assume that learners' beliefs about mathematics, and their mathematical and school-learner identities, will influence how problem spaces/boundaries are established, as will power and authority relationships among learners and between teachers and learners. Lastly, we assume that the establishment of problem spaces is an ongoing negotiation that takes place among learners, teachers, and "animated others" such as problem authors or representatives of the real world (e.g., people in a town, a business owner, etc.). Within this framework, we address the following questions:

4. How do mathematics teacher learners, engaged in an ill-structured contextualized problem, negotiate the problem space?
5. What boundaries do the teachers establish and how are they determined? How do the boundaries evolve throughout the problem-solving process?

Method of Study

Context and Problem Design

In Summer 2015, the authors taught an 80-hour mathematics content focused professional development (PD) course to 33 middle and high school mathematics teachers from three school districts in the Southwestern United States. Teachers spent most of their time during the PD working in small groups on problem sets and activities meant to highlight key ideas in middle grades and secondary mathematics.

One of these was the "Quantitative Reasoning Cards" activity, in which participants work in groups of four on a sequence of problems involving real-world contexts. Each problem consists of a statement and several pieces of information. For the problem analyzed in this study, both the statement and the information are on a single card given to the group member designated as the leader for the task. The text on the card is shown below:

> The town of Squareville (population 25,600) relies on a nearby lake for drinking water. The water has been tainted due to an industrial accident. The lake can be cleaned, but it will take about two weeks to do so. In the meanwhile, the state plans to use trucks to send clean water to Squareville from a town 23 miles away. How many trucks will the state need?
>
> In order to proceed to the next task, you (the person holding this card) must give a referee a convincing argument answering this question.

Figure 1. The Water Shortage Problem.

The leader may share the information on the card with other group members and help guide the discussion, but they are not permitted to write anything down nor look up any additional information. Other group members may write down their thoughts and mathematical work, but are not allowed to see nor touch the leader's card. Once the group reaches a consensus solution, the leader must explain it to one of the PD instructors (designated as the "referee"), who may then ask follow-up questions of the other group members. The design of the problem is meant to foster interdependence among group members (Cohen & Lotan, 2014); because the group leader cannot perform calculations, and other members have no direct access to the information on the card, group members must communicate about their overall problem-solving strategy as well as the details of the solution so that the leader can clearly describe the group's work to the referee.

The Water Shortage Problem, designed by one of the authors of this paper, requires participants to answer a practical question (how many trucks are needed to deliver water to a town) by analyzing rates of water consumption and delivery rather than absolute amounts. The information card intentionally leaves some essential questions unanswered, such as how much water a truck can carry, and how much water each person will need. The purpose of providing incomplete information is to stimulate discussion among participants about what quantities are relevant to the problem's solution, and to encourage participants to seek information from sources external to the activity.

The problem is designed to elicit thinking from participants about how to estimate quantities whose values cannot be determined exactly. For example, if a disaster preparedness website recommends that each person receive 2 to 4 gallons of water per day, should one assume that each person will receive 2 gallons, 4 gallons, or some amount between these two extremes? We have found in our own implementations of this and similar problems, given a range of possible estimates for a quantity, participants will often select an estimate at the middle of the range, even when a lower or upper bound might be more useful for the situation at hand.

The problem is also designed so that solutions that do not contain rate thinking (*e.g.*, thinking only about how many gallons total are needed for 2 weeks, rather than thinking about gallons per day) will likely lead to unreasonably large answers. This problem feature is intended to spur learners to reconsider their solutions and seek ways to decrease the number of trucks needed. For this to occur, participants must expand the problem space to include consideration of whether a given number of trucks is practically feasible; while a request for ten trucks is likely to be honored by an emergency management agency, a request for five thousand will almost surely be rebuffed.

Participants, Data Collection, and Analysis

During the problem implementation, we captured video and audio recordings of two groups of teachers working on the Water Shortage Problem. Each group consisted of four inservice secondary mathematics teachers. Group 1 consisted of three female middle school teachers and one male high school teacher; Group 2 consisted of one female high school teacher, one female middle school teacher, and two male middle school teachers. Group 1 spent 22 minutes on the problem, and Group 2 spent 30 minutes.

After the conclusion of the professional development course, the two researchers viewed both videos independently and made note of instances in which participants and instructors appeared to question or negotiate the boundaries of the problem. For each such instance, we attempted to identify factors in the group discussion, the instructor's comments, or the design of the task that may have influenced the group's decision about how to define the problem space. We repeatedly met together to compare analyses and come to consensus on any discrepancies. We report results of this initial work here; however, we intend to continue to refine our analysis process as we attempt to apply it to the data we have collected (video and audio) for small groups working on other contextualized problems.

Results

In both groups that participated in the study, the group leader read the task, and the group worked gradually toward a consensus solution, making assumptions about the situation described, making preliminary estimates, and refining these estimates to produce a reasonable and practically feasible solution. Along the way, each group confronted questions about which elements of the real-world situation should be taken into account and which considerations lay beyond their co-constructed boundaries. In this section, we analyze each group's negotiation of the problem space and observe how this space evolved over the duration of the group's work on the problems. All names used below are pseudonyms.

Shifting Responsibility for Boundary-Setting: The Case of Group 1

Vicki, the leader of Group 1, introduced the problem by reading her card aloud to her teammates Tina, Kenny, and Nalda. Shortly after reading the card, Vicki questioned whether the group was allowed to consider information not on the card. The question of how much discretion the group has in negotiating problem conditions and goals occurred again later, as Vicki asked whether the question was about "efficiency" or about how many trucks the state should send. Upon asking this, Vicki said, "I don't know how far we're allowed to take this," suggesting that authority for determining problem boundaries lay at least partially outside of the group itself. We hypothesize that many teachers' prior experiences with contextual problems (as teachers or learners) may consist mainly of problems for which the boundaries are largely pre-determined by the problem statement, or as structured by the teacher.

Table 1: Interactions Influencing the Problem Space for Group 1

Interaction	Action/response	Possible causes of interaction
Vicki: Are we allowed to extrapolate outside of what is on the card? We would need to know how much a truck could carry, average family size…	Tina begins to look up information on phone.	Contextual problems encountered in school often provide the information that is needed; no more, no less. In this setting the group must negotiate the boundaries of the problem space.
Nalda: Are we looking for realistic solutions to this? Because the state isn't going to pay for that many a day… each truck can make four trips…	Nalda's teammates assert that they are counting truckloads, not distinct trucks.	Nalda believes that in this case, issues of realism should at least be considered. Nalda uses the pronoun "we," while Vicki uses the pronoun "they."
Kenny: What if they don't have tankers, they have an average water truck?	Group considers both scenarios and produces an estimate for each.	The problem is ambiguous on the issue of which type of truck the state will use. The group does not have the resources to resolve this ambiguity, but is willing to manage it as a condition of the problem.
Tina: So did we answer the question? Vicki: I feel like we would need more parameters though to be able to really integrate the 23 miles. Nalda: I feel like they give us the 23 miles for us to estimate how many trucks.	Group begins to consider multiple trips per truck.	Task design: Tina cannot look at the card with the question on it. Nalda pushes the group not to set aside the mileage information. She reframes the problem so as to put group members inside the real-world situation.
Kenny: How long is a tanker? Nalda: I don't know. … Kenny: Where are they storing this? Nalda: Well, water towers. Kenny: I'm just thinking/ Nalda: /Half an hour to fill, half an hour to get there…	Tina pulls up a picture of a tanker on her phone and shows the group. Nalda redirects the group's attention to the calculation of the number of trucks.	Nalda seems to view Kenny's queries as outside of the problem's boundaries.

The group ultimately developed estimates for the number of trucks needed in two different scenarios: if the state sends large tanker trucks, and if it sends smaller water trucks. The group's initial approach assumed that each truck would make only one trip per day, and that each resident of the town would receive 90 gallons of water per day. This led to an estimate of 221 tankers per day. At this point Nalda raised the concern that sending 221 tankers per day would not be realistic, and suggested a model in which each tanker makes several trips per day. The group initially dismissed this suggestion, claiming that the problem was to estimate the number of truckloads, not tankers. However, by pointing out that the group had not used the information provided about the distance between the towns, which we interpret as an appeal to an external source (i.e., the problem author) in order to determine a problem space boundary, Nalda later persuaded her teammates to consider the possibility of allowing each truck to make several trips per day, and count the number of trucks rather than the number of truckloads of water. The group eventually produced an estimate of 56 tanker trucks.

Table 1 outlines some instances in which the problem space was negotiated, explicitly or implicitly, by members of Group 1. We note here that, for Group 1, interpreting the implicit intentions of the problem author appears to be a central part of their effort to negotiate the problem boundaries, and thus the problem space. At the same time, the group also attended to whether a particular approach or solution was realistic. In the data, we found multiple examples of this push-pull between school mathematics norms for contextualized problems (*e.g.*, figuring out what the problem author intends) and the desire to find a realistic solution. Importantly, we note that attention to realism may itself relate back to expectations about how we do mathematics in school when faced with contextualized problems for which some information is not given.

The Instructor's Role in Expanding the Problem Space: The Case of Group 2

Vince introduced the problem to his teammates Tobias, Darla, and Violet by summarizing the information on his card rather than reading it verbatim. The group immediately began searching the internet for information relevant to the problem and found that a water truck can carry 5000 gallons, and that the average American uses between 80 and 100 gallons of water per day. Based on this information, they obtained an initial estimate of 6450 trucks, which Violet deemed to be "excessive."

Spurred in part by the infeasibility of this estimate, the group then began to identify ways they could significantly decrease this estimate. Tobias suggested researching the minimum amount of water a person needs each day; based on his research, the group accepted a much lower estimate of 5 gallons per person per day. The group thus arrived at a more modest estimate of 358 trucks, still reflecting the implicit assumption that each truck will make only one run over the two-week period. The group presented this solution to Nancy, one of the PD facilitators. Nancy stated that the state did not have 358 trucks to spare, and that the group should try to determine the *minimum* number of trucks needed. After she left the group, Violet pointed out that the question did not ask for the *least* possible number of trucks, and Tobias claimed that Nancy had changed the question.

After this exchange, Tobias suggested considering how many trucks are needed per day (rather than for the entire two-week period); this brought the group's estimate from 358 down to 26. The group then gradually developed a plan in which six trucks take turns dropping water off at Squareville; at any given time, one truck is in Squareville dropping water, one truck is in the nearby town collecting water, and four other trucks are in transit between the two towns. Vince presented this solution to the other PD instructor, who endorsed it as an acceptable solution.

As the group worked on the problem, the problem space grew to encompass considerations of how much water a person needs during an emergency, and how much water an "average" truck can hold. However, only after Nancy visited the group and encouraged members to develop a more feasible solution did they seriously consider the possibility of having trucks perform multiple runs on the same day. This consideration entered the problem space at least in part due to Nancy's intervention. Table 2 below shows some instances in which Group 2 interacted to define the problem space and its boundaries, and our interpretations of possible causes of the interactions.

Table 2: Interactions Influencing the Problem Space for Group 2

Interaction	Action/response	Possible causes of interaction
Violet: Do we know how much water the trucks hold? Or how much each person needs? Vince: No.	Tobias starts to look for information on the internet using his tablet.	The problem cannot be solved without information that is not on the card.
Violet: Are they telling the people to limit the water? Because I feel like that would be beneficial.	Tobias determines that on average, a person uses between 80 and 100 gallons per day. The group doesn't pursue Violet's idea yet.	The group seems to feel that limiting water is beyond the boundaries of the problem.
Violet: It's 80-100 gallons per day, so do we just want to use 90?	Group calculated 90 x 14 x 25,600 = 32,256,000 gallons. Divided this by 5000 to obtain 6451 trucks.	Using the midpoint of a range as an estimator is possibly related to prior experience with school math problems; in this case, it may actually be worthwhile to use the lower end of the range in order to minimize the number of trucks needed.
Violet: 6451 trucks, that seems really excessive. Tobias: Let's see how much a person *needs* in a day. Violet: We don't know what limitations have been set for this town.	Group discusses different uses of water and eventually settles on 5 gallons per person per day, leading to an estimate of 358 trucks.	Initially, Violet seems to view the issue of water rationing as outside the boundaries of the problem. Eventually, the group shifts the boundaries to encompass this question.
Nancy: Yeah, well Circleville's also having a water issue, and I just don't have 358 trucks, so what's the minimum number I need? … Nancy: So think just a little more about how many trucks you *need*. Like what's the minimum number I can give you?	Group turns to the question of how long it takes for a truck to complete one cycle of loading, driving to town, unloading, and driving back.	Nancy observes that the group has not incorporated the possibility of trucks making multiple deliveries per day into the problem space; she uses the impracticality of a request for 358 trucks to encourage the group to reconsider the problem boundaries they have constructed.
Darla: How long does it take to unload a water truck? Violet: And how do you decide who gets water first? Are we figuring out the least? Is that the question? The least number? It just said how many trucks need to be sent, it didn't say least! Vince and Tobias: She changed it.	Group estimates how much time is needed for a delivery cycle and how many cycles are needed per day, and eventually decides upon 6 trucks.	Violet, Tobias, and Vince indicate their belief that they have been asked to enter a different problem space.

The push-pull between school mathematics and associated expectations of problem authors on the one hand and concerns for realism played out somewhat differently in this group. The group seemed initially unconcerned with limiting water, an aspect of the situation that would most certainly come into play in the real world. Yet, an unrealistic number of trucks did spur the group to reconsider water consumption, and this then became a part of the problem space. At this point, Violet raised another issue related to negotiating the problem boundaries, namely that the group did not know what limitations on water use had been put in place for the town in question. The group expressed frustration with the ambiguous nature of the problem space after the instructor questioned whether 358 was realistic.

Discussion

We offer this report as an initial analysis of the construct of "problem space" as it applies to contextual problems in mathematics. We make no claim that our findings generalize across all classes of mathematics problems and all groups of learners. The negotiation of a problem space may look markedly different in the context of a more closed-ended task, and may also vary according to the age and mathematical background of learners. One may argue that in this particular study the participants' shared familiarity with rate reasoning allowed them to devote additional attention to considering boundary issues such as which quantities in the problem should influence the problem's solution and which should not. Further study, with different types of tasks and with different populations of learners, is needed for a better understanding of how problem spaces develop in different settings.

The expectations that learners have of teachers, problems, and genres of mathematical tasks are central to the establishment of problem boundaries and spaces within them. If learners are accustomed to tasks in which all relevant and necessary information is explicitly provided, they may initially hesitate to consider external sources of information when presented with an open-ended problem. This may lead to learners attempting to work within a problem space that is too narrow to provide the intellectual resources necessary to construct a solution. At the same time, learners may make decisions to expand the problem space when faced with a problem that does not explicitly provide all the resources necessary for its solution. However, the boundaries defining the problem space cannot expand endlessly; learners must, at some point, accept that the situation they are attempting to analyze contains some details that are inaccessible to them and therefore cannot be modeled mathematically.

Our analysis of the negotiation of problem boundaries has implications for the practice of designing open-ended problems. In analyzing the groups' work on the Water Shortage Problem, we found that the problem worked as intended in at least one respect: both groups originally obtained infeasibly large estimates for the number of trucks needed, and thus were encouraged (without external feedback) to revise their assumptions. Both groups decided that the problem space should include some consideration of whether the solution obtained was fiscally responsible. Additionally, both groups decided to include some analysis of whether the solution obtained was physically feasible; for example, Group 2 developed a scheme in which six trucks rotate in and out of Squareville in succession, dropping off water as they arrive. Thinking about the problem at this level of detail helped the group develop confidence that a solution with six trucks was feasible and would deliver enough water. We posit that open-ended problems that contain supports for the development of detailed models and that encourage winnowing out unreasonable solutions may support learners in expanding problem spaces to include practical considerations.

Our analysis also has implications for the orchestration of open-ended problems. Both groups had questions that one could easily imagine asking of a teacher; for example, Vicki might have wanted to ask a PD instructor whether it was permissible to consider information outside of the problem statement. However, in the absence of instructor guidance, the group quickly decided that information from the real world lay squarely within the boundaries of the problem, since the information on the card was inadequate. Because problem spaces evolve over time, a teacher implementing an open-ended problem may wish to take an observer role initially and allow the problem space to develop according to the explicit and implicit demands of the problem.

We conclude this report by highlighting two ways in which the problem space of the Water Shortage Problem may communicate with the broader space of students' real-world experience. Since the time of the creation of this problem, serious water crises have occurred in places such as Flint, Michigan and Corpus Christi, Texas. In subsequent implementations of the Water Shortage Problem, the authors have noticed that teachers who have experienced water crises such as these sometimes interact differently with the problem; they are more knowledgeable about how water is actually distributed during a water crisis, and more attentive to logistical issues such as how a town should time and manage water collection. We offer this as an example of learners' real-world experiences interacting with the problem space. As an example of the problem space talking back to the broader world in which the learners live, consider the following comment from Violet: "If this [90 gallons per day] is what I use on a regular basis and this [4-6 gallons] is what I use in a disaster… like… I feel like *this* is the disaster!" Seeing the disparity between everyday water usage in the U.S. and recommended water usage during an emergency may heighten learners' awareness of the possibility of scaling back water consumption and using natural resources at a more sustainable rate.

Endnote

Alphabetical listing of author names is intended to indicate equal contributions to the paper.

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EXPLICATING THE CONCEPT OF CONTRAPOSITIVE EQUIVALENCE

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Abstract: This paper sets forth a concept (Simon, 2017) of contrapositive equivalence and explores some related phenomena of learning through a case study of Hugo's learning in a teaching experiment guiding the reinvention of mathematical logic. Our proposed concept of contrapositive equivalence rests upon set-based meanings for mathematical categories and negation, representing these sets by closed regions in space, and linking conditional truth to a subset relation between these regions in space. Our case study serves to portray that students must construct all of these elements to achieve a sense of necessity in the equivalence. This study thus contributes a set of learning goals for any introductory logic instruction using Euler (or Venn) diagrams, which has been little studied in the mathematics education literature.

Keywords: Reasoning and Proof, Problem Solving

Proof oriented mathematics instruction depends upon mathematical logic to ensure that students learn 1) to interpret mathematical statements the way mathematicians do and 2) to draw inferences that do not violate the mathematical community's norms. Prior research provides ample evidence that students' interpretations of mathematical conditionals (statements of the form "if…, then…") pose a number of difficulties (e.g. Durand-Guerrier, 2003) as does the logical equivalence of contrapositive (CP) statements (Stylianides, Stylianides, & Phillipou, 2004). There is little prior literature on how students are to come to learn CP equivalence (Yopp, 2017) or the meanings by which this can be a logical necessity. This paper seeks to fill this gap through a case study drawn from a larger series of teaching experiments guiding reinvention of mathematical logic through reflective use of mathematical language (Dawkins & Cook, 2016).

Logical Background

The CP of a conditional "If [P], then [Q]" is the conditional "If not [Q], then not [P]." Consider the statement, "If a triangle is obtuse, then it is not acute." Its CP is "If a triangle is acute, then it is not obtuse." Certainly these statements are both true, but how are their truths linked? In our prior studies we observe that students often reason about such statements using examples, properties, or sets (Dawkins & Cook, 2016). We encourage the reader to consider how both statements can be confirmed from the fact that any triangle has exactly one of the properties acute, right, or obtuse (a property-based strategy). Notice that so affirming both conditionals may not reveal the relationship between the two or why all conditionals with that relationship must share a truth-value. For this reason, we propose a distinction between a *CP inference* and *CP equivalence*. A student draws a CP inference when they infer a CP is true from the original conditional or when they use the original conditional to infer not [P] from not [Q] (*modus tollens*). CP equivalence instead entails students constructing a logical equivalence between any conditional and its CP rooted in generalizable meanings for conditional truth and reference.

We present our intended understanding of contrapositive equivalence in terms of Simon's (2017) explication of mathematical concepts. Simon explains, "A mathematical concept is a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (p. 7). This concept thus reflects our understanding of how a student might come to understand the necessity of CP equivalence. Simon clarified that concepts result from reflexive abstraction and thus are anticipations based on the learner's activity. This characterization

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of a concept helps distinguish our work from other relevant studies. Stenning (2002) explored logical reasoning as a broadly assessed without attending to students' meanings for particular concepts or their sense of logical necessity. Stylianides et al. (2004) assessed whether students use a CP equivalence rule they were taught when assessing arguments, and found that students frequently did not apply CP equivalence as intended. They did not study the students' meanings for conditional truth or how they entail CP equivalence's necessity. Hawthorne and Rasmussen (2014) explored students' meanings for elements of formal logic such as truth tables, and found that many learned such formalisms disjoint from their ongoing mathematical activity. They lacked necessity for the learned rules.

We articulate the concept of CP equivalence as follows:

> (Point 1) A mathematical conditional is true whenever the set of objects satisfying the *if* part is a subset of the objects satisfying the *then* part. (Point 2) These two sets can be represented as closed regions in space with points representing the mathematical objects. (Point 3) The negation of a mathematical category refers to the complement set of mathematical objects. (Point 4) Therefore, whenever a conditional is true, its CP must also be true because the complement of the larger region is contained in the complement of the smaller region.

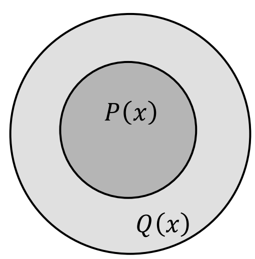
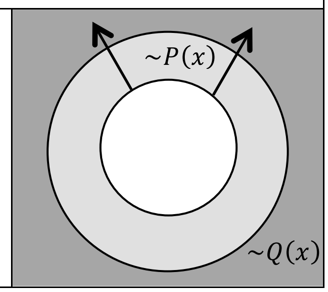
Original conditional		CP conditional	
"If for $x \in S$ $P(x)$, then $Q(x)$." is true whenever $\{x \in S \mid P(x)\} \subset \{x \in S \mid Q(x)\}$		"If for $x \in S$ not $Q(x)$, then not $P(x)$." is thus true because $\{x \in S \mid {\sim}Q(x)\} \subset \{x \in S \mid {\sim}P(x)\}$	

Figure 1. Euler diagrams portraying the subset meaning for a conditional and its CP.

Point 1) We propose the subset meaning for conditional truth as part of Dawkins and Cook's (2016) more general findings that set-based meanings were most propitious for students' reinvention of mathematical logic (over example and property-based meanings). Point 2) Logicians have long represented categories by closed regions (Stenning, 2002), but we find this is not always a natural step for participants in our reinvention. Students often prefer representations that maintain semantic meaning (e.g. the number line, example numerals, example shapes). Representing categories by closed regions reflects an abstract meaning for mathematical definition: any well-defined distinction among mathematical objects. Point 3) Dawkins and Cook (2016) explain that students' interpretations of negative categories do not always correspond to the complement set of examples, but this interpretation is necessary for interpreting CP equivalence diagrammatically. Point 4) CP equivalence is a necessary entailment of the topological relations portrayed in Figure 1. We do not claim that student understanding of CP equivalence must be mediated by visual diagrams (Yopp, 2017), but we anticipate that the necessity of CP equivalence rests upon isomorphic ways of reasoning across semantic content.

The Case of Hugo

In what follows, we shall explore the contours of our concept of CP equivalence through a case study of one student's participation in our guided reinvention teaching experiments. Hugo did not construct the concept of CP equivalence, though his interview partner did. We find Hugo's story of learning helpful because he made clear progress on diagrammatic reasoning about mathematical conditionals and in some way exhibited progress regarding each of the first three points of the concept. However, he still clearly lacked a sense of necessity for CP equivalence, though he observed the shared truth-values. We present this case both because it portrays the kinds of activity that we anticipate would foster students' abstraction of CP equivalence and the challenge inherent to such abstractions in mathematical logic. Furthermore, this case allows us to set forth three possible characterizations for logic learning in advanced mathematics, which we observe as an arena in need of clarification and disambiguation.

Methods

The methods of this study mirror those reported elsewhere regarding this series of teaching experiments guiding the reinvention of mathematical logic (Dawkins & Cook, 2016). Each teaching experiment involved pairs of volunteers recruited from Calculus 3 courses at a medium-sized, public university in the Midwestern United States. These students met with a teacher/researcher for 6-11 hour-long sessions. The sequence of activities consisted of presenting students with lists of statements of the same logical form (disjunctions, conditionals, and multiply-quantified) each with varied, familiar mathematical content. Students were asked to:

1. determine whether each was true or false,
2. formulate rules for when statements of the given form were true or false,
3. develop a method for negating statements, and
4. in the case of conditionals, explore the relationship between a conditional and its converse, inverse, and contrapositive.

Table 1: Sample Conditionals that Hugo and Elya Analyzed

1.	If a number is a multiple of 3, then it is a multiple of 4.
2.	If a number is a multiple of 3, then it is a multiple of 6.
3.	If a number is a multiple of 6, then it is a multiple of 3.
4.	If a number is not a multiple of 6, then it is not a multiple of 3.
5.	If a number is not a multiple of 3, then it is not a multiple of 6.
6.	If a triangle is not acute, then it is obtuse.
7.	If a triangle is obtuse, then it is not acute.
8.	If a triangle is not acute, then it is not equilateral.
10.	If a quadrilateral is a rectangle, then it is a parallelogram.
15.	If the sum of two integers x+y is even, then at least one of the numbers x and y is not odd.

The teacher/researcher generally provided minimal direct guidance besides clarifying mathematical facts about the content of each statement (e.g. 1 is not prime, a square is a rectangle), asking students to clarify explanations or compare claims about various statements, and asking partners to respond to one another's reasoning. Based on the earlier findings reported in Dawkins and Cook (2016), the teacher/researcher also explicitly guided Hugo and his partner Elya to focus on the sets of objects making each statement true or false. The interviewer attempted not to introduce any logical formalizations (i.e. notation, terminology, or diagrams) until the students seemed to recognize some relevant pattern or need to express their reasoning. All data was analyzed using the constant comparative method (Strauss & Corbin, 1999).

Results

Hugo and Elya studied conditionals during their third, fourth, and fifth experimental sessions. Elya was absent from the fourth session. During the third session, their initial task was to assign truth-values to (or *assess*) the conditionals. They assigned the same truth-values a mathematician would (*normative* truth-values). They did not exhibit set-based reasoning during this activity; the pair relied on examples and properties. For false claims, they recognized what constituted a counter-example to a conditional. For instance, Hugo denied statement #1 with the example 6 and explained this was sufficient for assessment: "So we came up with one case where it's false, so it is false." Hugo affirmed three statements using property-based reasoning. For instance, he reasoned about #8 saying, "not acute would mean either a right triangle or an obtuse triangle. Neither of those can be equilateral, so that would be true." In other cases when Elya affirmed a statement using properties, Hugo introduced examples. Regarding #15, Elya inferred that "one of [x and y] would have to be odd" to have an odd sum. Hugo chose 5 as the sum and considered the possible addends. It is unclear whether Hugo perceived this as a justification or simply an explanation, but it portrays Hugo's overall propensity toward example-based strategies for assessing conditionals even when Elya provided property-based explanations.

Point 1: The Subset Meaning of Conditional Truth

In the last 15 minutes of that session, the interviewer asked Hugo and Elya to consider sets.

I (1): Think about the set of all things that satisfy the *if* part and the set of all things that satisfy the *then* part. And tell me about the relationship between those two […]
H (2): I'd say, if the statement is true then the set for the first part—I'm sorry the set of the second part will be included in the set of the first part.
I (3): Okay. Why do you say that?
H (4): Um, because if we said that it's true then when we pick—something that's true for the first part, then it has to be included in the second part for the whole statement to be true.
I (5): […] So you're saying, if the statement is true then what was the relationship here?
H (6): Then the—*then* will be inside *if*.

Hugo's initial explanation (turn 2) suggests that he had not yet considered the sets of objects referred to by the categories in the given conditionals. It is possible that his reverse subset claim reflects attention to the set of properties in each statement. For instance #3 is equivalent to "If an integer is a multiple of 2 and 3, then it is a multiple of 3." The *then* property is "included" in the *if* property (turn 4), but the subset relation between the sets of integers goes the other way round.

When the interviewer asked Elya, she proposed the normative subset relation that the "*if* has to be in the *then*." She elaborated using statement #3, "all the multiples of 6 are contained in multiples of 3." The interviewer asked Hugo to respond using a particular statement.

H (7): Uh, you wanna talk about number 3. Um in like a circle, and multiples of 3—3,6,9,12. [draws a circle and inside of it writes the numbers he says aloud] Um, multiples of 6 will be included in that circle [draws smaller circle around 6 and 12]. Like 6 and 12 are multiples of 6. So there's an additional circle inside that includes some numbers but does not include others [completes the diagram in Figure 2].
I (8): Okay[…] which are you calling the *if* part and which circle are you calling the *then*?
H (9): The *then* part would be the bigger one [he labels the larger circle]. The inside would be *then*—sorry other way around. *Then* is on the outside. *If* is in [he labels the small circle].

By encircling his short list of examples, Hugo bridged his example-based representation into a set-based representation and acknowledged the normative set-based meaning for conditional truth. In this way, Hugo made initial steps toward the first point in our concept of CP equivalence, though his grasp was at times tenuous through the subsequent interviews.

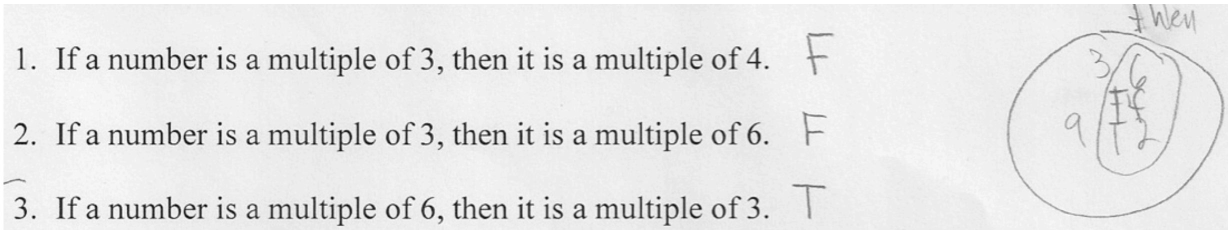


Figure 2: Hugo's first set diagram.

During this third session, Elya and Hugo recognized the syntactic relationships between conditionals traditionally known as *converse* (e.g. #6 and #7) and *inverse* (e.g. #3 and #4). They related CP statements as having undergone both transformations (e.g. #3 and #5, via #4). Using one of their subset diagrams, Elya provided an argument for why the original and CP statements should both be true in a manner compatible with Figure 1. We thus observe that she quickly and easily constructed our concept of CP equivalence from her fluency with set-based meanings and complement operations. Hugo showed little sign of following her reasoning, but he was exposed to a general explanation for why the CP must be true whenever the original statement is.

Point 2: Closed Regions and Their Topology

Elya was absent from the fourth session allowing Hugo to explore his understanding of conditionals and sets independently. Early in the session, Hugo considered statement #6. He appropriately explained that it was false because it failed the normative subset relation, "We said it was false 'cause our first set included… right triangles. So then it was only asking if it was only obtuse. So it could have been a right triangle or obtuse. Not just obtuse." Hugo went on to produce two diagrams to express his understanding of the two sets (Figure 3). The first reflected a traditional Venn diagram arrangement (with "O" standing for "obtuse"). As he unpacked the properties in the statement, he revised his diagram, "This is the *if*, "not acute" we said that could be a 90 triangle or obtuse. And then the *then* was, 'it is obtuse' so I guess that would be—this. So it's a little different than what I originally drew."

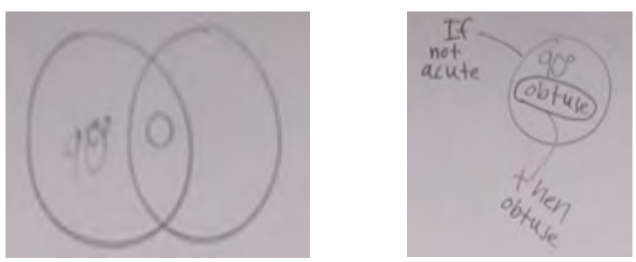


Figure 3: Hugo's two diagrams for statement #6.

Hugo recognized that one region of the Venn diagram did not contain any triangles and represented that in the topology of the two regions in his second, Euler diagram. In this instance, Hugo seemed to clearly make progress regarding the second point in our concept of CP equivalence by using closed regions to represent sets and using their topology to relate those sets.

We conjecture that Hugo's property-based reasoning the previous day influenced his diagram construction. He let the properties stand for the entire category without recourse to representative examples (as in his diagram produced the previous day). It is unclear the extent to which Hugo imagined the curves as encasing the sets of triangles imagined as points or whether they encased the words themselves that stood for the examples. One cause for questioning Hugo's interpretation of the circles and their reference arose when Hugo next considered statement #7. Though statement #7 contains the same categories, Hugo produced a completely new diagram like the second in Figure 3, except the *if* and *then* labels were reversed. The interviewer asked Hugo to compare the two diagrams, and, upon reflection, he said they were the same. He clearly did not anticipate this relationship.

Point 3: The Negation/Complement Relation

The interviewer invited Hugo to write the inverse statements to both #6 and #7 on the board. He then asked Hugo to assess these statements using the same diagram he produced for #6. Hugo considered #6's inverse, written "If acute, then not obtuse" in the following way:

> So "if acute" then we'd be talking about anything outside of the *if* circle, so everything outside of here, then "it is not obtuse"—right. 'Cause you're not—we're talking about everything except inside this circle. And obtuse is inside the circle. So that'd be true. We said this [statement #6] was false. So it was the opposite, or the negation.

Here, Hugo displayed two novel developments in his thinking. First, he associated the negation of a category with the complement of a closed region. Specifically, the region outside the larger circle represented acute triangles. Once again, we cannot be sure whether this inference was supported by 1) Hugo's knowledge that any triangle is exactly one of acute, right, or obtuse or 2) reasoning about the representational structure of the diagram. In either case, he used the *negation/complement relation*. Secondly, he did not affirm the inverse of #6 by the subset meaning, but rather notes that anything outside the large circle is not inside the small circle ("if *acute*, then not *obtuse*"). We call this the *empty intersection meaning for conditional truth.* This criterion is distinct from the subset meaning Elya used during the previous session, but formally equivalent to it. It depends upon the presence of *not* in the latter half of the conditional.

At the end of the previous quote, Hugo noted that the inverse statements had opposite truth-values, and anticipated this might be the case more generally. To explore this conjecture, the interviewer next asked Hugo about statement #7 and its inverse:

H (10): My guess is that it would be false 'cause it's—it'd be the opposite, but "if not obtuse," so anything that's outside of this little circle—then it is acute. That's not necessarily true because—that would—we still have 90 degree—triangles that are not obtuse but are still not acute. So that would still be false. Or that would be false.
I (11): Okay, now you anticipated it would be false. What was your basis for anticipating that it would be false?
H (12): That one. We took the inverse of this one, we got the opposite—the opposite truth-value.
I (13): […] What about the picture tells me which two [of the four statements] are true? […]
H (14): That if we limited it to this inner circle—the obtuse triangles. Then obviously we would not be talking about if it was outside of the circle. We're only talking about the inside
I (15): What about then this one [inverse of #6]? How can I see it in the picture, this one?
H (16): That if we're talking about anything outside of this circle, the bigger circle here, which would be all the acute. Then that would exclude anything inside the circle—obtuse triangles are only inside the circle. So then we would only be talking about the area out here.

In turn 10, Hugo denied the inverse of #7 because the non-obtuse triangles were not all acute. Hugo associated the negation of *obtuse* with the complement of the smaller region. He identified acute triangles (outside the larger circle) as counterexample to the conditional. Hugo noted that this example also affirmed his conjecture that inverse conditionals have opposite truth-values.

The interviewer noticed that Hugo was fluidly shifting between subset, empty-intersection, and counterexample meanings to assess the given conditionals. So, in turn 13 he asked Hugo to consider more generally which of the four statements (#6, #7, and their inverses) were true and how the diagram represented this. In turns 14 and 16, Hugo affirmed #7 and its CP both using the empty intersection meaning: the inner circle and outer complement were mutually exclusive. While this observation could provide a general sense of symmetry regarding the truth conditions for some CP statements, there was no evidence of Hugo abstracting this relationship at this point.

Point 4: CP Equivalence

To help Hugo see the syntactic relationship between the true statements he affirmed in the last interchange, the interviewer asked Hugo to specify all of the syntactic relationships among the four statements. Figure 4 shows the results of their discussion (Hugo and his partner used the term "switch" for converses). Using this diagram, Hugo noted that CP statements had the same truth-value: "if we have an *if-then* statement that's true, we take the inverse and the switch […] so far we've proved that it would be true [...] Well I'm observing but I'm trying to articulate why that is." He admitted that this was for him an empirical observation and he could not justify it.

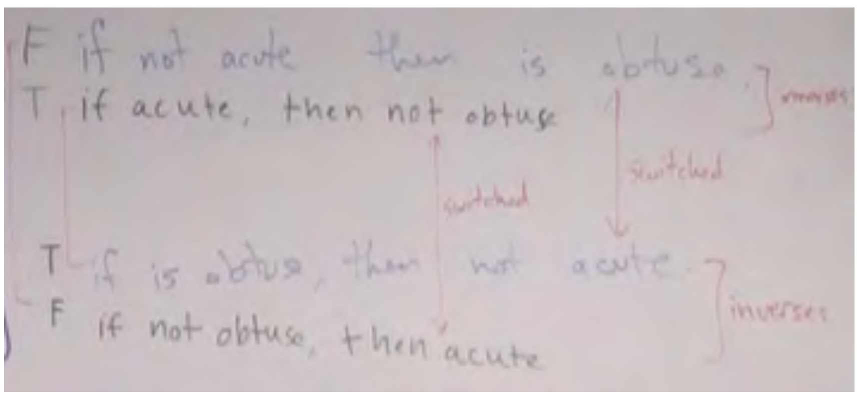


Figure 4: Exploring the syntactic relationship among a conditional quartet.

Not only did Hugo fail to see a general justification for CP equivalence, he was very inconsistent in his use of subset diagrams to assess conditionals about other topics. When asked to discuss the sets associated with statement #10 and its CP later in that session, Hugo drew separate and isolated circles above the words "rectangle" and "parallelogram" in #10. He recognized that all rectangles were parallelograms, but he did not use the topology of the regions to represent this. With prompting, he modified these diagrams to match the previous subset diagrams. Regarding the CP, Hugo began new circles rather than using the complements of the regions drawn for #10. Throughout the rest of that interview and the next, Hugo went on to consider at least three other quartets of conditional, inverse, converse, and contrapositive. Once prompted to produce a subset diagram, he consistently 1) affirmed the base conditional by the subset meaning, 2) denied the inverse and converse by counterexample or by failing to have a subset relation, and 3) affirmed the contrapositive by the empty-intersection meaning. However, he did not routinize creating such diagrams without prompting or begin anticipating the topological relations that would affirm a conditional and its CP. In short he did not construct the concept in such a way as to produce a sense of logical necessity for the shared truth-values.

Conclusions

Our goals in this paper were to 1) set forth our concept (Simon, 2017) of CP equivalence, 2) portray mathematical activity by which this concept could develop, and 3) convey the challenge these abstractions pose through Hugo's learning process. We claim that Hugo made progress regarding each of the first three points in the concept and empirically observed point 4, but did not perceive point 4 as a logical necessity. We presented evidence that Hugo understood that conditionals could be affirmed via a subset relation (Point 1). It is unclear whether this relationship was universal and reversible, or simply a sufficient condition. Hugo was able to represent the relationships between the categories in conditionals using closed regions and their topology (Point 2). At times he used these diagrams flexibly, as when he created the empty intersection meaning. He did not see such diagrams as a universal tool, judging by his alternating strategies and representations. He produced diagrams with different referential structures and often bypassed reasoning with the diagram by resorting to property-based inferences. While we appreciate that Hugo consistently connected his representation to the relevant mathematical categories (cf. Hawthorne & Rasmussen, 2014), he did not consistently use the diagram to draw new inferences about the mathematical categories. Hugo at times associated the negation of a category with the complement of either region in a diagram (Point 3), but he never coordinated two such complement regions simultaneously (as implied in Figure 1). Thus regarding each point of the concept, we see why Hugo's understanding did not support reflexive abstraction.

We intend for this analysis to emphasize the difficulty and nuance involved in constructing logical necessity in diagrammatic reasoning in the course of semantically-rich mathematical activity. We also propose that further literature on logic learning should clearly distinguish the kinds of understanding they intend. We propose three categories. *Reading* involves assessing mathematical statements in normative ways and drawing normative inferences. Hugo did this throughout. *Reflecting* involves finding general representations and criteria for assessing mathematical statements, such as Euler diagrams and the subset criterion. Hugo began this during the study. *Abstracting* involves reflexive abstractions yielding new insights from these representations and criterion. Elya, but not Hugo, displayed this kind of learning regarding CP equivalence. We anticipate that elaboration of these categories will facilitate future investigation.

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GENERALIZATION ACROSS DOMAINS: THE RELATING-FORMING-EXTENDING GENERALIZATION FRAMEWORK

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Generalization is a critical aspect of doing mathematics, with policy makers recommending that it be a central component of mathematics instruction at all levels. This recommendation poses serious challenges, however, given researchers consistently identifying students' difficulties in creating and expressing normative mathematical generalizations. We address these challenges by introducing a comprehensive framework characterizing students' generalizing, the Relating-Forming-Extending framework. Based on individual interviews with 90 students, we identify three major forms of generalizing and address relationships between forms of abstraction and forms of generalization. This paper presents the generalization framework and discusses the ways in which different forms of generalizing can play out in activity.

Keywords: Cognition, Learning Theory, Reasoning and Proof

Introduction: The Importance of Mathematical Generalization

The act of generalizing is at the core of mathematical activity, serving as the means of constructing new knowledge. Researchers have argued that mathematical thought cannot occur in the absence of generalization (Sriraman, 2003; Vygotsky, 1986). As a result, "developing children's generalizations is regarded as one of the principal purposes of school instruction" (Davydov, 1972/1990, p. 10). Researchers have studied the importance of generalization for promoting algebraic reasoning (Cooper & Warren, 2008), mathematical modeling (Becker & Rivera, 2006), functional thinking (Ellis, 2011; Rivera & Becker, 2007), and probability (Sriraman, 2003), among other areas. Despite the importance of generalization to success in mathematical reasoning, research on students' abilities to generalize has identified pervasive student difficulties. For instance, Rivera (2008) reported results of 5 years of performance assessments on generalization given to more than 60,000 middle and early high school students; these findings revealed a stable ceiling value of only a 20% success rate in the construction of a general formula. Other researchers have similarly documented students' difficulties in creating correct general statements, shifting from pattern recognition to pattern generalization, and using generalized language (e.g., English & Warren, 1995; Mason, 1996).

Although student difficulties are well documented, the instructional conditions necessary for fostering more productive student generalizing are not well understood. Complicating the matter, the bulk of research on generalization has occurred with algebraic patterning tasks, situating generalization as a type of, and route to, algebraic reasoning (Becker & Rivera, 2006; Cooper & Warren, 2008). There remains a need to understand how students construct generality in more varied and more advanced mathematical domains. The goals of this study are to investigate students' mathematical generalizing from middle school through the undergraduate level in the topics of algebra, advanced algebra, and combinatorics. In particular, our aim is to elaborate the nature of students' generalizing, contributing to the field's knowledge base by extending the investigation of generalization up the grade bands. Based on clinical interviews with 90 students from 6th grade through the undergraduate level, we introduce a framework characterizing three major forms of generalizing activity: *relating*, *forming*, and *extending*. We also introduce and discuss relationships between forms of generalization and forms of abstraction.

Theoretical Framework

Forms of Generalization

Definitions of generalization vary, with the most prominent situating generalization as an individual, cognitive construct (e.g., Kaput, 1999). More recent sociocultural definitions position generalization within activity and context, as a collective act distributed across multiple agents (Tuomi-Gröhn & Engeström, 2003). These perspectives attend to how social interaction, tools, and history shapes people's generalizing, recognizing generalization as a social practice that is rooted in activity and discourse (Jurow, 2004). We borrow from both the cognitive and the sociocultural traditions to define generalizing as an activity in which learners in specific sociocultural and instructional contexts engage in at least one of the following three actions: (a) identifying commonality across cases (Dreyfus, 1991), (b) extending one's reasoning beyond the range in which it originated (Radford, 2006), and/or (c) deriving broader results from particular cases (Kaput, 1999). We use the term *generalizing* to refer to any of these processes, whereas *generalization* refers to the outcome(s) of these processes.

Borrowing from Lobato's (2003) transfer framework, we take an actor-oriented approach to studying students' processes of generalizing. This approach represents a shift from the observer's (usually the researcher's) stance to the actor's (the student's) stance. In particular, it compels us to abandon normative notions of what should count as a generalization, instead seeking to understand the processes by which students construct relations of similarity that they experience as meaningful. Our framework also builds on Ellis' (2007) taxonomy of generalizations, which distinguishes between students' activity as they generalize, called *generalizing actions*, and students' final statements of generalization, called *reflection generalizations*.

Forms of Abstraction

The second line of research we rely on examines the role of abstraction in developing generalizations (e.g., Dorfler, 1991). Abstraction has been characterized in multiple ways, but we focus particularly on reflective abstraction and the interrelationships among the actions and operations that constitute students' construction of mental objects. In particular, we distinguish three types of reflective abstraction salient in informing students' generalizing activity: *pseudo-empirical abstraction*, *reflecting abstraction*, and *reflected abstraction* (Montangero & Maurice-Naville, 1997; Piaget, 2001). Pseudo-empirical abstraction is based on the observation of perceptible results, in which new knowledge is drawn not just from the properties of objects, but from how the student has organized the activities she has exerted on those objects. We further distinguish pseudo-empirical abstraction from other forms by noting that pseudo-empirical abstraction includes reflection on the *outcome* of one's activity. The focus is on the products of a learner's actions, rather than the coordination and transformation of actions themselves.

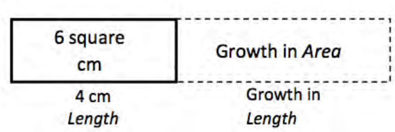
In contrast, reflecting abstraction includes reflection on one's actions, not merely on the outcomes of those actions. One can transfer to a higher plane what he or she has gleaned from lower levels of activity, leading to differentiations that imply new, generalizing compositions at that higher level. In reflected abstraction, one becomes conscious of his or her actions, bringing awareness of qualitative differences between his or her actions. Through reflected abstraction, one is able to formulate, formalize, and subsequently operate on his or her thought.

Methods

We conducted a series of individual semi-structured interviews with middle school (ages 12-14), high school (ages 14-17), and undergraduate students in the domains of algebra, advanced algebra, discrete mathematics, and combinatorics. The algebra and advanced-algebra topics included linear, quadratic, higher-order polynomial, and trigonometric functions, and the discrete mathematics and combinatorics topics included counting problems, combination and permutation problems, and the binomial theorem. We conducted 10 middle-school, 11 high-school, and 10 undergraduate algebra or advanced algebra interviews, and 19 middle-school, 13 high-school, and 27 undergraduate discrete mathematics (combinatorics) interviews.

During the interviews we presented the participants with domain-specific tasks to elicit both near and far generalizations, and we asked the participants to identify patterns and themes, discuss any elements of similarity they noticed, and, where reasonable, explain and discuss the generalizations they formed. All interviews were videotaped and we used gender-preserving pseudonyms for all participants. Table 1 presents a sample of the interview tasks across the mathematical domains.

Table 1: Sample Interview Tasks

Interview Task	Domain and grade level
The rectangle below grows along the dotted path as shown:  Complete the following statement: When the length of the rectangle grows by _____, the area grows by _____.	Algebra, middle school
You have a 1 cm by 1 cm by 1 cm cube, and all sides grow at the same rate. How much additional volume does the cube gain when the sides each increase by 1 cm?	Adv. algebra, high school
You have a deck of number cards numbered 1-6. You create a two-card hand by drawing a card from the deck, putting it back, and drawing a second card. Determine how many possible two-card hands you could get. How many times the number of two-card hands would you have if you had twice the number of cards?	Discrete math, middle school
Suppose passwords consist of (uppercase) As, Bs, and/or the number 1. How many such passwords are there that are *n* characters long?	Combinatorics, undergraduate

Analysis

We relied on the constant comparative method (Strauss & Corbin, 1990) to analyze the interview data in order to identify forms of generalization and abstraction. For the first round of analysis we drew on Ellis' (2007) analytic framework for categorizing students' generalizing actions and reflection generalizations, using open coding to infer categories of generalizing activity based on students' talk, gestures, and task responses. This first round led to an initial set of codes, which then guided subsequent rounds of analysis in which the project team met weekly to refine and adjust the codes in relation to one another. This iterative process continued until no new codes emerged. A final round of analysis was descriptive and supported the development of an emergent set of relationships between forms of abstraction and forms of generalizing, characterizing the evolving nature of students' mental activity as they generalized.

Results: The Relating-Forming-Extending Framework

Based on data analysis from the 90 interviews we developed an empirically-grounded framework capturing the broad range of generalizing activity across a variety of grade bands and domains. We present the results in two major sections. First we introduce the framework itself, which provides definitions, descriptions, and examples of each form of generalization demonstrated by the study participants (Tables 2-4). Due to space constraints, we do not elaborate on every form of generalizing, but we instead present a data episode identifying the interrelationships between the forms of abstraction and forms of generalizing. This episode is meant to be representative of the explanatory power of the framework, which we limit to one student due to space considerations. The Relating-Forming-Extending framework distinguishes between *inter-contextual* forms of generalizing, in which students established relations of similarity across problems or contexts, and *intra-contextual* forms of generalizing, in which students formed and extended similarities and regularities within one task. Following the actor-oriented perspective, we made the inter/intra distinction based on evidence of whether the student perceived the establishment of similarity or regularity he or she formed to occur across different contexts or situations, or to occur within the same context.

Table 2: Inter-Contextual Forms of Generalizing (Relating)

<table>
<tr><th colspan="2">Form of Generalizing</th><th>Example</th></tr>
<tr><td rowspan="2">Relating Situations: Forming a relation of similarity across contexts, problems, or situations</td><td>Connecting Back: Formation of a connection between a current and previous problem or situation.</td><td>HS Adv. Algebra Student: All the sides are the same length. The formula is generally the same [as the prior problem], you’re just adding one more side for the 4-dimensional one.</td></tr>
<tr><td>Analogy Invention: Creating a new situation or problem to be similar to the current one.</td><td>MS Algebra Student: The more seconds he has, he’ll slow down. And the less seconds he has, he’ll speed up faster. Int: Okay, and how come? Student: You know how, if you had less time to go into the grocery store to get the foods on the grocery list, you would go faster if you had like 1 second to do it in? You would, like, be in and out real quick. Same thing here.</td></tr>
<tr><td colspan="2">Relating Ideas or Strategies (Transfer): Influence of a prior context or task is evident in student’s current operating.</td><td>HS Adv. Algebra Student: So in this case it’d be P plus, let’s do V for valence because that’s one word I know for outer ring. Int: Cool, is that from chemistry? Student: Yep. Like the valence electrons…how much that equals plus the previous one, would equal your new answer.</td></tr>
</table>

The inter-contextual forms of generalizing all involved a type of *relating* activity. The intra-contextual generalizing, however, occurred in two major categories: (a) *forming* a similarity or regularity, in which students searched for and identified similar elements, patterns, and relationships (Table 3); and (b) *extending* or applying a similarity or regularity (Table 4). In the latter case, students extended established patterns or relationships to new cases.

We illustrate several intra-contextual generalizations and their relationships to forms of abstraction by presenting the work of Willow, a middle-school algebra student, who worked on the growing rectangle task (Table 1). Willow initially established a numerical relationship between the length of 4 cm and the area of 6 cm^2:

> Well, the area is 2 more than the length so I would think if, however, if they grew like the same amounts of, if this (points to the area) grew by 2 in the area, so it would be 8 and this (points to the length) grew by 2 and it would be 6, then it would always be 2 more if they grew in the same, like, the same amount.

Table 3: Intra-Contextual Forms of Generalizing (Forming)

Form of Generalizing		Example
Relating Objects: Forming a relation of similarity between two or more present mathematical objects	*Operative*: Associating objects by isolating a similar property, function, or structure.	*Und. Adv. Algebra Student*: [Comparing $x = \sin(y)$ with $y = \sin(x)$ graphs] They're both representing the same thing…with equal changes of angle measures my vertical distance is increasing at a decreasing rate [tracing graph]…here it's doing the exact same thing.
	Figurative: Associating objects by isolating similarity in form.	*HS Adv. Algebra Student*: How does the volume equation relate to this cube? Well the three numbers are getting one bigger and the three sides got one bigger.
	Activity: Relating objects or ideas based on identifying one's activity as similar.	*MS Algebra Student:* I think it would be 2 more than the 6. *Int*: Two more than the 6? Okay, how come? *Student*: Because, like, same as this one [points to the prior problem] you're just adding it.
Search for similarity or regularity: Searching to find a stable pattern, regularity, or element of similarity across cases, numbers, or figures.		*HS Adv. Algebra Student*: I don't think it goes up by the same amount each time. Does it? That goes up by 3, and that goes up by 5, and that goes up by 7. Three, 5, 7. Yeah, it goes up by…okay.
Identify a regularity: Identification of a regularity or pattern across cases, numbers, or figures.	*Extracted*: Extracting regularity across multiple cases.	*MS Combinatorics Student*: For every addition problem that we do, like 6 plus 6 equals 12, it is always one more added to that every time.
	Projected: Describing a predicted or known stable feature.	*MS Algebra Student:* You could do, you could do 1.5 times growth and that would get you, times the growth in the length and then that would give you the growth in area.
Isolate constancy: Focusing on and isolating regularity – a stable feature – across varying features.		*HS Adv. Algebra Student*: This is like the one thing that you started off with [circles the original rectangle]. It's like the only constant really. And so each time it changes a little bit so it's really one of these is being added each time and so that's not really taking it into account, the 15 that was already there.

Willow *identified a regularity* by stating "It (the areas) would always be 2 more (than the length)". Although Willow's generalization is incorrect, it represented a pattern that she saw as valid. We also suspect Willow's generalization relied on a pseudo-empirical abstraction, not because her generalization was incorrect, but because she appeared to generalize based on the outcome of her activity. Specifically, Willow's operation was to take the difference of the numbers 4 and 6, and she generalized the difference remaining constant. She made an additive comparison between numerical values that did not appear to be based in quantitative operations relating length to area. When asked what would happen if the rectangle grew by another 4 cm, Willow responded, "So it grew by 4…would the area have grown by 4 too? It could be, like, 10." Here Willow *extended by continuing* the "area = length + 2" relationship she had established to a new case. She then further generalized by stating, "If the length grew by x, then the area would be 2 more than the total length," which she expressed as “A = 2 + T”. Here Willow *extended by removing particulars* in order to algebraically express the relationship she had established. We maintain that this string of generalizations remained grounded in Willow’s activity of pseudo-empirical abstraction. Her focus remained on the result of her operation, the difference of 2, and at no time did Willow coordinate the growth of the rectangle simultaneously with varying measures of length and area.

Table 4: Intra-Contextual Forms of Generalizing (Extending)

Form of Generalizing		Example
Continuing: Continuing an existing pattern or regularity to a new case.		*MS Combinatorics Student*: [Moves from a 7-card case to an 8-card case]: It is like the last time. You don’t count (8, 8) twice.
Operating: Operating on an identified pattern, regularity, or relationship in order to extend it to a new case.	*Near*: Making a minor change to a regularity in order to extend it to a new case.	*HS Adv. Algebra Student*: [After having established a pattern of adding 8 square units for every additional rectangle]: And then plus 8, or I could just do plus, um, 8 times 5, right? And so that would be 40.
	Projection: Making a major change to a regularity in order to project it to a far case.	*Und. Combinatorics Student*: [After solving cases with 3 and 4 combinations]: So now I believe if you gave me something where if there was 20 combinations I could solve how many combinations there are without having to write them all out: 2^{20} and whatever that equals.
Transforming: Extending a generalization and, in doing so, changing the generalization that is being extended.	*Constructing a Quantity*: Constructing a new quantity or a relationship between quantities in order to extend a regularity to a new case.	*HS Adv. Algebra Student*: [Exploring the three sides of a rectangular prism, the interviewer asks the student to express one side in terms of the other.] So it’s *x* plus 1, right?
	Recursive Embedding: Embedding a previous situation into a new one as a key component of the new task.	*Und. Combinatorics Student*: Okay, we’re definitely using 1, so we’re limiting ourselves to only 2 possible states for the entire password, A and B, which means it’s basically no different than what we did in one of the earlier examples. So that I’ll probably just figure, okay, 2 to the 3rd equals 8.
Removing particulars: Extending a specific relationship, pattern, or regularity by removing particular details to express the relationship more generally.		*MS Algebra Student*: I was thinking, like, trying to put it in an equation I guess, so it kind of makes sense…Well it could be, area equals 2 plus total length [writes A = 2 + T].

Right when the interviewer began to remove the task in order to transition to a new problem, Willow suddenly evidenced a shift in her thinking, saying, “Unless it will start at 0?”

> Because if you start it at 0…to find out the actual growth, then, say this is like the first they grew and this, kind of, so this grew by 4 first (gestures along the length) and then this grew by 6 (gestures to the whole figure, along the area). So this (the length) could grow by 4 again, and this (the area) could grow by 6 again.

Willow appeared to construct a dynamic image of the rectangle growing “from 0”. She further explained, “Because it would always be plus 4 and plus 6, so if you said when the length grows by 8, the area grows by 12.” Willow imagined the rectangle growing in chunks, iterating twice. Willow therefore *identified a regularity* that if the rectangle started growing from 0, then for every 4-cm increase in length, the rectangle would increase in area by 6 cm^2. This regularity, unlike the first one

Willow identified, was based on an image of growth in which Willow was able to coordinate an increase in length with a corresponding increase in area. This image was informed by the operations of forming a ratio and iterating it. It was also a product of reflecting abstraction in that Willow reflected on her activity in order to coordinate iterating her formed ratio with the number of times it was iterated. Therefore, she could then state that the length would increase by 4 again, resulting in another increase of 6 for the area. Willow *extended by continuing* the relationship, and she did so by relying on her ability to coordinate growth in one quantity with growth in the other.

We take further evidence that Willow engaged in reflecting abstraction by what occurred next. Namely, she was able to *extend by operating* on the relationship she had formed, multiplying each term in the 4:6 ratio by 4, then by 10, ½, ¼, ¾, and 5/4 in order to generate new length:area pairs. This extension was significant because it included the use of both whole number and fraction values. It also suggests that Willow had reflected on her operation of forming a ratio in order to develop a flexible, generalizable relationship with which she could meaningfully operate. Willow ultimately developed a unit ratio, explaining, "Each time the growth in length goes up by 1, the growth in area, I think the growth in area equals [writes $A = 1.5 \times L$]." Thus Willow *identified a new regularity* and then *removed particulars* for this regularity. When she removed particulars, she reflectively abstracted a ratio from the phenomenological bounds in which it was created, and Willow's subsequent flexible use of this ratio with messy numbers is evidence that she could imagine it holding for any arbitrary value.

Discussion

The Relating-Forming-Extending framework identifies forms of generalizing based on data from multiple grade bands and mathematical domains, addressing the need to understand how students construct generality in more varied and advanced mathematical contexts. Willow's work provides evidence that students can and do generalize their reasoning on a variety of problems beyond typical patterning tasks. In particular, in contrast to much of the literature identifying how students inductively generalize patterns, Willow abductively (Peirce, 1931-1958; Radford, 2006) developed a generalization from just one case. Willow's reflective activity enabled her to develop, solidify and apply generalizations in two ways. Firstly, she generalized an additive comparison based on the numerical relationship she established between 4 and 6 (a pseudo-empirical abstraction). Secondly, she generalized by forming and operating on a ratio between quantities that was rooted in her image of the rectangle's length growing in tandem with its area (a reflecting abstraction).

The Relating-Forming-Extending framework extends prior work by distinguishing and characterizing three forms of generalizing activity and by coordinating these forms of generalizing with forms of abstracting. The case of Willow shows that students can engage in many forms of generalizing, such as *identifying regularities*, *extending by continuing*, and *removing particulars*, based on either pseudo-empirical or reflecting abstraction. Other forms of generalizing, such as *extending by operating* or *transforming*, appear to be more typically grounded in reflecting abstraction, as they often entail differentiations based on activity in order to support new compositions. By attending to both abstraction and generalization in students' sense-making, we can begin to characterize how students can leverage initial abstractions into first-pass generalizations that they can then reflect on and transform in further activity. Further analysis of these relationships between abstraction and generalization will inform a better understanding of the conceptual mechanisms driving generalizing activity in a variety of mathematical contexts.

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MATHEMATICAL MODELING: CHALLENGING THE FIGURED WORLDS OF ELEMENTARY MATHEMATICS

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This article is a report on a teacher study group that focused on three elementary teachers' perceptions of mathematical modeling in contrast to typical mathematics instruction. Through the theoretical lens of figured worlds, I discuss how mathematics instruction was conceptualized across the classrooms in terms of artifacts, discourse, and identity. I then highlight, through four themes, how mathematical modeling challenged the ways in which both the teachers and students understood what it means to know and do mathematics. Findings suggest that the practice of mathematical modeling allowed for access, empowerment, and real world connections that were typically not present in classroom instruction. In addition, it challenged student positioning in the classroom in terms of who was framed as capable of doing mathematics.

Keywords: Elementary School Education, Equity and Diversity, Modeling

Introduction

Mathematical Modeling, a standard of mathematical practice in the Common Core State Standards, is a process in which students use mathematical tools to reason about, represent, and make decisions surrounding a real world scenario (Lesh & Doerr, 2003). The process of modeling is cyclic and it begins when the modeler translates the scenario into the mathematical world by posing a question. Using knowledge and mathematical tools, the modeler proposes solutions and translates them back to the real world to determine if they are appropriate or if modifications need to be made. In this paper, mathematical modeling refers to the entire process rather than the end product.

Although mathematical modeling has traditionally taken place in secondary and college classrooms, researchers (Carlson, Wickstrom, Burroughs, & Fulton, 2016) have argued that it is equally as important for elementary students to engage in the process. Modeling supports mathematical literacy (Steen, Turner, & Burkhardt, 2007) and allows students to draw on their own backgrounds and experiences in framing the mathematical problem (English & Watters, 2005). Modeling also promotes productive attitudes toward mathematics (Lesh & Yoon, 2007), and fosters integration across mathematical content and practices (Lehrer & Schauble, 2007).

The study of mathematical modeling in the elementary classroom is a relatively new field of study. The purpose of this paper is to add to existing literature by describing an elementary modeling task and the ways in which it challenged teachers' and students' perceptions of what mathematics is and what it means to do mathematics as well as the students' and teacher's roles within the classroom. Through the theoretical lens of figured worlds, in this paper I address the following research questions:

1. How does mathematical modeling press on or extend the boundaries of what it means to know and do mathematics in the elementary classroom?
2. In what ways, if any, does mathematical modeling challenge positionality and roles in the elementary classroom?

Theoretical Framework: The Mathematics Classroom as a Figured World

This work is framed through the theoretical lens of Holland, Skinner, Lachicotte, and Cain's (1998) concept of figured worlds. They define a figured world as, "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others." (p.52) Holland et al.'s work addresses the idea that each individual's thoughts, behaviors, and ways of interpreting the world are often influenced by culture, power, and status. In addition everyday activities act as figured worlds that build, inform, and continually define individual's identities. In this paper, I argue that the mathematics classroom functions as a figured world. In the elementary classroom, there are routines that define what it means to know and do mathematics. In addition, both the teachers and students take on different roles and identities that are continually formed across the school year.

Figured worlds consist of three key elements: artifacts, discourse, and identity. I begin the paper by discussing a typical day in the teachers' classrooms in response to artifacts that contributed to the figured world of mathematics instruction, discourse surrounding how the three teacher's interpreted doing mathematics, and the identities and roles the teachers perceived in the classroom. Next, I identify and describe four themes that arose while mathematical modeling that challenged the established norms or figured worlds.

Methods

Participants

Three teachers participated into this study, Ms. A, Mr. B, and Ms. C. Ms. A was a fifth-grade teacher. Mr. B was a fourth-grade teacher and Ms. C was a third-grade teacher. The teachers were participants in a NSF-funded professional development on integrating mathematical modeling in the elementary classroom that took place in a school district in the Rocky Mountain West. As part of the professional development, the teachers attended a weeklong professional development on mathematical modeling and pedagogical practices in the summer. During the summer, they designed a modeling task to implement in their respective classrooms. Following summer professional development, the teachers participated in a teacher study group in which they met seven times across the fall semester to debrief and discuss the modeling task with a university faculty member. The three teachers were chosen for this study because they were grouped together in the same study group and enacted the same modeling task. I, the researcher, took on the role of their study group facilitator.

Modeling Task

This section is meant to give a brief overview of the task. Specific examples of students engaging in the task will be given in the results section. In designing the modeling task, each of the teachers discussed that they led some type of community-building lunch at the beginning of the school year for students to get to know one another. Instead of designing the activity themselves, they decided they would use this as a real-world scenario to engage students in mathematical modeling. Teachers presented the following scenario to students, "Building community in our classrooms is very important. The university has given us money to support a community building lunch." After presenting the scenario, they asked students to consider 1) What do we need to know? and 2) What tools could we use to help us? Students decided that they needed to address broad questions like "What should we have for lunch?", "What activities should be included to build community?", "Will what we want fit into our budget?". Students also discussed that there are other factors to consider like allergies and personal preferences in food selection.

The teachers worked with students on this task across 3-4 weeks, visiting the task a few times a week. Initially, the students worked on determining what should be served at lunch and quantity. Students primarily used surveys, multiplication, counting, and measurements as mathematical tools to aid them in making decisions. Once students had determined what should be served and how much, they needed to determine where the food would come from and if the meal was in budget. The teachers helped by providing grocery store and restaurant ads. Again, the students primarily used multiplication and repeated addition in determining the total cost.

Data Collection

I, the researcher, observed all three teachers across implementation of their modeling tasks visiting each teacher for 3-4 lessons. During observations, I took qualitative notes of what occurred in the classroom including what the teachers said or did, students' progress in the task, and students' remaining questions or concerns. In addition, I facilitated and video-recorded seven meetings in which the three teachers debriefed about the modeling task and their work as teachers. Following the fall teacher study groups, the teachers individually participated in a one-on-one interview that lasted for about 45 minutes. The purpose was for teachers to first describe the structure of a typical mathematics lessons including routines, student activities, and teacher activities. In the second half of the interview, the teachers were asked to describe their experiences enacting mathematical modeling. This included describing key features of and comparing mathematical modeling to a typical lesson. In addition, they were asked to describe what the process was like for their students and for them, as teachers. The primary data source for this article are the one-on-one interviews with observational notes used as triangulation.

Data Analysis

Interviews with teachers were audio recorded. Each of the interviews was transcribed verbatim resulting in about 10 pages of typed transcript per teacher. Classification and coding took place first related to the research questions and theoretical framework (Miles, Huberman, & Saldana 2014). I analyzed the transcripts first looking for statements that helped to contextualize and describe components of a typical mathematics lesson. Then, I analyzed the data looking for statements related to the three main components of figured worlds: artifacts, discourse, and identities. Finally, I looked for statements, in describing the modeling task, that were in contrast to the figured world of the mathematics classroom. I analyzed and grouped the statements to generate themes regarding the figured world of the mathematics classroom and the ways in which mathematical modeling challenged or reinforced this world.

Results

The Figured World of the Mathematics Classroom

Artifacts. Artifacts are objects that act as the "means by which figured worlds are evoked, collectively developed, individually learned, and personally powerful." (Holland et. al, 1998, p.61) Across the three classrooms, there were three artifacts that supported how mathematics instruction was conceptualized: group/carpet area, worksheets, and journals. Each of the teachers began the lesson by bringing the class together in a communal area or a carpet at the front of the classroom. The purpose was for the students to be able to express themselves in a communal environment. The teachers described that they first presented an idea or a problem to students in pairs or groups. Students were asked to compare and discuss their solutions with their partner and then solutions were shared allowed. In describing this part of the lesson, Mr. B stated:

> So they'll (the students) work independently, compare their answers with their learning partner and then I'll call, usually using the equity sticks, I will call several people up…often I just look and see a variety of different problem solving methods and always emphasize that the students learn more from each other than me.

Each of the teachers discussed this as time for students to voice multiple solution strategies, build community, and learn from one another.

The second artifact that shaped mathematics instruction was the worksheet. After students discussed and shared solution strategies, they moved back to their desks and were given different types of practice problems. All of the teachers used the worksheet as a way to individually assess if students understood the material as well as time to talk individually with students. Ms. C described, “I give them a few problems and I just want to see, I rotate from group to group to see if they are getting it.” The emphasis of this part of the lesson was for students to practice the mathematics and try out different strategies independently.

The journal was the final artifact that shaped mathematics instruction. All of the teachers either had students write in a journal or respond to a reflective prompt describing their learning and successes or challenges they faced. Ms. A described,

> After we have been doing any sort of activity…they open their journals and they reflect for a few minutes and I have them identify a success or a challenge that they had and we talk about it. Or sometimes I ask them to give advice to the next class on what they learned.

In examining the artifacts, students participated in mathematics through daily routines. Through the carpet space, mathematics was communal and open to discussion. Through the worksheets, mathematics became an individual endeavor in which teachers could examine students’ thinking and skills. Finally, mathematics became a reflective process through the mathematics journal.

Discourse. Discourse accounts for the ways in which people interact with one another and discuss a particular topic in their setting. Through classroom discourse, teachers and students are able to shape and define what it means to do mathematics. There were four themes that arose surround discourse from the teachers’ perspectives: student voice, multiple strategies, problem solving, and mistakes.

All of the teachers discussed student voice in defining what it means to do mathematics in their classrooms. The students were expected to share ideas with one another and this fostered the second theme of multiple strategies. When multiple students are able to share out, the teachers stressed that there are multiple ways to do a mathematics problem. For example, Ms. C described different student hand signals she employs so students can respectfully disagree or add to another students’ thought. She stated,

> As we talk we have different signals, like (one for) something to add when someone is explaining something or if they don’t think the answer is right. So there are different hand signals so that we establish a community that’s equitable and everyone’s voice is heard and we share different ways. So kids are prompted to think about if someone did in a different way and they can learn from each other.

For each of the teachers, mathematics was more than knowing facts or solving problems correctly, it was perseverance in problem solving. They each discussed that they wanted students to leave their classrooms with the confidence to attempt problems and apply what they knew. In addition, related to this theme, they also discussed making mistakes as part of problem solving. Mr. B stated,

> I want students to be sort of fearless. They don’t worry about failing. They just jump right in and, you know, my biggest thing is taking what you know, how can I approach a problem with what I know. So I want my students to be truly confident and believe that they can do this or don’t mind failing.

Across the four themes, mathematics was framed as an activity in which multiple voices should be heard and where multiple strategies could lead to a valid solution. In addition, students were encouraged to view mathematics as a problem solving activity in which it was normal to make mistakes.

Identity. Identity is the roles that teachers and students take on during mathematics instruction. The teachers each discussed that they expected students to take on different roles. They wanted students to learn from each other and be comfortable presenting mathematical strategies. Students took on roles of learner, presenter, and teacher with varying levels of engagement in each of these roles.

The teachers primarily envisioned themselves as facilitators rather than instructors. They discussed that they observed, listened to students, and had students explain their thinking rather than instructing students on how to solve the problem or having them complete several practice worksheets. Their role was to observe student reasoning over time and help students make progress both collectively and individually.

Pressing on and Extending the Boundaries of Mathematics through Modeling

As the three teachers engaged their students in the process of mathematical modeling, they described that the process was in contrast to typical mathematical instruction. Four themes emerged in relation to pressing on or extending boundaries: access, empowerment, real world connection, and positioning.

Access. Each of the three teachers discussed that the modeling task provided access and differentiation across the class that was not typically present during mathematics instruction. At the beginning of the modeling unit, when discussing the theme of a community luncheon, all students had questions and ideas that were important to them and they wanted to investigate. Based on past experiences, all students were able to contribute and posed broad ideas like we need to consider cost, likeability, and number of people but also more personal factors like food, allergies, and best places to shop. They each had experiences that they could draw from to start the conversation across multiple perspectives. In describing this, Mr. B stated,

> I think the thing that is most incredible about modeling is watching students use what they know to solve problems, you know? Watching each individual or each group come up with a completely unique way to solve a problem and bring their individual strengths to be part of the solution. That has been really powerful.

The teachers commented that the students, by grade level, determined the mathematics they would use and how far to pursue the task. For example, when planning the luncheon, the third-grade students decided to investigate the cost of a main and side dish while the fifth-grade students planned drinks, a main dish, a dessert, a game following lunch to help build community, and how they would allocate their time across each. Individually within grade levels, students could also access the task and apply mathematical concepts that were appropriate to their understanding. For example, in fourth-grade, students decided they wanted pizza for lunch. When determining the number of pizzas needed, some students used repeated addition while others used multiplication. In describing access, Ms. C stated, “It differentiates itself just by design and kids that are at different levels can be successful at it.”

Ownership and Empowerment. The second theme that emerged was the idea of ownership or empowerment. The teachers identified that the students were able to make choices in the process of modeling. For example, when determining what beverage(s) to serve at the lunch, students surveyed one another using Google documents and found that students wanted the following: 28% root beer, 25% orange soda, 17% lemonade, 10% Cool Aid, 10% apple juice, and 8% milk, and 2% Caprisun. At first, some students proposed that they should just serve root beer because it had the highest percentage. Other students disagreed and stated that less than half of the class wanted root beer, so they should have multiple choices. In the end, the class decided they would offer the top three choices so that more people would be happy in their beverage choice. In describing the process, Ms. A stated that she tries to give students choice, but the process of modeling provides greater opportunity for student choice and ownership. She stated,

> Well (modeling) it's all about choice. I mean they choose what path they want to do or take and how they go about solving it. I try to have a lot of choice in here (my classroom) but I can only have so much, right? And modeling is different because it (the choices made) are mostly theirs and when it wasn't theirs, they didn't know that. They had this empowerment that it was them controlling where they were going.

In addition to having choices, students understood that the choices they made mattered. Whatever they decided upon actually happened. For example, in the fifth-grade class, students decided on pizza for lunch, but ran out of Hawaiian pizza before everyone who ordered it was served. In describing the situation, Ms. A., stated,

> Like the little girl who didn't get her pizza. She just assumed her math was right and when I began to think about it, I just assumed that the kids had been sneaky, but maybe their math wasn't right and we didn't order enough of that kind of pizza?... It is pretty powerful and something that we could have a future discussion about.

Although this example highlights a negative outcome, through taking on the responsibility of planning, teachers commented that students felt confidence in making decisions and then seeing their decisions become reality.

Real World Connection. A third theme that emerged was the concept of math as reality. In describing the launch of the task, Ms. C stated, "I think a lot of them didn't realize they were doing math." Because the problem was situated in reality, all three teachers discussed that students engaged and related to the mathematics with more excitement and perseverance. Mr. B discussed that students were self-motivated when investigating the lunch problem. He stated, "for fourth-grade…our lunch was really successful, you know? It's been really motivating. There is no work I have to do, you know, no encouragement I mean. We just start the process and they are excited and want to attack the problem." It is interesting to note that the lunch modeling task took about a month to complete with students working on the task a few times a week. At no point did they loose motivation or interest to finish the project. Ms. C discussed that during a typical mathematics lesson there is limited connection to the real world. She felt that the process of modeling added an additional layer of meaning to the problem. She stated,

> The real worldliness of what we were doing was key. Because a lot of math that I teach on a daily basis I feel like has no connection to the real world. I mean, maybe you can stretch it to where we are talking about candy or in a story problem dividing it up, but it kind of loses something because it's not connected to a real-world thing that means something to the kids.

The process of modeling highlighted that mathematics is more than different strategies to a particular problem. It can be used to reason about issues students face.

Positioning. The three themes above, access, ownership, and real world connection, all involve how the students perceived and engaged with mathematical content. Mathematical modeling also served as a tool to question positioning in the classroom. All teachers described that it helped to challenge students who they labeled as gifted. The modeling process often takes time and there is no one right answer which was frustrating for some students. Ms. C. stated,

> I also like that it is challenging for the kids who are considered gifted. I have a couple (gifted) kids in my class that when they wrote their reflection, they were like, "I don't like this" because

> they are so used to, even though they are only third-graders, they are so used to being right and getting the right answer…and this was out of their comfort zone. I think that was a good thing.

The process helped to challenge the idea that mathematical thinking is not always about solving problems quickly and correctly. Mathematics can be interpretive. Some students commented that they usually did not like mathematics, but they enjoyed this process.

In addition, teachers commented that modeling fostered mathematics as a community activity. Everyone could feel included and that their ideas mattered. Students could bring knowledge and experiences from outside of the classroom in to help them make decisions about the task. In describing positioning in the classroom, Ms. A stated,

> The most amazing thing to me is that everybody is able, no matter who you are, can enter the (modeling) process where you need to enter it. I just, my entire life, as a person, I have always had a hard time not including everyone and not having everyone feel like they are valued and important. And, I've, when I decided to become a teacher, as much as we like to think public education is inclusive, it's not. We have groups, pullouts and things because we need to service everybody. I totally understand, but it has always made me a little uncomfortable because I see the dynamics because of that. Roles are created. Status is created within the classroom. It's just reality and so this was the first time that I had that that "aha" moment in the class this summer when we were reading those articles. If this is how math could be in my classroom where everyone was doing mathematics and didn't have a role in this or as the really smart kid or the not so smart kid. We would all just have a part in it.

All three teachers described that modeling allowed for all students to feel that they could actively contribute to solving the problem.

Discussion and Concluding Remarks

The teacher interviews and study group notes suggest that mathematical modeling can act as means to extend and redefine students' notions of what it means to know and do mathematics. In classrooms that already valued multiple solution strategies and community-based discussion, modeling acted as a means for all students to feel that they had something to contribute. This is similar to statements made by English and Watters (2015) that students draw on their own experiences to frame the problem. Students were also empowered to view mathematics as a tool rather than seeing mathematics as the practice of skills. The mathematical choices students made mattered. In addition, this study highlights that modeling pressed on the idea that mathematics is bound to classroom instruction. Students were able to see that mathematics could help them make choices about real world decisions. Lastly, modeling challenged perceptions regarding who was capable of doing mathematics and what it meant to be successful in solving a mathematical task. In closing, if mathematical modeling adds to students' understandings of what it means to know and do mathematics, it is important to investigate ways to provide opportunities for modeling across the k-12 system as well as investigate the impact from the student's perspective.

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PROBLEM DRIFT: TEACHING CURRICULUM WITH(IN) A WORLD OF EMERGING SIGNIFICANCE

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In this paper we frame our observations in enactivism, specifically problem posing, to propose the notion of problem drift as a method to analyze the curriculum generating actions of small group learning systems in relation to teacher interventions intended to trigger specific content goals. Teacher attentiveness to problem drift is suggested to be valuable in advancing the content goals in small group work.

Keywords: Problem Solving, Instructional Activities and Practices, Cognition

Small group problem solving has become a stalwart for classroom teachers attempting to occasion vibrant communities of mathematical communication and reasoning. Placing students in problem solving groups creates simultaneous short-range interactions, and results in a classroom ecology dense with opportunity for the teacher to curate productive mathematical insight. However, observing teaching and learning from an enactivist stance means a teacher cannot assume that the group will interact with the content goals of a lesson along a predictable pathway. Instead, cognition is a process of continually posing problems relevant in the moment (Varela, Thompson, & Rosch, 1991). A teacher may assign a task with specific content goals in mind, but, through their action with the task, groups bring forth multiple worlds of significance (Kieren & Simmt, 2009), each of which the teacher is then required to assess for relation to the original curriculum goals. After all, the ultimate responsibility of teaching any program of study is to deliberately impact learners. This leaves the teacher with the job of monitoring how a small group encounters targeted content outcomes within mathematically rich spaces during their course of interactivity as well as coupling with that interactivity with the intention of triggering interaction with targeted content outcomes.

Here, we suggest that the observation of the problem around which the group organizes their interactivity (the problem that is posed by the learners as relevant to addressing the task as currently understood by the group) can inform a teacher's intentional attempts to impact a group's mathematical action (Proulx & Simmt, 2016). This problem posing activity of a group signals the character of their knowing, of their world of mathematical significance. The ongoing re-posing of this relevant problem is termed *problem drift* (Banting, 2017), and can be thought of as a way of observing the emerging character of a group's curricular attention. We add to Proulx and Simmt's work by analyzing problem drift and its relation to the targeted content outcomes of a lesson. This frames our observations of the world of significance brought forth by a learning system (learners, teachers, and environment inclusive) in specific relation to the intended curricular outcomes. Doing so results in a pragmatic stance with(in) the learning system that includes more than just the interactions of the group surrounding content goals, but also the patterns of action in relation to the teacher interventions offered with content goals in mind.

In this paper we explore the notion of problem drift as a method for analyzing the dynamic bringing forth of meaning among members of small groups when given a mathematical task. We explore how problem drift can provide a focus for teachers (and researchers) whose goal it is to understand the nature of mathematical meaning and how they might be purposeful in their influence of it.

Establishing Problem Drift

For the enactivist, knowing is doing (Maturana & Varela, 1987); the process of knowing and its products are one in the same thing (Pirie & Kieren, 1994). The mathematical knowledge of a group (its knowing) is not a source through which the mathematical action (its doing) is resourced or initiated. Rather, knowing/doing emerges through interaction between a subject and their environment (Proulx & Simmt, 2016). Reciprocally, through these interactions from which knowing emerges, the environment is continually shaped (hence co-emerges), and, in return, triggers further possibilities for action. It is through this process that meaning is brought forth. Therefore, knowing emerges out of context. The coordination of learner and environment is not fully in the agent nor the environment, but emerging from the interaction between the two, constituting an emerging world of significance (Kieren & Simmt, 2009); this is the image of knowing and learning with(in) an environment. The learners do not interpret their context in multiple different ways, which would imply that the environment remains static as the learner constructs an impression of its character. Rather, through the mutual triggering of environment and agent, the learners' knowing—that is to say, doing—brings forth distinct worlds of significance. It is through this lens that content outcomes are recast in active terms. In other words, claiming that a specific skill (like "creating equivalent fractions") is *known* means that it must emerge as relevant; it must be enacted.

The mutual specification of problem (environment) and problem solver (learner) means that "we do not choose or take problems as if they were lying out there, independent of our actions, but we bring them forth" (Proulx, 2013). Action of a learner is not dictated or prescribed by the environment, but a learner's structure allows certain features of the environment to become problematic, curious, or interesting. These worlds are maintained through the ability of the learner, in this case, the learning group, to pose problems relevant to its needs at that moment (Varela et al., 1991), where that relevance is contextually and structurally determined (Maturana & Varela, 1987) and the action triggers a furtherance in the posing. In other words, for the enactivist, problems are not given (by the teacher); problems are posed (by the learner). It is through the posing of the problem deemed relevant in the moment that the problem environment and the learning action of the problem solver co-dependently arise. This evolution of the problem posing constitutes the problem drift of the learning system. Because problems are not ready-made, determining what problem has been posed as relevant provides the context in which the group is acting—what meaning they have brought forth. Problem drift details the relevant problem posed by a learning system in order to analyze its knowing action, a sort of trace of mathematical knowing.

It seems that we arrive at a fundamental tension between the responsibility (for the teacher) to provide problems pertaining to a specific set of outcomes as delineated by a curriculum document and the enactivist notion of problem posing—the recognition that learners enact the nature of the task by entering into interaction with it. It problematizes the role of problems to prescribe content outcomes. Rather, teachers design prompts in anticipation that the structure of the task will trigger action that is observed to be mathematical. Teachers then become fully complicit in bringing forth the world(s) of significance by participating in the meaning making (Proulx, 2010). In other words, the teacher does not stand aside and perturb the world of significance of the learners, the teacher participates in the becoming as a fully coupled agent. Conceptualizing knowing in this fashion co-implicates the teacher in the generation of significance. In this sense, the enactivist notion of problem posing does not consider the process of learning as helter-skelter and unbridled where the teacher has little-to-no influence. Such an image would be unapologetic to the project of schooling. Rather, enactivist cognition heightens the role of teacher as one who participates fully in the action—provoking, triggering, orienting, and influencing the learning system. The task of teaching becomes the tethering of the emergent problem posing the group undertakes to bring forth a world of significance to the anticipated content goals built into the task.

Problem drift sits at the crossroads between the enactment of a problem and the mathematical products required by a curriculum; it provides a method to analyze the curriculum a group brings forth. It allows an observer snapshots in the action that can be analyzed for the mathematical processes that were used to address the relevant issues that emerged through their interaction with the task. In short, problem drift allows us to observe the curricular outcomes emerge to address the problem(s) posed as relevant (Banting, 2017).

Methodology

The teacher participants for this study were recruited through previous professional relationships. Using a design research approach (Prediger, Gravemeijer, & Confrey, 2015) with specific attention on the tenants of enactivist methodology (Reid & Mgombelo, 2015), the daily instruction in two classrooms (belonging to two different teachers) was designed around small group tasks completed by students working in randomly-created groups of three. During the study, video data from the workstations of three randomly-chosen small groups was recorded on five separate occasions for a total of of fifteen accounts of group problem solving.

Three adults—a researcher, a classroom teacher, and a pre-service teacher partnered with the classroom teacher for the semester—took on the role of teacher during the classroom episode under analysis here. After each classroom session, a debriefing was audio recorded that included all three teachers. In it, they discussed what they observed in regards to the character of the groups, the interventions they offered, and the reasons behind their choices.

Portions of the action of Brock, Ria, and Sharla (pseudonyms used) is detailed as they worked together on the Tile Design task in their grade nine mathematics course which contained twenty-seven students, or nine groups of three. The Tile Design task asked students to create a series of shapes with colored square tiles to satisfy requirements given to them by a series of stage cards. The task was designed to provide occasions for students to work with two content outcomes:

- Creating equivalent fractions
- Comparing and reasoning about fractions in a part-whole model

Results

In what follows, we detail the action of Brock, Ria, and Sharla (along with the teachers) with interlacing dialogue, artefacts of their doing (images depicting the tile arrangements on their workspace), and description of teacher interventions with the intention of triggering interaction with the content outcomes of the task. For the sake of brevity, only a portion of their action (divided into three episodes) is provided.

In the first episode, the group was required to create a shape where one twentieth was yellow, one quarter was green, one half was blue, and the remaining was red. Dialogue begins after the group had arrived at an initial arrangement (Figure 1).

Teacher: Is that half blue?

After a quick glance of their arrangement, the group was unanimous that the shape was not half blue, and a student removed two blue tiles to leave the arrangement in Figure 2.

B	B	B	B	B	Y	G	G
B	B	B	B	B	G	G	G

Figure 1. Initial arrangement.

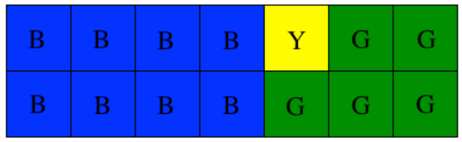

Figure 2. Altered arrangement.

Brock: This is only 14!
Ria: We need 20 tiles.
Sharla: Because the common denominator is 20?
Teacher: What were you going to say, Ria?
Ria: You need 20 tiles because the denominator, the common denominator is 20.
Brock: Yeah. It doesn't equal 20.
…
Ria: You add other blues.
Sharla: If you add, then you have to add in more. But then you have to add in another colour. Add another colour.
Brock: Wait, how many green do we have?
Sharla: We need to add in more green then if you added more blues.
Ria: Which means we have to add more yellow because it has to be equal.
…
Emma: Well, if blue is supposed to be one half, and green is supposed to be one-quarter, then green has to be half the size of blue because one half can be split up into two quarters and if this is one quarter it should be half the size of blue.

Shortly thereafter, a second episode was prompted when the group was provided with a new stage that required them to create a shape that was at least one half red, at least one quarter green, and no more than one sixth yellow.

Brock: At least means it has to at least be a half, right? Okay, so let's make it a common.
Ria: Let's just say this is half.
Brock: Let's do 12. Want to do 12?
Ria: Yeah, sure.
Brock: Okay. So half of 12 is 6. So we have to at least have 6 red.
Ria: Can I make the shape this time?
Brock: No more than one sixth is yellow. 2. And then at least one fourth is green.
Sharla: So that would be.
Ria: 3.
Brock: It has to be 3. Umm. Put one more yellow. It has to be like this. Make sure that's good.
Ria: So if its. We're using 12, then there can only be 2 yellow, right?
Brock: What?
Ria: We're using 12 and there can only be 2 yellow.
Brock: Yeah, but, at least. Oh yeah. K yeah, so that works.

The arrangement resulting from the second episode appears in Figure 3. The new stage did not trigger the group action away from the insistence of using a specific number of tiles, that equal to the lowest common denominator of the stage's requirements. In an attempt to trigger further action, a teacher removed a single green tile from the group’s arrangement (Figure 4) prompting a third episode.

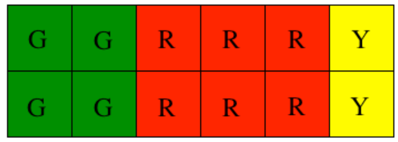

Figure 3. Initial arrangement.

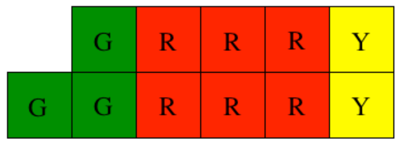

Figure 4. Teacher-altered arrangement.

Teacher: What if I did this? Does that still work?
Sharla: No, because he.
Ria: No, because this is less than a quarter now.
Sharla: No, it’s not.
Brock: No, it’s a quarter.
Ria: No, but it’s still not a quarter of this.
Brock: But it doesn’t equal 12.
Ria: Exactly, so you need that.

Analysis

The group’s action was analyzed using the notion of problem drift. That is, we, as observers, interpreted the relevant problem that was posed by the group as they worked through the stages of the Tile Design task. (For a more detailed explanation regarding the method of analysis, see Banting, 2017.) During the action detailed here, three relevant problems were determined to emerge and compose the group’s problem drift. In other words, we deciphered these episodes to each contain a unique problem posed as the group transformed—brought forth—mathematical meaning. The group’s action in the episodes detailed above was organized around the following pathway of problem drift:

- How can we meet the requirements by using 20 tiles?
- How does the *at least* requirement affect our 12-tile solution?
- Can we satisfy the requirements without using 12 tiles?

We now turn our attention to the important connection for the teaching of mathematics that emerged when the action of the group was analyzed by characterizing its problem drift. Analyzing problem drift allowed us to observe the group’s mathematical doing in relation to the teacher actions as they coupled with the learning system. Specifically, in the third episode detailed here, problem drift provides an explanation as to why a teacher intervention, while attempting to trigger mathematical action around a specific content goal, failed to do so.

Problem Drift Informs Content Goals

The problem drift reveals the group to be operating in a productive space with regards to the intended curricular outcomes. That is, the creation of equivalent fractions and the comparison and reasoning about the sections of a part-whole model have emerged as processes to address the problems posed as relevant in their context. We see this knowing/doing in two regards. First, the group consistently computes targeted common denominators. Specifically, they identify the lowest common multiple of the denominators in the stage's requirements, and execute the algorithm for creating equivalent fractions. This is how the group arrives at the 16-tile arrangement in Figure 1. The group converted one half, one quarter, and one twentieth into ten twentieths, five twentieths, and one twentieth respectively, only to lose focus on the size of the whole and assume that ten blue, five green, and one yellow would constitute the desired solution. We see it again in the 12-tile arrangement in Figure 3. These actions show that, when it is deemed to address a relevant problem, the group can execute this process.

Second, problem drift allows us to analyze the knowing at a deeper level by interpreting what problem caused the creation of common denominators to become relevant. The group calculation of equivalent fractions emerges while acting with the problem, "How can we meet the requirements by using 20 tiles?" The algorithm to create equivalent fractions is brought forth through a need to establish how many tiles they should use to create their arrangement. Before they begin to reason about the size of each part, they need to first establish the size of the whole.

Through the analysis of problem drift, we interpret that the group knows that equivalent fractions can be used to establish a whole because their doing surrounds the establishment of that whole. That is, the group understands the problem as one where a specific size of the whole (first twentieths and then twelfths) needs to be established. Creating equivalent fractions accomplishes that goal. The problem drift allows us to observe the curriculum outcome in the context from which it emerged as relevant (in active terms) and not simply as a skill that can be executed. It provides context as to why the group feels equivalent fractions are suitable; we begin to see the place of equivalent fractions in their emerging world of significance.

In the second episode the group was left with eleven of twelve tiles assigned a definite color after computing common denominators. They then began to address the remaining tile's worth of "empty space" by posing, "How does the *at least* requirement affect our 12-tile solution?" Throughout their action, there is no wavering in their understanding that they must work from a well-established whole. They reason about the possibilities of adding the different colors (attempting yellow, rejecting yellow, and eventually settling on the addition of a green) within the frame of the posed problem. In this sense, their reasoning about the size of fractions emerges within the context of filling in empty space left by the firm establishment of the 12-tile whole. The problem drift allows us to observe that the group knows that there are certain restraints in adding colors to an arrangement with a fixed whole. Throughout the episode here, analyzing the problem drift of the group allowed us to observe their dynamic mathematical doing tethered directly to content outcomes. Not only do we observe the direct execution of mathematical processes, but we also observe the world of mathematical significance in which these processes emerged as relevant.

Problem Drift Informs Teacher Action

The problem drift of the group in reaction to the teacher's decision to trigger action by removing one of the green tiles reveals the world of significance brought forth by the group. In the post-session interview, the teacher explained that they observed the group creating equivalent fractions, but their action never required them to compare the size of two fractions. With an eye to this content goal, the teacher hoped that removing a single green tile would trigger the group to compare the new fractions (now with a denominator of eleven) to the requirements of the stage. Having witnessed the group create equivalent fractions previously (creating equivalent fractions was a part of their structure), they assumed the intervention would result in a furtherance of this skill.

The third episode details how the group interacts with the intervention. The group not only fails to reason about the size of fractions using common denominators, they dismiss the new arrangement as impossible and the intervention as borderline nonsensical. Analyzing the group action through their problem drift reveals a possibility as to why. While the teacher assumed that the group would be triggered into further creation of equivalent fractions, they failed to take into account the context from which the creating of equivalent fractions initially emerged as relevant for the group. The group knows common denominators as an important process for establishing the whole; they do not know them as a process to compare unfamiliar fractions. This knowledge of equivalent fractions was never a part of their world of significance. In their understanding, equivalent fractions are a process to create the whole, not to compare sizes of constituent parts. That is why an intervention attempting to trigger action around such a concept was foreign. The teacher treated the creation of equivalent fractions as a skill held by the group, and did not interpret their action as brought forth with(in) a specific context. Problem drift allows us to re-conceptualize the reaction of the group to the trigger. They did not ignore the invitation because they were unable to execute the creation of a common denominator. As detailed previously, they do exactly this several times throughout the episodes. They ignore the invitation because the trigger was foreign to the world of significance they had brought forth.

For them, in their world of significance, the problem was one of establishing a whole and then assigning the parts to meet the requirements. This is evident through their problem drift. It is the symbolic equivalent to determining a fixed denominator and then adjusting the numerators of each section until the requirements are met. By removing a tile, the intervention asked the group to alter the established whole—to adjust the denominator instead of the numerator. Their reaction does not indicate a lack of skill, but rather that the anticipated interaction was not relevant to the problem they posed. The teacher’s anticipated vision was incompatible with the group’s world of significance. In other words, analyzing problem drift reveals the group’s preoccupation with establishing the whole, and why an intervention that suggests they do otherwise did not sponsor the desired action surrounding content outcomes. This does not suggest the intervention was an example of bad teaching (Towers & Proulx, 2013); it simply did not coordinate with their enactment of the problem—it was irrelevant.

Discussion

Problem drift is a method for analyzing the dynamic bringing forth of meaning. It provides a focus for teachers (and researchers) whose goal it is to understand the nature of mathematical meaning and how they might be purposeful in their influence of it. It focuses the search for emergent knowing on the nature of a learner’s doing.

The teaching intervention of removing a green tile did not have the desired curricular effect, but that is not to dismiss it as having no effect. Through their intervention, the teacher acting with(in) the world of significance provided a possibility by suggesting that shapes of different sizes could meet all the requirements of the stage. The structure of the group, and their world of significance evidenced through their problem drift, did not allow that possibility to become problematic, and it was treated as foreign or nonsensical. In order to trigger the desired content outcomes, the possibility of different sizes of shapes would need to first become relevant to their structure before becoming enacted into the group’s world of significance. Understanding this provides an avenue for the teacher to attempt further triggering.

We do not suggest that understanding problem drift will make the group response to a teacher intervention *predictable*. Instead, we suggest that identifying the problem that has focused action—

that is, become relevant—makes teacher interventions *accessible*. Problem drift speaks to what the group knows because it analyzes what they have done—what problems they have posed as relevant and their relation to the intended curricular outcomes. It is a way of viewing emerging knowing/doing and assessing whether the desired content outcomes have become a part of that action. It is a way of attuning to the group's structure and situating teacher inter-actions therein. Here, we propose that attuning to problem drift is a critical piece to inform teacher interventions designed to advance the content goals of a lesson. It switches the orienting question of teaching from "*How did they solve the problem?*" to "*What problem are they solving?*" The former assumes the problem was engineered to meet certain curricular requirements, while the latter sits at the crossroads between emerging meaning and curricular mandates—that which has become relevant in the moment.

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REFORMULATION OF GEOMETRIC VALIDATIONS CREATED BY STUDENTS, REVEALED WHEN USING THE ACODESA METHODOLOGY

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In this article, we report how a geometric task based on the ACODESA methodology (collaborative learning, scientific debate and self-reflection) promotes the reformulation of the students' validations and allows revealing the students' aims in each of the stages of the methodology. To do so, we present the case of a team and, particularly, one of its members who expresses the intention of reformulating validations for a mathematical conjecture besides showing evolution in the form of justifying.

Keywords: Reasoning and Proofs, Geometry and Geometrical and Spatial Thinking

Background and research problem

For some decades, the teaching and learning of proof have been studied by a number of mathematics education researchers to promote their learning (De Villiers, 2010; Hanna, 2000, among others) and identify and classify the procedures provided by the students when developing tasks in which they have to prove (Balacheff, 1987; Bell, 1976). The works by these authors consider that students in general show difficulties when they are asked to prove a mathematical statement or mathematically justify their statements. Proof is probably the only accepted way of validation among mathematics scholars. However, in a context of teaching and learning, students are not necessarily expert in the matter and will not become professional mathematicians (Legrand, 1993). This means that, when asking students to prove, we will probably find that what they consider will be far from what is accepted as proof by a professional or a scholar. Considering that, our research uses the term validation to refer to what a student may provide to justify a mathematical statement. We understand this validation as a dynamic process we expect to evolve according to the context in which the student works.

The aim of this work was to determine how the validation created individually by a student is reformulated and improved when working with tasks based on the ACODESA methodology proposed by Hitt (2007). To do so, we raise the following research question: How does a validation change from its written formulation by a student, its subsequent study by a team and debate by the class, to its final reconstruction in a process of self-reflection?

Theoretical References

In this study, we consider the term validation as a process through which a proposition or mathematical statement is validated, as Balacheff (1987) explains. He considers that proving is the intellectual activity, not entirely explicit, that deals with the manipulation of the given or acquired information to produce new one, aiming to ensure the truth of a proposition. The validation can be expressed in different ways: explanation, argumentation, proof or demonstration. All of them can vary or evolve, according to the context in which the student works. Therefore, as Brousseau (2002, p. 17) states:

> The didactical situation must lead them to evolve, to revise their opinions, to replace their false theory with a true one. This evolution has a dialectic character as well; a hypothesis must be sufficiently accepted—at least provisionally—even to show that it is false.

Accepting a form of validation will depend on the mastery of knowledge each student possesses as well as the social environment and how close he or she is to the validation criteria of the community where the validation occurs. This last element is of great importance since the validity of a proposition must be accepted by the student and his or her social environment. Above all, the validity must be guided by the validation criteria of the discipline—mathematics, in this case. The description of the different forms of validation, according to Balacheff (1987), are the following:

Explanation. Discourse with which the truth of a position or result previously acquired by the speaker is clarified.

Argumentation. Discourse aimed to obtain the listener's consent.

Proof. Explanation accepted by a community that can be rejected by another. It may simultaneously evolve with the advance of the knowledge on which it is based.

Demonstration. It is a series of statements organized according to a well-defined set of rules.

To identify and then categorize the validations produced by the students, we used the typology of levels and types of proof by Balacheff (1987):

Naive empiricism. It occurs when the student asserts the validity of a statement after verifying it in particular cases. The student's resistance to generalization is evident in this type of proof.

Crucial experiment. The student verifies with the least particular example he or she can manage. In this type of proof, the student explicitly generalizes from the example with which the statement is verified.

Generic example. The student provides an example representing the generality; that is, an example that is not considered a particular case but a representative of a type of cases for which the statement is true. In this type of proof, operations and transformations of the mathematical object explain why the statement is valid.

Thought experiment. The student explains the reasons through the analysis of the properties involved in the statement, decontextualizing it and taking it out from a particular representation.

Calculation on statements. Intellectual constructions based on more or less formalized or explicit theories, created in a definition or property. They are based on the transformation of symbolic expressions. This type of proof ranges from the thought experiment to the proof.

Balacheff (1987) groups the types of proofs, described above, in pragmatic and intellectual. On one hand, the naïve empiricism, the crucial experiment and the generic example are pragmatic proofs: they resort to action and concrete examples. On the other hand, the thought experiment and the calculation on statements correspond to intellectual proofs, given that they are supported by the formulation of mathematical properties set in play and the relationship between them.

Methodology

Students of a Master of Educational Mathematics participated in the study for two sessions of two hours each. In each session, two video cameras were used to record an overall view of the classroom and specific moments. Additionally, dialogs between the students were recorded using a voice recorder.

The task implemented was designed and organized according to the principles of the ACODESA methodology (Hitt, 2007), which allows promoting collaborative learning through social interaction and the use of technology. As a result, processes of conjecture, argumentation and validation are created in the classroom (Hitt, 2011; Hitt, Saboya, & Cortés, 2016). The ACODESA methodology has five stages:

1. Individual work. The student develops the task individually using paper and pencil.

2. Teamwork. The students work in teams of three or four members. Each member presents a solution and justifies the response to the problem in front of his or her peers to create a solution as a team.
3. Debate. Each team presents its proposal in front of all the class. The guidelines to the debate used in the ACODESA methodology must be in accordance with what Legrand (1993) stated.
4. Self-reflection. The students carry out a process of reconstruction of the task. This stage is important because, as Hitt y González-Martín (2014) and Hitt et al. (2016) state, the consensus obtained in the previous stage might be provisional for some students. Therefore, every student has to reconstruct the solution individually using paper and pencil, considering what has been done in previous stages.
5. Institutionalization. The teacher presents the institutional solution to the task in front of the students. To do so, the teacher summarizes what was done in previous stages and highlights the solutions proposed by each team.

The general aim of the task was to create a work environment in which the students could conjecture and then validate such conjectures both individually and in teams. With the activity, we sought to identify the types of validations provided by the students when working on the different stages of the ACODESA methodology. Then, we determined how their validations evolved from the individual formulation to the moment they were shared and discussed during teamwork in a plenary session and up to the moment when they were reconstructed by the student in the self-reflection stage. Due to space limitations, in this article we only report part of the results of the task. The statement in the task and the questions were as follows:

> A parallelogram is known to be a quadrilateral whose opposite sides are parallel. If you choose any given parallelogram and draw the respective diagonals, four triangles will be formed; then,
>
> - What can you say regarding the areas of the four triangles? Justify your response in detail and do not forget to mention which parallelogram you chose.
> - Are your responses above independent from the type of parallelogram you choose? Why? Justify your response in detail.

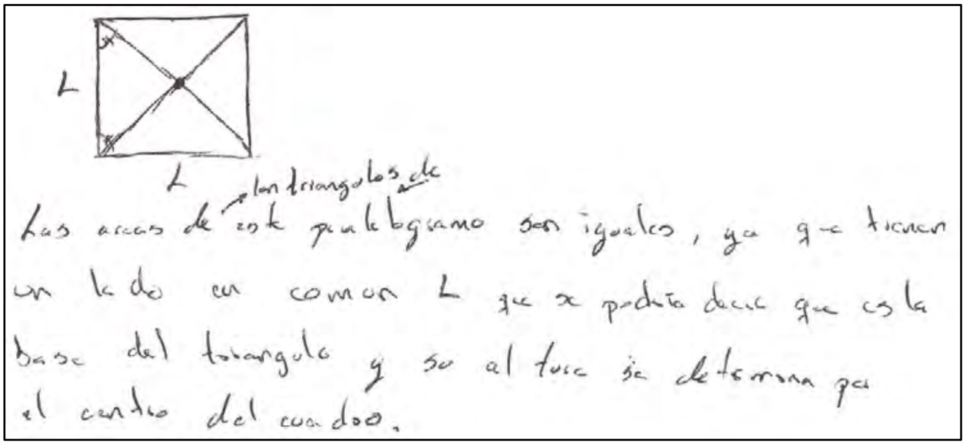
The first question aimed for the students to conjecture and validate for the particular case of a parallelogram. The objective of the second question was to make the students generalize their conjecture and then, validate it. To differentiate between their responses in the different stages, the students were asked to use a black ballpoint pen for the individual work, a red one for the teamwork and a blue one for the debate stage.

Analysis and Result Discussion

In this section, we present the case of Alex, a student whom we considered the most representative of the group in which the task was implemented. The analysis was carried out according to the stages of the ACODESA methodology.

Individual Work

Alex chose a square to formulate the response. To do so, he drew a general representative and assigned a measure L to each side (Figure 1). The student then conjectured that the areas of the four triangles formed randomly when drawing the diagonals were equal.

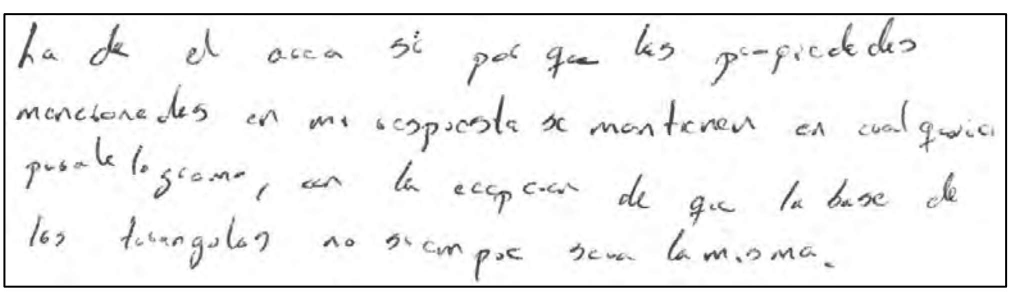


The areas of the triangles of this parallelogram are equal since they have a side L in common, which might be said to be the base of the triangle, and its height is determined by the center of the square.

Figure 1. Particular solution provided by Alex.

The student's strategy was to justify that the areas are equal because the four triangles have the same base and height. In his response, we identify two statements to achieve his objective: (1) the four triangles have equal base because L is a side of the square, and (2) the height of each triangle is determined by the center of the square.

The student justified the first statement through one of the properties of the square (equal sides). For the second statement, related to the congruency of the heights of the four triangles, Alex did not mention nor justified the property that allowed him to say the intersection point of the diagonals was the center of the square. Although his statements are valid, Alex omitted middle justifications (second statement). Regardless, we consider this validation to be a proof corresponding to the incomplete thought experiment, given that the student based the arguments on a general representation of the squares and presented (incomplete) justifications when he applied properties involved in the chosen parallelogram. In the response provided by the student for the second answer, in which he was induced to generalize, Alex claimed that his conjecture (regarding the square case) was independent from the type of parallelogram chosen; that is, the equality of the areas of the four triangles is met for any parallelogram (Figure 2).



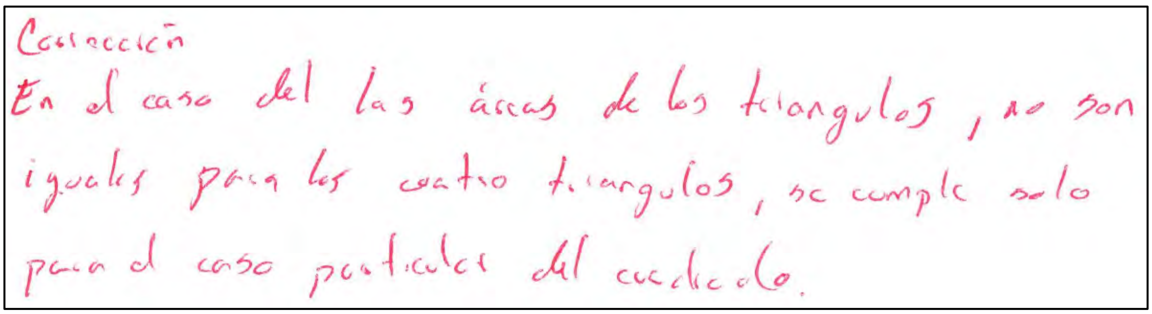
Yes to the one of the area because the properties mentioned in my response are maintained in any parallelogram, except that the base of the triangles is not always the same.

Figure 2. Generalization provided by Alex.

Alex correctly generalized the conjecture since the four triangles will always have the same area. However, when justifying the new conjecture—general conjecture, hereafter—we observe that the student supported his argument on the validation created for the case of the square. In consequence, we infer that the student believed that validation was also true for any parallelogram. Then, we have a generic example-type of proof because the validation of the general conjecture, which refers to all parallelograms, would be supported by the validation Alex created for the square case, which is only a representative of a type of the parallelogram family.

Teamwork

The other two members of the team, S1 and S2, created their own responses from a rectangle and a general parallelogram, respectively. Both students conjectured that only opposite triangles have equal areas and were adamant that the square is a particular case. Unlike Alex, they created thought experiment proofs based on the consistency of triangles. Although Alex conjectured that the four triangles would have equal areas in all parallelograms, he did not say so to his teammates and corrected his conjecture on the answer sheet (Figure 3), but did not create another validation.


Corrección
En el caso del las áreas de los triangulos, no son iguales para los cuatro triangulos, se cumple solo para el caso particular del cuadrado.

Correction
In the case of the areas of the triangles, they are not equal for the four triangles, it is met only for the particular case of the square.

Figure 3. Correction written by Alex during the teamwork stage.

Debate

During the debate, the students first discussed whether the four triangles or only opposite ones had equal areas. Once the teams presented their responses, the consensus of the debate was that, in any given parallelogram, the four triangles formed would always have the same area. The validation agreed on by all the students was based on congruency of triangles to justify the equality of the opposite triangles and the property of diagonals (the diagonals intersected each other) to justify the equality of the areas of adjacent triangles. After this, Alex corrected his response once more (Figure 4) and went back to his general conjecture—the one he had discarded during teamwork. He then expressed that the argumentation for such conjecture had to be changed. From his response, we infer that, after listening to different responses in this stage, Alex obtained more arguments to create a new validation for his general conjecture, although he only did so in the following stage: self-reflection.

Recorrección

La idea de que las areas eran iguales en los cuatro triangulos era correcta, (la argumentación cambia).

Se puede mostrar que son iguales tomando como base la parte de la diagonal de cada triangulo

Re-correction

The notion that the areas were equal in the four triangles was correct (the argumentation changes).

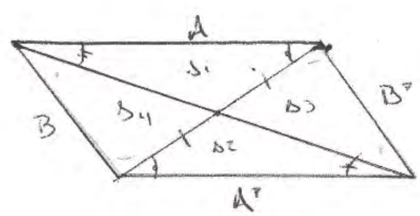
We can prove that they are equal based on the diagonal part of each triangle.

Figure 4. Correction created by Alex in the debate stage.

Self-Reflection

In this stage, we observed a more solid validation (Figure 5) than the one created by the student during the individual work stage.

Después de lo visto en clase, se puede considerar un paralelogramo general como el siguiente:



- Por intersección de paralelas tenemos que: $A = A'$ y $B = B'$
- Por el criterio de congruencia ALA (angulo lado angulo), concluimos que los triangulos Δ_1 y Δ_2 son congruentes. Es decir, $A\Delta_1 = A\Delta_2$.
- Análogamente se analizan los triangulos Δ_3 y Δ_4, concluyendo que $A\Delta_3 = A\Delta_4$.

Ahora bien, se sabe que el centro del paralelogramo divide a las diagonales en dos segmentos iguales, entonces, se observamos los triangulos Δ_2 y Δ_3 tienen la misma base y la misma altura por lo que $A\Delta_2 = A\Delta_3$

Por lo tanto $A\Delta_1 = A\Delta_2 = A\Delta_3 = A\Delta_4$ para cualquier paralelogramo

After what we checked in class, a general parallelogram can be considered as follows:

- By the intersection of parallels, we have that: $\boldsymbol{A} = \boldsymbol{A^T}$ and $\boldsymbol{B} = \boldsymbol{B^T}$.
- By the ASA (angle, side, angle) congruency criterion, we conclude that the triangles Δ_1 and Δ_2 are congruent, that is, $\boldsymbol{A}_{\Delta_1} = \boldsymbol{A}_{\Delta_2}$.
- Likewise, triangles Δ_3 and $\Delta_4$are analyzed, concluding that $\boldsymbol{A}_{\Delta_3} = \boldsymbol{A}_{\Delta_4}$.

That said, the center of the parallelogram is known to divide the diagonals in two equal segments, so, if we observe triangles Δ_2 y Δ_3, they have the same base and the same height, then $\boldsymbol{A}_{\Delta_2} = \boldsymbol{A}_{\Delta_3}$.

Therefore, $\boldsymbol{A}_{\Delta_1} = \boldsymbol{A}_{\Delta_2} = \boldsymbol{A}_{\Delta_3} = \boldsymbol{A}_{\Delta_4}$ for any parallelogram.

Figure 5. Validation created by Alex during the self-reflection stage.

Alex used triangle congruency as his first argument to justify the equality of the areas of opposite triangles ($A_1 = A_2$ and $A_3 = A_4$ in the parallelogram in Figure 5); such argument arose during the teamwork stage. The student then used the property of diagonals to justify the equality of the areas of adjacent triangles ($A_2 = A_3$ in the parallelogram in Figure 5); he built this argument during the debate stage. Alex returned to his general conjecture while his validation did not depend on a particular representation anymore. Then, it was a proof of the **type calculation on statements** since Alex based his statements on the definition and properties of the parallelogram.

Conclusions

In the individual work stage, we observed that the student created a thought experiment proof to validate his conjecture regarding the square case. However, when he generalized, his validation became a generic example proof. In the teamwork stage, Alex did not expressly wrote any reformulation to his validations, but did alter his general conjecture (Figure 3). During the debate stage, he went back to his general conjecture and gave indications that the validation had to change.

If we consider that the general conjecture created by the student (the four triangles have equal areas), the validation he created in the individual work stage corresponds to a generic example, a pragmatic proof. In contrast, during the self-reflection stage, the student built an intellectual proof for the same conjecture; it was a calculation on statements and included arguments that arose during the teamwork and debate stages. This revealed a noticeable change between his individual work and self-reflection to validate the same conjecture, given that his validation went from a pragmatic level to an intellectual one. We can credit this to the ACODESA methodology, as indicated by Hitt et al. (2016). An adequate environment was created for the students to conjecture and validate in a context of social interaction.

On the other hand, during the teamwork stage, Alex provisionally rejected his conjecture after listening to his classmates' arguments. This situation probably took place because Alex did not have the necessary arguments in that moment to persuade his team of the veracity of his conjecture. Regardless, this situation was overcome in the debate stage, during which all the students agreed that the four triangles would always have the same area despite the type of parallelogram used. In the previous chapter, we observed how the didactic situation led the students to evolve, as defined by Brousseau (2002), both in the initial conjecture and the arguments used for its validation.

Most of the students rationally justified their assertions, both in the individual and the group stages. Additionally, we observed an environment of discussion around the arguments used to defend the different statements that arose during the development of the task, especially in the debate stage.

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THE INTERSECTION BETWEEN QUANTIFICATION AND AN ALL-ENCOMPASSING MEANING FOR A GRAPH

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Quantitative reasoning plays a crucial role in students' and teachers' successful modeling activities. In a semester-long teaching experiment with an undergraduate student, we explore how her conception of a graph plays a role in her ability to quantify and maintain quantitative structures. We characterize here Lydia's conception of a graph as one in which the graph entails several quantities she identified in a given dynamic situation, contradicting the conception of a graph as a representation of a multiplicative object consisting of only two quantities. We also discuss her thinking about her graph in terms of figurative and operative thought during a session in which we support her in disembedding and graphically representing quantities.

Keywords: Geometry and Geometrical and Spatial Thinking, Curriculum, Modeling, Teacher Education-Preservice

Introduction

Quantitative reasoning is a crucial component to students and teachers establishing productive meanings (Thompson, 2013). Researchers, however, have found that students' meanings for functions and their graphs lack reasoning about relationships or processes between quantities (Dubinsky & Wilson, 2013; Lobato & Siebert, 2002; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994b), which ultimately influences students' representational activities. For example, Moore and Thompson (Moore & Thompson, 2015; Thompson, 2016) characterized students' non-quantitative graphing activities in terms of *static shape thinking* (i.e., treating graph-as-wire and focusing on physical features of situations and graphs). During a semester-long teaching experiment, we noted that one of our participants, Lydia, seemed to have a particular meaning for graphs that not only entailed remnants of the static shape thinking discussed by Moore and Thompson, but also included thinking of a graph as containing an abundance of information she perceived in a situation. This latter meaning became problematic as Lydia progressed through the teaching experiment. In this paper, we explore how Lydia's meaning for graphs influenced her reasoning and how quantification and establishment of a graph as a representation of *two* quantities supported her in reasoning quantitatively about the sine and cosine relationship and their graphical representations.

Background and Theoretical Framework

This paper focuses on the intersection of quantification and the consideration of a graph as a multiplicative object. It is important to note that as we define these words, we are operating under the assumption that knowledge is actively constructed in ways idiosyncratic to the knower (von Glasersfeld, 1995). Because of this perspective, we view quantities—conceptions of a specific quality of an object that entails the quality's measurability (Thompson, 1994a)—as personally constructed measurable attributes (Steffe, 1991; Thompson, 2011). Moore and Carlson (2012) highlight the significance of this perspective by arguing that the relationships an individual constructs between quantities depends on her understanding of the quantities and, relatedly, the transformational nature of her image of how these quantities constitute a situation.

Before determining relationships between quantities, one must establish quantities through a process called quantification. Quantification is "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, multi-linear) with its unit" (Thompson, 2011, p. 37). Consider the Ferris wheel in Figure 1; more specifically, let the object under consideration be the green cart (or more precisely, a point on the Ferris wheel that represents the location of the rider on the wheel). There are many attributes one could observe: colors, shapes, motions, etc. These attributes become a quantity when they are measurable; that is, a quantity is understood as a magnitude or amount-ness, such that it entails a unit or dimension and a way in which to assign numerical value to the magnitude or amount-ness. Note that the process of measuring does not need to be carried out in order for the attribute to be considered a quantity. Some quantities in the Ferris wheel situation include the distance the green cart is above the centerline, the arc length the rider is from the 3 o'clock position on the Ferris wheel, and the speed the rider has traveled. Reasoning about relationships between quantities is termed quantitative reasoning.

A graph is a way for students to represent the relationships between quantities they perceive in a situation. More specifically, a normative Cartesian graph defines a *pair* of quantities—directed lengths—via axes, and each point on the graph is a uniting of two quantity's magnitudes. A cognitive uniting of the two quantities in a given situation is necessary either to construct or to interpret a point on a graph in the aforementioned way. This cognitive uniting of magnitudes is what Saldanha and Thompson (1998) referred to as constructing a multiplicative object. This notion of a multiplicative object stems from Piaget's notion of "and" and as of a multiplicative operator (Piaget & Inhelder, 1963). For instance, the sine relationship can be considered as the cognitive uniting of the vertical distance above the horizontal diameter and the arc length traveled around a circle (both measured relative to the radius of that circle).

A conception of a graph as a multiplicative object along with a robust quantification process is necessary for thinking of graphs operatively (Moore, 2016). Piaget (2001) distinguished between two types of thought, figurative and operative thought. He characterized the former as thought constrained to sensorimotor experiences and perception and the latter as one that prioritizes the coordination of mental operations over figurative activity. For example, conceiving the sine graph as a multiplicative object is an example of operative thought due to the conception entailing the coordination of mental actions in the form of quantitative operations. Static shape thinking is an example of figurative thought, as such thinking is dominated by elements of sensorimotor experience and perception to the extent it does not necessarily entail a relation to Cartesian axes (Moore & Thompson, 2015).

Methods

The results of this study come from a teaching experiment (Steffe & Thompson, 2000), in which we worked with three students (two female, one male) across 10-11 videotaped teaching sessions lasting 1-2 hours. The sessions occurred over the course of a spring semester at a large public university in the southeastern U.S. We conducted two sessions with all three students present. All other sessions included one student with at least two research team members. The students were in their first semester of a four-semester secondary mathematics education program, enrolled in both a content course and a pedagogy course. The students had all completed at least two additional courses beyond a traditional calculus sequence with at least a C as their final grade. We selected students from their first content course based on the research group's analysis of their results on an adapted 1-hour version of Thompson's *Project Aspire* assessment, *Mathematical Meanings for Teaching Secondary Mathematics (MMTsm)* (Thompson, 2016), which focused primarily on questions related to rate of change, interpretation of graphs, symbolic notation, and proportion. The research group analyzed the assessments and identified three students who, from the researchers' perspectives, provided a range of responses and communicated their thinking clearly in their written responses. The three students then agreed to participate in the teaching experiment and were monetarily compensated for their time. The principal investigator of the project was the teacher-researcher at every interview. At least one other member of the research team served as the observer(s). The teacher-researcher and observers (heretofore referred to as researchers) took field notes and asked probing questions as necessary. All sessions and written work were videotaped and digitized.

The goal of the teaching experiment was to create models of individual's mathematics, specifically with regards to students' construction of graphs. Steffe and Thompson (2000) referred to these researcher models as the *mathematics of students* (cf. *students' mathematics*). In both ongoing and retrospective analyses efforts, we analyzed the students' actions using generative and axial approaches (Corbin & Strauss, 2008) in combination with conceptual analysis (Thompson, 2008). We first analyzed the students' observable and audible behaviors in order to develop tentative hypothetical models of their thinking. Then, we attempted to identify connections and consistencies across each student's activities with specific attention to her or his meanings for graphing and the extent to which these meaning entailed quantitative or covariational reasoning. Lastly, we made comparisons across students in order to construct more fine-grained models of the students' thinking. In this report, we focus on one student in the teaching experiment, Lydia, whose meaning for graphs enabled us to explore the intersection between the quantification process and her representational activity as we strived to support a conception of a point on a graph as a multiplicative object.

Results

We divide the results section into three parts: (a) Lydia's initial response to a graphing activity given a dynamic situation from her first interview, (b) a summary of our attempts to support quantitative and covariational reasoning of the sine and cosine relationships through reasoning about the relevant magnitudes in a simplified version of the Ferris wheel situation, and (c) Lydia's attempt to relate what she understood as the sine and cosine relationships in the situation and what she understood as the graph that represented those relationships.

The Ferris Wheel Task

One of the first tasks we presented Lydia was the Ferris Wheel Task, which includes a dynamic image of a Ferris wheel rider (green bucket) who travels at a constant speed clockwise from the bottom of the Ferris wheel (Figure 1) (Desmos, 2014). We first asked her to comment on what she observed in the situation, to which she stated there is "a function that would give us the shape of a circle." We subsequently gave her the prompt: "Graph the relationship between the total distance the rider has traveled *around* the Ferris wheel and the rider's distance from the ground." She then produced what she called her graph in Figure 2. When prompted, she indicated several different total distance and height pairs by pointing on a location on her drawn circle, tracing around her drawn circle from the bottom to indicate total distance, and motioning from the point to the denoted ground to indicate height. She also noted how the speed of the Ferris wheel would influence where along the circle she would be at a particular time, explaining "there has to be some physics formula for that, but I don't know it." Importantly, we inferred her drawn graph and comments to suggest that she conceived one particular curve to describe the Ferris wheel situation as a whole, and from that curve, she could isolate and discuss the quantities under question. As we describe below, this inference is important relative to her response to our prompting her to construct a graph in a normative Cartesian system.

Figure 1. Animation snapshots of the Ferris wheel task.

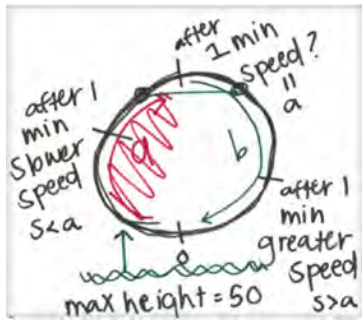


Figure 2. Lydia’s initial graph for the Ferris wheel task.

Partitioning Activities with Diagrams of Circles

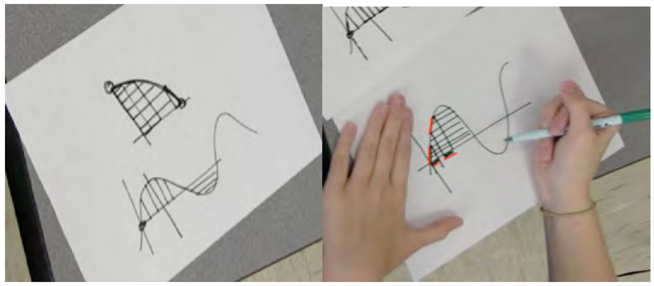
The teaching experiment tasks shifted away from constructing graphs shortly after this task in an effort to provide Lydia and the other participants of the teaching experiment with the opportunity to focus on reasoning together about the relationships between magnitudes in circular contexts. With considerable support from the researchers, the students engaged in partitioning activities (e.g., Figure 3) with a diagram of the circle to reason about amounts of change in horizontal/vertical distance from the vertical/horizontal diameter for equal changes in arc length. Lydia and another student related these quantities to sine and cosine graphs at the conclusion of the joint sessions. Lydia expressed the novelty of the partitioning activity to her, and it subsequently became a way for her to operate on images (diagrams or graphs) to explore relationships. However, as we argue in the following section, her conception of graphs described in the previous section constrained her ability to use this partitioning activity effectively to describe relationships between quantities.

Figure 3. Partitioning activities using a diagram of a circle from the joint sessions.

Sine Graph

In the next individual session, one week later, the researchers asked Lydia to return to the context of circular motion. Initially, they supported Lydia in drawing a diagram of a situation in which a point on a circle is traveling counterclockwise from the 3 o’clock to the 12 o’clock position (Figure 4a, top). She used her newly developed partitioning activities and constructed changes in horizontal distance for equal changes in arc length, and she compared these changes to conclude that the horizontal distance decreased by an increasing magnitude for an equal change in arc length as the point rotated from the 3 o’clock to the 12 o’clock positions (Figure 4a, top). Shortly after this description, we asked her to create a graph representing the relationship between the horizontal length and the arc length, with our intention being that she produce a normative graph for the cosine relationship. She produced the graph in Figure 4a (bottom), stating that the graph can be comparing “the *y*-height here [vertical segments in Figure 4a, bottom] and then also can be comparing the *x*-

distance [horizontal segments in Figure 4a, bottom].” In her initial description of her drawn graph, Lydia did not reference arc length. Thus, she seemed to indicate that the quantities she had conceived as “*y*-height and *x*-distance” in her diagram of the situation were both represented in her graph. This conclusion is consistent with her conception of her graph from the Ferris Wheel task in which she identified several quantities she thought were represented in her graph.



(a) (b)

Figure 4. (a) (top) Diagram of the situation highlighting amounts of change in “*x*-distance” and (bottom) resulting graph; (b) Equal changes in arc length denoted along curve and corresponding changes marked along horizontal axis.

The researchers subsequently asked Lydia to clarify how she was interpreting “*y*-height and *x*-distance” relative to her drawn graph. She first stated she would show the “*x*-distance” and then traced *along her curve* to indicate “increase in arc length.” Following this statement, she drew in horizontal line segments between the curve, starting nearest to the horizontal axis, and moving upwards (Figure 3b). She described these segments as decreasing as the “arc length” increased, which she argued was the same conclusion she had reached in the circular context. She made this statement while drawing in the vertical lines and motioning along two “arc length” and “*x*-distance” pairs, seemingly mimicking partitioning activities from a previous session (Figure 3). However, when asked to say more about how she was denoting “arc length” and “x-distance” on her graph, specifically about the vertical lines from and highlighted regions in Figure 4b, she questioned the efficacy of her actions. Soon afterwards, she switched from talking about horizontal distances to talking about height, stating, “[S]o we’re doing the arc length and height again [*labeling her axes with arc length on the horizontal axis and height on the vertical axis*]”. She then motioned along her horizontal axis, saying “and as I am going across my arc length” and shortly afterwards tracing along the curve starting from the first maximum in Figure 4b saying, “[O]ur arc length, as it increases, the height will decrease.” She then related this statement to her diagram by completing the first half of the full rotation on her diagram in Figure 4a (top).

To summarize, Lydia stated a directional covariational relationship between (i) “*x*-distance” and “arc length” and (ii) “height” and “arc length” using the same graph. Also, when referring to “*x*-distance”, she denoted horizontal segments that connected two points along her curve and conceived “arc length” as a distance along the curve. When referring to “height”, she conceived “arc length” as a magnitude along the horizontal axis and conceived height as vertical magnitudes between the curve and horizontal axis. After a researcher subsequently drew attention to Lydia’s reference to “arc length” as both a distance along the curve and a distance along the horizontal axis, she was perturbed and over the course of nearly 30 minutes attempted to rationalize the graph entailing the three quantities she had identified (“height”, “arc length”, and “*x*-distance/width”). About seven minutes into her efforts, she made the following statement:

Lydia: I don't know. I'm confused. That's what's going on. I like see the relationship, and I can explain it to a point, and then I get like – I confuse myself with the amount of information I know about a circle that I was just given to me by a teacher, and then what I've like discovered here [*referring to the teaching sessions*] and like how those – I'm like trying to find a connection, but I'm getting confused.

Her extended perturbation emphasizes the figurative conception she has of her graph, relying on perceptual features consistent with static shape thinking to relate it with her diagram. Due to the persistence of Lydia attempting to identify each of the three quantities in her drawn graph and the perturbations she experienced due to this attempt, the researchers decided to draw Lydia's attention back to the situation in hopes of isolating the three quantities she perceived in the situation. The researcher specifically asked Lydia to denote the three quantities (i.e., triple) she identified for various points along the circle (Figure 5).

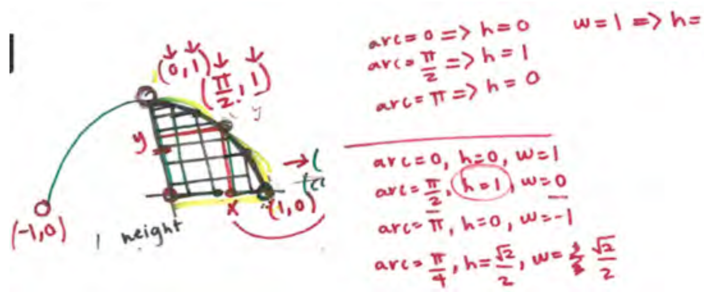


Figure 5. Lydia's diagram of the circle (left) and her triples for various points along the circle (*arc*= arc length, *h*= height, *w*= width).

Upon determining several triples (see Figure 5, bottom right), the researcher asked Lydia how the triples related to her graph (i.e., "[D]oes it represent all three [quantities]? Does it represent just two of them? Does it represent one of them?"). She then drew attention to the origin on her graph and explained:

Lydia: Because this is my – This is *x* – um, *x-y* plane, then here I'm saying at this point [*the origin*], my width is 0, my arc length is 0, and my height is 0.
Researcher: Width is 0, my arc length is 0 and my height is 0.
Lydia: Wait, but then I said at arc length 0, and [*laughs*] height is 0, then my width should be 1.
Researcher: And your width should be 1, right? What about at pi-halves? What should we have?
Lydia: Then I should have a height of 1 [*pointing to curve for an abscissa value of pi/2*].
Researcher: Okay.
Lydia: And then my width should be 0. So this graph does not do anything with the *x-y* plane.
[*Lydia summarizes this claim and then the researcher asks Lydia to consider an arc length of pi radians.*]
Lydia: Then my arc length on the *x*-axis [*motions across horizontal axis*] should be pi. My height should be 1 – or 0, and then my *x*-value should be negative 1. So this [*referring to her drawn graph*] just doesn't have – then this doesn't relate to the *x*, the width [*referring to width from the situation*], just this graph. So my whole circle talks about width and height and arc, but then this graph itself only talks about arc and height. [*speaking emphatically*] Done it. [*laughs*]

We infer that Lydia accommodated her meaning for her drawn graph, including how it related to the circle situation, during this interaction. Specifically, she came to understand that her drawn graph related two particular quantities—arc length and height directed horizontally and vertically, respectively—in a way compatible with the situation. She simultaneously held in mind that these two particular quantities were a subset of the three she understood to constitute the situation. We infer that her accommodation occurred when assimilating a point on the circle as containing a trio of information and then interpreting the point on her graph as entailing the three quantities' values as 0 (i.e., the arc along the curve, the abscissa value, and the ordinate value). This resulted in her experiencing a perturbation with her understanding of the situation (i.e., an arc length of 0, a height of 0, and a width of 1). Alleviating this perturbation required that she disembed two of the quantities from the situation, understand these quantities as represented orthogonally with respect to her drawn graph, and conceive a point on her graph as representing each quantity's value or magnitude simultaneously (i.e., as a multiplicative object). She tested the viability of this new model with two additional points before being confident in the efficacy of her actions; each point on her graph was a uniting of two, and only two, quantities. At this point, her thinking about her graph shifted from figurative to operative.

Discussion

We highlight four important findings from Lydia's activity. Firstly, we note the difficulty a student has in maintaining a consistent quantitative structure within and between a situation and its representation when (i) quantification and quantitative reasoning is a novel way of thinking and (ii) one has a conception of a graph as encompassing an abundance of quantities in a situation. For instance, Lydia had a graph on which she attempted to quantify based on the results of her partitioning activities in her diagram of the situation, but she did not maintain the quantities she believed the graph represented as evidenced by her switching arc length from axes to along the curve. Relatedly, the process of disembedding quantities was crucial for Lydia to view her graph as a representation of a multiplicative object, which was what shifted her thinking from figurative to operative. Thirdly, we have provided a more detailed example of Moore and Carlson (2012), who primarily characterized how the quantitative structure of the situation that the student constructed influenced their mathematical artifacts, including how the students conceived of a quantitative invariance between the two. We extend that work by providing a more detailed look into the students' activity by drawing figurative and operative distinctions, thus not presuming the students to have constructed and maintained quantities. Lastly, this study has important implications for the study of trigonometry in that students should understand the cosine (and sine) relationships as disembedding from the unit circle. Lydia's case shows that this disembedding process should not be taken as a given.

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THE PROMISE AND PITFALLS OF MAKING CONNECTIONS IN MATHEMATICS

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Making connections during math instruction is a recommended practice, but may increase the difficulty of the lesson. We used an avatar video instructor to qualitatively examine the role of linking multiple representations for 24 middle school students learning algebra. Students were taught how to solve polynomial multiplication problems, such as (2x + 5)(x + 2), using two representations. Students who viewed an explicit linking episode were more likely to make important connections, but less likely to exhibit problem-solving success than students who did not view the linking episode. Further, the quality of the connections made by the students was negatively related to subsequent problem solving and transfer. Thus, although focusing on connections may support rich understanding, it may decrease learning of solution methods. The results showcase the promise and pitfalls of making connections in mathematics.

Keywords: Problem Solving, Algebra and Algebraic Thinking, Instructional Activities and Practices, Technology

Introduction

Making explicit connections during mathematics learning instruction is a recommended practice (e.g., NCTM, 2000; Pashler et al., 2007). In fact, some researchers have even defined mathematics understanding in terms of the number or kind of connections that have been constructed by the learner (see Crooks & Alibali, 2014). One important type of connection to make is between multiple representations of the same concept or procedure (e.g., the graph of a line and its equation). In the current study, we used an avatar video instructor to examine the role that linking multiple representations during an algebra lesson had on connection-making and problem-solving performance. Our goals were (1) to compare the effects of a lesson that included a linking episode versus a lesson that did not include a linking episode on students' connection-making and problem solving, and (2) to examine how students' connection-making related to subsequent learning and transfer. We selected the domain of algebra because it functions as a "gatekeeper" to future educational opportunities (Moses & Cobb, 2001). Further, algebra is a focal point of reform efforts in mathematics education (e.g., NMAP, 2008).

Theoretical Framework

Mathematical ideas and representations are connected to and build upon other mathematical ideas and representations. The new *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, 2010) is explicit on this point: fundamentally, "mathematics is a connected subject" (p. 5). Understanding these connections is fundamental to having a deep, conceptual understanding of mathematics. Indeed, the notion of connecting mathematical ideas and representations emerges in many of the standards put forth by the National Council of Teachers of Mathematics (NCTM, 2000), one of which is the ability to "translate among mathematical representations" (p. 67).

The current study evaluated the influence of a lesson that explicitly linked multiple representations during instruction. We define *linking episodes* as segments of instruction during which the instructor seeks to make explicit links between ideas or representations (Alibali et al. 2014). For example, imagine instructing students on the concept of mathematical equivalence first using a balance scale, then using an equation, and finally by making the correspondences between the balance scale and equation explicit. Establishing the correspondences between the two representations would be considered a linking episode.

On the one hand, linking episodes during instruction should facilitate greater understanding for students because they point out conceptual links among ideas and representations (e.g., Crooks & Alibali, 2013; Rittle-Johnson & Alibali, 1999). For example, Hiebert and colleagues (1997) argue, "we understand something if we see how it is related or connected to other things we know" (p. 4). Further, there are many examples of students benefitting from connections made via a variety of instructional techniques, including direct comparison (Rittle-Johnson, Star, & Durkin, 2009), linking gestures (Alibali et al., 2013) and fading from concrete to abstract representations (Fyfe, McNeil, & Borjas, 2015).

On the other hand, making connections among representations can be cognitively demanding, requiring students to understand each representation as well as their correspondences (e.g., Gick & Holyoak, 1980; Nathan et al., 2011). For novices, this may overload their cognitive resources (e.g., Sweller et al., 1998). Indeed, learning from connections may be difficult for students with low background knowledge (e.g., Clark, Ayres, & Sweller, 2005; Kotovsky & Gentner, 1996). For example, one study found that making connections via comparison was beneficial for advanced students, but not for novices (Rittle-Johnson et al., 2009). Specifically, middle school students who did not know a method for solving the target equations benefitted more from studying two methods sequentially than from comparing two methods directly.

Thus, the inclusion of explicit linking episodes may help students focus on making rich connections between multiple representations. At the same time, it may detract from focusing on learning to work with each individual representation correctly, particularly for novice students.

Current Study

In the current study, we had two specific aims. Our first aim was to compare the effects of a lesson that included a linking episode versus a lesson that did not include a linking episode on students' connection-making and problem solving. Specifically, middle school students were taught how to solve polynomial multiplication problems, such as $(2x + 5)(x + 2)$, by an avatar instructor using an area-based representation and an equation-based representation (see Figure 1). Students in the *link* condition viewed a subsequent linking episode and students in the *no-link* condition did not. We expected students in the link condition to make more high-quality connections between the two representations than students in the no-link condition, but to have similar problem-solving performance. Our second aim was to examine how the quality of students' connection-making related to subsequent learning and transfer, regardless of condition. After the initial lesson and assessment, all students were exposed to an instructional linking episode and a posttest. This provided students an opportunity to use the knowledge they acquired from the initial lesson. We expected the quality of students' connection-making to be positively related to their performance on posttest items that tapped understanding of the links between the representations, but negatively related to their performance on posttest items that tapped understanding of individual representations. This work was part of a larger project that developed a teacher avatar (Anasingaraiu et al., 2016) and is investigating how variations in the avatar's behavior during linking episodes influences student learning. The present study focused on variations in the presence of linking episodes in the avatar lesson.

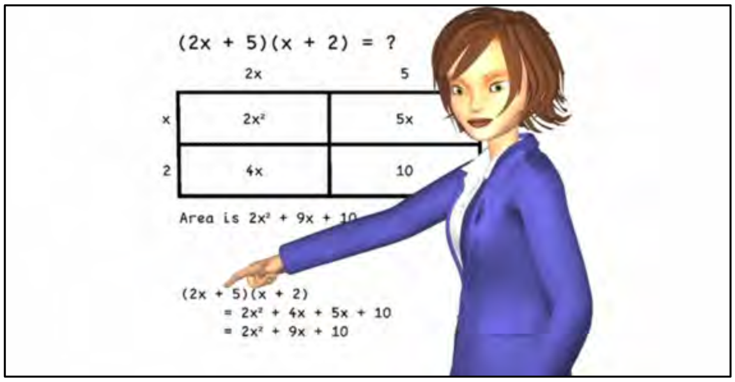


Figure 1. Image of the lesson with the area method (top) and equation method (bottom).

Method

Participants

Participants were 16 seventh-graders and 8 eighth-graders attending one of three middle schools in a mid-sized Midwestern city in the United States. Participants were predominantly White (75% White, 8% Asian, 4% Hispanic, 13% Other) and their mean age was 13.2 years (*min* = 11.5, *max* = 14.2). Sixty-two percent were male. An approved email was sent to all seventh- and eighth-grade students at the schools inviting them to participate in a project that would take place on the university's campus. Each student was compensated $15 for participating.

Design and Procedure

We used a pretest-lesson-posttest design. Each student participated in a single one-on-one session that lasted 45 minutes. Students completed a pretest to assess their background knowledge. Next, they viewed a lesson presented by an avatar video instructor. The lesson focused on multiplying binomials using a target problem: $(2x + 5)(x + 2)$. For the lesson, children were randomly assigned to one of two conditions (Figure 2): *link* (n = 12) or *no-link* (n = 12).

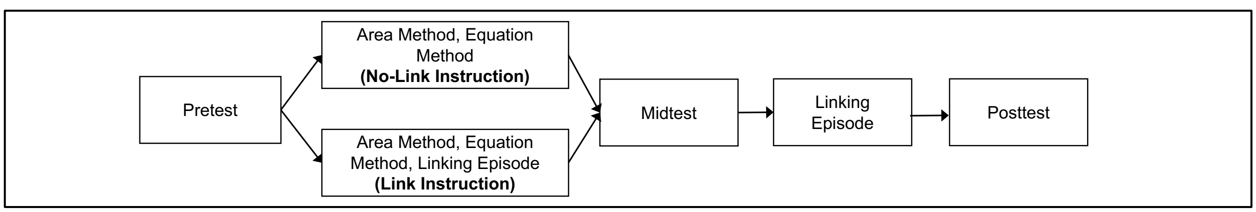


Figure 2. Sequence of activities in the experimental procedure.

The instructor described an area-based method and then described an equation-based method (Figure 1). In the *link* condition, the avatar instructor then provided a linking episode in which she delineated the correspondences between the two representations (e.g., "2x + 5 in the equation corresponds to the length 2x + 5 in the rectangle"). Students then engaged in an explanation of the target problem and solved the items on the midtest. The purpose of the explanation and midtest was to assess differences in learning between students who had viewed a linking episode and those who had not. After the midtest, all students in both conditions viewed the linking episode and completed a posttest. The purpose of the posttest was to evaluate how the quality of students' initial connection-making (as assessed on the explanation and midtest) related to their learning from subsequent instruction. Throughout the session, students were encouraged to think aloud so we could gain a richer account of their thought processes (Ericsson & Simon, 1993).

Materials and Measures

All items on all measures were presented one at a time on an interactive smart board.

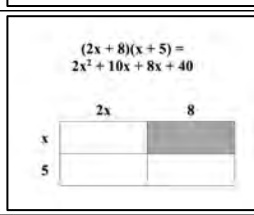
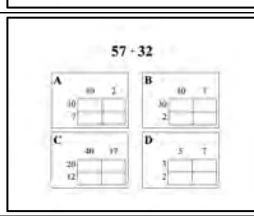
Pretest. The pretest included six items (see Table 1 for examples). The first five items tapped students' background knowledge of operating with variables and calculating area. The sixth item was a target polynomial multiplication problem.

Explanation. After the avatar lesson, students were shown the instructional problem and asked: "Here is the same problem you just learned about. Imagine that another student is seeing this example for the first time. Can you explain how to solve this problem?" Explanations were coded for whether students (1) exhibited a "trouble spot," defined as indicating confusion or displaying incorrect understanding (Alibali et al., 2013), (2) referred to one or both representations, and (3) provided a general solution strategy rather than a step-by-step procedure.

Midtest. The midtest included two items (see Table 1 for examples). The first item was a polynomial multiplication problem. The second item was a linking item.

Posttest. The posttest included seven items (see Table 1 for examples). Two were polynomial multiplication problems. Three were linking items. The final two were transfer items that tapped whether students could apply what they learned about multiplying expressions with variables to multiplying whole numbers. Items were scored as correct or incorrect based on students' written answers and on the verbal think-aloud reports they provided while solving.

Table 1: Example Items Presented on the Pretest, Midtest, and Posttest

Item Type	Example Item	Instructions	Example Responses
Background Knowledge Item (five on pretest)	**X + X**	Simplify the expression.	Correct: 2x Incorrect: 1, 1x, x, 2, x^2
Solve Item (one on pretest, one on midtest, two on posttest)	**(6x + 3)(y + 7)**	Simplify the expression by multiplying the terms 6x plus 3 and y plus 7.	Correct: 6xy + 42x + 3y + 21 Incorrect: 6xy + 10, 21*6xy
Link Item (one on midtest, three on posttest)		Circle the term in the equation that represents the area of the shaded rectangle.	Correct: 8x Incorrect: 40
Transfer Item (two on posttest)		Which area model(s) correspond to the multiplication problem 57 x 32?	Correct: BC Incorrect: Only B, ABC, D

Results

Pretest

Students did moderately well on the five background knowledge items (percent correct ranged from 42% to 88%). However, only one student (out of 24) correctly solved the polynomial multiplication problem: $(x + 2)(x + 1)$. The two most common errors on that problem were to add the two x's and add the two integers to get $2x + 3$, or to combine terms within parentheses to get $2x * 1x$. Conditions were well matched at pretest (M_{link} = 58% vs. $M_{no\text{-}link}$ = 54%). A median split on total percent correct yielded a low-background-knowledge group (n = 12, M = 36%) and a high-background-knowledge group (n = 12, M = 77%). All students but one were unsuccessful on the target problem and were thus novices; the knowledge groups differed in terms of the background knowledge necessary to learn about the target problem.

Explanation

Following the lesson, students were asked to explain how to solve a polynomial multiplication problem. Consider the explanations presented below:

> Student 1 in the link condition: "Basically what you would do is multiply each number by every other number that's in the different set. So 2x times x [draws line connecting the 2x and the x in the equation] is $2x^2$ [circles $2x^2$ in the area model]. 2x times 2 [draws line connecting the 2x and the 2 in the equation] is 4x [circles 4x in the area model]. Both of these are one side [circles the 2x and 5 across the top of the area model] so you don't have to multiply these. Then you do 5 times x [draws line connecting the 5 and the x in the equation], which is 5x [circles 5x in area model]. And 5 times 2 [draws line connecting 5 and 2 in equation], which is 10 [circles 10 in area model]. Then you would take all those answers together [circles all four terms in bottom equation] and simplify them. So 4x plus 5x is 9x. Then 10, and $2x^2$."
>
> Student 2 in no-link condition: "So first you would do what's in parentheses…you do 2x plus 5 [points to 2x and 5 in equation], which I think would be 7x [writes 7x under the 2x + 5 in the equation]. Then you do x plus 2, which would be 2x [writes 2x under x + 2 in the equation]. Then you multiply them I think. So, it would be 14x."

Student 1 provides an accurate explanation, mentions a general solution strategy ("multiply each number by every other number that's in the different set"), and refers to both representations. In contrast, Student 2 exhibits a trouble spot (i.e., incorrect understanding), provides only a step-by-step procedure, and relies solely on the equation-based method.

To capture these differences, we created an *explanation quality score*. Explanations received one point for each of the following features: (1) did not contain a trouble spot, (2) offered a general solution method, and (3) referred to both representations. One third of students scored a maximum 3 out of 3, and across all students the average explanation quality score was 1.8 (out of 3; SD = 1.1). This suggests that typical explanations hit about two of the three criteria for being high-quality. It was most common to provide an explanation that was free from trouble spots (19 out of 24 explanations). It was less common to refer to both representations (12 out of 24 explanations) or to offer a general solution method (12 out of 24 explanations).

Students with low background knowledge had difficulty in explaining. Compared to the high-background-knowledge group, they were more likely to exhibit a trouble spot (42% vs. 0%), less likely to state a general solution method (33% vs. 67%), and less likely to refer to both representations (42% vs. 58%). As such, students with low background knowledge had lower quality scores (M = 1.3) than students with high background knowledge (M = 2.3), and there was little variability between the two conditions among low-background-knowledge students.

However, within the high-background-knowledge group, explanations varied by condition. Compared to students in the *no-link* condition, students in the *link* condition were more likely to provide a general solution method (83% vs. 50%) and more likely to refer to both representations (83% vs. 33%). Indeed, for the high-background knowledge group, students in the *link* condition had higher explanation-quality scores (M = 2.7) than students in the *no-link* condition (M = 1.8).

Midtest

Over half of the students solved the target polynomial multiplication problem correctly at midtest (54%) and all but three students (88%) solved the linking item correctly. Students with low background knowledge were less likely than their high-background-knowledge peers to correctly solve the multiplication item (33% vs. 75%), and the linking item (75% vs. 100%).

As with explanation quality scores, condition differences were minimal for the low-background-knowledge group. For the high-background-knowledge group, performance on the linking item was at ceiling, but performance on the multiplication item varied. Students in the *link* condition were *less* likely to solve the problem correctly than students in the *no-link* condition (50% vs. 100%). All the high-background-knowledge students who solved the multiplication problem incorrectly also provided explanations focused on both representations, suggesting that a focus on linking potentially interfered with learning at least one method well. Overall, regarding our first research goal, we found that a lesson with a linking episode resulted in higher-quality connection-making among students with sufficient background knowledge, but lower problem-solving success relative to a lesson without a linking episode.

Posttest

The posttest occurred after all students viewed a brief instructional linking episode. It allowed us to evaluate how students' initial connection-making related to subsequent learning and transfer. Overall, performance was moderate on the polynomial multiplication solve items (M = 60%, SD = 44%), high on the three linking items (M = 88%, SD = 26%), and moderate on the two transfer items (M = 52%, SD = 35%). Most students demonstrated some learning by the posttest. At pretest, only one student (4% of the sample) solved a polynomial multiplication problem correctly, but 17 out of the 24 students (71%) solved at least one correctly at posttest.

Recall that students explained a target problem after the initial instruction and received an explanation quality score. These explanation quality scores were related to posttest performance (see Table 2). The correlations in Table 2 suggest that explanation quality scores were positively related to posttest linking scores, weakly related to posttest problem-solving scores, and negatively related to posttest transfer scores. We also examined these associations by splitting students into a high-quality explanation group (n = 8, scored 3 out of 3 on explanation quality) and a low-quality explanation group (n = 16, scored 0, 1, or 2 out of 3). Among students with high background knowledge, there were clear differences based on explanation quality. Compared to students in the low-quality explanation group, students in the high-quality explanation group had higher posttest link scores (100% vs. 89%), similar posttest solve scores (75% vs. 75%) and lower posttest transfer scores (41% vs. 67%). These differences lend credence to the idea that students who focus on making connections (and therefore have higher-quality explanation scores) do well on items that tap their knowledge of links, but not as well on items that tap their knowledge of the individual solution methods or representations.

Table 2: Correlations Between Explanation Quality Scores and Posttest Performance

	Whole Sample	Low-Background-Knowledge Group	High-Background-Knowledge Group
Posttest Link Scores	$r_s = .43$	$r_s = .50$	$r_s = .14$
Posttest Solve Scores	$r_s = .22$	$r_s = .29$	$r_s = -.16$
Posttest Transfer Scores	$r_s = -.18$	$r_s = .00$	$r_s = -.42$

To look at this more directly, we made one additional comparison. The explanation quality scores took into account trouble spots, the provision of a general solution method, and references to both representations. To more directly consider connection-making, we compared students who differed only on this last criterion: students who referenced both representations ($n = 12$) vs. students who referenced only one representation ($n = 12$). Students who referenced both representations had slightly higher posttest link scores (89% vs. 86%), slightly lower posttest solve scores (58% vs. 63%) and lower posttest transfer scores (41% vs. 63%). These transfer differences were particularly pronounced for students with high background knowledge (36% vs. 80%). Overall, regarding our second research goal, we found that students' initial connection-making was related to their subsequent learning and transfer. Connection-making seemed to support students' understanding of the links between the representations, but not their ability to solve familiar or novel problems about the individual representations.

Discussion

Educational opportunities for all learners expand as we come to understand the conditions under which teachers' connection-making during instruction affects student learning. The current results highlight the promise and pitfalls of including linking episodes during algebra lessons. For middle school students with sufficient background knowledge, a lesson with a linking episode led to higher-quality explanations than a lesson without one. That is, students who saw the linking episode were more likely to provide a general solution method that applied to both representations and to refer to both representations rather than one. This suggests they were developing rich connections necessary for mathematics understanding (Hiebert et al., 1997).

However, students who saw the link were also less likely to solve a target problem correctly than students who did not see the link – potentially because they were focusing on processing the two representations and their correspondences rather than solidifying their knowledge of a correct solution method. Further, regardless of condition, students' engagement in connection-making was related to their learning and transfer from a subsequent instructional episode. Specifically, higher-quality connection-making appeared to be positively related to performance on posttest items that tapped understanding of the links between the representations, but negatively related to performance on posttest items that tapped understanding of individual solution methods. Thus, linking episodes may help students focus on making rich connections, but may also detract from their focusing on learning each solution method correctly, particularly for novice students (see also Clark et al., 2005; Rittle-Johnson et al., 2009). This may represent a trade-off in the development of conceptual versus procedural knowledge (e.g., Crooks & Alibali, 2014). Improvements in understanding conceptual links may come at the expense of improvements in understanding key procedures. Importantly, these results support the recommendation that instruction should include linking episodes that highlight connections among mathematics ideas. We find connection-making can be supported by a video-based avatar and we identify trade-offs between building rich conceptual connections and performing representation-specific solution procedures.

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AN ETHNOMATHEMATICAL VIEW OF SCAFFOLDING PRACTICES IN MATHEMATICAL MODELING CONTEXTS

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In this work we report on the findings of our ongoing study of fostering validation and refinement of mathematical models in classroom contexts. Drawing on an ethnomathematical perspective we examined interactions within mathematical modeling contexts in two 11th grade pre-calculus classes to identify and analyze instances where students drew on their contextual and mathematical knowledge when solving problems. Our analyses indicate that in presence of teacher's responsive comments students develop deeper learning and engage more intensely in tasks and across contexts.

Keywords: Modeling, Classroom Discourse, Equity and Diversity, High School Education

With the adoption of the National Governors Association (2010) Common Core Standards for Mathematics (CCSM) there has been an increased interest in mathematical modeling in the United States. Indeed, the CCSM lists mathematical modeling as both a High School content standard as well as a Standard for Mathematical Practice. While the CCSM describes the mathematical modeling process it offers little guidance in productive means for teaching mathematical modeling. The absence of research-based data on how this standard might be effectively implemented in ways that learners' own voices are considered is hugely problematic (Cai et al., 2016). Evidence exists that facilitating modeling cognition among school learners involves intricate attention to cultural backgrounds and life experiences of children in ways that the desired mathematics is treated not as the cherished prize but rather a humanizing capital that permit for deeper reflection on these experiences by both the teachers and learners. As such, the relationship among learners' use of mathematics in making sense of real life events, teachers' reflections on what they learn from children as they attempt to facilitate their modeling work, and ways in which gulfs of knowledge are bridged demands meticulous scholarly attention, an area rarely explored in mathematics education research. The overarching goal of this research study was to address this gap by examining the reflexive ways that learners and teachers navigate chasms between mathematical and life experiences within modeling contexts. This work will assist in building a theoretical model regarding unique features of instructional demands in presence of modeling as well as offer methodological implications for the teaching of mathematical modeling.

Background and Significance

Blum and Leiss (2007) offer an overview of the mathematical modeling process where a problem situation, couched in the real-world transitions into the mathematical world through the process of mathematization; is work on mathematically and mathematical results are produced. These results are then analyzed and interpreted against the context being modeled then finally validated with respect to the situation model and the cycle could be re-engaged (p. 226).

While there has been considerable research in examining the mathematization processes of students (Lesh, Galbraith, Haines, & Hurford, 2010); there has been significantly less research reports that examine how validation and refinement of models is mediated in the process of teaching mathematical modeling, a call voiced during the 2016 International Congress on Mathematical Education. Indeed, Cai et al. (2016) outlined a need for extensive research examining the nature of classroom discourse that supports students in becoming successful modelers and in particular examine and report on what mathematical modeling looks like during classroom instruction (p. 4).

Currently there is limited research on teachers' scaffolding practices in mathematical modeling contexts, and the current perspective in the literature seems to align with the view that instruction must shift students' views of problem contexts to an idealized solution (Tropper, Leiss & Hanze, 2015; Schukajlow, Kolter & Blum, 2015; Stender & Kaiser, 2015). Such an elitist view has been challenged by researchers for whom epistemological and ontological perspectives on learning are grounded in ethnomathematics, building a case for listening to the voices of the oppressed in designing instruction (Rosa & Orey, 2016). Primarily absent from the literature are cases where scaffolding involves sensitivity to students' views of particular contexts and formal mathematizations of those views. Previously we have argued (Manouchehri & Lewis, in press) that when engaged in mathematical modeling processes students rely heavily on their intuitions and personal experiences when working on such contexts, and these views impact both the perceived utility of their models and whether they consider the need to validate and refine them.

This ongoing study examines the nature of classroom interactions that occur during the implementation of mathematical modeling across two 11th grade pre-calculus classrooms and considers the particular ways that the teacher responds to students' interpretations of contexts, their solutions to their perceived problems in these contexts and ways that mutual negotiations take place when validating and refining models. In particular we examine the following research questions: (1) How do teachers and school learners co-construct practices associated with developing mathematical modeling over the course of an academic year? (2) What function do these practices serve in facilitating mathematical modeling process? How are these practices different from other mathematical practices?

Theoretical and Methodological Orientation

D'Ambrosio (1895) defines ethnomathematics as the "mathematics which is practiced among identifiable cultural groups" (p. 45) and distinguishes it from what he defines as academic mathematics, or the mathematics taught in school. Orey and Rosa (2016) outline "ethnomathematics embraces the mathematical ideas, thoughts and practices as developed by all cultures across time and space" (p. 6). This view of mathematics affords the ability to treat the experiences and views of a particular context as being viable, and unearth the mathematical validity of those statements in concert with those cultural ways of knowing. This affords a view where Western mathematical forms are not the only viable means of representing mathematical thinking. Further Orey and Rosa argue that when cultures meet, there are three possible outcomes: (1) that one culture eliminates the other completely, (2) that one culture is absorbed by the other, or (3) that the two cultures come together to produce a third distinct culture (p. 18), and as such argue for fruitful and equitable means of cultural merging as being aligned to the third view where both perspectives are honored.

We argue that a classroom in itself consists of multiple views motivated by the various cultural backgrounds of its members. Further we notice that the teacher serves as a representative of one cultural group with one mediating function being to bring learners into the practices of academic and curricular mathematics. Similarly we recognize that students across classrooms come from a variety of background and experiences and draw on various funds of knowledge (Moll et al. 1992) that that informs their intuitions. We argue that with these funds of knowledge informing interaction across tasks, it becomes paramount to consider the means by which these interactions frame the views of the teachers as well as their students, and examine how these experiences with contexts are treated in establishing the culture of the classroom and advancing learners' cognition.

Research Design and Methods

In examining the teacher's practices and interactions with students and their ideas in mathematical modeling contexts we employed a micro-ethnographic study (Bloome et al., 2010) to trace the types of interventions effective in fostering validation of these models. Bloome et al. (2010) articulates a relationship between theorizing, discourse analysis and the event in which analysis occurs where in triangulating of the generated theories they must be validated in the context of event through a discourse analysis, and validated by the members of the space. According to the authors, this process informs the overarching theory through these iterative processes in multiple cycles.

Data Collection

Preliminary data collection for this ongoing study commenced in the fall 2016 across two different pre-calculus classes taught by the same teacher. Our field site was a private academy in the Midwestern United States. Prior to formal classroom observations our cooperating teacher identified her primary instructional goal engaging her students in mathematical modeling on a regular basis, and to support them in both generating and answering their own contextual questions through mathematics.

Our cooperating teacher implements modeling tasks approximately once per week and focuses primarily on students generating and responding to questions about a particular context of interest. In addition to capturing these events through audio and video recording we also examined the nature of general classroom instruction to look for transfer of the mathematical modeling contexts used in class during the designated modeling time. In framing our analysis, we focused on the various ways that students drew on their funds of knowledge and intuitions over the course of instruction which became emergent through their interactions in the classroom.

Additionally a focus group of students was selected for additional audio recording to capture small-group interactions during classroom instruction and small-group/individual work. Written field notes were collected by the research team to identify points of interest for further analysis as well as serving as a log of instructional interactions. Regular reflective interviews were conducted with the teacher in order to get their perspective on classroom instruction for that particular day as well as capture in the moment thinking about the nature of the daily instruction. The research team also implemented daily reflection (either written or audio/video recorded) to help in identifying key events to analyze.

All video and audio records of the sessions are first transcribed. In our analysis we seek instances of interactions and methods that students use when solving problems. Particular attention is paid to the means by which learners' intuitive and experiential knowledge informs interaction with peers and contexts under study. Finally we look across these analyzed cases to seek productive ways that the teacher helps to scaffold students through this process, and where and how these interactions inform future interactions in other mathematical contexts. We characterized these experiences as meaningful instances of scaffolding.

Results and Conclusions

Our preliminary analysis of key events that occurred across five different modeling contexts over the course of three instructional units revealed that when the teacher was sensitive to students' interpretations of problems within each context that this triggered deep mathematical learning and intense engagement in these tasks. For example, in one instance during which students were considering the relationship between revenue, cost and profit models in an economic context, students challenged the viability of the particular model presented, and offered insights into optimizing the business growth potential, and drew on significant and robust mathematical ideas such as increasing revenue through raising prices, or evaluating staffing models of the business. We observed that it was through students drawing on their funds of knowledge and the teacher recognizing that these ideas had both contextual and mathematical merit that these issues were brought to bare in the classroom. Through the teachers responsiveness to these ideas the teacher was able to facilitate a mathematical discussion that helped the students gain insight into this context and unpack the mathematical ideas that they were considering.

In our session, drawing from event maps and illustrations of exemplary interactions, we will argue that in viewing student interpretations and ideas as viable sources of knowledge affords the ability for a responsive teacher to engage their students in deep mathematical and contextual learning that builds on these views. We offer that our analytical interpretations of these events provide a means for theorizing the types of teacher interventions that may be productive in developing a modeling disposition.

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CHARACTERIZING SOPHISTICATION IN REPRESENTATIONAL FLUENCY

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Representational fluency is key to understanding mathematical ideas is deeper ways. This research advances a framework for characterizing sophistication in representational fluency that builds on a structure of observed learning outcome (SOLO) taxonomy. Drawing on an analysis of interviews with 9th-grade algebra students solving equations using a computer algebra system and paper-and-pencil, I discuss four levels of sophistication: prestructural, multistructural, unistructural, and relational. This characterization is sensitive to the nature of students' creations, interpretations, and meanings. Future directions for research are noted.

Keywords: Problem Solving, Algebra and Algebraic Thinking, Technology

Introduction and Research Purpose

Representational fluency—"the ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations" (Zbiek, Heid, Blume, & Dick, 2007, p. 1192)—is key to understanding mathematical ideas in deeper ways. Modeling representational fluency is often approached through examinations of the number and type and representations student use (Dreyfus, 1991; Selling, 2016), and the nature of connections students make across representation types (Fonger, 2011). However, little research accounts for how to model early phases of students' learning and representing, including students' cursory use of representations to make progress in solving problems. For example, students may make progress toward solving identity or contradiction equations with multiple representations, yet struggle to articulate clear meanings of solution sets across symbolic, graphic, and numeric representation types (Huntley et al., 2007). Indeed, solving equations with multiple representations is cognitively complex, for "solution set" takes on differing forms when solving single-variable equations in symbols, as graphs of functions in a Cartesian plane, and as systems of equations in two variables. Examining students' equation solving with multiple representations with technology such as computer algebra systems (CAS) is especially important, for CAS can both afford and constrain students' activity and equation solving techniques.

The purpose of this research report is to advance characterizations of students' sophistication in representational fluency that are sensitive to the nature of students' mathematical activity and meaningful use of multiple representations in solving problems. I focus on the context of students' solving problems involving linear equations in a CAS environment.

Theoretical Framework and Background

A key theoretical assumption in this study is the notion that students' learning and representing are complexly intertwined processes. From this stance, Dreyfus (1991) advances four stages for learning processes from lesser to greater sophistication: using one representation, using more than one representation (in parallel with others), linking multiple representations, and integrating the use of multiple representations in a flexible way. Fonger (2011) elaborated the notion of connecting representations in an analytic frame that examines the number of representations connected and how. More recently, Selling (2016) modeled students' change in learning and representing as use of: use of a single type of representation, use of different types of representations, use of multiple representations for the same problem, and connects different representations of the same problem.

Across these studies, key components of learning, and arguably change in representational fluency, include: the number of representation types used to solve a problem (one or several), and how students reason about connections.

Building on this, and recalling the definition of representational fluency, in this study, I take translation and transposition as key indicators of representational fluency. A student *translates* among representations by creating and interpreting the meaning of a target representation from the source representation (e.g., symbolic to graphic); a *transposition* involves creating and interpreting a single representation type (Janiver, 1987). In students' interpretations, I focus on students' meaning of a mathematical scenario or entity (e.g., how a graph represents solution sets to equations). I take students' creation and interpretation of representations as the "assessed element" for examining representational fluency. With this unit of analysis, I followed an adapted version of the structure of observed learning outcomes (SOLO) taxonomy (Biggs & Collis, 1982) to articulate students' mathematical activity with respect to using one or more representations, connections between representations, and mathematical meanings students make in solving problems. I focused on four levels of the SOLO taxonomy: (a) *prestructural*—lacks knowledge of the element being assessed; (b) *unistructural*—focuses on only one element being assessed; (c) *multistructural*—focuses on several elements being assessed, may not be able to relate them; and (d) *relational*—correctly completes and relates more than one aspect of a task.

Methods

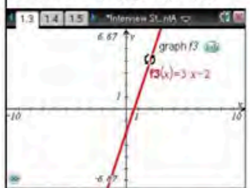
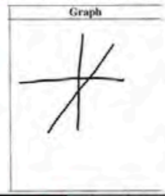
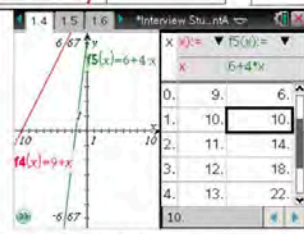
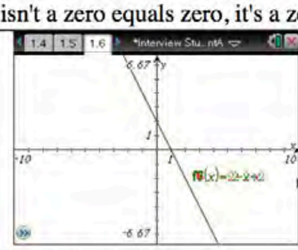
The data for this study are from two semi-structured task-based interviews I conducted at the close of a teaching experiment with 9th-grade algebra students (age 14-15). The participants included 2 students (of a class of 31) who participated in the 25-day collaborative teaching experiment taught by a ninth-grade teacher. Through this collaboration, we introduced equation solving with CAS and paper-and-pencil by building on students' experiences in comparing and justifying the equivalence of expressions. Students identified solutions to equations such as $ax + b = cx + d$ by comparing functions $f(x) = ax + b$ and $g(x) = cx + d$. Activities were designed to support students in creating and interpreting CAS graphs and tables as comparisons of functions before symbolic manipulations. Interview tasks involved solving linear equations given in symbols like $ax + b = cx + d$ for x (given integer parameter values for a, b, c, and d). Students solved equations with one, no, and infinite solutions. Additionally, students solved a set of two equations in two variables, $y = ax + b$ and $y = c$, solving for the variable x.

Data sources include one 45-minute video-recorded interview with each of Annie and Bryon (pseudonyms). I followed a constant comparative method of coding the video-recorded interviews (Glaser & Strauss, 1967). During a first round of coding, two external coders were trained using an adaptation of the analytic scheme for representational fluency as introduced by Fonger (2011). With 81% agreement across coders, all code disagreements were discussed until the discrepancy was resolved in either a re-assignment of a code or an adaptation to the framework. I drew on the initial round of analyses and additional literature to refine the framework, and recode all data. I again followed a constant comparative method of coding until a stabilized coding framework was established, and all data were coded consistently.

Results: Sophistication in Representational Fluency

In Table 1 I introduce an adaptation of a SOLO taxonomy for characterizing sophistication in representational fluency in four levels: prestructural, multistructural, unistructural, and relational.

Table 1: Levels of Sophistication in Representational Fluency

Level Definition	Data Example
Prestructural. Student demonstrates some procedural work within a given representation type but no understanding is conveyed. Student is not able to create and/or interpret representations of a given type with success to complete a task.	*Annie*: (Types $f3(x) = 3x - 2$ into CAS Graphs, moves cursor along the line,   moves horizontally to the y-axis, uses pen and hovers over quadrant one of graph drawn on her paper) [y=] four is right there. If only I could find that [solution] on my [paper] graph (Sighs).
Multistructural. Student moves between more than one representation type to make progress toward or successfully complete a task but demonstrates no or incomplete understanding of how these representation types are related or what they mean to solve the problem.	 *Annie*: [Starting from $9 + x = 6 + 4x$ Annie creates a split screen CAS graph and table relating $f4(x) = 9 + x$ and $f5(x) = 6 + 4x$] It tells me that it will intersect soon (gestures along positive *y*-axis of the CAS graph [below left]) and with my table it tells me right there (points to $f5(1) = 10$ in table [below right]). […] My solution is either ten or one. I'd have to find out, I really don't know right now.
Unistructural. Student creates and/or interprets representations within a single type to successfully complete a task but demonstrates no or limited understanding of connections to other representation types.	*Bryon*: [Task: Solve $x + 2 + 2x = 5 + 3x - 1$ for *x*. With paper-and-pencil Bryon wrote $3x + 2 = 4 + 3x$, subtracted 2 from both sides, wrote $3x = 2 + 3x$, subtracted $3x$ from both sides, and wrote $0 = 2$] *NF*: So, what's the solution to this equation? [$x + 2 + 2x = 5 + 3x - 1$] *Bryon*: Um, it's got none. Well there's zero *x*'s and it equals two, so it'd be, … if there's no *x*'s there's no solution. Unless it's a zero equals zero. But there isn't a zero equals zero, it's a zero equals two. So that's a no solution.
Relational. Student relates at least two different representation types together with a more sophisticated understanding of how they are related, what they mean, or of their significance to the whole. To *connect* multiple representations, a student gives a correct interpretation of an invariant feature across multiple representations or types.	 *Annie*: I need a graph for this. Because if I take both of them (points to $2 - 2x$ and $-2x + 2$ in the equation $2 - 2x = -2x + 2$) which is probably going to be equal in the graph, it may help (types the expressions into a graph $f6(x) = 2 - 2x$ and $f7(x) = -2x + 2$). (Looking at graph takes a deep inhalation) That's my problem! […] they have infinite solutions.

At the *prestructural* level, students may attempt to create or interpret representations to solve a problem without success. A typical prestructural level of representational fluency is characterized by either no attempt at the problem, or incorrect transpositions within one representation type. For example, asked to find the solution to $y = 3x - 2$ when $y = 4$, Annie struggled in working within the graphic representation type across CAS and paper-and-pencil. She found the point (2, 4) on a CAS graph, but could not identify that point as a solution on a hand-drawn graph. Second, students' *multistructural* level of representational fluency is characterized by students' movement between more than one representation type in either creation or interpretation of representations (but not both). At this level, the student makes progress toward successfully completing the task, but has difficulty conveying a relationship or connection between these representation types. For example, in solving $9 + x = 6 + 4x$ for *x*, Annie was successful in creating a split-screen graph and table, yet could not discern if the solution was 10 or 1. In this case, Annie's uncertainty about the solution being "ten" or "one" reflects a cognitive complexity in identifying solutions sets (i.e., $x = 1$ in a one-variable equation, or (1, 10) in a system of equations), and the meaning of solutions across representational forms.

At the *unistructural* level of representational fluency, students are successful at creating and interpreting multiple representations within one representation type, yet multiple representation types and the connections between them are not considered. For example, Bryon demonstrated a

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

unistructural level of representational fluency, when given a one-variable equation in symbolic form, he wrote symbolic equations to reach 0 = 2, explaining "if there's no *x*'s there's no solution. Unless it's a zero equals zero. But there isn't a zero equals zero, it's a zero equals two. So that's a no solution." Here Bryon compared equations as equivalence relations. Finally, at the *relational* level, students correctly translate between representations to solve a problem, and may make a connection by correctly identify the solution as an invariant feature across representation types. For example, consider Annie's work. The task listed an original equation, $2 - x - x = x + 8 - 3x - 6$ with a next step of combining like terms ($2 - 2x = -2x + 2$), adding $2x$ to both sides ($2 = 2$) and subtracting 2 from both sides ($0 = 0$) with a solution of $x = 0$ and $x = 2$. In examining the truth of the claimed solution set Annie graphed the functions $f6(x) = 2 - 2x$ and $f7(x) = -2x + 2$ to claim "they have infinite solutions." Annie treated the equation as a relationship between expressions.

Discussion

In the proposed levels of sophistication, *prestructural* and *multistructural* denote lesser sophistication in representational fluency, while *unistructural* and *relational* convey greater sophistication. At the lower levels students conveyed cursory understandings of solution sets, while at higher levels students conveyed meanings of relational equality. It is notable that both students demonstrated sophistication across these levels, which varied by task type. Other frameworks that build on SOLO (e.g., Fonger, 2011) did not sufficiently account for the cursory understandings students expressed in this study at lower levels of sophistication, which may be important for developing greater sophistication over time. Characterizing students' sophistication in representational fluency with this lens can afford a window into students' emerging development of learning and representing. A fruitful direction for research on representational fluency is to investigate how students' meanings of mathematical ideas evolve over time in relation to how students create and interpret representations. The nature of the complex interplay between learning and representing with multiple representations is a rich area to investigate.

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CONCEPTIONS OF MODELING REPORTED BY INSTRUCTORS IN TEACHER PREPARATION PROGRAMS

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With attention on modeling in the Common Core State Standards for Mathematics (CCSSM), research has turned to supporting preservice teachers in preparing to teach mathematical modeling, especially addressing their diverse perspectives of modeling. This study explores conceptions of modeling, especially those not aligned with mathematical modeling as described in CCSSM, as reported by instructors of mathematics and mathematics education courses required by teacher preparation programs. With the analysis of interview data collected from five teacher preparation programs, we found varying modeling conceptions. Specific examples of the conceptions are described, as well as implications for mathematics education and the preparation of future secondary mathematics teachers.

Keywords: Modeling, Teacher Education-Preservice, High School Education

Engaging in mathematical modeling can enable students to learn mathematics as a complex, integrated, and coherent subject, as a powerful tool for making sense of the real world and everyday life, and as a critical numeracy lens for exploring embedded mathematics in media and political reports (Lesh & Doerr, 2003). Mathematical modeling is recommended by *Common Core State Standards for Mathematics (CCSSM)* as both a Content Standard and a Standard for Mathematical Practice (National Governor's Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). This dual role of modeling – as a vehicle for learning mathematics as well as a mathematical goal on its own – has been noted by researchers, along with other potential meanings of modeling that may cause misunderstandings in both research and practice (e.g., Anhalt & Cortez, 2015; Smith, 2015).

Principles and Standards for School Mathematics [PSSM] (National Council of Teachers of Mathematics [NCTM], 2000) stated that "it is not surprising that the word [model] is used in many different ways in discussions about mathematics education" (p. 70). In addition to connecting mathematics to the real world, *PSSM* included the meanings: (1) physical materials, such as manipulatives, and (2) simulations in which a teacher shows how to solve a problem to students. Smith (2015) noted that language around modeling can be imprecise, and that care should be taken in distinguishing *instructional models* from *mathematical modeling*. Smith identified *instructional models* as "visible tools we all use to make invisible and abstract mathematical ideas accessible to students" (p. 8), but *mathematical modeling* as the process of making assumptions and simplifying an ill-formed situation to "where we can do some mathematics to answer our initial question" (p. 8). Smith cautioned that the distinction can be unclear. As an example, he explained that a numerical representation can be used both as an instructional model and as a tool supporting mathematical modeling. Similarly, Anhalt and Cortez (2015) described that their mathematics preservice secondary teachers (M-PSTs) expressed initial views of modeling in terms of instructional models, but their views developed to be closer to mathematical modeling after engaging with full modeling tasks.

CCSSM defined mathematical modeling as a complex, iterative process for "choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (NGA & CCSSO, 2010, p. 72). Within the process of mathematical modeling, *CCSSM* recommended that students make assumptions to build a model, test the model mathematically, interpret the results, and then use the interpretations to validate the model or revise the model. *CCSSM* focused on students developing "expertise as well as creativity" as they engage in the modeling process, and focused on making and investigating conjectures as they make sense of the mathematics found within a given situation. M-PSTs need experiences developing their conceptions of mathematical modeling as they engage in modeling tasks and distinguish between the meanings of modeling (Anhalt & Cortez, 2015; Smith, 2015).

As we explore experiences that support M-PSTs in developing their own conceptions of mathematical modeling, we aim to answer the following research question: *What conceptions of modeling, different from those described in CCSSM, were reported by instructors who prepared secondary mathematics teachers to learn and teach algebra?*

Methods

Data used in this paper come from a larger research project *Preparing to Teach Algebra*. A part of the project investigated M-PSTs' opportunities to learn about mathematical modeling at five universities that we call Great Lakes University (GLU), Midwestern Research University (MRU), Midwestern Urban University (MUU), Southeastern Research University (SRU), and Western Urban University (WUU). We interviewed 48 instructors of required mathematics courses, mathematics for teachers courses, mathematics education courses and general education courses. We asked instructors to describe modeling opportunities they provided for students.

The interview data were transcribed and checked by another project team member. Two researchers individually coded the interview transcripts for instances when instructors described modeling in ways different from the conception of modeling described in *CCSSM*. They then met to compare coded transcripts to resolve discrepancies. The first author read all the coded data, summarized the main idea(s) of each coded item, and organized them by similar conception(s) of modeling. Two other authors then individually reviewed the categories and examples from the transcripts. Three authors met to discuss the coded items and categories until consensus was reached. Four themes emerged from this process: (a) generating mathematical notations, (b) constructing proofs, (c) using representations to solve non-contextualized problems, and (d) using manipulatives without solving a problem.

Findings

In considering instructors' examples of modeling opportunities that revealed conceptions different than those described in *CCSSM's* "Model with Mathematics" (NGA & CCSSO, 2010), we describe each theme below, using examples provided by instructors.

Generating Mathematical Notations

Linear Algebra and Abstract Algebra instructors described writing mathematical notations for algebraic concepts as examples of modeling. A Linear Algebra instructor at SRU provided an example of writing equations of spanning sets for $\mathbb{R}^n$. He compared physical representations of $\mathbb{R}^2$ or $\mathbb{R}^3$ with representations of $\mathbb{R}^7$ or $\mathbb{R}^n$ where M-PSTs looked for an abstract model of an abstract algebraic idea. Similarly, an Abstract Algebra instructor at MUU mentioned modeling as generating mathematical notation to describe certain properties. When she introduced an example of a group, she reported her students' struggle to use general mathematical notation describing properties in a specific context. Students either forgot to replace *G*, which represents an arbitrary group, or represented elements incorrectly, or used irrelevant operations. She concluded, "It's being able to interpret what these general statements mean from this particular example that they're looking at. In that sense, I think I'm using modeling." These examples of modeling tasks that rely on using mathematical notation to model abstract mathematical ideas, reveal a difference in conception from that described in *CCSSM* which focused more on mathematical application than abstraction; that is, on quantities and relationships in “physical, economic, public policy, social, and everyday situations” (NGA & CCSSO, 2010, p. 72).

Constructing Proofs

A Reasoning and Proof instructor at SRU and an Abstract Algebra instructor at MUU described constructing proofs as modeling. When the Reasoning and Proof instructor considered the *CCSSM* modeling process during the interview, he said, “Some modification of this goes into proof design.” He described several steps (e.g., identifying variables, formulating a model) of the *CCSSM* modeling process with respect to constructing and testing a proof. He explained, “This [modeling process] doesn't apply as much to formulating a specific problem, as it is to formulating and modeling the style of an argument.” Similarly, when an Abstract Algebra instructor at MUU considered the *CCSSM* modeling process, she mentioned that her students “had to formulate a model in terms of a proof by determining what representation worked.” She described students’ “validating the conclusions and possibly improving the model” in a classroom activity in which M-PSTs constructed a sketch of a proof and then exchanged their proof with another student in the class to receive feedback. These explanations revealed that the *CCSSM* modeling process can be interpreted as processes of proof for algebraic concepts.

Using Representations to Solve Non-Contextualized Problems

For geometry courses, instructors reported that M-PSTs learned about algebra as they learned geometry through modeling. The Geometry instructor at SRU demonstrated a modeling activity in which M-PSTs found the shortest distance between points on the surface of a sphere. This activity required M-PSTs to use a model of a sphere and consider the surface metric; he explained, “When we were first starting to talk about the idea of lines and originally thinking of it in terms of shortest distance… Here's a model that we did, it's sort of like spherical geometry [drawing], not quite so round…” Finding the distance between two points is often used in real-life situations, such as selecting the shortest path to go from one place to another. However, the focus in this model is representing abstract mathematical ideas using a more concrete model, without a specific real-life context. Without support, M-PSTs may struggle to connect these abstract ideas with the real-world, as needed to teach modeling as described in *CCSSM.*

Using Manipulatives without Solving a Problem

Another conception of modeling was the use of algebraic tools. The Student Teaching Seminar instructor at SRU provided an example of modeling where M-PSTs discussed the use of manipulatives, such as algebra tiles, pennies, and counters: “We talked about …models like using pennies or counters that were green and red, or something like that to talk about integers.” The instructor considered modeling as use of tools to discuss algebraic concepts, such as integers, but did not connect it with problem solving.

One major difference between these instructors’ modeling conceptions and those from *CCSSM* is that the instructors did not necessarily considered modeling as involving real-life problem solving. All the examples provided here, however, could be extended to include real-life contexts. For example, for the “Using Manipulatives” conception, if a real-life problem can be represented and solved using algebra tiles, pennies or counters, the problem could be considered as a modeling task described in *CCSSM*. Without such extensions, these reported conceptions entailed a distinct nature that is not evident in the description of modeling from *CCSSM*.

Discussion and Conclusions

It would be ideal if instructors of mathematics and mathematics education courses required by teacher preparation programs were supported in collaboratively sharing conceptions of modeling and preparing M-PSTs to learn about and to learn to teach modeling. It might not be realistic, though, to force all stakeholders to share a single conceptualization of modeling when the term *model* is interpreted in several ways. Along with (1) *using manipulatives*, which we described earlier as part of using instructional models (NCTM, 2000; Smith, 2015), and (2) engaging in mathematical modeling as described in *CCSSM* (NGA & CCSSO, 2010), participating instructors provided examples of three conceptions of modeling that differ from *CCSSM*: (3) generating mathematical notations, (4) constructing proofs, and (5) using representations to solve non-contextualized problems. It is worth recognizing these different meanings of a model or modeling when teacher educators discuss mathematical modeling in their classrooms because M-PSTs may possess varying conceptualizations of these terms either from previous mathematics or from mathematics education courses.

As we investigate several conceptions of a model or modeling from literature and additional meanings of these terms found from data collected from the teacher preparation programs, we conclude this paper by proposing a systematic way to interpret diverse meanings of a model or modeling. When we consider the term *model*, without mathematics in mind, it can be interpreted as manipulatives or imitation, as described in *PSSM* (NCTM, 2000) or by other researchers (Anhalt & Cortez, 2015; Smith, 2015). When we discuss *model with mathematics*, *CCSSM* started with the following statement in their Standards for Mathematical Practice: "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" and provided several real-world examples to support this idea. If connections to the real-world are necessary in *model with mathematics,* the first three conceptions reported by participating instructors are not necessarily consistent with this idea. They do, however, share mathematical processes (e.g., identify variables, interpret and validate results) that can be extended beyond modeling to generating mathematical notations, constructing proofs, or representing non-contextualized problem-solving. Connecting such mathematical processes can be an opportunity, not an obstacle, for M-PSTs if they have opportunities to learn from instructors who clarify ways in which these processes can be used for learning and doing mathematics; including, but not limited to, modeling. The various views of modeling held by instructors of teacher preparation programs have the ability to influence the way secondary M-PSTs learn to model with mathematics; therefore, instructors' awareness of these views need to be recognized and discussed in teacher preparation programs.

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EDUCATIVE EXPERIENCES IN A GAMES CONTEXT: SUPPORTING EMERGING MATHEMATICAL REASONING

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Reasoning is critical to students' success in mathematics, yet reports on its development in elementary school are scarce. An action research project with grade 5 and 6 students investigated how growth in mathematical reasoning occurred within abstract strategy games. Results show that Dewey's educative experience emphasizes the importance of students' active engagement and noticing their own reasoning. Through characteristics of continuity and interaction, students analyzed moves, generalized toward strategies, and convincingly justified effective approaches through accepted structures of reasoning. Using educative experiences as practice/theory intersection demonstrates that all students could have equity in accessing high-level mathematics—such as proof and proving—later in school mathematics given opportunities in elementary school to engage in the verbs and strucutres of mathematical reasoning.

Keywords: Reasoning and Proof, Elementary School Education, Instructional Activities and Practices, Classroom Discourse

Reasoning is crucial to students' mathematical learning (Nunes et al., 2012). The possibility of abstract strategy games to support growth in reasoning is relatively unexplored (c.f. McFeetors & Palfy, 2017; Houssart & Sams, 2008). The research question was: How can we understand elementary school students' development of mathematical reasoning through commercial games? Our purpose is to offer a finely-nuanced understanding of a games-based context's contribution to students' foundational experiences of reasoning. Students' data illuminates an intersection between Dewey's theory and a novel classroom practice of using strategy games. The research at this intersection informs a potential practical route for the use of games in mathematics class and a potential theoretical route to understand how informal opportunities to learn impact students' interest and capabilities for future mathematical learning.

Perspectives: Experience and Reasoning

Dewey (1938/1997) saw experience as the essence of education. Experience is comprised of activity and reflection. Students learn by acting in and on the world around them. Reflective thought as an interpretive act is needed to transform activity into an experience. Educative experiences open up possibilities for further learning, determined through the characteristics of interaction and continuity. Interaction involves the interplay between the internalization of an individual and the situation. For situation, both context (ideas, physical environment that is material and natural) and other people are included. Continuity considers each experience as growing out of another experience and leading toward future experiences. Viewing interaction and continuity over a student's school mathematics learning could direct attention to earlier experiences as being foundational. *Foundational experiences* are students' (inter)actions and reflections that prepare them for future, more formalized mathematics learning opportunities.

We view mathematical reasoning as a systematic and logical pattern of behavior (Reid, 2002). Verbs characterizing this behavior include specializing, conjecturing, representing, generalizing, investigating, explaining, justifying, refuting, modifying, and convincing (Lannin, Ellis, & Elliott, 2011; Mason, Burton, & Stacey, 2010; NCTM, 2000). Reasoning structures include deductive and plausible, with subcategories of induction, generalization, specialization, and analogy (Polya, 1954), metaphor, metonymy, mental image (English, 1997); abductive (Conner et al., 2014); and transformational (Simon, 1996). Previous research has identified a developmental trajectory of mathematical reasoning across the grades which mirrors the development of conceptual understanding, moving from analysis of specific cases to systematic generalizations and proof (Stein & Burchartz, 2006; Tall, 2014). Contexts which promote the development of reasoning are familiar, stimulating, and motivating. Teachers' prompting for explanations (Mueller & Maher, 2009) and setting discursive contexts supports mathematical reasoning (Ko et al., 2016). Game play is familiar, motivating and social.

Mode of Inquiry

This report focuses on Pollard School, one of three schools in an action research project. Three grade 5/6 classes were cross-grouped for station-based learning in three micro-cycles. 45 of the 60 students and 3 teachers participated in data collection. Each week for one hour, over three months, one-third of the students participated in the classroom intervention with games. Four games were used: Gobblet Gobblers, Tic Stac Toe, Othello, and Go. The games contain no element of chance. Students played in pairs against another pair to encourage interaction.

While students played, and developed strategies the research collaborators (teachers, researcher, and assistants) prompted them to elicit reasoning. Data collection methods included: verbatim recording of student statements, student record sheets to explain their strategies and reasoning with words and drawings, field notes, photographs of students and game configurations, 20-minute student pair interviews and 30-minute teacher interviews.

We grouped data for analysis by individual student, and read through to develop a holistic view of each student's reasoning (McFeetors & Palfy, 2017). Data was first coded by reasoning structure used: inductive, deductive, indirect, metaphoric, analogic, imagistic, and informal and next by reasoning verbs: specializing, conjecturing, representing, generalizing, investigating, explaining, justifying, refuting, modifying, and convincing. After coding data individually in each phase, we compared data coded to maintain reliability to result in cohesive data sets within categories of both verbs and structures of reasoning. The high quality of students' reasoning caused a return to interrogate processes in which reasoning developed.

Results: Data Interpretation

Although initially students played with little justification for moves, they moved quickly to . conjecturing possibilities, generalizing strategies, and convincing others using varied reasoning structures. Reasoning developed in students through educative experiences, where continuity and interaction were present and contributed to mathematical learning.

Mathematical Reasoning and Interaction

The games were purposefully selected because the possibility of winning elicited a high degree of engagement which led to meaningful interactions. Students used the games to *investigate* moves, *explain* to a partner, *refute* other students' claims, and to *conjecture*. During play they *investigated* strategies and *explained* reasons for moves. In Othello, Renée and Eve *convinced* each other with whispered comments like "go here because we would get the side."

Partners also questioned as they *evaluated* each other's reasoning and *integrated* new strategies. In Gobblet Gobblers, Floyd developed a "checkmate" position, engaging in *analogic* reasoning by using a familiar term from chess. While Eric initially tried to *refute* the claim, he declared in a later match "I have Floyd in the checkmate position." Floyd *explained* "Because then if I put it here he can just go here. If I put it here he goes there. And if I put it over here then he can go there." Floyd's chains of if-then statements are indicative of *deductive* reasoning.

Frequently students *explored* opponents' strategies when their opponent won repeatedly. Students *analyzed* by noticing patterns in plays and moved toward *generalizations*. In Tic Stac Toe, Alex *analyzed* Robyn's moves and stated, "I'm trying to figure out a strategy to do a horizontal diagonal win, like Robyn." Robyn's wins were *convincing* enough for Alex to *investigate* why Robyn was effective and form a *conjecture* around a *modified* way to win.

Collaborators interacted with students' reasoning through verbal prompts to encourage them to enhance reasoning. Furthermore, Esme mentioned to her teacher, "The questions you asked me actually helped me play the game better. They made me think about the strategies."

Mathematical Reasoning and Continuity

The cyclical design of the study occasioned continuity of experiences from playing less complex games toward a highly strategic game by the end. Students' reasoning became more sophisticated as the complexity of the games increased, as well an increased eagerness to improve reasoning emerged. Continuity is best viewed by a focus on one student's development. We use Renee's thinking to illustrate how integral continuity was to students' progression.

Beginning with Gobblet Gobblers, Renee built on her prior Tic Tac Toe strategy of "playing in the middle of the board helps you because then any other place you go you can get two in a row." Renee *generalized* a middle position as a winning strategy and *justified* her answer using evidence from her previous experiences with Tic Tac Toe and Gobblet Gobblers. She combined *analogic* reasoning in connecting two games and *inductive* reasoning in moving from many specific instances to a global strategy.

For more complexity, Renee and her peers moved to *investigate* strategies in Othello. Renee quickly realized that both the sides and corners were powerful spots to control on the board: "I would put white in spot 13 because then I would get the side. The blak [sic] might then go on the side on square 12 and then I would sandwich them on the square to the left of square 12. I would then have 3 on the side". The importance signified by her chain of if-then statements indicates the development of a game tree of mental images or *imagistic* reasoning (Pirie & Kieren, 1994).

For the duration of the project, Renee challenged herself to learn a new game, Go. She demonstrated an emergent strategy after two matches to make an offensive move to "make a wall" in a diagonal line. Renée *conjectures* and *explains*, but does not justify her strategy of "making a wall". Her continued experiences gave her the confidence to advise others in the final class through the tips: "Making dimonds [sic] to capture the other player and marking territory" and "Making diagonal lines so that the other player has a hard time capturing you.

Renée *identifies* two strategies where naming board arrangements is an abstraction from *generalizing*, and now also *justifies* each strategy. The accompanying images point to her use of *imagistic* reasoning. Orally, Renee *justified* the first tip stating, "I like to make rhombus shapes…because it's my territory and nobody can put one in there. And, it's difficult to capture." Renee also commented on how game experiences opened up the possibility for further learning:

> It also helps learn how to make up your own strategy. So when you are doing problem solving–you use–what you did for learning a strategy and try to connect it to problem solving that you are doing. … I find it helps playing with other people. 'Cause then you can learn how their strategies affect the games. And also you get to know about the way that they think. And I think that it is important when you are with people, learning how they think so that you can use that to help you –

Renee sees connections to future experiences with problem solving. She identifies interpersonal learning in figuring out how others think. Finally, Renee introduced Go to her family, carrying forward learning from mathematics into familial, recreational activities outside of school.

Discussion: Significance and Conference Theme

As a potential new route to explore, commercial abstract strategy games like the ones used in this study can be used to support and investigate children's development of mathematical reasoning. What our research suggests is that the games were an authentic context, and what we observed were moments of reasoning that moved from nascent toward more robust. The verbs of mathematical reasoning students enacted while playing the games were broad and varied. Future research could include exploring more games and exploring a new site of parent-child interactions which could enrich home-school connections.

At the intersection of theory and practice, Dewey's notions of interaction and continuity are helpful in framing pedagogical decisions. Our research demonstrates that students can have educative experiences when teachers plan for interaction and continuity. These *foundational experiences* are required early in children's mathematical learning so that they can be consolidated over time and prepare them for success in secondary and post-secondary studies.

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EFFECT OF QUANTITATIVE REASONING ON PROSPECTIVE MATHEMATICS TEACHERS' COMPREHENSION OF A PROOF ON REAL NUMBERS

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This study investigated the effect of quantitative reasoning on prospective mathematics teachers' comprehension of a proof on the decimal representation of real numbers. A proof comprehension test prepared according to the dimensions suggested by Mejia-Ramos, Fuller, Weber, Rhoads and Samkoff (2012) was used as both the pre-test prior to and the post-test upon completion of an instruction given to 19 prospective mathematics teachers. Also, one hour long post-clinical interviews were conducted with six of them. Wilcoxon Signed Rank Test results showed a significant difference between the pre and the post proof comprehension test on the performances of prospective teachers. Together, the results from the interviews suggest that once prospective teachers engage in quantitative reasoning during instruction their proof comprehension might develop.

Keywords: Reasoning and Proof, Rational Numbers, Number Concepts and Operations

Introduction

Proof and proving are the important branches of mathematics education. However, previous research showed that students face with difficulties related to proof and proving (Moore, 1994). Moore (1994) summarized the sources of difficulties considering proof practices: lack of knowledge about definitions of terms and statements and how to use them in the proof; lack of understanding the concepts and also mathematical language; inadequate concept images; and failure to generate or to use examples about the proof statement. For the elimination of these difficulties, researchers focused on different aspects during teaching using informal proofs such as visuals, diagrams, and generic examples (Gibson, 1998). As much the use of empirical examples are important, research also suggested that along with examples, the use of quantitative reasoning (Thompson, 2011) could be critical for students' thinking about proof (Weber, Ellis, Kulow & Özgür, 2014) since students who focus on generalizations through quantitative reasoning can perform better in constructing proofs than students who focus on empirical examples while obtaining generalizations (Ellis, 2007). Besides the importance of eliminating students' difficulties about proof and proving during teaching, it is essential to consider how to evaluate students' performances. Instead of using 'state and prove' format, Mejia-Ramos et.al. (2012) developed an assessment model of proof comprehension at undergraduate level with seven dimensions: **D1:** Meaning of terms and statements, **D2:** Justification of claims, **D3:** Logical status of statements and proof framework, **D4:** Summarizing via high-level ideas, **D5:** Transferring the general ideas or methods to another context, **D6:** Illustrating with examples, and **D7:** Identifying the modular structure. Considering research both on eliminating students' difficulties about proof and proving and using proof comprehension model, this study investigated the effect of quantitative reasoning on prospective teachers' comprehension of a proof about decimal representation of real numbers because research has shown that students have misconceptions and difficulties regarding real numbers (Voskoglou & Kosyvas, 2012), which is proposed to be taught through quantitative reasoning (Karagöz-Akar, 2016). The research questions of this study were as follows: What are the current levels of prospective mathematics teachers in the pre-and post-proof comprehension test according to the proof comprehension dimensions? Is there any significant difference between pre and post proof comprehension test results of prospective mathematics teachers who took an instruction about construction of real numbers as decimal expansion via quantitative reasoning? How do prospective mathematics teachers reason on the post-proof comprehension test upon completion of an instruction on the construction of real numbers as decimal expansion through quantitative reasoning?

Method

The sample of the study (N=19) was the prospective primary and secondary mathematics teachers* (from now on will be called as students) at a university in Turkey in which the medium of language was English. The design of the study was mixed methods, specifically an embedded experimental study since the quantitative data obtained from proof comprehension tests were supported by the qualitative data via conducting interviews. Quantitative part of the study was one group pre-test post-test design and the qualitative part consisted of semi-structured interviews conducted after the post-test.

For the data collection, a statement and its proof about the decimal representation of real numbers was used (Usiskin, Peressini, Marchisotto & Stanley, 2003) The authors individually developed the test questions considering the proof comprehension dimensions (Mejia-Ramos et.al., 2012). Eventually, the test consisted of 13 questions. After checking validity and reliability issues, the test was provided to students prior to and after the instruction. The instruction included two teaching sessions which focused on the decimal representations of real numbers via quantitative reasoning. Then, semi-structured interviews, with 6 students were conducted to collect further data on how students reasoned about and justified their answers to the proof comprehension test questions.

For the analysis of the data for research question 2, Wilcoxon-Signed Rank test was used due to the low number of participants. For the analysis of the data for research question 1, the percentages of correct, incorrect and partially correct responses for each dimension and each question were obtained. This informed whom to interview. Particularly, if the student answered all the questions in the dimensions correctly, his/her performance was classified as correct answer. If they could not answer or incorrectly answer all the questions in the dimensions, their performance was classified as no/incorrect answer. If they gave correct answer to one of the questions in the dimension and no answer or incorrect answer to the other question, the performance in the dimension was classified as partially correct. This classification was maintained both for the pre-test and the post-test performances. Then, the transitions between the answers from the pre-test and the post-test were illustrated (See Figure 2). For instance, for proof comprehension dimension 1, if a student was classified as partially correct in the pre-test and correct in the post-test, for the analysis, the case was taken as from partially correct to correct answer for Dimension 1. The focus of analysis for the interviews was students' reasoning quantitatively on the answers to the post-proof comprehension test questions.

Results

Quantitative Results

Regarding research question 1, when the percentages for correct, partially correct and incorrect answers in terms of dimensions were examined (See Figure 1), there is a decrease in the percentage of incorrect answers from the pre-test to the post-test. Therefore, percentages of partially correct and correct answers increased for almost all the dimensions, except for D3, asking about the method of the proof. For that, the percentage of incorrect answers in the post-test was higher and the percentage of the partially correct answers in the post-test was lower than the pre-test. For the research question 2, the Wilcoxon Signed rank test showed that there was a significant difference between the pre-and the post-proof comprehension test results ($z=-3.731$ and $p<0.05$). Additionally, 18 students' post-proof test scores were higher than the pre- test scores. One of the students' post-test score were the same as the pre-test score. Also, there were no students with a lower score in the post-test than the pre-test.

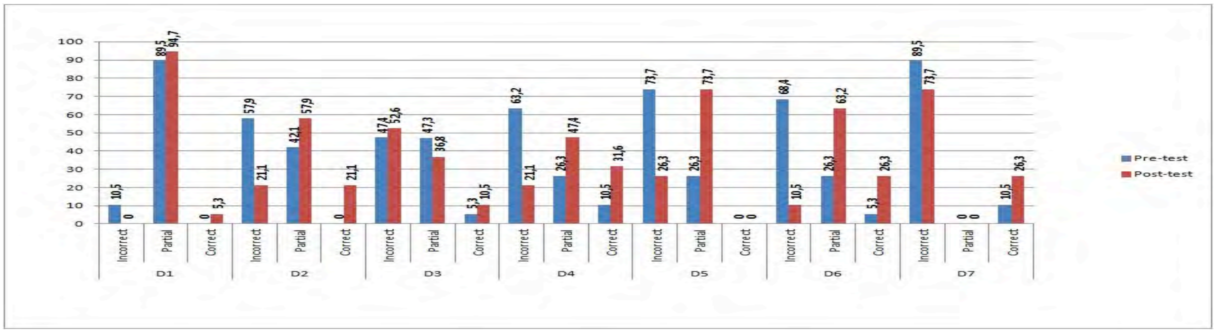

Figure 1. Response percentages for proof comprehension dimensions.

Qualitative Results

For research question 3, the following outline was used: As the data analyzed from 19 students, 5 different transitions from the pre-test to the post-test occurred (See Figure 2).

Interviewees	Pre-test D1	Pre-test D2	Pre-test D3	Pre-test D4	Pre-test D5	Pre-test D6	Pre-test D7	Post-test D1	Post-test D2	Post-test D3	Post-test D4	Post-test D5	Post-test D6	Post-test D7
S1	Partial	Partial	Correct	Partial	No/incorrect	Partial	Correct	Correct	Partial	Correct	Partial	Partial	Partial	Correct
S5	Partial	Partial	No/incorrect	No/incorrect	Partial	Partial	Correct	Partial	No/incorrect	Partial	Correct	Partial	Partial	Correct
S7	Partial	Partial	No/incorrect	No/incorrect	No/incorrect	Partial	No/incorrect	Partial	Correct	No/incorrect	Correct	Partial	Partial	Correct
S8	Partial	Partial	No/incorrect	Partial	No/incorrect	No/incorrect	No/incorrect	Partial	Correct	Partial	Partial	Partial	Partial	No/incorrect
S9	Partial	Partial	Partial	Partial	No/incorrect	No/incorrect	No/incorrect	Correct	Correct	Correct	Partial	Partial	Correct	Correct
S15	No/incorrec	Partial	No/incorrect	No/incorrect	Partial	No/incorrect	No/incorrect	Partial	Partial	No/incorrect	Correct	Partial	Partial	No/incorrect

Meanings of the colors		Transitions between pre-test and post-test		
(yellow)	There is no change	No/incorrect answer--> Partially correct answer	No/incorrect answer--> Correct answer	Partially correct answer --> No/incorrect answer
(green)	There is a positive change	Partially correct answer --> Correct answer	No change	
(pink)	There is a negative change			

Figure 2. Interviewees' profile.

For all of the five transitions, there was data showing that students used quantitative reasoning when needed. Yet, due to the page limitation, a sample data from Student 8 (S8) representing the transition from partially correct answer to correct answer in D2 was shared: S8 answered the Question 3(Q3) in both the pre and the post test correctly but he had no answer for Question 4(Q4) in the pre-test. Yet, in the post-test for Q4 (See Figure 3) he used both the diagrams and symbolic expressions to justify how he got the inequality asked in the question.

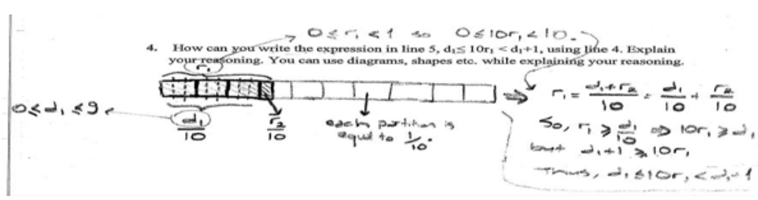

Figure 3. S8's answer for the Q4 in the post-test.

The data in the figure seems to suggest that he reasoned in the following way: S8 continued reasoning on the previous example he used in the test, 10/3, in question 1. He had divided 10 with 3 and came up with 3+ 1/3 using the division algorithm. Then, continuing with the remaining part, 1/3, he first partitioned a whole into 10 equal pieces. Then, he repartitioned each equal piece into 3 more equal pieces. Thinking 10 of those 1/30 th pieces, he obtained 10/30. This allowed him to determine the number of 1/10th s in 1/3 such that he grouped the pieces (each being equal to 1/30) by 3 and then counted how many of those 1/10 th there are in 1/3 rd. That is he measured 10/30 th with 1/10 th. In this way, he was able to find $d_1/10$ as being equal to 3/10 such that there were 3 times 1/10 th in

10/30 and $r_2/10$ was equal to 1/30. This way he was able to justify the inequality in the Q4. During the interview his reasoning supported his written work:

> S8: Now r_1 could be between zero and 1. I told you. For this reason, $10r_1$ is between 0 and 10. Now d_1 is $0 \leq d_1 \leq 9$ in this way. Now it is easier to define over the figure. We reach $r_1 = d_1/10 + r_2/10$. r_1 is greater than $d_1/10$. Why? we said here that from the piece of 10/30 that is 1/3 there are some remaining parts 1/30 portion, I mean. The non-terminating part I mean $r_2/10$. Here too. r_1 is greater than $d_1/10$. Therefore, $10r_1$ is bigger than d_1. d_1+1 is bigger than $10r_1$ because it is $r_1<(d_1+1)/10$.(He indicates the expression). That is d_1+1 means that. It exceeds this part (shows $r_2/10$ part). For this reason it (referring to 1/10) exceeds the portion 1/30.

As the data indicated S8 thought through an example using quantities and shifted to symbolic expressions to justify his answer through quantitative reasoning. Particularly, since r_1 was equal to the sum of $d_1/10$ and $r_2/10$, he was able to reason that r_1 had to be bigger than $d_1/10$. In this way, he reasoned that $10r_1$ was bigger than d_1. For the other side of the inequality, i.e. $10r_1$ is smaller than d_1+1, he again thought through the diagram such that he knew that the amount referring to $d_1/10+1/10$ exceeded the portion $r_2/10$.

Conclusion

The analysis of the results showed that once students reasoned quantitatively their comprehension of a proof on real numbers have developed: Particularly, not only there was a significant difference between the pre-and-post test results but also both in the post-test and during the interviews, for all dimensions students had a tendency to use diagrams and think through quantities while elaborating and justifying their answers. Specifically, descriptive results showed five different transitions from the pre-test to the post-test. Results showed that students had a development in proof comprehension dimensions, except for the D3, in which the name of the proof was asked. In fact, for all of the five transitions, data from the interviews showed that all students used quantitative reasoning while justifying their answers. These results align with Weber et.al. (2014) and Ellis' (2007) study arguing that quantitative reasoning is beneficial for proof and proving. Therefore, extending the earlier research results this study showed that students were able to make generalizations through quantities and even explain the abstract expressions in the proof comprehension test. Though, because of the number of students (N=19) participated in this study, we propose the effect of quantitative reasoning on proof comprehension regarding both the real numbers and other concepts to be investigated further.

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LOGICAL IMPLICATION AS THE OBJECT OF MATHEMATICAL INDUCTION

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Proof by mathematical induction poses persistent challenges for college mathematics students. We use an action-object framework to analyze ways that students might overcome these challenges. We conducted three pairs of interviews with students enrolled in a proofs course. Tasks were designed to elicit student understanding of logical implication and components of proof by induction. We report results from one student, Mike, who had constructed logical implication as an object, and who invented a quasi-inductive proof.

Keywords: Reasoning and Proof, Instructional Activities and Practices, Post-Secondary Education

Mathematical induction relies on two defining properties of the natural numbers: 1 is a natural number; and if k is a natural number, then k+1 is also a natural number. Any set with these two properties contains the natural numbers. In particular, if we define S as the set of natural numbers n for which an open proposition P(n) holds, then we can show S=N by showing the following: (1) P(0) is true, and (2) for any natural number k, if P(k) is true then P(k+1) is also true, written P(k)→P(k+1). In other words, P(n) holds for all natural numbers n if S satisfies the two defining properties of the natural numbers. The inductive implication P(k)→P(k+1) can be treated in one of two ways: as an inductive step from the inductive assumption, P(k); or, as an invariant relationship between P(k) and P(k+1) for any k. We apply an action-object framework to the study of logical implication and its use in proofs by induction.

Action-Object Theory

Piaget (1970) distinguished logico-mathematical knowledge from other forms of knowledge via its objects of study and, specifically, how they are created. Following Piaget, we define mathematical objects as coordinated mental actions. Our distinction between logical implication as a transformation and logical implication as an invariant relationship builds upon Piaget's action-object theory of mathematical development. Dubinsky made a similar distinction between actions and objects in APOS theory, which is also derived from Piaget's genetic epistemology (Dubinsky & McDonald, 2001). Within that framework, Dubinsky (1986) conjectured that understanding implication as an object could empower students in mastering proof by induction. Our study is an investigation of this claim within our own action-object framework.

Methods

The first author conducted clinical interviews with students from an Introduction to Proofs course taught by the second author. The course is a junior-level mathematics course designed to prepare mathematics majors for rigorous expectations in subsequent proofs-based courses. Three students volunteered for the study, and all three students were invited to participate in a pair of clinical interviews—one interview before mathematical induction was taught in class and one after. All of the interviews were video-recorded, and they lasted about 45 minutes. Each interview consisted of the students responding to tasks that were designed to elicit their reasoning and understanding. These tasks consisted of three types: logical implication, components of mathematical induction, and formal proof by induction (see Table 1).

Table 1: Sample Tasks

Task Type	Sample
A. Logical Implication	Suppose the statement is true: "If two topological spaces are homeomorphic, their homology groups are isomorphic." Evaluate whether the following statements are **true**, **false**, or **uncertain**. 1) [converse] 2) [contrapositive] 3) [negation]
B. Components of Mathematical Induction	Suppose P(n) is a statement about a positive integer n, and we want to prove that P(n) is true for all positive integers n. For each scenario, decide whether the given information is enough to prove P(n) *without induction*, *with induction*, or whether the given information is *not enough*. P(1) is true; there is an integer k≥1 such that P(k) implies P(k+1). P(1) is true; for all integers k≥1, P(k) implies P(k+1).
C. Formal Proof by Induction	Prove the following claim: For every positive integer n, $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$.

Results

We report results for a student named Mike from his first interview. We focus on Mike because he seemed to possess an understanding of logical implication as an object. Mike quickly recognized the three transformed statements in Task A1 as the converse, contrapositive, and negation of the original statement. For example, in evaluating Task A3, he responded, "if the first statement is true, then that has to be false." When asked to justify his response, Mike replied, "it's the negation of the implication." He seemed to recognize how the original statement had been transformed into the three statements he was evaluating, and he did not have to rely on any formal notation in order to do so. We characterize this recognition as an assimilation of the statements within a single structure for comparing and transforming them.

In contrast, Mike did not seem to readily assimilate the scenarios in Task B1. However, he could make critical distinctions between them, particularly with regard to quantifiers like "there exists" and "for all." For example, consider his comparison of Tasks B1 and B3. Note that "M" refers to Mike, and "R" refers to the researcher (first author).

M: [reads Task B1 aloud, then pauses] That's interesting. I don't *think* it's enough because that's only the truth of two, unless you are meant to assume to know that that means that you could like replace that k+1 with some other integer--you know, j--that was… and then j+1 is also true, and so on. But I think it's just not enough.
R: Okay, so what *would* you know from this one?
M: I would know that there's at least... P(1) is true; P(k) and then P(k+1) is true. And that's all I really know, from this.
R: So you would know P(1) is true, P(k) is true, and P(k+1) is true.
M: Yeah.
R: But that's not enough to prove…?
M: That's not enough to prove all positive integers.
R: So, you know P(k) is true, for which k's?
M: Just, there is a k. Just one k.
R: Is that the problem, then?
M: I think that's the problem.
R: [shows Task B2 on paper]
M: [looks at statement and immediately responds] Yeah, I know this is what, this is what it would take. P(1) is true and for all integers, k, greater than 1, P(k) implies k+1 is true. So,

that is the like sort of recursive thing that I was talking about, up there, where like… If you know that, yeah, so… P(1) is true and for all integers greater than 1, P(k) implies P(k+1), so that just grows to encompass all reals.

R: How does this recursive thing work?

M: Well, recursive is the wrong word. I mean, it just like… If you know P(k). Let's say k is equal to 2. So then P(2) implies P(3). But since it's for all integers k greater than 1, then P(3) also implies P(4), and so on.

Despite having never seen proof by induction, Mike was able to engage in the cognitively demanding task of analyzing its components in Task B. His objectification of logical implication seemed to free up his cognitive resources for focusing on *where,* not *how,* to apply the inductive implication. However, Mike's struggle with the increased demands of Task B were evidenced by his numerous pauses and minor mis-statements (e.g., "k greater than 1").

Upon reading Task B2, Mike immediately recognized what had been missing in Task B1. The new scenario allowed the implication to be applied to all values greater than or equal to 1. Mike began to recursively apply the logical implication, P(k)→P(k+1), in a manner consistent with what Harel (2002) called "quasi-induction." In both Tasks B1 and B2, Mike was eager to successively apply the inductive implication to consecutive pairs of integers. After sorting through the cognitive demands of the tasks, he was able to do so correctly.

Mike subsequently applied his quasi-inductive reasoning to Task C. Although the task did not explicitly call for induction, Mike independently attempted to prove the claim that way.

M: Well, I'm sort of thinking here that like, start with n equals 1 because you know, the simplest to add all of them up. So, you have 2 is equal to 2 to the second minus 2, just… And I was thinking, if you could write it in sort of like a symbolic way where you have like the next one where n equals 2, then you have 2 plus 2 squared is equal to 8 minus 2. And then I was thinking maybe you could plug like the 2 squared minus 2 in for this initial 2 and get it to like [moves right hand in circular motion], you know, build on itself.

R: Oh. Okay. Um, I think that's a good idea. Um, does that relate to any of [the Task B scenarios]...

M: Yeah. Yeah, it does. That's what sort of gave me the idea... is like for the k+1.

R: Okay. These gave you the idea for doing that? Which of these scenarios would it best fit?

M: Well, hopefully [Task B2].

In Task C, P(n) is the statement "$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$." Mike's approach was to use P(1) to build the equation in P(2) by substituting the right hand side (2^2-2) of the equation of P(1) for the initial 2 in the left hand expression in P(2). In this way, Mike used an inductive approach to show P(1)→P(2). Mike did not complete a formal proof by induction because he did not know how to write the inductive implication in a "symbolic way" that would generalize from any case, k, to the next case, k+1. However, Mike was conceivably on his way to generalizing his quasi-inductive argument into a formal proof by mathematical induction.

Conclusions

Piaget (1970) characterized logico-mathematical thought as grounded in composable and reversible mental actions. He described mathematical objects as coordinations of such actions. Dubinsky (1986) conjectured that treating logical implication as an object enables students to reason in more powerful ways, specifically with constructing proofs by mathematical induction. We investigated and affirmed Dubinsky's conjecture through our interviews with college mathematics students, like Mike.

Mike entered our study with an understanding of logical implication as an object. He coordinated actions on logical implications as objects to organize components of mathematical induction into inductive arguments. He used implication across particular pairs of cases (e.g. P(2) implies P(3)) in a manner that fits Harel's (2002) description of quasi-induction. Mike's struggles were limited to the following: (1) determining the cases in which the object applied; (2) symbolizing the inductive implication in a way that generalized to all valid cases.

Mike's first struggle relates to the role of (hidden) quantifiers and students' difficulties in differentiating between "there exist" and "for all" statements (Shipman, 2016). He recognized that Task B1 was existentially quantified, but struggled with the difference between an arbitrary variable and a fixed, unknown value. However, because Mike had logical implication as a mental object, he was able to resolve the details of the quantification. Mike's second struggle was apparent in his pre-interview in that he could not symbolically state the inductive implication for an arbitrary k. However, by the post-interview, he easily formalized a general inductive implication, possibly due to instruction on mathematical induction.

Mike did not seem to experience difficulty with other common challenges reported in prior research on mathematical induction. For example, he did not conflate the inductive assumption with assuming the proposition he was supposed to prove (cf. Avital & Libeskind, 1978; Ron & Dreyfus, 2004). We argue that students like Mike, who understand logical implication as an object, avoid this pitfall of conflation by treating the inductive assumption as a component of a larger object. For them, the inductive assumption is not an independent claim, rather it exists within the implication that must be established.

Piagetian theory offers a lens through which Dubinsky (1986) identified the objectification of logical implication as a potentially critical aspect of mastering proof by mathematical induction. Our results suggest ways that instruction can build upon such understanding. One such approach is the task sequence used in our study (Task A, Task B, and then Task C), which seemed to guide Mike to nearly invent mathematical induction. His independent formulation resembled Harel's (2002) quasi-induction, which Harel had recommended as an instructional approach. Instructional methods that separate the inductive hypothesis from the inductive step may inadvertently discourage students from engaging in quasi-inductive reasoning.

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PUSHING TOWARD THE PINNACLE: SUGGESTIONS FOR ASSESSING PROOF UNDERSTANDING

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Proof is considered the pinnacle of mathematics, but students commonly fail to understand proof. Research programs have sought to understand how students develop an understanding of proof and looked for ways to help improve students' understanding. Showing significant gains on measures of proof understanding is difficult. We suggest that the way proof assessments are scored is a cause. Namely, they focus on whether or not students can create acceptable proofs, and fail to capture students' informal means of reasoning. We describe three different proving practices we have observed that appear similar in their understanding of proof from the perspective of many rubrics, but when analyzed more closely show differences in understanding.

Keywords: Assessment and Evaluation, Reasoning and Proof

Proof has been called the pinnacle of mathematics, but most students fail to learn how to prove (see Harel & Sowder, 2007). Mathematics education researchers have posited several causes. No doubt a problem this widespread is likely the result of many factors. However, one issue we raise is that proof is often conceived of as a concept that one has either learned or not. Instead, we posit that proving is a complex process that even when understood well may not lead to success in all proving situations.

When proof understanding is assessed, students' proofs are often compared to teacher- or researcher- generated proofs. If we maintain the metaphor of proof being the pinnacle – the highest point – of mathematics, then it is clear there is an arduous journey from the bottom to the top. However, we are only measuring whether or not students have reached the mountain top and do not capture where they fall along this journey. We hope to provide insights into understanding students who are on their journey to the top.

Summary of Previous Research about Proof and Proof Understanding

In this section, we elaborate our meaning for proof and describe how various studies have attempted to assess proof understanding in order to highlight our critical issue. Namely, that these assessment rubrics focus on whether students have or have not developed deductive reasoning, and failed to parse the many diverse ways of reasoning informally.

Defining Proof and Proof Schemes

By proof we mean an argument that removes the doubts for a conjecture. In mathematics, proof is often reserved for specific types of arguments. To avoid confusion, we will clarify proofs by their types and allow others to decide if the arguments meet their definition for proof. Harel and Sowder (2007) described proof scheme as a way of thinking about proof. We have found their framework to be useful in categorizing the different types of thinking individuals use in proving situations. Harel and Sowder (1998) identified three kinds of proof schemes; external-conviction, empirical and deductive. Those who use the external-conviction proof scheme remove doubt by referring to an external source, such as a book, teacher, or form of an argument. Empirical proof schemes are marked by the use of examples, and deductive proof schemes meet three criteria: they prove for all cases, rely on valid forms of logical reasoning, and show operational thought (Harel & Sowder, 1998). Each proof scheme has many sub-schemes. Three are relevant to our discussion; the ritual proof scheme (external-conviction), the perceptual proof scheme (empirical), and the contextual proof scheme (deductive).

Assessing Understanding of Proof

The National Assessment of Educational Progress (NAEP) includes items that asked for a proof (National Assessment Governing Board, 2008). These items determine whether or not students understand deductive proof. For one released item students received full credit for including the full number of "statement-reason pairs" used in the assessor-generated proof, and varying degrees of partial credit for the number of statement-reason pairs they completed (NAEP Questions Tool, 2014). This rubric however would treat a student who reasoned empirically (demonstrating an understanding of the conjecture) and a student who left the item blank (demonstrating no understanding) as the same because both used zero statement-reason pairs.

More robust assessments, such as, Knuth and colleague's (2009) used a rubric broken into four categories. Level 0 included no attempt at proof, a proof based on an outside authority or based on an unsupported claim (external-conviction), Level 1 included empirical arguments, Level 2 included students who attempted to prove, but the arguments "fall short of being acceptable proofs" (p. 155), and Level 3 arguments were acceptable proofs – a term we adopt. Their rubric distinguishes students who no longer hold the empirical proof scheme, but still have not developed the deductive proof scheme. However, these students are all treated the same.

The most relevant evidence for our claim comes from Quinn (2009). As a result of breaking empirical arguments into two categories, she showed improvement in her students' ability to prove. Would similarly including more categories to Knuth and colleagues' Level 2 lead to a greater likelihood in showing interventions were effective? These studies all measured the differences between mathematicians' and students' views of proof. Weber and Mejia-Ramos (2015) suggested that one issue with these assessments is the ambiguity of the word "convince" because it has two interpretations, absolute and relative conviction. Absolute conviction occurs when someone "has a stable psychological feeling of indubitability about a claim" (p. 16). Students' inconsistent responses are a result of researchers' assumptions that participants have gained absolute conviction when they have only gained relative conviction..

Examples of Individuals Transitioning to Deductive Proof Schemes

We now highlight three different ways of reasoning we observed in our participants who would be considered as reasoning similarly by the many assessments for proof. Creager (2016) analyzed 13 pre-service secondary teachers (PSST) knowledge of geometric proof using a survey and two separate hour-long, cognitive interviews. Zeybek (2014) analyzed pre-service elementary teachers' (PSETs) conceptions of proof using a survey that consisted of open-ended proof items and cognitive interviews. Both studies asked participants to create, evaluate, and prove conjectures and evaluate researcher-generated proofs.

Proof Method

Ten out of 13 PSST in Creager's (2016) study appeared to have a procedural understanding of a method for proof in geometry. When asked to prove figures congruent, their method included the following steps; find two triangles that are congruent, prove those triangles congruent, then use triangle congruence to show the original figures congruent. These PSSTs used this method to create acceptable proofs, but there were five pieces of evidence that suggested they only had a procedural understanding of this method for proof.

First, this was the only method of proof that was used. For example, Mike said, "I don't know if I can [prove triangles similar]." Sadie said, "I don't remember how we prove lines parallel." Second, when they arrived at a point where they could not use their method, they either assumed an additional property was true (whether it was or was not) or claimed the conjecture was false. For the conjecture, “The median of a triangle creates two smaller triangles of equal area”, the PSSTs in this category could not use their proof method so, they said the conjecture was false. Third, they did not gain absolute conviction from their arguments. These PSSTs hedged their arguments by calling them “informal proofs” or “demonstrations”. Fourth, they focused on the form of the argument when asked to evaluate researcher-generated proofs. They preferred arguments that did not use examples and included “statements and reasons”. However, their meaning was not aligned with accepted uses. They accepted arguments that were based on examples that did not have any special characteristics (e.g. an acute triangle versus an isosceles triangle) and they felt that any reason was valid. Finally, in situations where proof would be a reasonable, but not explicit expectation, they relied on the empirical proof scheme. Mike for example, verified empirically that the diagonals of rectangles create two congruent triangles. When he was asked if he was absolutely certain, he agreed, but when asked how he would prove that he said, “Well, I guess I’d try to prove two triangles congruent.” Eight of the thirteen PSETs in Zeybek’s (2014) study held a similar procedural understanding of proof.

Interestingly though, the PSST and PSETs created several acceptable proofs. In a way, they are more like individuals with a ritual proof scheme. They seem to know what to do, but not why they would do it. However, they would be considered to have improved in their understanding of proof on many assessments. Our perspective however is that these students have learned to play their teachers’ game more effectively. Something we consider in the discussion.

Ritual Proof Scheme: The case of Alana and Elizabeth

Alana, a PSST, often reasoned similarly to those with a method for proof, but created arguments that were not “acceptable” proofs. To prove the diagonals of isosceles trapezoids are congruent, Alana marked the figure in a way that would suggest a proof, but claimed two different triangles were congruent. Thinking this was an error, she was asked for clarification. At this point Alana said, “Nope, nope, nope. Hold on here. No, I just know. I don’t know how to prove it but, I just know it’s diagonals are equal.” At that point she was okay with simply accepting the conjecture as true and moving on. Alana regularly displayed evidence of having the ritual proof scheme, but rejected empirical proofs regularly, despite the fact that she used them and found them convincing. Because of this, she would be considered as improving in her understanding of proof on many rubrics.

Zeybek (2014) similarly felt that PSET Elizabeth had a ritual proof scheme. She primarily created arguments to meet her perceived image of how proofs should look. When evaluating a false researcher-generated argument in the two-column format, Elizabeth claimed it was valid proof despite recognizing a flaw in the argument.

Contextual Proof Scheme: The Case of Lacy

Lacy, a PSST, worked on several conjectures about special segments of triangles (medians, altitudes, and angle bisectors). In every case, Lacy only considered the special segment that was drawn from the apex angle of a triangle drawn with a horizontal base. When Lacy used a Geometer’s Sketchpad to construct an isosceles triangle, the figure did not have a horizontal base, and she asked the researcher to fix the figure. Because this was a consistent behavior, it seems that Lacy’s conception of special segments of triangles was only those drawn from the apex angle. Because of this she “proved” several false conjectures.

Discussion

Lumping these types of student thinking into one category is problematic because it limits our ability to show gains in proof understanding. However, some rubrics seem too lenient. Are we to consider individuals like Alana or Elizabeth an improvement? Understanding the limitations of empirical reasoning could be a motivating factor in the search for more secure ways of justification. However, Alana consistently rejected empirical proofs, but gained absolute conviction by using the empirical proof scheme. Is it the case that she truly understands the limitations of empirical reasoning or is it that she has learned it is not part of the game called proof? Simply rejecting empirical arguments seems to be insufficient evidence to claim an improvement over those with the empirical proof scheme because it was Alana's dominant proof scheme. We have found success in differentiating between students like Alana and those who understand the limitations of empirical reasoning by asking participants to evaluate false proofs.

The differences in these students' thinking is probably best highlighted by the differences in actions one would take as their teacher. Lacy showed great skill at creating proofs, appreciated them, and even offered them spontaneously. It seems Lacy needs help in making sense of the segments of a triangle. We have found this type of thinking to be common. Proof might even be a vehicle to help these students develop concepts as it uniquely requires one to be explicit about their assumptions – Lacy's problem. Therefore, determining whether participants have knowledge of the concepts proofs rely on is a necessary part of assessing proof.

Is there an increase in sophistication in the reasoning of those with a proof method over people like Alana and Elizabeth? Consider for example a student who can set up a proportion, cross-multiply and solve. Few researchers would suggest this student has proportional reasoning. We can similarly argue that even though the students with the proof method were more successful than Alana and Elizabeth at creating proofs, the fact that they did not see those proofs as providing absolute conviction and only used proof when asked to do so suggests that these students have not developed a sufficient appreciation for proof. We hope our theoretical argument sparks a discussion about how to evaluate these students in terms of their understanding of proof.

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REASONING WITHIN QUANTITATIVE FRAMES OF REFERENCE AND GRAPHING: THE CASE OF LYDIA

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In this report, we draw from a teaching experiment to report a prospective secondary mathematics teacher's reasoning within frames of reference in quantitative contexts. We pay specific attention to her graphing of relationships between two quantities within non-canonical coordinate systems and the interplay of figurative or operative thought in her committing to directionality of measure.

Keywords: Modeling, Geometry and Geometrical and Spatial Thinking, Teacher Education-Preservice, Cognition

Quantitative reasoning (Thompson, 2011) is important in the development of numerous K-16 mathematical ideas such as function and rate of change (Ellis, 2007; Confrey & Smith, 1995; Thompson, 2011; Moore & Carlson, 2012). Coordinate systems are used to coordinate sets of quantities by establishing frames of reference and obtaining a representational product space in which quantities are coordinated. Researchers and educators often assume students have internalized coordinate systems and thus—whether intentional or not—provide little attention to how students reason within frames of reference implied (to the observer) by a coordinate system (Lee, 2016). In this report, we draw from a teaching experiment to report how Lydia, a prospective secondary mathematics teacher, reasoned within frames of reference in quantitative contexts and engaged in graphing relationships between two quantities within non-canonical coordinate systems. Specifically, we present shifts in Lydia's commitment to directionality of measure across three tasks and provide implications and future research directions.

Theoretical Framework

We refer to *quantity* as a conceptual entity an individual constructs as a measurable attribute of an object (Thompson, 2011). Joshua, Musgrave, Hatfield, and Thompson (2015) offered a theoretical model of mental actions involved in conceptualizing a measurable attribute within a *frame of reference*: committing to a unit, committing to a reference point, and committing to a directionality of measure. We draw from this framework the notion of reasoning within quantitative frames of reference. Among the three mental actions involved, we narrow in on Lydia's committing to directionality of measure comparison.

We also draw from Piaget's distinction of figurative and operative thought. Figurative thought is based in, constrained to, and dominated by perceptual elements and sensorimotor experience; operative thought foregrounds the coordination and re-presentation of mental actions and the transformation of those actions (Montangero & Maurice-Naville, 1997). In developing models of students' graphing activity, Moore (2016) provided examples in which students' graphing activities are dominated by either figurative or operative thought. Drawing on Moore's work, we use figurative and operative distinctions to describe Lydia's graphing activity and her commitment to directionality of measure. In this paper, and emphasizing the notion of re-presentation, we use the term *graphing* to include both constructing graphs and determining whether a pre-constructed graph represents the appropriate relationship between two quantities.

Methodology

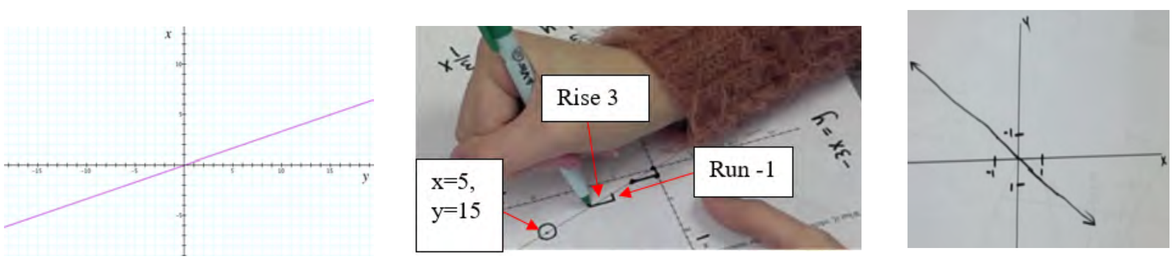
The data we present and analyze is from a semester-long teaching experiment (Steffe &

Thompson, 2000) at a large public university in the southeastern U.S. with three prospective secondary mathematics teachers. Our goal in the teaching experiment was to investigate how the participants conceived of situations quantitatively and represented quantitative relationships under particular coordinate system constraints. Among the three participants, we focus on Lydia, who participated in 12 videotaped sessions, each lasting approximately 1-2 hours. Through ongoing and retrospective analyses (Steffe & Thompson, 2000) we analyzed Lydia's activity graphing relationships between two quantities with specific attention to the interplay of figurative or operative thought in her committing to directionality of measure.

Analysis and Findings

Lydia's interplay of committing to directionality and figurative thought

In Task A, the teacher-researcher (TR) provided Lydia the following prompt: "You are working with a student who happens to be graphing $y = 3x$. He provides the following graph (Figure 1a). How might he be thinking about the situation?" As shown in Figure 1a, the student's graph was presented on a non-canonical coordinate system with the horizontal and vertical axes each representing y and x, respectively. Lydia rotated the paper clockwise 90-degrees such that the x-axis was horizontal from her perspective. She then concluded the "slope" of the line was "rising negative three." After picking two points on the line, Lydia drew a horizontal line segment left one unit and a vertical line segment up three units (Figure 1b). Lydia explained, "If I were to rise here…I'm rising this three…and then I'm running negative one, which would then [be] three over negative one x still equals negative three x." As such, Lydia related the amount of increase in the y-values to the amount of change in the x-values that she perceived of.



(a) (b) (c)

Figure 1. (a) Graph provided in Task A; (b) Lydia's actions on graph (a); (c) Lydia's $y = x$ graph.

Noticing that Lydia associated moving to the left with "running negative one," the TR asked Lydia the coordinates for the circled point on the line (Figure 1b). Lydia responded $x = 5$ and $y = 15$; she was aware the quadrant in which she was working entailed positive values for both x and y and thus held in mind a spatial orientation that entailed positive values compatible with that designed by the research team. Nonetheless, Lydia maintained her meaning for moving horizontally to the left with a decrease in the quantity's value. Her commitment was different from committing to the quantitative organization as we implied by the axis and the quantity's measures. From Task A, we inferred Lydia's committing to a directionality of measure to be figurative in that it was dominated by sensorimotor elements of graphical representations (e.g., moving to her left/right implying decrease/increase in values).

Rotating the Graph and Reconsidering Directionality

In Task B, Lydia revisited a task in which she had constructed the sine graph on a conventional Cartesian coordinate plane. We asked her to determine if that graph, when rotated (including axes) in different orientations, shows the appropriate relationship of how arc length and height change together. Lydia rotated the paper 90 degrees clockwise. Moving her finger from origin down along the vertically oriented arc length axis, Lydia said "And then here we're increasing in arc length." Moving her finger from origin to the right along the horizontally oriented height axis, Lydia explained "We're having a positive distance in arc length, having a positive distance in height" and concluded that the graph showed the appropriate relationship of how arc length and height change directionally together; Lydia attended to the direction of change in quantities along the axes of the coordinate system. This was different from her engagement in Task A in that her direction of change in magnitude was divorced from perceptual/physical orientations (e.g., going left/right implying increase/decrease in quantities). From this we infer that Lydia has shifted from committing to figurative directionality to committing to the quantitative organization as implied by the axes and quantities' measures.

We highlight, however, that Lydia continued to verify if graphs, when rotated, represented the same quantitative relationship in different orientations by purposefully and sequentially sweeping her finger/hand along the axes; when the TR asked her to predict for each case, Lydia was reluctant to conclude that the relationship was maintained without carrying out the physical activity of rotating the paper, moving her finger/hand along the axes, and considering how the displayed graph corresponded to increasing or decreasing values. As Lydia said, "If I'm being skeptical, I don't want to say it's going to get…guarantee that it's going to follow the pattern unless I rotate it and like can visualize…" We interpret this to mean that it was necessary for Lydia to carry out sensorimotor activity and instantiate each case using the perceptual material of the graph when the physical orientation of the axes changed. Lydia was yet to anticipate that each graph represented an invariant relationship no matter how rotated.

Lydia's Committing to Directionality Supported by Operative Thought

In Task C, after constructing the graph of $y = x$ on a non-canonical coordinate system in which the y coordinates decrease as one moves upward along a vertical axis (see Figure 1c), Lydia reasoned about the "slope" of the line and the relationship between the two quantities. In contrast to her activities in Task A, when determining the "slope" of the line, Lydia did not physically re-orient the graph such that the y-axis behaved in the conventional way, nor did she adhere to the visualization of "what we're used to a negative slope looking like." Instead, Lydia reasoned about the quantities and associated changes by maintaining an awareness of directionality in accordance with the axes as we designed them. For example, she explained, "So if x is 1, then y should be 1, which this looks like it has a negative slope, but it doesn't have a negative slope… Because if I rise one, then I'm like going down, but I'm rising a value, like a positive value." Further, Lydia explained, "even though [a graph] looks like a negative slope," for any linear function with positive slope "As an x increases, the y should increase" and vice versa. This stands in contrast with her activity in Task B, in which she engaged in sensorimotor activity to verify the invariance of the quantitative relationship for different orientations of the same graph. From this we infer that Lydia's directionality of measure has shifted from dominated by figurative thought to operative thought. That is, Lydia's directionality of measure and her claims of "slope" here were rooted in explicit attention to magnitudes organized in a directed, operative system and its transformations.

Discussion and Implications

Through our analysis, we presented how Lydia made sense of quantitative relationships represented on non-canonical coordinate systems. Specifically, we focused on Lydia's commitment to directionality of measure and its interplay with figurative and operative thought. Based on her activities, we found Lydia's notion of directionality to be critical in her reasoning within frames of reference and quantitative reasoning. Over the course of the teaching experiment, Lydia gradually attended to the direction of change in quantities along the axes of the coordinate system and established quantities in ways that sustained an awareness of directionality in accordance with the axes as were implied by the coordinate system.

We hypothesize that the questions from the TR to consider graphs in various orientations, her physical enactment of sweeping along axes, and her rotating graphs supported reorganizations in her notion of directionality. We also propose that the use of non-canonical coordinate systems afforded Lydia opportunities to operationalize directionality independent of the perceptual features of coordinate systems and particular sensorimotor activity. As suggested in Moore (2016), we emphasize the importance of opportunities that afford students engaging in and differentiating between figurative and operative frames of reference.

We envision that further investigations into students' conceptualization of frame of reference with attention to figurative and operative thought may provide insight into instructional support that afford powerful tools for reasoning about quantitative relationships independent of coordinate systems.

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REVIEWING STRATEGY TYPES, PARTICIPANT CHARACTERISTICS, AND TASK TYPES FOR EFFECTIVE MATHEMATICS PROBLEM SOLVING

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Mathematics problem solving has evolved significantly since Polya's and Schoenfeld's problem solving frameworks. This review examines research on strategy use during mathematics problem solving in terms of the strategies investigated, context in which problem solving occurs, characteristics of the participants, and efficacy of the strategy use on problem solving solutions. We are in the process of reviewing 164 studies that examined mathematics strategy use during problem solving. Preliminary findings indicate that domain-general strategies proliferate the current literature, contexts in which mathematics problem solving is investigated are diverse, studies mainly examine younger learners, and positive effects of strategy use on mathematics problem solving is conditional. We offer suggestions for future directions of research dealing with strategy use and mathematics problem solving.

Keywords: Problem Solving, Metacognition

The current study aims to review research related to mathematics problem solving in terms of: types of problem solving strategies investigated (e.g., pictorial, metacognitive); characteristics of the learners (e.g., grade level); methodologies of the study (e.g., naturalistic, controlled intervention); measures of strategy use; types of problem-solving tasks; and salient results of the strategy (ies) used and their impacts on problem solving. Using a systematic literature review process, this paper seeks to shed light on what contextual and individual characteristics lead to more effective problem solving. Additionally, this review will allow an examination of areas (e.g., pictorial mathematics strategies) that require further investigation.

Theoretical Framework

One influential framework for mathematics problem solving was Polya's (1957) *How to Solve It*. Polya proposed a general framework of mathematical problem solving which consisted of four steps: understanding the problem, devising a plan, carrying out the plan, and looking back. This framework showcased 67 heuristic strategies as general rules of discovering mathematical solutions which would guide their thinking and reasoning processes towards discovering of mathematically meaningful knowledge in mathematical problem solving.

Schoenfeld introduced his own framework (e.g., Schoenfeld, 1985; 1992) that retained many of the same elements of Polya's. Schoenfeld (1985) discussed two types of decision-making during problem solving: tactical decisions and strategic decisions. The former were similar in style to Polya's and included which algorithms and heuristics to implement, while strategic decisions referred to actions that guide, monitor, and control problem solving. The latter refers to metacognitive or self-regulatory processes. Schoenfeld (1985) indicated that the latter had received greater attention in the research literature. A study of students' cognitive processes in mathematical problem solving revealed that experts' use of monitoring and control processes to guide their use of heuristics lead to successful outcomes. In essence, while the use of heuristics played a significant role in mathematics problem solving, the importance of metacognitive processes seemed to dominate over that of heuristic thinking in mathematics problem solving.

Schoenfeld (1992) reframed mathematics knowledge as a science of patterns in place of the traditional view as a body of facts. To develop a broader framework of mathematical cognition for pattern-seeking, Schoenfeld defined mathematics as:

> …an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns – systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in system defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world object ("applied mathematics"). (p. 335)

In this review, we sought to examine if these assumptions of mathematics, and mathematics problem solving more specifically, were evident in empirical investigations of strategy use (i.e., procedures that are "purposeful, effortful, willful, essential, and facilitative"; Alexander, Graham, & Harris, 1998, p. 130). We divided strategy use into two general categories: *domain general strategies* (i.e., processes invoked during problem-solving activities useful in any domain; Alexander & Judy, 1988) and *domain specific strategies* (i.e., processes invoked during problem solving that are specific to the relevant domain; Alexander & Judy, 1988). Domain general strategies consisted of *metacognitive strategies* (Flavell, 1979), *self-regulatory strategies* (Zimmerman 1989) and *heuristic strategies* (Polya, 1957). Domain specific strategies are comprised of *pictorial strategies* (Stylianou & Silver, 2004) and *symbolic strategies* (Zazkis, Dubinsky, & Dautermann, 1996) to represent mathematical relationships or patterns using visual-spatial properties or symbols.

According to Schoenfeld (1992), mathematics also involves social activity and tasks that allow pattern finding to occur. Hence, we decided to also investigate the context in which problem solving occurred, either in a more *naturalistic context* (i.e., study takes place in a typical learning context such as a classroom or museum) or a more *controlled context* such as a laboratory environment. Additionally, if the study included instruction or intervention we examined if it was: *one-on-one instruction*, *group instruction*, or *peer instruction*. We also examined whether the tasks were *well-structured tasks* (i.e., have a defined computational path with one acceptable solution; Simon, 1977) or *ill-structured tasks* (i.e., not well structured and may contain symbols with multiple acceptable solutions; Simon, 1977).

Finally, we are sensitive to the idea that low achieving students or students with disabilities may influence how effective these strategies are. Hence we also examined characteristic of the participants of each study that included age and if the participant were typically developing or if there were special development considerations (e.g., cognitive delay). Thus, with these issues in mind, four questions guided this review:

1. What types of strategies (i.e., metacognitive, self-regulatory, heuristic, pictorial, or symbolic) are most prevalent in empirical investigations of mathematics strategy use?
2. What contexts (i.e., settings, instruction types, and task types) are most prevalent in empirical investigations of mathematics strategy use?
3. What learner characteristics (i.e., age and typicality of development) are most prevalent in empirical investigations of mathematics strategy use?
4. What conditions or joint conditions regarding type of strategy, context, and learner characteristics promote the most effective problem solving outcomes?

Method and Data Sources

To investigate these questions we undertook a systematic literature review. First, we identified appropriate studies for the study pool by searching for the terms *mathematics*, *problem solving*, and *strateg** in the abstracts of the PsycInfo database. Further, we narrowed the results to peer reviewed empirical studies and English text between 1985 (when Schoenfeld's framework went to press) and 2013. A total of 164 studies met these criteria.

After the initial identification, we developed codes for the type of strategy, measure of strategy use, nature of the sample, type of development, context of the study, and the problem solving task discussed previously in the introduction. Additionally, in our table we decided to include descriptions of the strategy, the problem-solving outcome measure, and salient results of the study demonstrating a relation between the strategy under investigation and student outcomes on the problems given. All authors jointly coded five studies, then each independently coded 10 studies and compared the results. Once we were satisfied that the coding and descriptions were similar, the first author then coded the remainder of the studies (one-third of them currently).

During the coding process, we found and excluded studies that did not examine learners' strategy use during problem solving. Most of these studies examined teaching strategies during mathematics.

Results

Our preliminary findings indicate that studies have primarily: studied strategies that are domain general; encompassed both laboratory and naturalistic settings with diverse instructional approaches; involved typically developing elementary children; and, suggested fairly constrained findings on the effectiveness of the strategy on problem solving based on study conditions.

With regard to types of strategies, domain general strategies were the focus of 73% of the studies. Far fewer studies examined domain specific strategies on open sentence problems. With regard to settings, emphasis in the literature was much more evenly distributed among the categories. Approximately 45% of the studies were in a laboratory or laboratory type situation, whereas the rest where classroom based environment (48%) or computer-based environments (7%). Instructional approaches (one-on-one, group, or peer instruction) were also diverse. However, the types of tasks that students were asked to do were primarily well-structured (76%), with only 20% ill-structured and 4% that did not involve a task. With regard to learner characteristics, studies primarily focused on younger children. Elementary-aged students accounted for 52% of the studies, with 10% middle school, 7% high school, and 28% undergraduate or adult learners. 83% of studies focused exclusively on typically developing students, 14% with students with a cognitive impairment, and only one study of English language learners.

With regard to salient findings of the studies, a few common themes were found among these studies. First, strategy instruction did not always result in students' use of those strategies. Second, metacognitive and self-regulatory training can have positive effects on problem solving success. Last, success of strategy use is often subjected to certain prior conditions (typically aspects of the problem) in order to be effective.

Discussion

While the analysis is ongoing, it is apparent that the earlier focus on domain-specific strategies (Schoenfeld, 1985) has shifted more toward domain-general strategies. Certainly, metacognitive, self-regulatory, and more heuristic strategies are now the lion's share of the research agenda. This may be due in part to Schoenfeld's urging (among others who have echoed this since; e.g., De Corte et al., 2000) and also a broader reflection of the increasing trend in research on metacognition and self-regulation (Dinsmore et al., 2008).

Lastly, while we did not set out to examine how strategies were defined in each study, it was interesting to note what these strategies were called and how they were related to other constructs. For example, while each studied had something labeled "strategy", there were seemingly synonymous terms concurrently used such as skills (Davis et al., 2009), types of thinking (Groth, 2005), ways of thinking (Webb et al., 2009), and mathematical thinking (Wood et al., 2006). While we certainly would not expect identical use of terms, it may be helpful to clarify the differences in these constructs moving forward. It is our hope that an illumination of these and other issues and trends in this review can enable more strategic and coordinated research efforts moving forward.

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UNDERGRADUATE STUDENTS' REASONING ABOUT MARGINAL CHANGE IN A PROFIT MAXIMIZATION CONTEXT: THE CASE OF CARLOS AND MARK

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The purpose of this exploratory study was to investigate how business calculus students reason about marginal change (marginal cost, marginal revenue, and marginal profit) when solving optimization problems in an economic context. To carry out this investigation, task-based interviews were conducted with 12 pairs of students and this study reports on one of the pairs of students (Carlos and Mark). Analysis of Carlos and Mark's reasoning about marginal change in a profit maximization task (PMT) revealed that these students were able to define or even recall facts about the idea of marginal change and its relation to the fundamental principle of economics (FPE), which states that maximum/minimum profit occurs at a production and sales level where marginal cost equals marginal revenue. However, these students conflated marginal cost and marginal revenue with total cost and total revenue respectively in another PMT.

Keywords: Post-Secondary Education, Instructional Activities and Practices

Much of the existing research on students' reasoning about optimization problems has focused on problems that lack a real-world context (e.g., Borgen & Manu, 2002; Brijlal & Ndlovu, 2013; Heid, 1988, Swanagan, 2012) or problems that have a physics or life science context (e.g., Klymchuk et al., 2010; Maharaj, 2013; White & Mitchelmore, 1996). Research on students' reasoning about optimization problems that are situated in economic contexts are scarce, which is the motivation for this study. More specifically, there is a dearth of research on what students' reasoning about optimization problems in an economic context reveals about their understanding of marginal change, an important idea in economics. Marginal cost refers to the cost per additional unit produced, marginal revenue refers to the revenue generated per additional unit sold, and marginal profit refers to the profit per additional unit produced and sold. Mathematically, marginal change can be calculated as an average rate of change where the length of the interval of change is one unit. Marginal change can be approximated using the instantaneous rate of change. The importance of investigating what students' reasoning about optimization problems that are situated in economic contexts reveals about their understanding of marginal change cannot be overemphasized. First, there is a large number of students (more than 300,000) who enroll in business calculus nationwide every year (Gordon, 2008), and yet we do not know how these students reason about marginal change, an important idea in any business calculus course. Second, understanding optimization problems and marginal change in an economic context is vital in several fields such as marketing, managerial accounting, supply chain management, finance, and economics. The investigation of students' reasoning about optimization problems in an economic context reported in this study was guided by the following research question: What does business calculus students' reasoning about economic-based optimization problems reveal about their understanding of marginal change?

Evidence from research (e.g., Ärlebäck, Doerr, & O'Neil, 2013; Flynn, Davidson, & Dotger, 2014; White & Mitchelmore, 1996) shows that engaging students in solving mathematical problems that are situated in real-world contexts helps to reveal students' conceptual understandings and difficulties/misunderstandings of certain mathematical concepts and ideas. In their investigation of undergraduate students' understanding of average rates of change in the context of a discharging capacitor, Ärlebäck et al. (2013) found that "a focus on the context made visible students' reasoning about rates of change, including difficulties related to the use of language when describing changes in the negative direction" (p. 314). Flynn et al. (2014) reported on sophomore students who conflated rate of change with accumulation in an engineering context. More specifically, these students confused the rate flow of water into a roof drain with the total amount of water accumulated. The current study reports on what students' reasoning about economic-based optimization problems reveals about their understanding of marginal change.

Theoretical Framework

This study draws on the theory of realistic mathematics education (RME) which is both a theory of teaching and learning in mathematics education. As a theory of learning, RME emphasizes that students should be asked to solve problem situations that are not only realistic in the sense of being connected to a real-world context, but also "problem situations which they can imagine" (van den Heuvel-Panhuizen, 2000, p. 4). The economic context as it relates to optimization problems is not only realistic, but also experientially real for some students taking business calculus. This is especially true for students who take business calculus after having taken high school or college economics classes. Providing students with opportunities to reason about realistic optimization problems in an economic context has the potential to reveal their understanding of the idea of marginal change as it relates to the FPE, something that was fostered by the design of the study reported in this paper.

Methods

This study followed a qualitative design. A task-based interview was conducted with a pair of undergraduate students (Carlos and Mark) who had successfully completed a business calculus course in the spring semester of 2015 at a research university in the Unites States. In conducting the interview, I followed the principles and techniques (e.g., encouraging free problem solving) suggested by Goldin (2000). There were four tasks in total (Mkhatshwa, 2016). This study reports on the students' reasoning on one of those tasks:

> The following table shows the marginal revenue (MR) and marginal cost (MC) at various production and sales levels (q) for SciTech, a company that specializes in producing and selling computer chips. The company knows that total revenue is greater than total cost at all the production and sales levels shown on the table.

q	400	401	402	403	404	405
MR	58	56	55	54	53	51
MC	52	54	55	57	60	62

> What advice can you give to the management of the company about when to increase or decrease production and sales of computer chips?

Mark and Carlos were high achieving students. They both were management majors, sophomores, and had grade point averages (GPAs) over 3.5 in a 4.0 scale. Mark earned an A grade in the business calculus course while Carlos earned a B+ grade. While both students had previously taken a college economics course, herein referred to as ECN 200, Carlos had also taken two economics classes in high school (AP microeconomics and AP macroeconomics). The task-based interview was video-recorded and lasted for about one hour and 45 minutes. The interview was transcribed for analysis. Work written by students during the interview was also collected as part of the data. Analysis of the interview transcript and students' written work focused on students' understanding of the FPE. The findings of this analysis are presented in the next section.

Results

There are two findings from this study. First, drawing from their economics background, Carlos and Mark demonstrated a good understanding of the relationship between marginal cost and marginal revenue at a profit maximizing quantity, that is, the FPE. More specifically, when asked about the advice they would give to the management of the company mentioned in the task, Carlos indicated that "because marginal cost equals marginal revenue and at that point [402 units], that's the most profitable you going to be". Mark added, "that's where the marginal cost equals the marginal revenue so it doesn't make sense to create more just like in the skittles experiment where you keep eating them but you are not getting any satisfaction from them, so there is no point taking them." Prior to working on the task presented in this study, both students had recalled examples related to the FPE that were discussed in an economics class (ECN 200) they had previously taken together. In particular, Carlos recalled that a company has to produce until marginal cost equals marginal revenue. He went on to give an example, "like the example, which I kind of like going back to, if you have free pizza at the Cafeteria you will produce or eat pizza until the point of having like one more slice of pizza is not so appetizing or producing one more unit costs more than it would if you sold it." Mark recalled another example related to eating skittles, "same like eating skittles. That's the example that our professor [ECN 200] gave us." He added, "he [ECN 200 professor] was like he brings back a packet so there is a point where I just don't want to eat skittles and that's where …marginal cost equal marginal revenue." Carlos and Mark concluded by advising the management of the company mentioned in the task to increase production and sales of computer chips up to 402 units and then decrease production and sales of computer chips afterwards.

Second, I remark that although these students, especially Carlos, correctly explained what marginal cost and marginal revenue means, their understanding of these ideas are not robust. When asked to comment about marginal cost and marginal revenue (from a graph) at a break-even point in another profit maximizing task (Mkhatshwa, 2016), Carlos stated that "so marginal revenue, going back to economics, I know is the benefit or the revenue taken from producing one additional unit and marginal cost is the cost from producing one additional unit." On probing Carlos and Mark further about their understanding of the idea of marginal change, it appeared that their understanding of these ideas was vague. More specifically, Carlos and Mark incorrectly claimed that "marginal cost equals marginal revenue" at a break-even point: - a production and sales level where total cost equals total revenue. However, marginal revenue exceeded marginal cost at the break-even point. Thus, Carlos and Mark conflated marginal cost with total cost and marginal revenue with total revenue.

Discussion and Conclusion

This study makes a major contribution in what we know about students' learning of business calculus at the undergraduate level. More specifically, findings of this study indicate that while students can memorize the definition of marginal change, recall the FPE (relationship between marginal change and profit at a profit-maximizing quantity), their understanding of marginal change is not robust in that the students conflated rate with accumulation when they conflated marginal cost with total cost (and marginal revenue with total revenue). As noted earlier, undergraduate students' tendency to conflate rate with accumulation was reported by Flynn et al. (2014) in an engineering context. Given that the students reasoned correctly about marginal change from a numerical table but not so in a graphical task, it might be important for business calculus instructors (applied calculus in general) to provide sufficient opportunities for students to reason about marginal change in multiple function representations (algebraically, textually, graphically, and numerically) and in different real-world contexts. Future research might examine the opportunity to learn about marginal change via course lectures and business calculus textbooks.

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CHALLENGES IN MODELING WORD PROBLEMS

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In 2008, after the National Mathematics Advisory Panel of the U.S. Department of Education acknowledged the necessity of teaching early algebra, the focus in mathematics education shifted toward teaching students how to create models as tools for solving word-problems. Some educators concentrate on teaching students how to present word problems using symbols (Dougherty & Slovin, 2004), others teach *schemata* approach (Jitendra et al., 2015), which is not always mathematically strict. There is no agreement in approaches, which are necessary to increase students' ability to model word problems. We undertook research to analyze what challenges children have when creating simple models and what could be done to help them overcome those challenges.

Twenty elementary and middle school students attending a suburban learning center participated in this study. We taught participants schemata for creating (a) verbal model—the description of each value involved in word problems; (b) spatial-visual representations; (c) number equation model; (d) algebraic model, including defining symbols. We analyzed students' assignments and individual verbal responses. The mistakes of the students were thoroughly analyzed following an open and axial coding method.

Students demonstrated misconceptions when defining measurable attributes. Young students frequently mixed-up units with measurable attributes and had difficulties with verbal models for the problems involving difference. Many erroneously wrote, *3 is the difference between girls and boys*. Children demonstrated difficulties with interpreting symbols used in equations. They frequently identified symbols as equivalent to objects, units, or words, which was consistent with other studies (MacGregor & Stacey, 1997). Students benefited from a 4-model approach. This approach pinpointed weak areas for each student, helped create models for simple problems, and prepared students for modeling complex word problems.

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CONSTRUCTING MATHEMATICAL HABITS OF MIND WITH LATTICE LAND

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Wilensky and Papert proposed a theory of *restructurations*, where different computational representations can create new ways of thinking (2010). In this paper, we propose a *restructuration* of geometry that replaces the Cartesian plane with integer points on a square lattice. We created a constructionist environment called Lattice Land, a dynamic geometry system (DGS), and designed complementary activities for geometry classrooms (Sally & Sally, 2011). Now that new computational tools are reshaping the way scientists and mathematicians understand the world, we need to train students to learn and think in increasingly computational ways (Orton et al, 2016).

Cuoco and Goldenberg showed DGS have been used to shift the focus from low-level details to high-level thinking, to develop *mathematical habits of mind* (1996), but these DGS did not change the fundamental *structuration* of Euclidean geometry. Lattice Land curriculum tackles many of the same proof-based habits of mind we seek to develop, but with two key differences. (1) Using discrete geometry, Lattice Land taps into our earliest experiences with counting, making it a natural entryway into continuous geometry (Piaget, 1960). Additionally, discrete thinking resonates with computational thinking. (2) The curriculum uses empirically based inductive approaches to reasoning, which Schoenfeld has shown to strengthen deductive approaches to geometry, as well as helps to break students' tendencies to compartmentalize knowledge (1986).

Pei implemented and video recorded Lattice Land curriculum in 4 geometry classes at a large non-selective urban public high school: 81% Hispanic, 9% African American, 7% White, 2% Asian, 2% other, 94% low-income. Students in the study relied on a variety of techniques that demonstrated a range of mathematical habits of mind. They *tinkered* with the software and *experimented* to find multiple methods of dissection; built *definitions* from findings and their intuitions; *visualized* problems; *generated and collected data* about some subset of polygons; and used *inductive reasoning* to build sensible and testable formulas from *patterns*. Based on our analysis of 2 groups of 4 students, and pre/post-interviews, we show how the restructuration using an integer focus allowed students to build higher-order abstract thinking without requiring a well-developed sense of the density of real numbers. All students showed content gains. Based on the students' initial reticence and the teacher's surprise at their high levels of engagement and success, these habits of mind do not typically manifest themselves in everyday classroom practices. Because the teacher's role is critical—celebrating experimentation and mistake-making, steering conversation, summarizing—we plan to continue our study with classroom teachers facilitating.

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EXPLORING STUDENT PERSPECTIVES ON THE TRANSITION TO PROOF IN COLLEGIATE MATHEMATICS

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For many students, experience with proof prior to college is limited (Jones, 2000). Students' prior competence (and interest) in mathematics is typically centered on producing accurate answers to easily recognized tasks. These abilities do not support, and may interfere with, students' work to conceptualize, write, or evaluate proofs. This study complements prior work that has focused on students' reasoning on specific problem-based tasks—typically proof construction and evaluation (Selden & Selden, 2003). We explored how students (n=14) describe their experiences in one "transition to proof" course at a mid-western university. In targeting students' experience, we have focused on what is different for students in elementary proof work and how students reorganize their learning to meet the challenge. These issues are important foci for all efforts to assist students in understanding the new challenges at this crossroads in their mathematical experience as well as supporting students in addressing them. The present study builds on Smith & Star (2007) in its focus on the following dimensions of students' experience of the introduction to proof course: (1) The nature of the course and how it differed from prior courses, (2) the reasoning involved in proving, and (3) the learning activities that support success. It also drew on that study's use of student-constructed graphs to assess changes in students' confidence.

Interviews were transcribed and coded using thematic codes based our interview protocol and the dimensions of Smith & Star (2007). We found that students were quite articulate about how the course differed from their prior mathematics course experiences, but their focus and emphasis varied. There were three main characterizations of difference: (1) Work in the transition to proof course explained why mathematics worked the way it did, where prior work had focused mainly on producing answers, (2) the course valued process over answers, and (3) the course changed students' conceptions about proof and writing in mathematics, noting that mathematical terms have more specific meanings than in everyday language. All but two students explained that they employed different practices in completing their homework than they had in prior courses (e.g., working with classmates, asking questions in mathematics class, using the university's mathematics learning center). The methodological technique of asking students to construct confidence graphs was fruitful, allowing us to uncover several contrasting profiles of students' affective experience over the course of the semester.

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EFFECTS OF A MATHEMATICAL WRITING TREATMENT ON CHILDREN'S CONCEPTIONS OF EQUIVALENCE

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Research suggests that students' engagement in writing in mathematics has a positive effect on mathematics achievement (e.g., Cross, 2009). Recently, evidence has emerged suggesting a relationship between elementary children's mathematical argumentative writing and their early algebraic reasoning (Kosko & Singh, 2016). To explore the phenomenon further, this study examines whether engagement in writing and early algebra tasks (i.e., equivalence) affects second grade students' mathematics achievement.

Data were collected from 46 second grade students across three teachers' classrooms. Students were assigned to one of three classroom conditions: writing and problem-solving (WPS), problem-solving (PS), and a control (C). The WPS and PS classrooms engaged in weekly tasks focusing on number relations (e.g., examining equations of the form $a+b=c+d$), with accompanying manipulatives (e.g., Cuisenaire rods). Students in WPS and PS conditions engaged in small group and whole class discussions, but students in WPS were also tasked with describing and justifying their mathematics in writing journals. Students in all conditions completed Rittle-Johnson et al.'s (2011) assessment on conception of equivalence at three time points across six months. Additional data collected includes classroom observations and teacher interviews across classroom conditions and over time. Data was examined using Helsel and Frans's (2006) Regional Kendall Trend Test (RKT). Applied to classroom comparisons, the RKT produces a tau coefficient representing the proportion of 'up moves' to 'down moves in time, adjusted by students within a condition. Results indicate statistically significant growth for students in WPS (τ=.66, p<.001) and PS (τ=.34, p=.02) classrooms, but not for students in MO classrooms (τ=-.04, p=1.00). Additionally, the WPS growth slope was statistically significant from both PS (Z=1.71, p=.09) and MO (Z=4.12, p<.001) classrooms, and the PS growth slope was statistically significant from MO (Z=2.41, p=.02).

These results indicate that students in the WPS classroom demonstrated larger gains in their conception of equivalence scores than students in both PS and MO classrooms. Further, while students across all conditions demonstrated statistically significant growth between the first and second assessment points, only students in WPS demonstrated growth between the second and third time points. This suggests that engaging in argumentative writing may have helped students to maintain and expand upon mathematical relationships developed early in the school year.

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INSTRUMENTAL VS. CONCEPTUAL UNDERSTANDING IN CALCULUS

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Keywords: Cognition, Advanced Mathematical Thinking, Post-Secondary Education

Calculus is a gatekeeper for many aspiring STEM (science, technology, engineering, and mathematics) students. My overarching goal is to help students who are blocked from STEM fields due to limited ability in mathematics to find a pathway to achieve their goals. This exploratory study is the first step in understanding what knowledge of calculus students are constructing. Working from the assumption that students construct knowledge to make sense of their experiences (von Glasersfeld, 1992), I am exploring what kinds of procedural and conceptual knowledge (Hiebert & Lefevre, 1986) these students are able to construct. Many researchers and educators have been concerned that K-12 education is too heavily focused on rote, procedural teaching, learning, and assessment (Hiebert & Lefever, White & Mitchelmore, 1996).

My conjecture, in line with some of the past research (Engelbrecht, Bergsten, & Kagesten, 2009; White & Mitchelmmore, 1996) is that through traditional instruction, many students construct only procedural understanding at the expense of a conceptual understanding and that is problematic for students' continued participation in the STEM fields. My study is differentiated from the other studies in that I am looking specifically at non-mathematics majors, students who may have struggled with mathematics in the past and whose trajectory into the STEM fields may be blocked by their ineffective construction of calculus concepts. My research question is "What kind of procedural and conceptual knowledge do students in a college level Elements of Calculus class construct of the concept of the derivative."

Research Methodology

This study will employ a task-based clinical interview with undergraduate, non-math majors at a state university in New England. In describing the clinical interview, Ginsburg (1981) claims that there are "three basic aims: the discovery of cognitive processes; the identification of cognitive processes; and the evaluation of competence" (p. 10). It is appropriate then, to use a clinical interview to discover the cognitive process students use to make sense of calculus concepts.

Findings

In this study the vast majority of the data (about 70%) provided evidence of only procedural knowledge. Another finding worth discussion is the students' lack of precision, in particular, precision in verbal communication of mathematical concepts.

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INTERPLAY OF REPRESENTATION, BELIEFS, AND PROBLEM SOLVING PERFORMANCE

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Keywords: Problem Solving; Affect, Emotion, Beliefs, and Attitudes

Problem solving is regarded as a crucial aspect of mathematics learning (NCTM, 2000; Schoenfeld, 1985). Past research has clearly highlighted the paramount impact of affective factors on problem solving performance (Buchanan, 1987). Despite the close link between representations and mathematical cognition, less is known about the relationship among the use of mathematical representations and mathematical problem solving performance (Krawec, 2014). This study examined an individual's problem solving practices to explore the following research question: How do mathematical resources from which the individual draws when confronted with tasks impact their problem solving performance?

A college student not involved in STEM area served as the subject for the study. Since a major goal of the study was to trace the impact of the use of representations when solving problems, data collection consisted of two phases. First, the type of representations he accessed during the sequence of 4 different tasks from discrete mathematics, measurement, and geometry were documented through 4 problem solving interviews. During the second phase, to investigate the potential impact of additional representation on his work, a second set of 4 interviews took place during which the same tasks were used. During these second sequence of interviews, the researcher asked the participant to use a certain tool which included GeoGebra software, a calculator, a measuring cup, or a physical paper cylinder. A total of 8 sessions were observed and used as sources for data analysis to classify different modes of representation presented in past publications such as Applied Mathematical Problem Solving (Lesh, Post, & Behr, 1987). These modes are contextual, physical, visual, verbal, and symbolic representations. Interviews were transcribed and analyzed using open coding of 157 responses into 18 categories.

It was found that the individual's use of representations and overall performance were strongly influenced by his self-concept as a mathematics learner, persistence towards solving problems, beliefs about mathematics, the desire to be correct, and reliance on his preferred representations and heuristics. The individual's preference for verbal as well as symbolic representations using paper and pencil persisted although different tools, which could potentially assist in solving the problems, were offered. In most cases, the participant's performance was positively related with his use of verbal and symbolic modes of representations.

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STUDENTS' CRITERIA WHEN EVALUATING SOLUTIONS FOR A PROOF TASK

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Despite the importance of proofs in mathematics, research has shown students are more likely to construct empirical arguments than deductive proofs when asked to prove a statement (e.g., Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009). One possible explanation is that the empirical arguments reflect what students are mathematically capable of producing, not what they believe constitutes a proof (Healy & Hoyles, 2000). However, both of the studies mentioned used written assessments as their primary data source, which did not allow for follow up questions to ascertain students' reasoning for their answers. To fill this gap, this study sought to uncover what criteria Algebra 1 students use when they evaluate given solutions for a proof task.Semi-structured interviews were conducted with 10 ninth graders enrolled in an accelerated Algebra 1 class. At the time of the study, no students had received formal instruction on mathematical proofs. In the interview, students were asked to prove that if you add any three odd numbers together, your answer will be odd. After students constructed a written argument, they evaluated five sample solutions for the previous task: a correct algebraic proof, an examples-based argument, a generic example/visual proof, a paragraph proof that drew on the definition of even and odd numbers, and a circular algebraic "argument". Students were asked to explain their evaluations for each solution and then select the solution they preferred the most.

Out of the five sample solutions, students overwhelmingly evaluated the paragraph proof and (correct) algebraic solution as being proofs (8/10 for each) and were least likely to consider the examples-based argument as a proof (2/10). However, of the eight students who thought the algebraic solution was a proof, half justified their response by saying the solution provided examples of why it worked. This criterion suggests that some students did not attend to the generality inherent in the algebraic argument when evaluating the solution. When justifying their responses, students consistently talked about whether the given solution "made sense". This was particularly true when explaining why they thought the paragraph argument was a proof. Even though none of the students had been formally introduced to proofs, 7/10 students preferred one of the deductive arguments (algebraic, visual, or paragraph) the most.

These findings extend those from prior studies in two ways. First, not all students who select algebraic proofs do so because they recognize the generality within the argument. Second, some students are able to recognize non-traditional deductive arguments as proofs when evaluating solutions based on whether they "make sense". One way that teachers can increase students' access to proofs is through leveraging their potential intuition to prefer mathematical arguments that make sense. This can be achieved by placing more instructional emphasis on making sense of the mathematical content in proofs than on writing proofs in the "proper" form.

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STUDENTS' UNDERSTANDING OF PERIODICITY

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The New York State mathematics curriculum has not included extensive mention of periodicity, with it appearing only in the context of trigonometric functions (New York State Education Department, 2005; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Concepts of periodicity extend beyond trigonometry into other areas in mathematics and science. I designed a qualitative study, utilizing a questionnaire and interviews, to gather data on secondary students' broader understanding of periodicity. Specifically, I studied the characteristics students attend to when identifying representations as periodic and determining their periods, and how they reason about periodicity in a context.

Shama (1998) and Gerson (2010) studied student understanding of periodicity. The authors' studies demonstrate that secondary student conceptions of periodicity were situated in trigonometry, which led to limited conceptions of periodicity, such as periodic functions needing to be waves or symmetric. Shama concluded that students' conceptions of a period of a periodic phenomenon were restricted to the fundamental period. In Gerson's study, the precalculus student's "decisions about periodicity [were] not based on the formal definition, but instead upon properties he has generalized from prototypical functions in his concept image" (p. 33). Gerson further concluded that the student had an understanding of periodicity that was compartmentalized within graphs, meaning that the student "was able to make conjectures about periodicity only within the graphical representation" (p. 35). The findings from my study confirm and extend the findings of both of the above studies.

Three teachers in a small suburban high school administered my questionnaire in class to their students who agreed to participate – five classes of algebra 2/trigonometry students (n = 95) and one class of AP calculus students (n = 14). Two girls from the algebra 2/trigonometry classes and two girls from the AP calculus class volunteered to participate in the task-based interviews.

My analysis of the questionnaire data revealed that the majority of students had a basic understanding of periodicity as repetitive patterns. There were instances of misconceptions particularly for tabular representations and for discontinuous graphs. Overall, the calculus students had more robust understandings than the algebra 2/trigonometry students. All students expressed understandings of periods as exclusively fundamental periods; they did not demonstrate a conception of periodic phenomena as having multiple periods. My analysis of the interviews revealed that calculus students engaged in periodic reasoning in a context, but the algebra 2/trigonometry students did not. For all students, the task allowed use of contextual vocabulary to discuss periodic concepts, which made the concepts more accessible.

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THE EFFECT OF ITEM MODIFICATION ON STUDENTS' STRATEGIES FOR NEGOTIATING LINGUISTIC CHALLENGES IN MATHEMATICS WORD PROBLEMS

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Some researchers have recommended item modification as a means for accommodating English language learners in mathematics tests prepared in English (e.g., Kieffer, Lesaux, Rivera, & Francis, 2009). Item modification as a measure is based on the notion that complexity in the natural language is construct irrelevant (Messick, 1989). In other words, linguistic challenges resulting from the use of a natural language, such as English, are not part of the concepts a mathematics test is meant to measure. Indeed, "the fundamental notion of test validity is that low test scores should not occur because of factors that are irrelevant to the construct an instrument intends to measure" (Solano-Flores &Trumbull, 2003, p. 3).

Research has shown differentiated performance between students taking modified test items and those taking the original items (e.g., Abedi & Lord, 2001). In these research studies, students taking modified test items have performed better, and have expressed preference of the modified test items over their original counterparts. While many studies have investigated differences in performance between students taking modified and original mathematics test items, there seems to be no research investigating how the strategies students use to negotiate linguistic challenges may vary between students taking modified test items and those taking original ones. Knowing the differences and similarities in how students negotiate linguistic challenges while taking modified and original test items can help inform researchers about: (a) the effectiveness of item modification, (b) any additional linguistic challenges other than those emanating from the natural language, and, (c) how students' problem solving may be supported.

This poster will use data collected among eight Kenyan high school (ages 17 or 18) students who were assigned modified and original Kenya Certificate of Secondary Education (KCSE) mathematics test items. KCSE mathematics is a Kenyan high-stakes examination prepared in English, a foreign language, and taken by students completing high school. The original aim of this study was to investigate the linguistic challenges students face while solving selected KCSE word problems, and ways in which they negotiate these challenges. The findings of this study were presented at the 38th PME-NA conference. For this poster, we will compare the strategies students used to negotiate the linguistic challenges across the modified and original test items. This poster will thus present the salient differences and similarities between strategies students used while negotiating linguistic challenges in modified and original test items, as well as the implications for future research and practice.

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Chapter 8

Preservice Teacher Education

Research Reports

Brief Research Reports

Posters

AN EXPLORATION OF HOW ASPECTS OF A NOTICING INTERVENTION SUPPORTED PROSPECTIVE MATHEMATICS TEACHER NOTICING

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Numerous studies, including our own, have documented that teacher noticing interventions can be effective in developing teachers' abilities to notice salient aspects of the mathematics classroom. In this study, we explore how specific aspects of one such intervention may have supported three prospective teachers in learning to notice high-potential instances of student mathematical thinking. The findings provide evidence that it was not one particular aspect of the intervention that was effective in supporting their noticing, but a combination of factors that include the use of a noticing framework, interactions with their peers and a facilitator, and targeted learning-to-notice activities.

Keywords: Classroom Discourse, Instructional Activities and Practices, Teacher Education-Preservice

The ability of a teacher to attend to and make sense of important events or aspects of the classroom – *teacher noticing* – is recognized as an important component of teaching expertise (Sherin, Jacobs, & Philipp, 2011). However, "the noticing required in teaching is specialized, it is not a natural extension of being observant in everyday life" (Ball, 2011, p. xx) and thus is a skill that must be learned. Fortunately, research has shown that noticing interventions, in a variety of forms, can be successful in helping teachers notice salient aspects of the classroom. For instance, interventions have been found to help prospective mathematics teachers become more focused on students' mathematical thinking (e.g., Mitchell & Marin, 2015), more discriminating about what is important to attend to in a classroom (e.g., Sherin & van Es, 2005), and better able to make connections between teacher actions and student learning (e.g., Roth McDuffie et al., 2014). In general, research suggests that teachers can become more attentive to whatever aspect of instruction is the focus of an intervention.

While it is clear that targeted interventions can be effective in scaffolding noticing, it is less clear why particular interventions work. Researchers have hypothesized a range of explanations, including the use of specific frameworks or targeted prompts (e.g., Roth McDuffie et al., 2014), discourse among participants (e.g., Mitchell & Marin, 2015), and multiple opportunities to engage in noticing activities (e.g., Santagata, Zannoni, & Stigler, 2007). Facilitation has been found to play a critical role in video analysis, a common feature of noticing interventions, since the facilitator must support teachers to "not only see what is worthwhile but how to dissect the details of the interactions represented in this video…to draw informed interpretations of teaching and learning" (e.g., van Es, Tunney, Goldsmith, & Seago, 2014, p. 352).

Our own work with prospective teachers (PTs) has documented that our noticing intervention has helped PTs become more focused on individual students' thinking, better able to articulate the specific mathematics underlying that thinking, and more capable of identifying instances of student thinking that have significant potential to be used to support students' learning (Stockero, Rupnow, & Pascoe, 2017). Like others, we have hypothesized aspects of the intervention that supported PT learning: using a framework, interacting with peers and a facilitator, and requiring a response template to structure PTs' reflections. We also suspect that some of the learning took place as a result of many opportunities to engage in noticing activities over time. The purpose of this study is to begin to explore how specific aspects of a noticing intervention may have supported the changes in noticing we have documented, and is thus at the crossroads of past teacher noticing research that focused on whether interventions can work and future research that is necessary to understand why such interventions work. To do so, we examine the cases of three PTs who formed one cohort that engaged in the noticing intervention. Specifically, this exploratory study focuses on the question: How do particular features of a noticing intervention support PTs' ability to notice high-potential instances of student mathematical thinking?

Theoretical Framework

Although teachers need to attend to a variety of classroom features while enacting a lesson, we focus our work on the noticing of students' mathematical thinking. This choice is grounded in our goal of helping teachers learn to enact *ambitious teaching* (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010), teaching that is intentionally responsive to students' current thinking as a means of helping all students develop a deep mathematical understanding. We adopt Jacobs, Lamb and Philipp's (2010) definition of *professional noticing of [students'] mathematical thinking* to include the skills of attending, interpreting, and deciding how to respond. In this study, we focus specifically on the first two components. We hold the perspective that not all instances of student thinking should be given equal attention, however, since they do not all have the same potential to enhance student learning. We focus specifically on noticing instances of student thinking that have significant potential to be used during a lesson to support mathematical learning. We use the MOST Analytic Framework (Leatham, Peterson, Stockero, & Van Zoest, 2015) as a tool to identify such instances—those that occur at the intersection of student mathematical thinking, significant mathematics, and pedagogical opportunity. In the framework, each of these three characteristics has two criteria. Student mathematical thinking requires inferable *student mathematics* and an associated *mathematical point*; significant mathematics requires that the mathematical point is *appropriate* and *central* to student learning goals; and pedagogical opportunity requires that the instance of student thinking creates an *opening* to build on student thinking and that the *timing* is right to take advantage of the opening at the moment it occurs (for more detail about the framework, see Leatham et al., 2015). We prioritize the noticing of MOSTs because they are instances that have significant potential to advance students' mathematical understanding if built upon by a teacher – that is if made "the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36).

Methodology

This study is part of a larger research project focused on supporting PTs' ability to notice MOSTs that surface during a classroom lesson. In this study, we focus on the last of five iterations of the intervention; this iteration was selected because it was found to be the most successful in supporting PT noticing (Stockero et al., 2017).

Intervention

The intervention took place during a one-semester early field experience course at a Midwestern US university. The participants were three PTs who comprised the fall 2014 cohort. Each PT completed weekly observations in a local, secondary school mathematics classroom, with the PTs taking turns recording a lesson in their classroom each week. The common full-length classroom video was analyzed individually by the PTs and by the research team each week. The research team used the PTs' and their own analyses to strategically select video instances to discuss at a weekly group meeting among the members of the cohort, facilitated by the first author (see Stockero et al., 2017, for more detail about the instance selection processes). The participants analyzed 9 different videos and attended 11 weekly meetings.

At the start, all of the PTs' video analyses focused on identifying "mathematically important moments that the teacher should notice"; they tagged such instances on a video timeline and annotated each instance with their reason for its selection. The PTs were introduced to the MOST Analytic Framework (Leatham et al., 2015) at the end of the Week 2 meeting to define important mathematical instances and provide focus for their video analysis. In Week 3, the PTs used the framework to re-analyze the two videos they had analyzed in the prior weeks. Subsequently, they used it to analyze each new video for MOSTs. In the Week 9 of the intervention, the PTs completed an activity focused on identifying in a set of statements those that represented a *mathematical point* that a given instance of student thinking could be used to work towards and rewriting those that did not. When analyzing the last two videos (Weeks 10 and 11), the PTs were provided a template to structure their video annotation by requiring them to address all six MOST criteria in their reasoning about instances.

Data Collection and Analysis

The data for the study include the PTs' individual video analysis timelines, the research team's weekly meeting plans, and video of each group meeting. In all of the data analysis, the data were first coded individually by two or three members of the research team. The team then met to compare coding and discuss coding differences until consensus was reached.

The unit of analysis for the PTs' video timelines was each instance marked by a PT, including their annotation. These instances were analyzed in two ways. First, each was coded according to three characteristics: *agent* (who was noticed), *specificity* (the level of detail with which the mathematics was discussed), and, for instances where the agent included student(s), *focus* (what about the student(s) was noticed) (adapted from van Es & Sherin, 2008; for more detail see Stockero et al., 2017). This coding was used to analyze changes in specific characteristics of the PTs' individual noticing in relation to our noticing goals: individual student agent, specific mathematics, and a focus on noting or analyzing student mathematics. Second, each instance was coded according to whether it aligned in the video with a MOST identified by the research team and whether the PTs' reasoning was consistent with what made it a MOST, an indication of whether the PTs were noticing high-leverage instances of student thinking.

The meeting videos were also analyzed in two ways. The first analysis focused on identifying instances of what we call *analytic discussion*. Here the unit of analysis was a segment of the meeting discussion that focused on a single topic; for example, making sense of the student mathematics in an instance. Informed by Lohwasser's (2013) concept of *accountable talk* in teacher professional learning communities, we focused on identifying segments of dialogue that were likely to advance the PTs' learning—those that went beyond sharing their thinking or agreeing with one another. Instead, analytic discussion included making sense of ideas, critiquing the thinking of others, and providing alternative perspectives. In short, it is discussion that has the potential for "developing and creating usable…knowledge for teaching" (Lohwasser, 2013, p. 141-142). In the second analysis, each individual facilitator move was coded according to its purpose, using a coding framework that was informed by the facilitation moves described by van Es et al. (2014). In the analysis reported here, we focus specifically on *probing* and *challenging* moves—the moves that were most likely to directly influence PT learning. In a probing move, the facilitator pushes for more detail or specificity about a PTs' thinking. In a challenging move, the facilitator may point out a discrepancy in reasoning, or push the PTs to consider an alternative explanation or point of view, critique another PTs' explanation, or make a firm decision about the value of an instance.

The data analysis involved analyzing the coding to compare changes in the PT's individual noticing to key features of the intervention as documented in the meeting notes and group meetings. This analysis focused on determining whether changes in the PTs' noticing could be explained by particular aspects of the intervention.

Results

In the following, we first briefly describe the cases of the three participants – Claire, Aaron, and Ruth – in terms of the overall trajectory of changes in their noticing. We then use these cases as background for considering the extent to which particular aspects of the intervention appeared to support the PTs' noticing.

Cases of Learning to Notice

Claire's baseline noticing data showed that she was focused on the important mathematical ideas in the lesson, but in isolation from the students. For example, her annotation of one instance said, "Opposites added together always equal zero." None of her instances in the baseline data were coded as being consistent with MOSTs. After the introduction of the MOST framework, Claire quickly changed her focus and was able to maintain a productive focus throughout the remainder of the intervention. Beginning from the first week she used the framework, Claire consistently focused on the students' mathematics, discussed this mathematics in a specific and often analytical way, and noticed instances that satisfied at least some of the MOST criteria. Claire is a case of a PT who was quickly able to make sense of the framework she was provided and use it to identify high-potential instances of student thinking.

Aaron's initial noticing was focused on the teacher and non-mathematical issues, such as "getting everyone involved". It was thus inconsistent with MOSTs. After the introduction of the MOST framework, Aaron began to focus more on students, but moved back and forth between a teacher and a student focus for several weeks. Aaron was similar to Claire in that he began displaying analytical behavior early on. His noticing generally focused on instances that satisfied some of the MOST criteria beginning in Week 5, and his noticing became entirely aligned with MOSTs by the last two weeks. Aaron is a case of a student who took some time to make sense of the framework, but at the end of the intervention was displaying productive noticing skills.

Ruth's baseline noticing had a mixed focus on the students as a collective and on the teacher. Over half of her noticing was non-mathematical in nature and all of it was inconsistent with MOSTs. The introduction of the framework allowed Ruth to shift her focus to the students in the video and thus allowed her noticing to become more consistent with MOSTs. She also began to discuss some of the mathematics in a specific way, although much of such discussion was still at a very general level. She was slower than Claire in developing the ability to identify MOSTs; it was not until Week 8 that the majority of Ruth's noticing consistently focused on MOSTs. Ruth had the most difficulty with the interpreting aspect of noticing, as she only had instances coded as analyzing student mathematics in the last two weeks of the intervention, and even then only a single instance in each video reached this level. Ruth is a case of slow growth over time and of a student who may have benefitted from a longer intervention.

Supports for Noticing

Noticing framework. We first considered how the use of a framework supported the PTs' noticing. To understand its immediate impact, we analyzed the PTs' noticing in Week 3, when the MOST framework was used to reanalyze the videos from the first two weeks, and Week 4 when the framework was used to analyze a new classroom video.

The data suggests that the MOST framework immediately and effectively supported Claire's noticing. During her first use of the framework, her noticing shifted from teacher to students, and to instances that were MOSTs. Impressively, Claire's annotations were coded as analyzing student mathematics in over two-thirds of the instances she identified in Weeks 3 and 4, and nine of her ten identified instances were MOSTs. To give a sense of the type of noticing Claire engaged in during her early use of the MOST framework, consider her annotation of an instance that occurred when students were being introduced to Pythagorean Triples:

Student: ‘Times 13^2’. One student thought that $5^2 + 12^2$ should be multiplied by 13^2 to find out the hypotenuse length. [T]his concept is not especially difficult, that it should [be] equal to 13^2, but when this is just being introduced, it might be difficult for a student to understand how to know if a 5, 12, 13 triangle is a Pythagorean triple. At this point it is important to understand that they just need to plug the values into the equation $a^2 + b^2 = c^2$.

In this instance, Claire not only noticed an important error made by a student, she also hypothesized why the concept might be difficult (Pythagoreans triples have just been introduced) and explained what mathematical idea she would want the student to understand.

Aaron and Ruth, on the other hand, seemed to take more time and need more support to make effective use of the framework. Although about half of their noticing was focused on MOSTs in Weeks 3 and 4, each displayed key inhibitors to their noticing. Aaron’s was his continued focus on the teacher, despite the fact that the first MOST criteria is student mathematics (e.g., “The question is how to compute the hypotenuse given two legs. The goal is to be able to use the Pythagorean Theorem to do this. [The teacher] explains the central goal in detail so the students will understand this concept”). For Ruth, it was her vague explanations that lacked evidence that she was engaging in analysis and interpretation of the student’s thinking (e.g., “The students are all getting the problem wrong, and you can tell what they are thinking mathematically by their misspeaking or wrong answers.”). Thus, although the framework provided some focus to Aaron and Ruth’s noticing, it was not sufficient to focus them on noticing and interpreting MOSTs.

Group discussions. The data indicated that, on average, the PTs engaged in 14 episodes of analytical discussion in each weekly meeting, with a range from 8 to 19 episodes. The PTs’ participation in these discussions was found to be evenly distributed, so they all had equal opportunity to engage in discussions that were likely to promote their growth in noticing.

The Week 1 and 2 analytical discussions focused largely on distinguishing between teachers’ noticing and their use of prior knowledge to make instructional decisions, as well as on making sense of what it means for something to be mathematically important (versus important for some other reason). Inferring the student mathematics (SM) in the video was a primary focus of analytical discussions nearly every week, as was articulating a mathematical point (MP) that the student mathematics could be used to work towards, after this concept was introduced with the MOST framework. Other topics that were the object of analytical discussion were definitions of specific MOST criteria and considering these criteria within specific contexts (such as what it means for an idea to be central to student learning if it is not the focus of the current lesson, or what the SM is if two competing student solutions have been shared). During the second half of the intervention, proposing building moves was also a significant topic of discussion, although not the focus of our current analysis.

The SM and MP criteria are the most mathematical of the MOST criteria and are typically the components that require the deepest level of analysis to identify whether an instance is a MOST. They appear to be the components that were most challenging to Ruth and Aaron and thus affected the advancement of their noticing skill. There is evidence that a sustained focus on these topics during the meetings was effective, however, since Ruth and Aaron both continually increased in their focus on noting and analyzing the student mathematics and their ability to identify MOSTs. Aaron did so more quickly than Ruth as his noticing was coded as either noting or analyzing the student mathematics in nearly all instances beginning in Week 5, and by Week 6 he noticed mostly MOSTs. Ruth gradually increased in the percentage of instances coded as noting student mathematics and that were MOSTs, but only showed evidence of analyzing student mathematics in the last two weeks of the intervention. To give a sense of how the analytical discussion may have supported the PTs in learning to infer the SM, consider an excerpt from the Week 5 meeting in which the PTs were discussing a video instance in which a student said that the coordinates of the x- and y-intercepts of a graph "both have a zero":

Facilitator: What do you think the student is saying there?
Claire: I think she is noticing that the x-intercept is when y is zero and the y-intercept is when x is zero. That's not what she says. She says they both have a zero, but I think that's what she's getting at.
Aaron: She just didn't find the right way to explain it, a way that everyone else would understand what she meant by saying it, which is why [the teacher] later went into explaining when looking for the x-intercept, it's when y=0, when it crosses the x-axis. She goes over a few ways to explain it. Actually explaining it in a way that everyone else would understand makes sense. It makes sense for her to interject there.
Facilitator: I heard Claire say that she didn't really say that. Is there enough there to infer the student math?
Ruth: I feel like she wasn't entirely clear on what the correct answer was. She just made the observation that each coordinate has a zero. That's what I thought.
Facilitator: So that's all you're willing to infer, then? That each one has a zero in it.
Ruth: Yeah, she was just making that observation and the teacher then elaborated on it.

In this excerpt, we see Claire and Aaron make an inference about the SM that went beyond what should be reasonably inferred based on the student's brief statement. Ruth contradicted their inference, however, providing the opportunity for them to reconsider their assertion about the SM. In fact, later in the discussion Claire noted that, "We've kind of thought that she made an observation that might or might not be correct, so [the teacher] needs to elaborate or figure out what she really means before it can be opened up to the class to discuss," indicating that Ruth's comments had caused Claire to reconsider her original inference.

The meeting data showed that Ruth struggled to articulate MPs related to the students' ideas through much of the intervention. For example, a Week 7 meeting discussion focused on an instance in which a student had suggested that to convert the fraction 1/10 to a percent, you could just use the denominator, so it would be 10%. When the facilitator asked, "Is there a mathematical point associated with that? In other words, what would you want him to understand?", Ruth replied "That's not how you do it,"—a response that typified her articulation of MPs and lacked the level of interpretation that was our goal. Despite participating in numerous discussions where the other PTs had articulated MPs, Ruth struggled to do so. This was a primary reason that she was unable to reach what we considered the most advanced level of noticing, where she was able to identify and also make sense of students' mathematics.

Facilitation. Related to the *analytic discussion* findings, an examination of the role of the facilitator suggests that the meeting facilitation was also important to the PTs' learning. In the ten meetings that focused on analyzing video instances (all except Week 9), 84% (119/155) of all analytic discussions were supported by either *probing* (79) or *challenging* (6) facilitator moves, or by both (34). This was relatively consistent across meetings, with between 69% and 92% of analytic discussion supported by such moves. Additionally, 77% (40/52) of all *challenging* facilitator moves during the intervention meetings coincided with analytic discussion, indicating that such moves were effective in causing the PTs to grapple with ideas. Together, these results suggest that the meeting facilitation played a key role in supporting changes in the PTs' noticing during the intervention; in particular, the facilitator's moves appeared to support discussion among the PTs that was likely to advance their learning.

Targeted activity and template. Prompted largely by the observation that Ruth was having difficulty advancing her noticing due to her inability to articulate MPs—an important part of the identification of MOSTs—in Week 9 we engaged the PTs in an activity where they worked on articulating MPs associated with instances of student mathematical thinking that they were provided. Following this activity, we also provided a template that prompted the PTs to address all six of the MOST criteria in their instance annotation beginning with the Week 10 analysis. Because these activities occurred simultaneously, it is difficult to separate their impact on Ruth's noticing. There is evidence, however, that together they had a positive effect on her noticing.

In Week 8, a typical annotation by Ruth addressed the SM and the MP as follows: "The student math is that she discovered what the pattern is for getting the inverse [of a matrix]. Her point was closely related to what they're learning because it was the teacher's next part of the lesson." Note that this response alludes to the SM and to a MP, but does not precisely articulate either. In Week 10, however, her response addressed the same two criteria as follows:

> Student Math: She said the absolute value of -5 was 5 because it's the opposite of 5.
> Mathematical Point: The absolute value of a number is always the distance that number is away from 0. This is because absolute value is a measure of distance and distance is always positive.

Here, the level of detail and precision are both much improved from the prior example. Thus, it seems that the Week 9 activities and template did advance Ruth's noticing. Even at the end of the intervention, however, she did not display the same level of analysis of student mathematics as her peers, with only a total of two coded instances reaching this level. In this case, a longer intervention may have allowed Ruth to continue to develop her noticing skill.

Conclusions

This exploratory study—at the crossroads of past teacher noticing research that focused on whether teacher noticing interventions could work to future research focused on understanding why they work—examined how various aspects of a noticing intervention that have been hypothesized to support teacher noticing appeared to help three PTs learn to recognize and interpret MOSTs in classroom video. As hypothesized, a provided framework did in fact support changes in their noticing, although the changes for two of the PTs were neither immediate nor drastic; in other words, the framework did not serve as an 'answer key' and was not by itself sufficient to support noticing. Rather, the PTs' noticing appeared to develop over time though participation in regular group discussions that allowed them to grapple with components of the framework. Moreover, the evidence suggests that facilitation that probed or challenged the PTs' thinking was important in supporting their engagement in analytic discussions that pushed their thinking and thus promoted learning. In the case of one PT, a direct intervention to support her in becoming more analytical of the mathematics underlying the student thinking was necessary to improve her noticing, but even then there was room for improvement. The results of this study highlight the significant effort required to develop skills in a practice as complex as noticing student mathematical thinking. Consistent with research on teacher professional development (e.g., Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010), this study suggests that brief or minimally supported interventions are unlikely to fully develop such a practice.

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DIRECTING FOCUS AND ENABLING INQUIRY WITH REPRESENTATIONS OF PRACTICE: WRITTEN CASES, STORYBOARDS, AND TEACHER EDUCATION

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We discuss affordances and liabilities of using a storyboard to depict a written case of a teacher's dilemma that involves race, opportunity to learn, and student community. We rely on reflections by the teacher educator who authored the written case and later depicted it as a storyboard to use it with his preservice teachers (PSTs). The analysis involved, first, organizing the signifiers in each of the two representations of practice into what we call concentric spheres of stratification, and secondly, contrasting the various meanings attributed to signifiers by both the author and his PSTs. We suggest that the resources of storyboard allow for more inquiry and alternative narratives than is available from the single modality of text in the written case.

Keywords: Equity and Diversity, Teacher Education-Preservice, Technology

Introduction

Motivated by the increasing use of multimodal representations of practice (e.g., video, animations, storyboards) in mathematics teacher education, we discuss affordances and constraints of such media and propose a framework for analyzing them. This contribution helps address the challenge of anticipating what meaning a group of preservice teachers (PSTs) will attribute to the complex system of signifiers included in a representation. The framework addresses this issue by offering a way of organizing those signifiers and suggesting that teacher educators could consider where and how the meanings that they and their PSTs attribute to a given representation might differ. To demonstrate the value of the framework, we use it to analyze a teacher educator's use of a written case and a storyboard representation of the same classroom scenario. The scenario was used to discuss a teaching dilemma that involves race, opportunity to learn, and student community – the decision to move a student to a more advanced mathematics class. The data comes in the form of the reflections by the teacher educator (and co-author) who initially wrote the case and then created a storyboard for use with his PSTs. We examine issues of representation illustrated by that data and that concern the use of representations of practice to support practice-based and inquiry-oriented teacher education.

Perspectives and Theoretical Framework

The notion of *representation of practice* has been a key in developing a practice-based approach to professional education (Grossman, et al., 2009). Representations of practice using the written narrative modality have been common in professional development (e.g., Stein, Smith, & Silver, 1999). Written cases are useful because they can help focus attention on important aspects of practice. But inasmuch as written cases use abstract symbols (words!) to represent individuals,

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settings, and actions of practice, they are less effective in immersing their audience into key aspects of classrooms such as the simultaneity and temporality of classroom events (Doyle, 1986; see also Herbst et al., 2011). As video technology became more accessible and approaches to teacher education as inquiry more common, teacher educators have been using video records to immerse novices in practice (Brophy, 2004; Lampert & Ball, 1998). Much has been written describing the affordances of having teachers watch and discuss classroom video to promote noticing and reflection (Rich & Hannafin, 2009; Sherin, Philipp, & Jacob, 2011). The capacity of video to record simultaneous multimodal communication (gesture, inscription, voice, movement, etc.) by a diversity of individuals has been noted as advantageous for creating an increased sense of presence (e.g., in comparison with text; see Kim & Sundar, 2016). That capacity also provides key affordances to allow for expanded inquiry pursuing a variety of foci, though the camera always directs attention in some way (Hall, 2000). This is not always what teacher educators need or prefer, as quite often their students latch onto aspects of a video not particularly germane to the goals of their instructors (Star & Strickland, 2008).

To bridge the gap between the capacity of video to immerse and the capacity of text to focus, professional educators have started to explore other media (e.g., animations and storyboards with cartoon characters) to represent practice (see Herbst et al., 2011; Tettegah, 2005). Art critics and scholars of visual communication have for long used language as a metaphor in examining visuals (Barthes, 1972). But more recent progress extending the systemic functional linguistics approach to language (Halliday & Matthiessen, 2004) to a variety of sign systems (Kress & van Leeuwen; 1996) and levels of realization (including the register of classrooms; Christie, 2002) has brought us closer to actually being able to examine the affordances of the comics medium as a language using similar approaches and resources as how SFL examines the uses of language. This paper contributes to an examination of how the multimodal resources in the comics and animation medium permit the production of complex messages about classrooms and the way in which they also enable a degree of openness (Weiss, 2011) that allows for inquiry and alternative narratives. Some of these features of comics will be exemplified through a comparison of a comics-based representation and a written representation of a classroom story.

Mode of Inquiry

We collaboratively examined the interaction of a teacher educator (Lawrence Clark) with successive versions of a technology for depicting classroom interaction in an effort to translate a written case to a storyboard. This examination allows our field to learn about affordances of storyboard technology for the representation of teaching practice. *Depict* (a tool included in Lesson*Sketch*; www.lessonsketch.org), allows users to upload and manipulate graphics and provides cartoon characters designed to create classroom visual meanings. Just as language has the words *student* and *teacher* to represent roles in a classroom, and other linguistic resources to describe how people feel (e.g., the student was *happy*), where they are (e.g., the student was *at the board*), or what they are doing (e.g., the student was *solving an equation*), *Depict*'s graphic language has resources to represent those meanings (e.g., see Figure 1).

Notably, if words report simultaneous happenings, the graphic medium allow us to show this simultaneity. Video can also do that, but it might also include other messages about the action that might be less relevant to convey (e.g., the style of clothes popular at the time the video was captured). In designing a semiotic system for representing classrooms, the developers of *Depict* wanted to make available graphic elements and software features that help communicate classroom meanings of particular relevance for the study of teaching practice. One important set of considerations has been the dimension Herbst et al. (2011) called *individuality*, or the extent to which the set of graphic resources enables distinguishing individual differences among classroom participants and settings. In this regard, it is worth distinguishing between *enacted* individual differences (e.g., the possibility that one student would do or express something that others would not) and *enduring* individual differences (e.g., based on individual characteristics that recur across enactments, such as body size, race, class, or gender). While *Depict*'s character set (Figure 1) was originally conceptualized as a cast of characters with some resources to express enacted individual differences (e.g., facial expressions, body orientation), very few resources were originally provided for enduring individual differences (e.g., no affordances for body size, hairstyle, or skin color). *Depict*'s characters were nondescript characters whose role was to depict practice rather than individuals. In that context, Chazan and Herbst (2011) had described the affordances of the cartoon characters as comparable to variables in algebra and distinct from diagrams in geometry: As Laborde (2005) has noted, diagrams have the liability that they spatiographically assert properties that are not always theoretically necessary—e.g., a diagram of a rectangle will likely show two sides longer than the other two sides. Other properties of rectangles, however, are not only visible but also necessarily true (e.g., opposite sides are congruent). Students often latch onto spatiographical properties as they use diagrams to learn geometry. While video is like diagrams in geometry in that it enables the observation of ancillary events, Chazan and Herbst (2011) argued that *Depict*'s cartoon language is like the generic language of algebra in its capacity to make assertions about practice, as opposed to assertions about specific individuals. Yet they also thought of the graphic resources as a developing language that would progressively incorporate new semiotic systems to represent more aspects of practice. The extent to which considerations of race enter in the way teachers relate to students in practice offered an important opportunity to further explore the possibilities of phasing in new graphic resources to increase the representation of individuality.

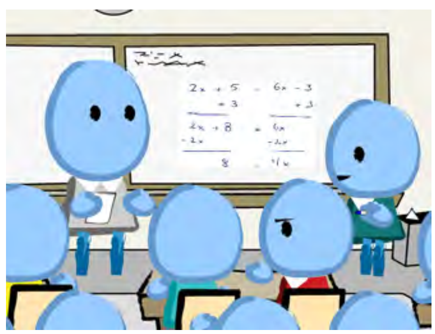

Figure 1. A frame from a storyboard using Depict.

New resources have been recently added to *Depict*'s graphic language to allow representation of some enduring individual differences. The *complexion* system, operationalized by color wheels that enable the user to pick skin tones for characters, affords the user the ability to choose whether to use the default blue skin or to choose freely from the color wheels. We wondered whether this particular affordance supported the work of a teacher educator in representing a case that he wanted to bring to his PSTs, and the extent to which the depicted representation allows for alternative inquiries and narrative. We examine the use of *Depict* by comparing Clark's Case of Mya (see Chazan, et al., 2016, p. 1059) and a storyboard Clark constructed to represent the case. The comparison is of interest because the storyboard demanded more graphic resources than were prescribed in the text (e.g., things unsaid in the text needed to be depicted to visualize the classroom scene) although it was created after the written case; generating the possibility that alternative stories could emerge from the engagement of the readers with the media.

The Case of Mya

The Case of Mya describes a dilemma faced by a middle school mathematics teacher, Scott Johnston, in his effort to provide a more rigorous and challenging mathematical learning environment for one of his students, Mya. Johnston was employed in a middle school and district where 8th graders were assigned to one of four mathematics courses (from lowest to highest level of rigor): Math 8, Pre-Algebra, Algebra, and Advanced Algebra. A potential byproduct of grouping students by performance is grouping students by race and social class: Racial and class gaps in performance have persisted throughout the history of mathematics education in the U.S. The Case of Mya acknowledges and incorporates these complexities. Scott Johnston commits his efforts to providing Mya, an African American female eighth grade student enrolled in one of his lower level mathematics classes, a more rigorous school mathematics experience. Based on Mya's social and intellectual performance in the course she is currently enrolled in, Scott takes on the work of enrolling her in a higher-level course.

Clark was also recruited as a fellow for a project that supported the creation of multimedia representations of practice for use in teacher education. He took on the challenge to represent the Case of Mya as a storyboard. He describes his challenge thus:

> When faced with moving the written Case of Mya to a storyboard, numerous considerations came into play. The first consideration revolved around how I might illustrate the complexities of the case context. I grappled with questions like
>
> - How can I depict the larger forces at play (context) that shape and direct a mathematics teacher's classroom decision-making and instructional practice?
> - As race and class are 'in the mix' of students' access to mathematics opportunity, how can the storyboarding tools serve to illustrate these phenomena?
>
> For example, in the written case, I had stated:
>
> > [S]he (Mya) was unquestionably the most engaged, inquisitive, and mathematically confident student in the entire [Pre-Algebra] class, and she consistently outperformed her classmates on tasks and assessments. She thrived in her position as one who I could call on to assist struggling students. Her ability to communicate her mathematical thinking and problem solving approaches in front of the entire class was unmatched. Mya fared less well in other academic areas, but it was evident to everyone, including herself, that she was comfortable and in her element when interacting in the [Pre-Algebra] class. (Chazan, Herbst, & Clark, 2016, p. 1059)
>
> I had not gone into the specifics of her mathematical thinking in the written case. But I did go into specifics of her mathematical thinking when afforded the opportunity to create the storyboard. I designed the following scenes in the depiction and asked readers to explore and comment about Mya's mathematical thinking in the lower level class:

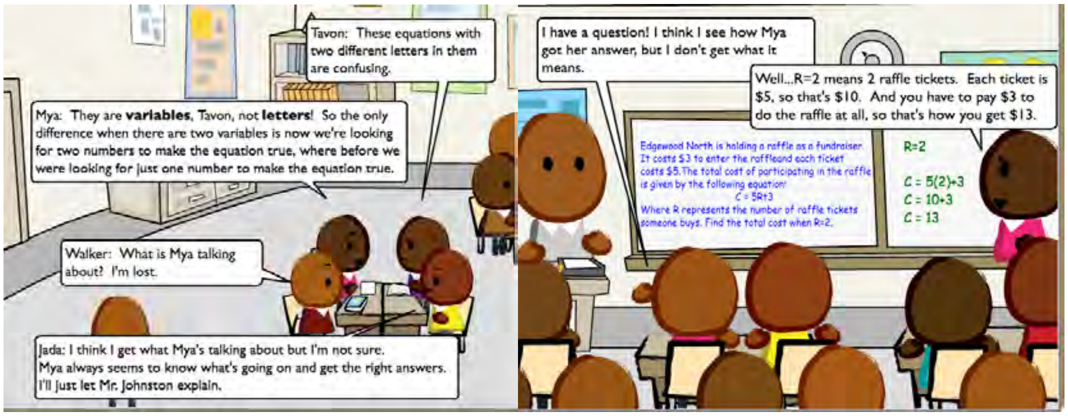

Figure 2. Mya's mathematical ideas, represented by speech and boardwork.

It was also important that both depictions of Scott and Mya reflect their African descent through skin tone, so I chose to depict Scott and Mya using dark skin tones. The written text of the Case of Mya explicitly refers to Mya as African American, however the text does not explicitly refer to Scott as African American, so the reader of the case may (or may not) see both Scott and Mya as African American. This shared racial characteristic can be explored in the discussion of the depiction to gauge the reader's perception of Scott and Mya's shared racial characteristic as relevant or salient to Scott's decisions. For some readers, Scott and Mya's shared racial characteristic may suggest some form of connection, allegiance, and loyalty. In the written case I described the Advanced Algebra class to which Mya transferred as being populated predominantly by Caucasian students. Identification of skin tones on the color wheel that represent Caucasian students was challenging. I tended to rely on pinkish or creamy skin tone shades. I had to create multiple pilot scenes to determine if pilot viewers saw the Advanced Algebra classroom as populated predominantly by Caucasian students. In some cases, viewers mentioned that the students in Advanced Algebra were unnaturally pink. Furthermore, when choosing pinkish or creamy tones for skin color, the color of the outline edge of the character created confusion in the viewer. For example, a light pink character possessed a dark pink edge. A creamy character possessed a brown or tan outline edge. During depiction design, I incorporated scenes of small group work in the Advanced Algebra class that showed Mya as the only student of color in the group. In these scenes, Mya's mathematical ability is questioned and challenged by others. The purpose of including these scenes is to further explore viewers' perspectives on whether race could be a salient and relevant influence on student-student interaction around the mathematics task at hand.

Results and Discussion

From the above example, one may see affordances of a semiotic system embedded in the storyboarding environment. While building a storyboard the availability of the empty, but editable, whiteboard may make the creator wonder what should be written on the whiteboard in the case of Mya. It is more difficult to see those opportunities while writing a case. Similarly to the first example described earlier, from this second example one may generalize the affordances that a semiotic system could offer in the storyboarding environment. When creating a storyboard in *Depict*, the default skin tone of the students is blue (see Figure 1). One can change the skin tone of one character, but then it is likely that one will feel the need to assign skin tones to all characters. The teacher educator felt compelled to represent Scott's skin tone – which, as he suggests, might motivate a reader to infer social relationships between Mya and the teacher.

Table 1. Meanings attributed to the written and depicted case of Mya.

Excerpt from representation	Author meaning/intent	Reader/viewer interpretation (E) - expected; (U) - unexpected
Written case: *Excerpts from written case transcribed above (p. 4)*	Mya is successful when completing complex mathematics tasks and assessments in the pre-algebra classroom Mya has a strong mathematics identity in the Pre-Algebra course, yet may not possess a strong general academic identity (across all subjects). Unlike the majority of her peers in the Pre-Algebra class, Mya possesses confidence and comfort in communicating her mathematical thinking publicly. Mya may be viewed as a peer resource by other students. Mya's explanations may support the development of other students' understanding.	PSTs remarked that Mya possesses power and agency in the Pre-Algebra course. (E) PSTs remarked that Mya is a resource for other students in the Pre-Algebra course. (E) Some PSTs felt that the teacher should not move Mya from the Pre-Algebra course to the Advanced Algebra course. (U)
Depiction: See Figure 2	The Pre-Algebra class is composed predominantly of students of African descent as signified by student skintone; the Advanced Algebra class is composed predominantly of Caucasian students. Mya's demonstrates an understanding of the concept of variable. Mya's demonstrates the ability to solve equations in one variable and equations in two variables given a value for one of the variables. Mya may be able to reason through identifying a set of values of the two variables that solve the equation without being given one of the values.	PSTs focused on many classroom signifiers when describing differences between the two classrooms (seating arrangements, calculator use, etc.), yet were hesitant to mention racial differences between the two classes. (U) A PST interpreted Scott Johnston's physical distance from students in the Pre-Algebra class as a classroom management strategy ('he needs to be able to see all students at all times due to behavior problems') (U) PSTs remarked that Mya understood that parallel lines have the same slope. (E) PSTs remarked that Mya would be able to solve equations in two variables only when given a value for one of the variables. (U) PSTs remarked that Mya would be able to solve systems of equations. (U) When asked how Mya is perceived by her peers, a PST remarked that her peers may view Mya as unrelatable, intimidating, and a 'know it all'. (U)

The case of Mya provides the grounds for a distinction in the kind of storytelling afforded by the storyboarding tool. While originally developed to represent practice, *Depict* also permits to tell character-centered stories that happen in practice; indeed, to some extent it uses the latter to flesh out the former, much in the way that specific diagrams can represent generic figures – diagrams convey important intuitions that help generate geometric theory, might scenes with cartoon characters do the same for teaching? By character-centered stories we refer to stories that are focused on the individualities of one or more characters and what happens to them as they go through episodes in their lives; this contrasts with environment-centered stories that are focused on what happens in specific places as different characters interact. The original design of *Depict* supported the representation of practice in classroom-centered stories. The storyboarding of the case of Mya challenged *Depict*'s graphic language and required the development of a framework for us to examine character-centered stories.

The Framework

We suggest that one can think of the signifiers in a representation of a character-centered story as developed in concentric spheres, each of which includes a stratum of graphical elements available to choose from in order to graphically communicate strata-specific meanings. The innermost sphere in such representations consists of signifiers associated to the characterization of the protagonist(s) of the story. These could include signifiers of physical, cognitive, or emotional individual traits. The second sphere consists of signifiers of the immediate context of the protagonist(s) at various points in the story--resources to represent relations to other characters or to the immediate physical environment. A third sphere consists of signifiers of the more general environment in which the whole story takes place. Earlier, we described and compared the author's textual and storyboard representation of the case of Mya in which Scott proposes that Mya be moved from the Pre-Algebra class to the Advanced Algebra class. In that story, the second sphere will consist of characteristics of those classrooms and the third sphere consists of characteristics of the school community. These different strata form the first dimension of the framework. We argue that the value of the first dimension of this framework is that it helps one organize the many signifiers in a (character-centered) representation of practice.

A second dimension includes consideration of whose meanings one is attending to. The value of the second dimension of this framework is that it encourages considering what meanings different people attribute to a representation, in particular those attributed by the author and by readers, which we suggest are the source of alternative inquiries and narratives. As suggested earlier, we argue that storyboards allow for more alternatives, as they include not only text, but also other communication modalities. A comparison of Clark's intended interpretations of various signifiers in the two representations of the case of Mya and those made by his PSTs is used as example. In some instances, PSTs interpretations were aligned with Clark's meaning or intent; in other instances PSTs generated interpretations that were unexpected (see Table 1). We focused here on signifiers in the innermost sphere of the representation, similar such tables could be used to consider and compare interpretations of the signifiers at each sphere of stratification.

Conclusion

The multimodal resources in the comics medium permit creation of complex messages about classrooms that allow for inquiry and alternative narratives by different readers (Weiss, 2011). These alternative forms of representation also have liabilities. A main gleaning from this paper is that the graphics communicate as a system, both for the author and for the reader. The creation of materials for the study of teaching is not only a creative endeavor but also an analytic one that involves composing with a language and examining the systems of choice with which that language is built. The *individuality* dimension (Herbst et al, 2011) in representations of teaching is one aspect in which this semiotic system can be built, and complexion is one subsystem that contains affordances both for focusing the message and for enabling inquiry.

Indeed while written cases can focus attention on important aspects of practice and video can immerse students in the complexity of practice, a graphics-based semiotic system can be used to scaffold this complexity, combining inquiry with direction. The case of Mya shows that the translation from written case to storyboard included the opportunity to show some interesting nuances in the visibility of race, all of which have a lot to do with mathematics education if we think of mathematics education as an institutionalized practice: Students are learning mathematics in classes and with other students with whom they have particular kinds of social relationships, they are being taught by teachers who could be their advocates or mentors, and these relationships are mediated by race as well as other factors.

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EVALUATING PROOFS AND CONJECTURES CONSTRUCTED BY PRE-SERVICE MATHEMATICS TEACHERS

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This study focused on investigating the ability of 58 pre-service mathematics teachers' (PSMTs) to construct-evaluate mathematical conjectures-proofs in a mathematics course. The combined construction-evaluation activity of conjectures-and-proofs helps illuminate pre-service mathematics teachers' understanding of proof. The result of the study demonstrated that the number of instances where the PSMTs constructed conjectures were less than the number of instances where they constructed arguments to prove/disprove assertions during the semester. Additionally, the PSMTs usually constructed conjectures when they were explicitly asked to do so. The majority of the arguments that were constructed by the PSMTs attempted to provide an explanation for why the assertion held true which may show that the explanatory role of arguments indeed held an essential criterion for the PSMTs.

Keywords: Teacher Education-Preservice

Proof is viewed as a cornerstone of mathematics and an essential element for developing deep understanding (e.g., Ball & Bass, 2000; NCTM, 2000). Yet, research indicates that students of all levels tend to have limited understanding of proof and struggle with constructing proofs (e.g. Harel & Sowder, 1998). Many researchers demonstrated that the empirical reasoning is pervasive among school students including advanced or high-attaining secondary students (e.g. Coe & Ruthven, 1994; Healy & Hoyles, 2000), university students including mathematics majors (e.g., Goetting, 1995) as well as prospective and in-service teachers (e.g. Morris, 2002; Simon &Bume, 1996).

Despite the importance of teachers' understanding of proof, relatively little research has investigated aspects of prospective or practicing teachers' understanding of proofs (Goetting, 1995; Morris, 2002; Stylianides, Stylianides, & Philippou, 2007). Furthermore, previous studies focus solely on teachers' understanding of the distinction between deductive and empirical arguments by asking them to evaluate researcher generated arguments. Stylianides and Stylianides (2009) criticized that there has been limited research about how instructions can help pre-service teachers' develop their understanding of proof. Thus, this study aims to contribute to literature on pre-service teachers' understanding of proof by reporting on pre-service mathematics teachers' (PSMTs)processes of constructing-evaluating mathematical conjectures-proofs during a course in which PSMTs specifically engaged in proving tasks.

Functions of Proof in Classrooms

Traditionally the function of proof has been seen almost exclusively as being to verify or justify the correctness of mathematical statements (e.g. Ball & Bass, 2000). The "verification" function of proof is often interpreted in subjective terms, establishing the truth of a statement with an individual's belief in the truth of a statement and thus allocating proof a role in the subjective acquisition of such belief. However, as Bell (1976) argues, proof is not necessarily a prerequisite for conviction; proof is essentially a social activity of validation or establishing results, which follows reaching a conviction. Duval (2002) argues that a proof can change the logical value as well as epistemic value of a statement. That is, a proof may logically validate a statement, but it can also affect the belief of the cognizing subject as to the truth of the statement. These two functions of

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proof—to convince individuals and to establish results in the field—are by no means the only functions of proof in mathematical activity.

Researchers have contributed to such elaboration on the functions of proof both by reflecting on its many roles in the discipline of mathematics and by identifying its roles in mathematical understanding. These roles are identified by de Villiers by building on the work of others (Balacheff, 1988; Bell, 1976; Hanna, 1990; Hersh, 1993) as follows: (a) verification (concerned with the truth of a statement), (b) explanation (providing insight into why it is true), (c) systematization (the organization of various results into a deductive system of axioms, major concepts and theorems), (d) discovery (invention of new results), (e) communication (the transmission of mathematical knowledge), and (f) intellectual challenge (de Villiers, 1990, p.18).

Hersh (1993) argues that the role of proof in the classroom and the role of proof in mathematical discipline could be different, stating that the purpose of proof in mathematical discipline to be to convince, while in a classroom it should be to explain. Knuth (2002) has echoed this theme, arguing to teachers that proofs are valuable because they can help students understand mathematics. Hanna (1990) distinguishes between “proofs that prove” and “proofs that explain”.Thus, the development of proofs in the course where the study took place served two related functions: (a) as means for explaining why an assertion was true or false by showing how the statement of a theorem coheres and connects with the key properties of the concepts involved in the proof, which will be referred as Type P_1proof and (b) as a means for justifying that an assertion was true thereby promoting conviction, which will be referred as Type P_2 proof in the study.

Methodology

In this section, the context in which the research reported here took place, the research participants and the data collection and analysis processes will be described.

Participants

Participants of the study were 58 pre-service mathematics teachers (PSMTs) who are certified to teach mathematics in grade 5- 8. The PSMTs enrolled in a mathematics course during the semester of spring of 2016. The course was worth three university credits, and so the class met 3 hours per week for a semester. The course was designed to cover a wide range of mathematical topics in three major mathematical domains (algebra, geometry and number theory). The PSMTs were offered various opportunities to engage with mathematical proofs including constructing-evaluating proofs, representing them in different ways (using everyday language, algebra, or pictures), and examining the correspondences among different representations.

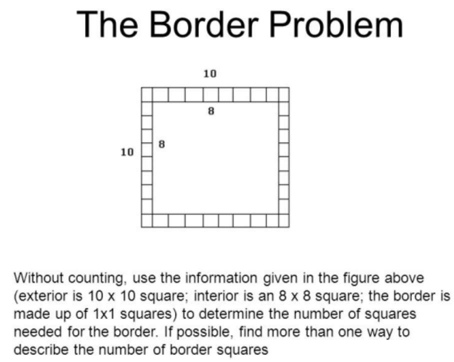
Tasks

A sample of proof tasks in which PSMTs were engaged in during the semester will be presented here in order for readers to better conceptualize PSMTs’ conjecture/proof construction and evaluation processes (see Table 1).

Data Collection Process

The participants were engaged in a course where they were required to work in groups of 6. The participants were engaged in solving tasks that were adopted from existing literature (see Table 1). All instructions were videotaped during three hours of the instruction time for 14 weeks in the semester. The video camera was located at the corner of the classroom where the board was captured. These videos served as the main data source for the study. In addition to the class videos, the PSMTs’ written responses to some of the tasks and their class assignments were also collected.

Table 1: Sample of the Tasks

Task A was adopted from Wilburne (2014). The task was as follow: A fast food restaurant sells chicken nuggets in packs of 4 and 7. What is the largest number of nuggets you cannot buy? How do you know this is the largest number you cannot buy?	**Task B** was adopted from (Weber, 2003). The task was as follows: For every odd integer n, n^2 - 1 is divisible by 8.
Task C was as follows: Justify that the area formula of a kite is $\frac{d_1 d_2}{2}$, where $d_1$and d_2 are the diagonals of the kite.	**Task D** was adopted from Boaler and Humphreys (2005). The task was as follows: 

Data Analysis

The data analysis started with reviewing the videos of the instructions first. After the first review of the videos, the parts where the PSMTs were engaged in construction-and-evaluation of mathematical conjectures-and-proofs were selected and transcribed. Later, the selected segments and the transcript of these segments were viewed again and the PSMTs' proof constructions were coded in one of the following categories: Type P_1: valid general argument that explains why an assertion was true by standing of the underlying mathematical concepts, Type P_2: valid general argument that proves that an assertion was true but did not provide any insight into why it might hold true, Type P_3: general argument that fall short of being acceptable proofs , and Type P_4: unsuccessful attempt for a valid general argument (invalid, unfinished, or irrelevant responses (or potentially relevant response but the relevance was not made evident). Categories TypeP_1 through TypeP_4 represent four different arguments constructed by the PSMTs in decreasing levels of sophistication (from a mathematical stand point), with Type P_4 representing the least sophisticated argument. The construction of conjectures was coded in one of the following categories: Type C_1: conjecture that was constructed as a response to a requested wish (usually by the instructor) in a given context, Type C_2: conjecture that was constructed spontaneously as a natural extension of a task, and TypeC_3: incorrect conjectures. As opposed to categories for proofs, categories Type C_1 through Type C_3 for coding PSMTs' conjectures were not listed in hierarchical levels of sophistication.

Results

General Findings

Table 2 summarizes the distribution of proof-conjecture constructions during the semester. As it was evident in the table, the majority of the proofs constructed during the class were Type P_1 proof, valid general argument that explains why an assertion was true or false by showing how the statement of a theorem coheres and connects with the key properties of the concepts involved in the proof. Of the remaining 29 proving occurrences, 18 of them were Type P_2 proofs, valid general arguments that established that an assertion was true thereby promoted conviction, but provided little or no explanation for why it held true.

Type P_4 proofs, unsuccessful attempt for a valid general argument (i.e. incorrect, invalid, unfinished, or irrelevant responses-or potentially relevant response but the relevance was not made evident in the argument), were proposed 8 times during the semester; however, it should be noted that the PSMTs were aware of the limitations of these arguments. Therefore, they were able to evaluate those arguments as not proofs or as not correct argument during the class discussions. Of these 8 unsuccessful attempts to prove the class tasks, 2 arguments were empirical arguments. The PSMTs who proposed these empirical arguments as well as the others in the class were able to state the fact that generalizing from specific cases was not a valid mode of argumentation.

The number of the cases where conjectures were constructed during the class happened significantly less than the number of cases where proofs were proposed (13 vs. 57). Additionally, the majority of the cases where the conjectures were constructed occurred as a response to a request made usually by the instructor (Type C_1 conjecture). Incorrect conjectures were proposed 3 times during the instructions and after these conjectures were proposed the other PSMTs in the class were able to refute these conjectures by providing a valid counterexample.

Table 2: Distribution of Proof-Conjecture Constructions During the Class

Proofs				Conjectures		
Type P_1	Type P_2	Type P_3	Type P_4	Type C_1	Type C_2	Type C_3
28 (49%)	18 (31%)	3 (5%)	8 (14%)	6 (46%)	4 (30%)	3 (23%)

Classroom Episodes That Represent Different Types of Proof-Conjecture Constructions

In this part two classroom episodes will be shared to exemplify some of the codes used to codify the PSMTs' proof –and-conjecture conjectures.

Episode 1: Chicken tender task. In this episode, the PSMTs were engaged in working on chicken tender task in their groups.

Orhan: Our group has decided that the numbers that can be represented as 28k+27, where k is an integer, cannot be bought in the packets of 7 and/or 4.
Instructor: Ok. Where did 28 and 27 come from?
Orhan: 7 times 4 is 28, so 28 can be bought in packets of either 7 or 4 or its multiples.
Instructor: OK, if k=1 then how many nuggets do you think you cannot buy, umm, 55?
Orhan: Yes
Instructor: Can we have 55 nuggets in the packets of 7 and/or 4?

Merve: Yes. We can have 5 packets of 7s and 5 packets of 4s. So, we can get 55 nuggetsin total.
Instructor: Ok, so the numbers that are represented as 28k+27 can be bought in packets of 7 and 4. Anybody else have an argument? Selman?
Selman: (Writing numbers on the board). You can represent all numbers by adding 4. For instance, if you add 4 to 11, you will get 15; if you add 4 to 12 you will get 16; if you add 4 to 14 you will get 18 and it will continue like this. These numbers cannot be bought in packets of 7 or 4 (highlighting the numbers underlined below). Umm, I needed to check 21 because it is to add 4 to 17 and I know 17 cannot be represented as addends of 4 (or multiples of 4) and 7 (or multiples of 7). However, I have found that 21 is 3 times 7, so it is okay too. Now you will continue this pattern for all numbers 21+4=25, 22+4=26,23+4=27,24+4=28...etc.

1	2	3	4	5	6	7	8	9	10
11	12	<u>13</u>	14	15	16	<u>17</u>	18	19	20
21	22	23	24	25	26	27	28	29	30

Ayse: When you get modulo 7, the residue classes will be 1,2,3,5,or 6 (4 cannot be counted here because we can get the packets of four). When you get modulo 4, the residue classes will be 1,2,and 3. When you add all residue classes up you will get the answer-17.

In this episode, the conjecture proposed by Orhan was coded as Type C_3: incorrect conjecture. The instructor posed a counterexample as a necessary condition for the realization of falsity of the conjecture. The PSMTs were able to explain how the example 55 contradicted the conjecture and refuted Orhan's conjecture. Selman's argument was coded as Type P_1 since it was built on the properties of numbers. It was a correct argument to justify that 17 was the highest number of chicken tenders that could not be bought in packets of 4 and/or 7. Ayse identified all residue classes of modulo 7 except 4 (since it could be a possible answer) and added them up to reach the answer of 17. However, her argument did not include a justification for the assertion that the residues would always be the highest number that could not be bought in the packets of 4 or 7. Indeed, when her argument was applied to different numbers such as packets of 6 and 4, it would not give the correct response. Therefore, her argument was coded as Type P_4.

Episode 2: Area perimeter task. In this episode, the PSMTs were engaged in working on geometry task-investigating the relationship between area and perimeter of rectangles. The instructor asked the PSMTs to construct conjectures about area and perimeter of rectangles. Cihat proposed the following conjecture: "With *the same perimeter, the smaller the difference between the side lengths of a rectangle, the biggest the area*". The instructor asked the PSMTs to evaluate the conjecture and prove whether it was correct.

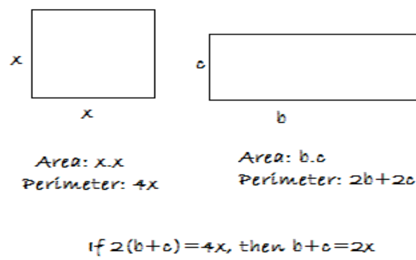
Merve:(Drew three rectangles with the side lengths of 12 by 6, 15 by 3, and 9 by 9). It is true. These rectangles have the same perimeter, 36. But, the area of the square is bigger than the other two rectangles.
Instructor: Do you think that Merve proved Cihat's conjecture?
PST: No, she just demonstrated for those rectangles.
Instructor: What is missing in her argument?
PSTs: It is not general
Instructor: We mentioned that providing examples do not suffice as mathematical proofs. How many examples can I draw with a perimeter of 36?
PSTs: 5? (Said as if they were asking if it was true). Infinitely many?

Cihat: Infinitely many, because in between whole numbers, there are infinitely many rational numbers

Instructor: So we can draw infinitely many rectangles with the perimeter of 36, will you be able to try all of these rectangles out like Merve attempted to do here?

PSTs: No!

Yılmaz: (Volunteered to share his argument). Now we have the lengths of b,c (referring to the long and short sides of a rectangle in this order) and x (referring to a side length of a square). They should have the following relationships: b>x>c. Thus, x^2>b.c. Let's assume that x=n and c=n-1 and b=n+1. Therefore, $x^2=n^2>b.c=n^2-1$



Instructor: Why does b have to be bigger than x and x has to be bigger than c?

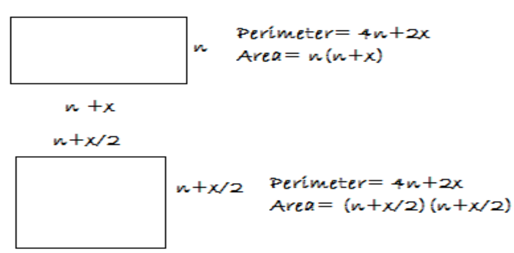
PSTs: If these two rectangles have the same perimeter, than this relationship should hold.

Instructor: Ok, but why should x between b and c?

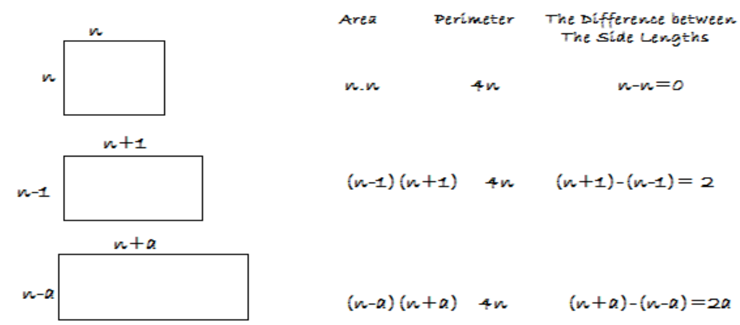
Cihat: b and c should be different in lengths, because we consider the rectangles that are not squares, so b≠c. Then, we know that $x=\frac{b+c}{2}$ since the perimeter of the two shapes should be equal. Thus x should be between b and c. We know that $x=\frac{b+c}{2}$ so, $x^2=\frac{b^2+c^2+2bc}{4}$. We know that b-c>0, so $(b-c)^2>0.b^2-2bc+c^2>0 \Rightarrow b^2+c^2>2bc$. If $b^2+c^2>2bc$, Then $\frac{b^2+c^2+2bc}{4}$ should be bigger than b.c (the area of the rectangle).

Instructor: Ok, great.Zeynep, would you like to share your method with us?

Zeynep: (Writing her argument on the board).The perimeters of these rectangles should be the same.$A_1= n^2+nx$ and $A_2= n^2+nx+x^2/4$. Thus, it is obvious that A_2 should be bigger than A_1 since x≠0.



Aysegul: (Writing her argument on the board).



Cihat's conjecture was constructed as a response to the Instructor's request. Therefore, it was coded as Type C_1. Merve provided three examples that demonstrated that Cihat's conjecture was true. Since Merve used an invalid mode of argumentation-inductive argument-, her argument was coded as Type P_4. Stylianides (2007) argues that the main difference between empirical arguments and proofs lies in the modes of argumentation: invalid versus valid modes of argumentation. Empirical arguments provide inconclusive evidence by verifying the statement's truth only for a proper subset of all covered by the generalization, whereas proofs provide conclusive evidence truth by treating appropriately all cases covered by the generalization. When asked to evaluate the argument, both Merve and the other PSMTs in the class were able to state this limitation of the argument. Stylianides & Stylianides (2009) argued that construction-evaluation tasks can better identify prospective teachers' who seem to posses the empirical justification scheme. Unlike Merve, Yilmaz attempted to construct a deductive argument. However, his argument did not provide justification for some of the assertions he used in his argument (i.e. $b>x>c$). Additionally, Yilmaz's argument was constructed based on a condition- the side lengths of the rectangle and the square should be consecutive. Yilmaz's argument was coded as Type P_3. Cihatwas able to provide the justifications for each step of his argument. Thus, his argument as well as Zeynep's and Aysegul's arguments was coded as Type P_1.

Conclusion and Discussion

Given that teachers' ability to teach mathematics depends on the quality of their subjectmatter knowledge, a necessary condition for the realization of the importance of mathematical proofs as stated in the current curriculum reforms (NGA/CCSSO, 2010; NCTM,2000) is that teachers of all levels have good understanding of proofs (Stylianides & Ball, 2008). This study reported pre-service mathematics teachers' engagement with proof-and-conjecture tasks.The results of the study demonstrated that the number of instances where the PSMTs constructed conjectures, which isreferred as one of the essential parts of the process of making sense of and establishing mathematical knowledge (Stylianides, 2008), were limited and usually occurred when asked explicitly. Constructing arguments to prove and/or disprove assertions, on the other hand, occurred more often. Furthermore, the majority of the arguments constructed highlighted the explanatory aspect, which is consistent with the results of many studies that claimed that in mathematics classrooms, it would be more useful to use proof as a tool to explain than to convince (Hanna, 1990; Knuth, 2002).

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EXPLORING ELEMENTARY MATHEMATICS TEACHERS' OPPORTUNITIES TO LEARN TO TEACH

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Mathematics teacher education is faced with the challenge of preparing new teachers for ambitious instruction, but we have limited understandings of what happens within the courses where this preparation occurs. This paper draws on interview data from of a larger investigation of novices' enactment of ambitious instruction in elementary mathematics and language arts across six teacher preparation programs. Findings describe the application of the framework developed by Grossman et al. (2009) to opportunities to learn to teach in elementary mathematics methods courses and associated field experiences, focusing on the range of activities described and their relation to the framework.

Keywords: Teacher Education-Preservice

A current challenge facing teacher educators is preparing new teachers to engage in ambitious mathematics instruction (Lampert et al., 2013). Yet, despite recent research related to specific pedagogies for mathematics teacher education (e.g., Lampert et al., 2013), we still know very little about the range of instruction and opportunities available to teacher candidates in methods courses (e.g., Clift & Brady, 2005). In response to these challenges, as part of a larger study seeking to understand the relationships among teacher characteristics, features of teacher preparation programs, and novice teachers' enactment of ambitious instruction, we interviewed elementary mathematics methods instructors and program coordinators across multiple teacher preparation programs. Our goal is to characterize the opportunities to learn provided through methods courses. Specifically, we focus our investigation on the opportunities teacher candidates have to learn to teach, as compared to opportunities to learn mathematics content (Schmidt, Bloemeke, & Tatto, 2011).

Grossman and colleagues (Grossman et al., 2009) developed a framework based on *pedagogies of practice* "to describe and analyze the teaching of practice in professional education programs…" (p. 2055). Here, we use their framework to continue that same work. Specifically, in this paper we share findings from our efforts to use the Grossman et al. framework (2009) to categorize the activities described by instructors and coordinators. In so doing, we explore the range of activities shared, note the ways in which these activities do and do not fit the Grossman et al. (2009) framework, and consider the importance of variations in the enactment and sequencing of activities in terms of the opportunities to learn they might offer.

Theoretical Framework

We frame our study of opportunities to learn in elementary mathematics methods courses using Grossman and colleagues' (2009) framework that established three primary pedagogies for teaching relational practices, including teaching, to novices – representation, decomposition, and approximation. In this paper, we are focusing specifically on representation and approximation, both of which we see as also involving aspects of decomposition. Grossman and colleagues define *representations* as, "the different ways that practice is represented in professional education and what these various representations make visible to novices," (Grossman et al., 2009, p. 2058). They noted that representations can vary not only in what they do and do not make visible for teacher candidates, but also in their "comprehensiveness and authenticity" (p. 2065). *Approximations* are defined by Grossman and colleagues as, "opportunities for novices to engage in practices that are more or less proximal to the practices of a profession," (p. 2058). They suggest that, similar to representations, approximations can vary along a number of dimensions, including the nature of the aspect of teaching practice being approximated, "how closely the activity approximates actual practice", and "the role of the [teacher] educator" (p. 2079). This framework focuses attention not only on how novices learn to enact teaching practices, but also on how novices learn the knowledge and skills that underlie those practices.

More recent studies have taken up this framework to design and characterize pedagogical interventions in methods courses and to understand the relationship between these pedagogies and novices' learning outcomes. For example, Amador and colleagues (2016) explored the differences in teacher candidate noticing of teacher practices in the context of a representation and an approximation of practice. Ghousseini and Herbst (2016) focused on the importance of sequences of representations and multiple approximations for teacher candidates' opportunities to learn to lead classroom discussions. Here, rather than focusing on a specific activity or series of activities, we are investigating the range and sequences of activities across multiple teacher preparation programs, with a specific focus in this paper on those activities involving teacher educators' enactment of representation and approximation pedagogies.

Methods

The findings reported here are part of a larger study of novices' enactment of ambitious instructional practices in elementary mathematics and language arts. The larger study investigates (a) how a purposively sampled set of six teacher preparation programs in three states supports elementary teacher candidates to develop ambitious math and language arts instruction and (b) factors that are associated with how graduates of these programs enact math and language arts instruction as first- and second-year teachers. This investigation includes surveys of approximately 150 teacher candidates from the set of six teacher preparation programs during their final year of the program and their first two years of teaching. Additionally, we observe these graduates multiple times as they teach mathematics and language arts as first-and second-year teachers.

For this specific study, we focused on interviews with 12 elementary methods instructors and 9 program coordinators to better understand their perspectives on the opportunities to learn to teach provided in elementary mathematics methods courses and associated field experiences across the a subset of three of the six teacher preparation programs in the study. Each participant was interviewed once during 2015-2016 for approximately 45-60 minutes. Data was audio recorded and later transcribed. Interviews were semi-structured based on a protocol designed to solicit information about the backgrounds, instructional activities, and philosophies of method instructors and their respective programs. Questions asked included the following:

1. How would you characterize the overall approach to teaching that you seek to develop among the teacher candidates through the course?
2. What major instructional strategies do you want teacher candidates to learn and know how to enact? Why do you focus on these strategies?
3. How do you engage teacher candidates in learning these strategies? What kinds of activities do you use to help them learn about these strategies?

Analysis

Our analyses focused on making sense of the interview data from methods instructors and program coordinators. Our process involved iterative cycles of coding during which we both developed emergent codes from the data and built from theory. Specifically, we began by applying grounded theory techniques of initial coding (Saldaña, 2015) to examine a broad sample of our interview data looking for common issues discussed across the group of methods instructors and program coordinators. Next, we refined our initial codes by looking for similarities and differences across the codes and comparing our emergent ideas to those categories present in existing theoretical frameworks (e.g., Darling-Hammond & Bransford, 2005; Grossman et al., 2009). Subsequently, we continued multiple rounds of this iterative process to clarify our codes. In particular, we sought out examples of interview excerpts that were not well captured by previous versions of codes in order to identify those features of opportunities to learn to teach that our codes did not yet capture. Finally, we generated code definitions and selected representative examples.

Ultimately, our analyses resulted in a multi-leveled codebook that distinguishes: (a) what knowledge, practices, or content teacher candidates have opportunities to learn, (b) how those opportunities to learn are made available to teacher candidates, (c) who provides the opportunities to learn, (d) teacher preparation program capacity for opportunities to learn, (e) teacher preparation program structure, (e) program and teacher candidate evaluation, and (f) reasoning behind particular opportunities. Here we focus on a subset of the codes related to *how* opportunities to learn are made available to teacher candidates.

Findings

In the following sections we share initial findings from our early coding work focused on interviews with methods instructors and program coordinators. Broadly, we found the theoretical constructs of approximations and representations to be a useful starting point to interpret the opportunities teacher candidates have to learn to teach during methods courses. Additionally, we identify a number of interesting dilemmas with regards to parsing the work of methods instructors into these discrete categories. Here we share our codes, summarized in Table 1, along with representative excerpts of interview data to illustrate these dilemmas and describe what we have learned.

Table 1: Codes for How Opportunities to Learn are Provided to Teacher Candidates

Code Name	Description
Representations	Opportunities for teacher candidates to watch examples of the work of teaching
Approximations	Opportunities for teacher candidates to experience deliberate practice immersed in activities of actual teaching.
Do a math or literacy task	Opportunities for teacher candidates to engage with specific content knowledge through tasks
Learning to learn from teaching	Opportunities for teacher candidates to learn how to be reflective of their work as teachers and to use their teaching experiences as a means to grow as professionals
Reflection	Opportunities for teacher candidates to reflect about their teaching in writing or aloud
Formative feedback	Opportunities for teacher candidates to receive formal or informal feedback about their work
Problems of practice	Opportunities for teacher candidates to learn from challenges that arise during instruction.
Examining classroom artifacts	Opportunities to examine samples of student work or other classroom artifacts (generated or authentic) as a specific focal point for reflection.
Connect to other coursework/ knowledge	Opportunities for teacher candidates to build on work from previous classes or to connect to ideas that will be the focus of courses later in the teacher preparation program
Field Experiences	Opportunities for teacher candidates to learn to teach through field experiences (e.g., practicum placements, student teaching)

This paper explores only our findings related to the codes *representations* and *approximations*. In our efforts to understand how to organize and apply these two constructs, however, we identified a number of additional ways in which methods courses provide opportunities for teacher candidates to learn to teach. Thus, an important initial finding is that representations, approximations (and decomposition) as proposed by Grossman and colleagues (Grossman et al., 2009) are not sufficient to address of all of the opportunities provided in methods courses. Although this paper focuses on applying and clarifying definitions of the constructs of representations and approximations, we note that there exists an important course of future research to explore the nuances of opportunities to learn to teach presented through our other codes in Table 1, especially learning to learn from teaching and field experiences.

Representations

Following Grossman et al. (2009), we define representations as opportunities for teacher candidates to observe examples of the work of teaching. A classic representation of teaching practice might be a video recording of a teacher teaching a lesson to a class of students. In our initial analyses we did find some discussion of instructors using videos of practicing teachers as representations of quality practice. Early coding revealed, however, that instructors more often discussed representations of practice other than those involving video, including teacher candidates observing their cooperating teachers in action and observing their methods instructors modeling particular instructional practices.

For example, the following excerpt, from an interview with a mathematics methods instructor, is representative of some of the issues we encountered related to identifying representations of teaching practice.

> Instructor: Part of the modeling is that I model. If we're going to do something, like for example, we have a giant number line that I made and they had to put decimals and fractions on it. And one of them facilitated it. But, I made the materials and then we said, how does this help to model?
> Interviewer: Oh, to model a concept or something?
> Instructor: Yes, model a concept… but modeling how you model a concept. [laughs]
> Interviewer: Right
> Instructor: And so that's another thing that, in terms of conversation, a lot – I always have them sit in groups. And if there's something that I think is important, I will either model it or I'll set it up so someone else [a teacher candidate] can help to model it. […] So I try to set up experiences for them to experience things like modeling and then *they* talk about it. And I will facilitate their discussion or I will say to somebody else [a teacher candidate], could you please facilitate the discussion on… So, that they are doing as much as possible.

In this excerpt the methods instructor described her strategy of using her own instruction during the methods course to model the kinds of teaching activities and strategies she was presenting to the teacher candidates. That is, she used her own instruction as a representation of the kind of ambitious instruction that she would like her teaching candidates to learn.

This excerpt also highlights some of the complexity we found in determining the boundaries of representations in methods courses. It was not always clear when teacher candidates were engaging in opportunities to observe representation of practice as compared to other opportunities to learn, such as approximation of practice, or developing content knowledge. For example, in the excerpt above, the instructor described how she represented quality teaching practice while simultaneously engaging students with mathematical content (e.g., locating decimals and fractions on the number line) and providing opportunities for teaching candidates to try out pedagogical strategies (e.g., facilitating discussion). This was typical across our data set, and thus a dilemma arose for us as to how to bound the idea of a representation of teaching practice. Could we consider any teaching the method instructor does (e.g., share PowerPoint lectures, ask thoughtful questions) a representation of teaching practice? Since not all interactions with teaching are necessarily supportive of learning to teach (e.g., Lortie 1975) nor intended to serve as representations of teaching, we limited considerations of representations to only those instances in which the interviewee explicitly discussed the pedagogical reasoning related to the representation. For example, to be considered a representation, a methods instructor would need to explicitly describe how their actions provided an opportunity for teacher candidates to learn about teaching practice. Additionally, in instances where teacher candidates had opportunities to observe the work of teaching in the field (e.g., practicum, student teaching) without explicit discussion of the pedagogical purpose, we coded those instances as opportunities to learn through *field experience*, not representations.

Approximations

Building from the work of Grossman and colleagues (2009), we define approximations as opportunities for candidates to experience *deliberate practice* (Ericsson, 2002) as they engage in activities of actual teaching practice. Through approximations, teacher candidates can experiment with their new teaching skills, knowledge, and ways of thinking. Unlike representations, which focus on observing practice, approximations necessarily involve teaching candidates in doing aspects of the work of teaching. As described by Grossman and colleagues (2009), approximations may be simplified or scaffolded versions of practice (e.g., only facilitating a small group discussion instead of the whole class, analyzing assessment results with extensive instructor support), or they might engage teacher candidates in more explicitly elaborated versions of practice (e.g., detailed unit plans). Since approximations are filtered version of reality, they typically involve intervention from instructors and/or cooperating teachers.

A typical example of an approximation we observed in the data was related to engaging teacher candidates in planning for a lesson. The following excerpt was from a mathematics methods instructor describing how she engaged her teacher candidates in learning teaching strategies. She explained,

> So, they will do lesson planning, and they will present a lesson, and they will create an assessment that they use with students in their field site and then reflect on that. So, you know, reflection is a big part of it. They do journal writings and the journal writings are specific to the students in their field sites, whether it be just observing what the teacher does and reflecting on that, or when you work with the students, what happens there.

The instructor described a characteristic methods course activity in which teacher candidates plan a lesson, teach that lesson to a small group of students, assess students' learning, and then reflect on what they learned from the experience. This type of activity was pervasive throughout our interviews with methods instructors. Lesson planning could be considered a canonical approximation of teaching practice. It was simplified in that teacher candidates had to only prepare a single lesson plan, usually with extensive feedback from instructors, and often the lesson was taught to only a single class or a small group of students.

Methods instructors also described such lesson planning activities as involving more details and complexity than might otherwise occur in a regular classroom. For example, another mathematics methods instructor described the major assignment in her course in the following way,

> And even though in some ways it feels silly to have one lesson plan count so much, we really, I feel like we use that as a vehicle for learning all kinds of other things. Because we use it as a vehicle – it's essentially an annotated lesson plan, with a lot more required than would be typical,

> and I continue to tell them that. You will never write this much in a lesson plan as long as you live.

This quote highlights one of our findings that opportunities to learn through approximations, even typical lesson planning activities, do not always follow a clear trajectory of moving from simplified to more complex activities. Our interviews with methods instructors revealed that the purposes instructors have for engaging teacher candidates in an approximation affect how complex or simplified that approximation of practice might be. For example, the instructor in the preceding excerpt explained that she used the lesson planning approximation to support nearly all of the learning to teach activities that occurred throughout her course over an entire semester. Teacher candidates were expected to examine content and practice standards, consider student development, plan for thoughtful pedagogy and intentionally use instructional strategies. Subsequently, teacher candidates also taught the lesson to a group of students, video recorded their teaching, and then later reflected on the video recording of their instruction. This level of detail and time commitment to a single lesson would likely be impossible in the real world of a classroom teacher. The approximation of practice, however, allowed the methods instructor the space to support teacher candidates to deeply explore a range of components that go into a single lesson.

Grossman and colleagues (2009) noted "approximations may require more elaborated versions of practice than what novices will enact in their careers" (p. 2077) citing detailed unit plans as an example. In addition to more elaborated versions of planning activities, our initial analyses also revealed additional examples of approximations with added complexity, including elaborated investigations of students' communities. For example, a mathematics methods instructor described an assignment that required teacher candidates to immerse themselves in the community of their teaching placement to better understand the specific challenges and resources of their placement location. The methods instructor explained,

> [teacher candidates are] not just talking to their mentor teacher. They're talking to parents. They're talking to shop owners. They're exploring the space around school and spending time in coffee shops and groceries stores and the community where their school's situated, despite the fact that they might live in [another town].

Practicing teachers may not always immerse themselves in the community where their students live; however, an important part of ambitious instruction involves building on students' knowledge and experiences. This example of an elaborated approximation of practice illustrates another way methods instructors provided opportunities for teacher candidates to learn about teaching in ways that might have been more complex than their counter-parts outside a methods course.

Another interesting complexity we found related to approximations was the ways they juxtaposed with representations of practice. The illustrative excerpt below was taken from an interview with a program coordinator, during which the coordinator described the opportunities to learn to teach provided to teacher candidates during their student teaching placements.

> Mentors are trying to help them learn to teach, but I think that they are helping them learn to teach in the way that they teach their district teachers. So we do encourage them to do co-planning, co-teaching, but in terms of the philosophy of teaching, this year we talked about our own visions of teaching, how those match with their mentors. But I worry sometimes our interns go out and just try to imitate with what they are seeing the mentor doing and consider that good teaching or not good teaching.

The coordinator described opportunities for teacher candidates to learn from their mentor teachers, including opportunities to co-plan and co-teach.

By engaging in co-planning and co-teaching, teacher candidates have the opportunity to actively participate in the work of teaching, engaging collaboratively alongside their mentor teachers. Thus, we would consider opportunities to co-plan and co-teach as approximations. This quote highlights, however, that participating in co-planning and co-teaching also provides opportunities for teacher candidates to observe a representation of practice, the practice of their mentor teacher. In this case, the coordinator emphasized the authenticity of this representation, in that mentor teachers teach in the ways supported by their district, not necessarily in the model suggested by the university teacher preparation program.

Although co-planning and co-teaching fit within the category of a scaffolded approximation of teaching practice, we found that the opportunities to learn provided to teacher candidates are not easily delineated to separate an approximation from a representation. When in the field, any interactions teacher candidates have with practicing teachers offer potential representations of practice. The ways in which teacher candidates interact and engage with those examples of practice offer potential approximations of practice. Methods instructors and mentor teachers make use of these opportunities to learn in specific and sometimes overlapping ways.

Discussion and Implications

We found that methods courses provide a range of opportunities for teacher candidates to learn to teach, and Grossman and colleagues' (2009) framework provided a useful starting place to interpret and organize these opportunities. These findings reveal that applying the constructs of representations and approximations methods course data is not straightforward and that we needed to refine and supplement the definitions. Likewise, these findings build on literature that encourages interrelationships between pedagogies (e.g., Darling-Hammond & Bransford, 2005) by illustrating how methods instructors and mentor teachers use representations and approximations intertwined with one another.

Additionally, this paper highlights the need for a more detailed taxonomy of methods course activities as they fit within the broad pedagogies of practices. Thus, we seek to identify possible sequences of learning activities within methods courses. One model of using approximations to learn to teach might be that methods instructors move teacher candidates from more distal examples of practice, with more scaffolds, to more proximal examples of practice, dropping the scaffolds as they progress. For example, teacher candidates might move from facilitating a single small group discussion to facilitating a full class discussion. Alternatively, we found that approximations of practice do not necessarily progress along a trajectory of complexity, but rather the degree of complexity of an approximation may be tied to an instructor's learning goals for a specific activity. For example, some methods instructors discussed how they used a variety of approximations within the course to support teachers' candidates learning of specific instructional practices (e.g., asking higher-order thinking questions; learning about students' communities), but did not discuss the progression of these approximations from one activity to the next, suggesting that they may be thinking more closely about the alignment of individual approximations to instructional goals than the progression of approximations over time. Given the similarities we observed across the methods courses, we wonder then if there are also underlying trajectories of particular pedagogical activities that support learning to teach, as found by Ghousseini and Herbst (2016) with respect to learning to teach through discussion. Relatedly, we seek to explore instructors' rationales for using particular pedagogies with teacher candidates. Additional research is warranted into opportunities to learn to teach including further exploration of approximations and representations, as well as additional opportunities we identified beyond the scope of the Grossman et al. framework (2009).

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INTERPRETING AND REPRESENTING STUDENTS' THINKING IN THE MOMENT: PRESERVICE TEACHERS' INITIAL NUMBER STRING LESSONS

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Being able to effectively interpret students' thinking and respond effectively in the moment are important skills that preservice teachers (PSTs) need to learn. This study zooms in on 14 PSTs' planning, teaching, and reflections involved in number string lessons, and investigates to what extent PSTs anticipate their students' strategies and incorrect answers. Further, through a lens focused on PSTs' talk moves, we explore how PSTs supported students' understanding of the strategies and concepts, and handled incorrect answers and unanticipated responses (strategies). Results show that PSTs were capable of anticipating many answers and strategies but need more improvement in making full use of various talk moves in their questioning. Implications are provided for mathematics methods instructors to plan activities to better support PSTs.

Keywords: Teacher Education - Preservice, Classroom Discourse, Instructional Activities and Practices

The ability to know students' thinking is a key factor in mathematics teachers' ability to conduct group discussions and improve their classroom teaching (Stein, Engle, Smith, & Hughes, 2008). Especially when teaching practices emphasize students' freedom of exploration and use of mathematical processes, the importance of facilitating discussion around students' solutions to problems are paramount (NCTM, 2006). Teachers' roles in facilitating discussion rely on being able to anticipate what students will say and have strategies for dealing with unanticipated responses. Our study helps characterize this process with preservice elementary teachers during number string lessons, which involve presenting a series of related arithmetic problems that highlight an overarching strategy or pattern (Stein et al., 2008).

Theoretical Framework

Responding to Students' Thinking in the Moment

Even though much literature focusing on PSTs offers teacher educators resources to educate PSTs before they actually teach, responding to students' spontaneous thinking during number string lessons is a challenge for PSTs and a demanding task for teacher educators (Bofferding & Kemmerle, 2015). Responding to students' spontaneous thinking is complex, involving PSTs' mathematics knowledge, students' mathematics knowledge, the classroom learning culture, teacher-student interactions, and instructional representations (Ghousseini, 2015). Further, PSTs must make quick actions to respond to students' ideas, especially when unanticipated (Stein et al., 2008), under the condition where they lack a rich knowledge-and-experience base to handle everything in the classroom (Jacobs, Lamb, & Phillip, 2010). Because of this complexity, there are no textbooks to tell PSTs how to anticipate or determine the best way to respond to the variety of situations they might encounter (Zeichner, 2012).

Before responding to students' thinking in the moment, PSTs must unpack and interpret students' reasoning, a process that is aided by the use of restatement and other talk moves to ensure mutual understanding among the class (Chapin, O'Connor, & Anderson, 2009). They make use of symbols, pictures, gestures and mathematical discourses to represent chosen students' thinking and assess students' strategies (Leinhardt & Steele, 2005). During this process, PSTs must judge the relative importance of different ideas. The judgment is particularly pertinent to what Ghousseini (2015) defined as "mathematical sensibilities" (p. 343), i.e., PSTs' mathematical knowledge and abilities to attend to what proper interpretations to make and how representations are selected and used.

Anticipating Students' Problem-solving Strategies

There are a few steps PSTs should take before engaging students in solving a mathematics problem. They should know the correct answer (or answers) and should at least solve the problem themselves. However, knowing only one solution to a problem is insufficient to teach students with a wide array of answers and strategies for the same problems (Stein et al., 2008). Full preparation for possible emergent student thinking is beneficial (Jacobs, Lamb, & Philipp, 2010); without such preparation, PSTs might be inclined to feel unprepared and have limited ideas about ways how to handle unanticipated answers (Smith, 1996). Therefore, it is important for PSTs to anticipate multiple student strategies (both correct and incorrect), different reasoning supporting those strategies, and representations utilized by students at different levels of mathematical understanding (Stein et al., 2008).

Talk Moves

Aside from anticipating possible student strategies and reasoning, PSTs also need to utilize this knowledge together with appropriate questioning prompts in order to structure the sharing of students' mathematical thinking to the class (Stein et al., 2008). To solicit students' thinking in the moment, PSTs have access to some discursive tools that are helpful in encouraging students to say more about their ideas, support positive teacher-student interactions, and align with instructional goals. Much literature has documented such discursive tools. Chapin, O'Connor, and Anderson (2009) presented five talk moves that support students' thinking and help maintain a collective learning environment, i.e. teacher revoicing, student restating, applying one's own reasoning to someone else's, further prompting opportunities and using of wait time. In a similar vein, Ghousseini (2015) elaborated the discursive tools from a perspective of discourse routines and divided them into five categories, which serve as our lens for analysis. These categories are revoicing (the teacher repeats, rephrases and translates students' saying), orienting (the teacher puts someone's idea on the spot and asks other students to comment on and contribute to that idea), pressing (the teacher pushes students to talk more about their reasoning), negotiating (the teacher connects different students' strategies and tries to involve students in the discussion about the similarities and differences), and making certain aspects of the discourse explicit.

In the face of errors or mistakes, many teachers cannot help correcting students (Stein et al., 2008) and tend to use show-and-tell discourse (Ball, 2001). Especially when these errors are unanticipated, PSTs might feel out of control and at a loss for how to appropriately respond to students' ideas (Son, 2016). This study is aimed to contribute to the literature by looking at the intersection of PSTs anticipating and use of talk moves in number string lessons. Investigating PSTs' actual and spontaneous responses to students' strategies and identifying what discursive tools PSTs usually fall back on when encountering unanticipated student errors will advance our understanding of this intersection. The following research questions guide us in this exploration.

1. To what extent do PSTs anticipate their students' mathematical strategies and incorrect answers during number string lessons?
2. What talk moves do PSTs use, and what potential do they have for supporting students' understanding of the mathematical strategies and concepts?
3. How do PSTs handle incorrect answers and unanticipated responses (strategies), if they occur?

Methods

Participants and Setting

Participants in the study included 17 female PSTs taking an elementary mathematics methods course at a Midwestern university. Of these, we selected 14 for further analysis by excluding PSTs who worked with only one student, a life skills class, or did a topic besides addition, subtraction, or multiplication. Between two sections of the methods course, twelve PSTs came from one section and two came from the other. As part of their course, the PSTs each spent nine afternoons teaching, doing interviews with students, helping out, and teaching three mathematics lessons and two to three science lessons at an elementary school. Our focus—a number string lesson—was one of the three mathematics lessons. All but two of these PSTs shared a classroom with a second PST, and their placements varied from Kindergarten to Grade 5.

Design and Materials

The methods course aimed to raise PSTs' awareness of students' strategies and attune them to ways of building on students' mathematical strategies through talk moves and the use of representations. In the class, PSTs were asked to read chapters from two books to help them think about students' arithmetic strategies: Fosnot and Dolk's (2001) Young Mathematicians at Work: Constructing Multiplication and Division and Wright, Stanger, Stafford and Martland's (2006) Teaching Numbers in the Classroom with 4-8 Year Olds. With the purpose of interpreting and representing students' thinking in number strings, PSTs read about students' strategies for solving addition, subtraction, and multiplication problems. Drawing on these reading materials, PSTs designed their number string teaching plans (as negotiated with their placement teachers in the elementary schools).

Their plans, which were co-written if they shared a classroom with another PST, required them to identify the targeted strategies they hoped students would use in the lesson or pattern they hoped students would notice, list the problems they would pose, anticipate strategies and answers (both correct and incorrect) for each problem, draw possible representations PSTs would use to illustrate students' thinking, and list connections they anticipated making among students' strategies. Given PSTs' instructor's written feedback to the teaching plans, some PSTs also revised their plans. PSTs' teaching plans and revised versions serve as one data source.

Then PSTs implemented their teaching plans in elementary schools, and lessons lasted around 15-20 minutes. They audio-recorded their lessons and took pictures of any representations they made on the board during their lessons. The course instructor (the second author) was on site with PSTs and noted down their representations and students' uses of strategies. Together, the transcripts of PSTs' teaching and the course instructor's notes served as a second data source. Finally, PSTs each wrote a teaching reflection focused on the actual strategies students used, how the PSTs handled errors, how they used representations, and any changes they would make to the lesson. Therefore, the data includes 8 lesson plans, some with revisions, audio and pictures of the representations used from the lessons, and 14 lesson reflections.

Expected Take-Away from Course Readings

Fosnot and Dolk (2001) and Wright et al. (2006) discussed in depth strategies of solving addition, subtraction and multiplication used by K-5 students (see Table 1 for addition examples). We expected PSTs to think about these strategies when they anticipated what their students would do as part of their teaching plans and draw on them during their number string lessons as well as their reflections. Aside from the strategies discussed in the readings, PSTs also had their own strategies, such as the use of standard algorithms, to use.

Table 1: Strategies of Solving Addition Mentioned in Course Readings

Strategies	Explanations
Counting by ones	14 + 5: Count all or count on starting at either number.
Doubling	7 + 7: Double same numbers
Jump Strategies	14 + 7: Start at one number and make strategic jumps to add on, e.g., 14 + 6 = 20, 20 + 1 = 21.
Split Strategies	24 + 13: Split apart the numbers into tens and ones and add similar place values, then add the results together, e.g., 20+10=30, 4+3=7, 30+7=37.
Split-jump	24 + 13: Split apart the numbers into tens and ones, add the tens, and then make jumps to add on the ones, i.e. 20+10 = 30, 30+4=34, 34+3=37.
Compensation	26 + 17: Change one number to make the addition easier, and then adjust for it later, e.g., change 17 to 20, then do 26 + 20 = 46, 46 – 3 = 43.
Manipulatives/tools	Use ten frame, double ten frame, fingers, cubes, pictures, number line, etc. to help counting

Analysis

With PSTs' teaching plans and actual teaching transcripts available, we began our analyses by identifying the anticipated vs. unanticipated strategies and answers. We looked at PSTs' planning documents for answers and strategies they anticipated students would have for each problem, and compared these to the answers and strategies students used in the lessons. We organized these by anticipated answers (anticipated strategies or unanticipated strategies) and unanticipated answers (anticipated strategies or unanticipated strategies) with the purpose of revealing the consistency between PSTs' anticipation and actual student action. We looked at totals for each category as well as how many of the anticipated and unanticipated answers generated were incorrect. Correct answers were assumed as anticipated if not explicitly discussed in PSTs' lesson plans.

To better understand PSTs' talk moves around anticipated and unanticipated answers and strategies, we focused on two addition number strings and coded each talk turn within each lesson episode using Ghousseini's (2015) framework. An episode is defined as a round of teacher-student exchanges (talk turns) in which a student's reasoning for his/her answer is sought out in detail. Episodes occurred after initial answers were elicited, and certain problems had more episodes if there was more variety in how students solved the problems (i.e., the teacher asked several students how they solved the problem). If a PST asked a student a question, the student responded, and the PST followed up before moving onto a different strategy, that was three talk turns and one episode. Some talk turns had no codes if no talk moves were used, and some had multiple codes if more than one talk was used. One researcher completed the coding, and a second researcher checked a portion of the codes; disagreements were discussed until there was agreement. Finally, we compared and contrasted the frequency of different talk moves in order to get a general profile of PSTs' discourse when they responded to students' ideas.

Results

Discrepancies Between PSTs' Anticipated and Unanticipated Answers and Strategies

First of all, we focus on the results of PSTs' anticipation of students' answers and strategies across the addition, subtraction and multiplication lessons. As Table 2 shows, 53.5% of students' strategies (54 out of 101) were anticipated by PSTs while 33.7% (34 out of 101) were not anticipated by the PSTs' regardless of their successful anticipation of students' answers. PSTs were able to anticipate answers most of the time but missed out on the variety of strategies students used to get the same answer. In other words, the PSTs focused mainly on one way to get an answer, even though thinking about multiple ways was modeled heavily in their methods course. For instance, for 8+8+4+4, PSTs in the 3rd grade addition class succeeded in anticipating that students might use doubles to get 16 and 8 and then add 16 to 8 to get the final answer; however, the PSTs did not anticipate a popular strategy, 8x3, arising from the third graders' recent focus on multiplication. Even though PSTs could anticipate a general strategy, they did not typically anticipate all variants of the strategy. In the case of 19+21, PSTs expected students to use a compensation strategy, i.e. 20+21-1=41; 41-1=40. One student, nevertheless, knew the answer of 19+22 and thus compensated differently.

Table 2: Anticipated Answers versus Unanticipated Answers

Strategies / Grade Levels	Anticipated Answers		Unanticipated Answers	
	Anticipated	Unanticipated	Anticipated	Unanticipated
2nd Addition	8 (1) *	2	1 (1)	3 (3)
3rd Addition	6	9	3 (3)	0
Kinder Subtraction 1	3	3 (3)	0	1 (1)
Kinder Subtraction 2	6	7	0	1 (1)
Kinder Subtraction 3	5	2	0	2 (2)
3rd Subtraction	0	7	0	0
3rd Multiplication	16 (1)	1	1 (1)	1 (1)
5th Multiplication	10 (1)	3	0	0
Overall	54	34	5	8

Note:
*: "8" means the number of student strategies PSTs anticipated; the number "1" in the bracket refers to the number of those that were incorrect answers.

In consideration of PSTs' different responses to correct and incorrect answers, we found that they mostly anticipated correct answers. A plausible explanation might be that PSTs applied the strategies they had learned in the methods class to their teaching plans. When predicting the split-jump strategy, PSTs in the second grade addition class anticipated two different usages of this strategy and were able to smoothly handle students' reasoning. By contrast, PSTs without a perception of students' strategies in advance had a more difficult time making connections when questioning students. For example, in the third grade addition class, two co-teaching PSTs did not expect students to use multiplication (or arrays) in addition problems. When four students claimed that they used arrays in three addition problems, PSTs took a lot more talk turns to figure out what students' arrays looked like. For example, given 16+16+4+4, a student claimed to use an array of 8 down, 5 across, i.e. 5x8. PSTs repeatedly asked questions about this to ensure they understood what the array looked like. Once getting the idea of 8 down, 5 across, PSTs concluded the conversation and moved to the next one without making a reasonable connection between the addition and the multiplication. The

PSTs later reflected, “Arrays are a drawing strategy taught by teachers to encourage multiplication.” Overall, PSTs only anticipated 6 out of 19 incorrect answers. Another 5 incorrect answers resulted from students’ improper uses of PSTs’ anticipated strategies and the remaining 8 were unanticipated answers preceded by unanticipated strategies.

PSTs’ Talk Moves in Addition Problems

PSTs had to utilize different talk moves for anticipated versus unanticipated answers and strategies. The two groups that did addition number strings demonstrated an overwhelming use of revoicing and pressing talk moves in both classes, i.e. 29 in the 2nd grade class and 62 in the 3rd grade class (see Table 3). Also PSTs used more talk moves all together in situations of anticipated answers than those in situations of unanticipated answers (2nd: 19>16, 3rd: 45>22). Since the number string lessons were meant for PSTs to elicit and represent students’ mathematical thinking, PSTs were advised not to tell students how to solve the problems. Consequently, they often repeated students’ answers or strategies and pressed for more information to get clarity from students. This is consistent with Baxter and Williams’ (2010) results that PSTs deprived of their most familiar show-and-tell teaching mode likely lean toward silence and avoid telling students anything, expecting students can make use of their questioning to discover the right way to correctness.

Table 3: PSTs’ Talk Moves in Two Addition Classes

		Anticipated Answers		Unanticipated Answers		Total
Strategies		**Anticipated**	**Unanticipated**	**Anticipated**	**Unanticipated**	
2nd Addition	revoicing	10	1	3	5	19
	pressing	4	2	3	1	10
	orienting	0	0	0	0	0
	negotiating	2	0	2	2	6
	Total	**19**		**16**		**35**
3rd Addition	revoicing	11	16	7	0	34
	pressing	5	12	11	0	28
	orienting	0	1	2	0	3
	negotiating	0	0	2	0	2
	Total	**45**		**22**		**67**

The aforementioned situation was applicable when students themselves could clearly articulate their reasoning. Then PSTs could pretend not to know the strategy and press students to articulate their reasoning. In the 3rd grade addition class, for example, a student proposed multiplication (8x3) to solve 8+8+4+4 because of three “8’s.” A PST argued that she did not see three 8’s, pressing the student into saying more about the strategy. The student then made it clear that two 4s were put together first in his strategy. This is a successful episode where the PST used pressing to make the student’s reasoning clearly accessible to the whole class. On the other hand, revoicing and pressing did not move students’ reasoning forward when their reasoning was ambiguous. Students with difficulty presenting their strategies logically and meaningfully often revealed more helpful information when PSTs oriented the class to these students’ ideas or negotiated with them around alternative strategies. Solving 4+4+4+4+2+2, one student relied on “8 times 3” to get his 24 but could not offer where three 8’s came from. Then the PST negotiated by asking, “So you thought it was like the last problem (8+8+8)?” A “yes” answer was given. Under such a circumstance, the PST was able to look far beyond students’ reasoning and associate possible mathematical evidence with students’ reasoning so that more information was elicited from the student through the PST’s appropriate questioning.

Similar to the function of negotiation mentioned above, PSTs were inclined to turn to orienting and negotiating talk moves when they addressed students' mistakes. In the 3rd grade addition class, there were three episodes dealing with wrong answers and PSTs either invited the involvement of the class to discuss the problems in question, negotiated an alternative with students, or negotiated by soliciting other students' explanations.

Conclusions & Implications

Overall, most PSTs did anticipate a majority of students' answers and strategies after the instructional reading in their methods course. But there were still many alternative strategies to the same problem that PSTs failed to anticipate. Therefore, we as teacher educators need to guide students to try out various solutions to a problem, in particular multiple ways to get the same answer, especially considering students' knowledge at different grade levels (i.e., such as the third graders knowing multiplication). The results from this study also indicate that the revoicing and pressing talk moves prevailed in PSTs' classroom discourse. PSTs need to analyze the benefits and use cases of the talk moves in more depth; students' problem-solving procedures were well explored by means of revoicing and pressing but student reasoning was ignored to some extent. Admittedly, revoicing and pressing helped PSTs continue on in the exploration of student reasoning when they encountered something unexpected; but these strategies are far from being sufficient to elicit students' thinking and create learning opportunities that students can take up later. In this regard, the negotiating and orienting talk moves provide students with opportunities to reconsider and compare their strategies with others.

In the case of errors, the fact that most incorrect answers were not anticipated by the PSTs during their planning called our attention to the need for PSTs to think more about common incorrect answers, especially how students might make mistakes using strategies they anticipated. When students presented incorrect strategies, they could not easily jump out of their reasoning and discover the expected path to correct strategies when the only talk moves PSTs used were revoicing and pressing. One possible way to better support students is for PSTs to use negotiating and orienting question prompts to target students' reasoning, and bring students face to face with their emerging strategies. These discursive interactions with students allow PSTs to help students notice unproductive steps in their strategies. Further, an important next step is to support PSTs in helping students move beyond addressing their mistakes to building on their mistakes (e.g, when a student did 19+19 by adding only 9 to 9, PSTs should move beyond eliciting what the student was thinking when making the mistake, have students build on what they did successfully (9+9), push students to fix their mistakes (help them figure out how to make use of the tens).

When we delve further into PSTs' interpretations and representations of students' thinking, it is evident that there is room for teacher educators to orient the methods course with how to build on student mistakes and encourage PSTs to try the orienting and negotiating talk moves. Therefore, teacher educators can encourage PSTs to embrace the uncertainty of what students will say and see if the class can collectively make sense of and build on places where students' strategies are breaking down. This could be done through such pedagogical approaches as rehearsals (Grossman, Hammerness, & McDonald, 2009), creating opportunities of modeling real classroom teaching to help PSTs practice dealing with some specific mathematical topics like unanticipated strategies and student mistakes. By means of these rehearsals, PSTs could possibly enhance their teaching skills and mathematics expertise when teacher educators focus PSTs' attention on those specifics of teaching as well as those "variations of the practice as it relates to particular students and mathematical goals" (Lampert, et al., 2013, p. 238).

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MATHEMATICAL MAKING IN TEACHER PREPARATION: WHAT KNOWLEDGE IS BROUGHT TO BEAR?

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In this paper, we describe an experience within mathematics teacher preparation that engages pre-service teachers (PSTs) in Making and design practices that we hypothesized would inform their conceptual and pedagogical thinking. With a focus on the design of new tools to support mathematics teaching and learning, this Learning by Design experience has PSTs exploring at the crossroads of content, pedagogy, and Making. We report our findings of the variety of forms of knowledge that PSTs brought to bear on their design work. As the engagement and advancement of these forms of knowledge is essential to effective mathematics teaching, these findings suggest the promise of a making-oriented experience within mathematics teacher preparation coursework.

Keywords: Technology, Teacher Education-Preservice, Teacher Knowledge

Preservice elementary teachers typically come to teacher preparation with limited conceptions of mathematics (Association of Mathematics Teacher Educators, 2013) and a model of mathematics teaching based solely on their own classroom experiences as students (Lortie, 1975). These models can be characterized by appeals to rules and procedures (Ball, 1990; Ma, 1999; Thompson, 1984), problems whose solutions are predetermined and predictable (Schoenfeld, 1992; Thompson, 1984), and teaching in which mathematical information is imparted from teacher to student with unquestioning acceptance (Lampert, 1990; McDiarmid, Ball, & Anderson, 1989). This is a problem because this model of mathematics teaching is not consistent with a pedagogy that is viable for learning mathematics with understanding. Consequently, as part of their preparation for elementary teaching, prospective elementary teachers must be presented with opportunities to challenge their current models of mathematics teaching and learning that engage them with both the problems of mathematics and the problems of children's learning of mathematics.

At the same time, the proliferation of spaces for digital design and fabrication suggests new opportunities to teach and learn new mathematical things in new ways and to even think in new ways about what teaching and learning mathematics might look like. However, research is only beginning to identify the mathematical thinking and reasoning that these technologies might make possible. And there is no research that explores what these technologies might offer to support the preparation and professional development of teachers. As such, this proposal presents a novel Making-oriented experience within mathematics teacher preparation that tasks pre-service elementary teachers with designing, fabricating, and evaluating new manipulatives (Post, 1981) aimed at engaging and advancing learners' mathematical thinking and reasoning. Thus, this project addresses the need for better preparation of elementary mathematics teachers through education research that seeks to understand the processes and potential benefits of teacher learning in a Maker context.

Proceeding from the hypothesis that Making and doing lead to new ideas and experiments in embodied (Johnson, 2007), networked (Latour, 2005), and tool-centric (Vygotsky, 1978) engagement that, in turn, will lead to powerful innovation in mathematics teaching and learning, this project seeks to address the following question: *What forms of knowledge are brought to bear on pre-service elementary teachers' design work as they make new manipulatives to support the teaching and learning of mathematics?*

Theoretical Framework

In the context of math-focused exhibitions in the designed informal learning environments (National Research Council, 2009) of science centers and museums, investigators have identified evidence of visitors engaging in algebraic (Pattison, Ewing, & Frey, 2012) and spatial reasoning (Danctep, Gutwill, & Sindorf, 2015), and also demonstrating qualitative, intuitive understandings of slope (Nemirovsky & Gyllenhall, 2006; Wright & Parkes, 2015). Within Makerspaces (Peppler, Halverson, & Kafai, 2016), where activities are designed with a variety of learning goals in mind, some research suggests that in order to see and support opportunities for mathematical activity, it is necessary to look beyond the content and use a more broadened conception of mathematics – "including mathematical dispositions, habits of mind and identity" – to identify the mathematics in which learners engage (Author et al., 2016a). These findings of mathematical engagement in informal settings point to the possibilities that semi-structured design-centered experiences can offer in relation to mathematics teaching and learning.

As for K-12 educational settings, Shaffer's (2005) use of design tasks in a microworld (Papert, 1980) to teach transformational geometry, and Cochran and colleagues' (2016) suggestions about how middle school teachers can use 3D printing as a context to promote geometry understanding, lend further credence to the proposition that Making can provide a gateway to meaningful interaction and deepened understanding of both content and pedagogy by engaging preservice teachers (PSTs) in the design of new manipulatives and corresponding tasks that generate environments for mathematical thinking and learning. Research can shed light on the creative and participatory practices associated with teachers' Making experiences and how those experiences inform their knowledge and their identities as elementary mathematics teachers.

Teachers as Designers

In investigating the experiences of PSTs designing for mathematical learning, we connect with other researchers' conceptions of *teachers as designers* (Kalantzis & Cope, 2010; Maher, 1987). Svihla et al. (2015) refer to "teachers as designers of learning experiences to emphasize teacher involvement in designing from pre-instructional designing of lessons, activities, units and learning environments to their design work that continues into the classroom" (p. 284). When teachers are given agency to craft their own manipulatives and corresponding curricular materials, they assume ownership over these materials and the learning environments they generate, thereby coming to see themselves as agents of curricular and pedagogical reform (Leander & Osborne, 2008; Priestley, Edwards, Priestley, & Miller, 2012). In doing so, they find themselves moving toward more legitimate forms of participation (Lave & Wenger, 1998) as they develop their identities as designers of mathematical instruction.

Learning Teaching by Design

The premise of this project follows from the proposition that it is productive to develop teacher knowledge within a context that honors the connections between its constituent forms of knowledge. Accordingly, we took somewhat of a Learning by Design approach (Koehler & Mishra, 2005; Koehler, Mishra, Hershey, & Peruski, 2004; Mishra & Koehler, 2003), a methodology that was developed as a means to advance teachers' technological pedagogical knowledge, or TPCK (Koehler & Mishra, 2010). An environment is created in which teachers naturally confront content, pedagogy, and technology so that the connections between are honored and maintained. Within this environment teachers assume the role of designers of technology and work collaboratively in small groups to develop technological solutions to authentic pedagogical problems. "By participating in design, [they] build something that is sensitive to the subject matter (instead of learning the technology in general) and the specific instructional goals (instead of general ones). Therefore, every act of design is always a process of weaving together components of technology, content, and pedagogy" (Koehler & Mishra, 2005, p. 95).

Our own Learning by Design approach to mathematics teacher preparation is grounded in several principles. First, constructionism (Harel & Papert, 1991) is the theory of learning that undergirds the Maker movement's focus on problem solving and digital and physical fabrication (Halverson & Sheridan, 2014, p. 497). Second, Piagetian constructivism, a theory of learning that is well suited to the way learning works in an environment of mathematical inquiry (Author et al. 2016b), informs the pedagogy. Indeed, the power of manipulatives lies in their capacity to support the construction of abstract mathematical concepts from sensorimotor engagement with concrete tools (Kamii & Housman, 2000; Piaget, 1970; Vygotsky, 1978), a process grounded in the theory of constructivism. Third, knowledge of the content to be taught and a variety of ways in which that content may be presented, represented, and experienced (Ball & Bass, 2009; Ball, Thames, & Phelps, 2008; Shulman, 1986) informs the mathematics. Finally, Dewey's (1938) and Pinar's (2012) broadened conceptions of curriculum that frame learning as the product of play, experimentation, and authentic inquiry align with our conception of curriculum. Still, the rich scholarship devoted to teacher knowledge reflects the complexity of the question of precisely what forms of knowledge might actually be brought to bear on PSTs' design work (Ball, 1990; Borko & Livingston, 1989, 1990; K. F. Cochran, DeRuiter, & King, 1993; Grossman, Wilson, & Shulman, 1989; Hill, Ball, & Schilling, 2008; Ma, 1999; Shulman, 1986).

Methods

The study took place in two sections of the first of two required specialized mathematics content courses for pre-service elementary teachers at a large public university in the northeastern United States. Our Making-oriented experience began with PSTs' inquiries into the principles that ground our Learning by Design approach and that are among the standard course goals and objectives for this course. Specifically, these include providing PSTs with opportunities to reconceptualize the content of K-6 school mathematics (including number, arithmetic, and algebraic thinking) while also promoting an inquiry-oriented pedagogy by fostering an understanding of the nature of mathematics, assimilating a constructivist theory of learning mathematics, acquiring a model of how learning works in interaction with manipulatives and other technologies, designing instructional tasks that both promote and reveal students' understanding of mathematics, and developing an understanding of the way in which students' content knowledge develops over time, as well as the struggles they're likely encounter. Concurrently, as PSTs learned to use 3D design and fabrication technologies, they engaged in an iterative "Design Thinking Process" (Stanford University Institute of Design, 2004).

As students were permitted to work either individually or in groups on a design project, the twenty-six students who consented to participate in the study comprised a total of twenty-one groups. The data corpus consists of the following three components of each group's "design case" (Boling, 2010): 1) a "Project Idea Assignment," which describes the group's initial thoughts about a manipulative they want to work on; 2) a "Project Rationale Assignment," which provides an account of why and how a group thinks their project should work from a mathematical learning point of view as well as how their design reflects an understanding of what mathematics is and of how learning happens; and 3) a "Final Paper and Design Show," which includes a short research paper about the project and a PowerPoint that describes the intended purpose of the manipulative, the corresponding tasks that were created, and the group's findings from an intended user's manipulative-mediated engagement with those tasks.

To initiate the analysis of that data, we chose three design cases at random. Three researchers individually analyzed the components of those cases and generated codes (Corbin & Strauss, 2008) that identify forms of knowledge that were revealed in the elements of PSTs' written work. Then, the three researchers got together to generate a cumulative list of codes and clarifying definitions (Table 1). Next, each of the researchers used those codes to analyze all of the remaining design cases. As the analysis continued, new codes were also introduced and then shared among the researchers.

Table 1: Analytical Framework for Coding Knowledge Types

Code (Knowledge of…)	Description
Mathematics Content	Common content knowledge of mathematics.
Specialized Mathematical Knowledge	Variety of ways mathematical ideas can be expressed and explained.
Content and Students	Common student struggles and misconceptions; planning for student thinking.
Standards and/or Curriculum	Acknowledgement of Common Core and/or curricular materials as an important aspect driving instruction; knowledge at the mathematical horizon.
Distinction between Concrete & Abstract	Abstract ideas are abstracted from concrete representations.
Constructivism	Knowledge is constructed; model of knowing as understanding; role of exploration and experimentation; relevance of prior knowledge.
Research on Student Learning	Use of mathematics education research literature.
Task Design for Problem Solving and/or Assessment	Tasks designed for use with a manipulative require challenge/productive struggle, but can also play a dual role of learning and assessing.
Personal Experiences	Students' personal mathematical experiences (both as learner and teacher) inform their design.
Student Affect	Importance of designing tools and tasks that make learning engaging and fun.
Mathematical Tools	Knowledge of currently available tools (e.g., integer chips, base ten blocks, number lines).
Manipulatives	General comments about how learning works with manipulatives; as embedded representations of mathematical ideas.

Intercoder reliability was calculated using percentage of agreement. Since three coders participated in the analysis, each coder was compared to one another in a pairwise manner. Thus, every coding decision had a total number of three pairs to check for agreement. The number of agreements was noted, and ultimately divided by the total number of possible agreements in order to calculate the percentage of agreement. The data presented here had a percentage of agreement of .82, well within the standard put forth by Neuendorf (2002).

Results

Our analysis showed that students used a variety of forms of knowledge in the course of their "Design Thinking Process," as demonstrated in the table of codes provided above. These knowledge types ranged in frequency of occurrence from 68% to 100% (see Table 2).

Table 2. Code Frequencies and Match Percentages

	Mathematics	Specialized Mathematical	Content & Students	Standards and/or Curriculum	Concrete/ Abstract	Constructivism
Coder 1	21	12	20	12	11	19
Coder 2	20	20	20	17	20	20
Coder 3	21	20	19	14	19	20
Total	62	52	59	43	50	59
Match %	0.97	0.75	0.97	0.78	0.71	0.87

	Research on Student Learning	Tasks for Problem Solving & Assessment	Personal Experience	Student Affect	Mathemati cal Tools	Manipulatives
Coder 1	17	12	11	7	11	21
Coder 2	13	19	19	9	12	21
Coder 3	14	18	17	8	14	21
Total	44	49	47	24	37	63
Match %	0.78	0.71	0.68	0.87	0.75	1.00

From this analysis, we see that every group drew on both Knowledge of Mathematics and Knowledge of Manipulatives in their design process. Comments such as "the manipulative will aid students in learning geometry because they will be able to turn, rotate, and reflect on the shapes that they will make" and "the main idea behind the design of our tool… is to help students determine the area and perimeter of two similar figures" demonstrate that knowledge of mathematics content was an extremely important aspect of their design thinking. In particular, these excerpts demonstrate some of the ways the groups expressed their knowledge of manipulatives as embedding mathematical principles.

In the course of thinking about content in this way, each of the groups also leveraged their Knowledge of Manipulatives. We saw a diversity of thinking about manipulatives, ranging from the more generic ("Manipulatives are defined as concrete objects that aid in classification.") to the more sophisticated ("Manipulatives not only allow students to construct their own cognitive abilities for abstract mathematical ideas and processes, but they also provide a concept and common language behind it."). Other students professed a more nuanced understanding of the role of manipulatives in instruction, acknowledging they are best used with other teaching techniques: "Fraction circles are a simple, clear 'physical tool' for teaching this challenging concept, and when used in conjunction with other [fraction contexts] (equal sharing, part-whole, etc.) can be very illustrative." One group drew on their own review of the research literature to inform and support their thinking about manipulatives, writing that "These concrete materials are meant to assist children at all levels of education including understanding processes, communicating their mathematical thinking, and extending their ideas to higher order thinking levels (Balka, 1993)."

Evident in the PSTs' Knowledge of Manipulatives is the related Knowledge of Constructivism as a learning theory that can inform design decisions. Phrases such as "help students construct the idea," "children can tinker with the board and the pieces to find the relationships between the pieces and the groups," and "create a way to teach even and odd numbers that does not revolve around memorizing," all demonstrate the ways the groups were thinking about making tools that allowed for exploration and discovery, both hallmarks of the pedagogical implications of a constructivist theory of learning. In this way, students began to seriously consider not only features of an inquiry-based pedagogy, but also the ways in which tools can be seen to support those implications in classrooms.

Although their design assignments hadn't explicitly called for students to make connections between their design ideas and the coursework, almost every group conceived of their design and the learning it aimed to promote through the lens of their Knowledge of Content and Students. They drew on class readings, the math education literature, and their own experiences as learners of mathematics to anticipate concepts that students would be likely to struggle with. These considerations were evident in statements like, "Since some children have non-anticipatory coordination between groups and shares, my manipulative serves as a way for students to utilize the pieces to see the distribution of shares to each group."

We also saw evidence of other knowledge categories, though with less frequency than those elaborated above. These forms of knowledge include considerations of the relationship between concrete and abstract representations, knowledge of task design for problem solving and/or assessment, knowledge of currently available mathematical tools, and the importance of considering student affect in their designs.

Conclusion

At the crossroads of digital fabrication technologies, human-centered design practices, and constructivist orientations to mathematical thinking and learning, students and teachers are afforded a host of new possibilities. As researchers exploring how these technologies might be used to engage teachers and students in new forms of learning, we hypothesized that a making-oriented approach to pedagogical and curricular change aligned with the kind of progressive, inquiry-oriented pedagogy we aim to cultivate in students preparing to teach mathematics. Accordingly, we developed an approach to nurturing students' inquiry-oriented pedagogy that leverages design practices and digital fabrication technologies as a resource for their learning. While we recognize that teacher preparation is complex and that pedagogical change is difficult, that we identified in PSTs' design work a variety of forms of knowledge whose advancement is essential to mathematics teaching, these findings suggest the promise of a making-oriented experience within mathematics teacher preparation.

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ROLE OF REPRESENTATION IN PROSPECTIVE TEACHERS' FRACTIONS SCHEMES

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This research report explores relationships between fractions' task representations (discrete, rectangular, or circular) and elementary prospective teachers' (PTs) fractions conceptions. Studies show PTs' conceptions of fractions are centered on a part-whole understanding, which may be problematic when teaching children about improper fractions. We studied PTs' conceptions of fractions using a task-based written assessment. The assessment also included PTs' rankings of task difficulty. We found that PTs' responses involving circles representations aligned best with the empirical trajectory of children's developing understandings of fractions. We discuss implications for supporting PTs in conceptualizing fractions as measures.

Keywords: Rational Numbers, Teacher Education-Preservice, Teacher Knowledge

Introduction

Although the conception of a fraction as a *measure* (Lamon, 2007) is emphasized early in the Common Core State Standards (CCSSM, 2010), many middle grades students and prospective teachers (PTs) remain focused on *part-whole* meanings of fractions (Newton, 2008; Norton & Wilkins, 2010; Olanoff, Lo, & Tobias, 2014). In this research project, we build off the learning trajectory for PTs' understandings of fractions as measures described by Lovin, Stevens, Siegfried, Norton, & Wilkins (2016). Our study aligns well with the PME-NA Conference Theme of "Synergy at the Crossroads" because our approach involves looking across both fraction representation and fraction task structure to better understand the role of each in understanding and supporting PTs' understanding of fractions.

Background and Purpose

Norton and Wilkins (2010) designed written assessments of middle grades students' fractions' schemes using rectangular and circular representations of fractions. A subset of these (rectangular) items were validated via clinical interviews with sixth-grade students (Wilkins, Norton, & Boyce, 2013). Lovin et al. (2016) used the written items to assess the fractions schemes of PTs before and after instruction in mathematics courses for elementary PTs. In examining PTs' written responses, Lovin et al. (2016) noted that PTs often set up proportions and used division to correctly solve fractions tasks in ways that would not be available to elementary or middle grades students. They mentioned that PTs' use of such procedures to find equivalent ratios may have been confounding researchers' assessments of some of the PTs' fractions understandings (potentially producing both false positives and false negatives). This could lead to difficulty in assessing and supporting PTs' learning, particularly for fostering PTs' self-monitoring of their understanding of the mathematical goals and constraints that their prospective students may face.

In this paper, we report on results of modifying Norton and Wilkins' (2010) items and methods to explore relationships between the form of fractions' task representations (discrete, rectangular, or circular) and elementary PTs' fractions conceptions prior to instruction in their college course. Our aim in this study is to understand how PTs' ways of operating with fractions in different representations are connected, so that we, as mathematics educators, can plan to introduce and moderate perturbation (von Glasersfeld, 1995) that will help them to construct more powerful fractions schemes. With this aim, we modified the assessment approach of Norton and Wilkins (2010) and Lovin et al. (2016) to isolate differences in PTs' responses to fractions tasks.

Theoretical Framework

We adopt a radical constructivist epistemology in modeling PTs' fractions meanings as the product of their organizing mental structures (schemes) to fit their experiences (von Glasersfeld, 1995). The construct of scheme refers generally to the way researchers model how individuals operate mentally in service of a goal. A scheme consists of three parts – recognition of a situation, operations (mental actions), and an expected outcome. Individuals' schemes become established as they become refined and generalized through their use, via processes of assimilation and accommodation (Piaget, 1970). When a scheme is interiorized, the situation, operations, and anticipated result of operating are experienced altogether as a unified and connected structure (a concept) that can itself be operated upon (Piaget, 1970).

Fractions Schemes

We focus on four specific schemes pertaining to fractions identified by Steffe and Olive (2010) – the parts-out-of-wholes fraction scheme (PWS), the partitive unit fraction scheme (PUFS), the general partitive fraction scheme (PFS), and the iterative fraction scheme (IFS). The PWS involves *partitioning* a whole into discrete pieces that can be disembedded (removed from the whole without modifying the whole) and double-counted to form a numerosity of part(s) within a numerosity of a whole. The PUFS builds upon the PWS as the individual conceives of the *size* of a disembedded part and its relation to the size of the whole (i.e., that *iterating* the amount of $1/n$ n times results in the size of 1). The PFS extends this notion to the size of a composite (but proper) fraction. An individual with an iterative fraction scheme (IFS) understands the size of an (im)proper fraction (m/n) as the result of coordinating mental operations to include partitioning the size of '1', disembedding a unit fractional size ($1/n$), and iterating the disembedded fractional unit m times.

In order for an individual's fraction scheme to become interiorized as a fraction concept, his or her fraction scheme must be *reversible*. For instance, an individual with a *reversible* IFS could reverse his or her ways of operating to determine the size of '1' from a given improper fraction size. Reversing the PFS involves forming the size of '1' from a given (composite) proper fraction size, and reversing the PUFS involves forming the size of '1' from a given unit fraction size. Reversing the PWS involves forming the numerosity of the whole from a given proper fraction (e.g., reasoning that if three parts represents the fraction 3/7, then the whole must be 7 parts). In the next section we describe specific examples of these four fractions schemes.

Task Structures

Figure 1 displays four task structures, two involving proper fractions (Task PFS1 and Task PFS2) and two involving improper fractions (Task IFS1 and Task IFS2). Consider that if a task involves a discrete representation for a unit fraction (e.g., a dot or a chip) then forming a size (via a counting measure) is often indistinguishable from forming a numerosity. Thus, Task PFS1 theoretically requires a PWS in the discrete representation and a PFS in the bar and circular representations. In each of the three representations, the correct response is 2/5, but an individual with a PFS might instead respond with a slightly different fraction (such as 4/7 or 3/8) in the bar and circle models if rulers or protractors are not available to make precise measurements.

Task PFS2 theoretically requires a *reversible* PFS in the bar and circle representations. To form the size of a unit fraction from the proper fraction requires intermediately forming the size of the whole. In the dots representation, the task requires a reversible PWS.

Theoretically, Task IFS1 requires a reversible IFS, as it asks for the size of '1' from a given improper size. To form this size, one could partition the given amount into nine equivalent one-fourths and then iterate that amount four times. However, a PT could potentially solve the task in the discrete representation using a ratio understanding. A PT might coordinate partitioning and iterating to solve the task in the bar representation, but not the circular representation, because of an established understanding of a circle as necessarily the size of ‘1’.

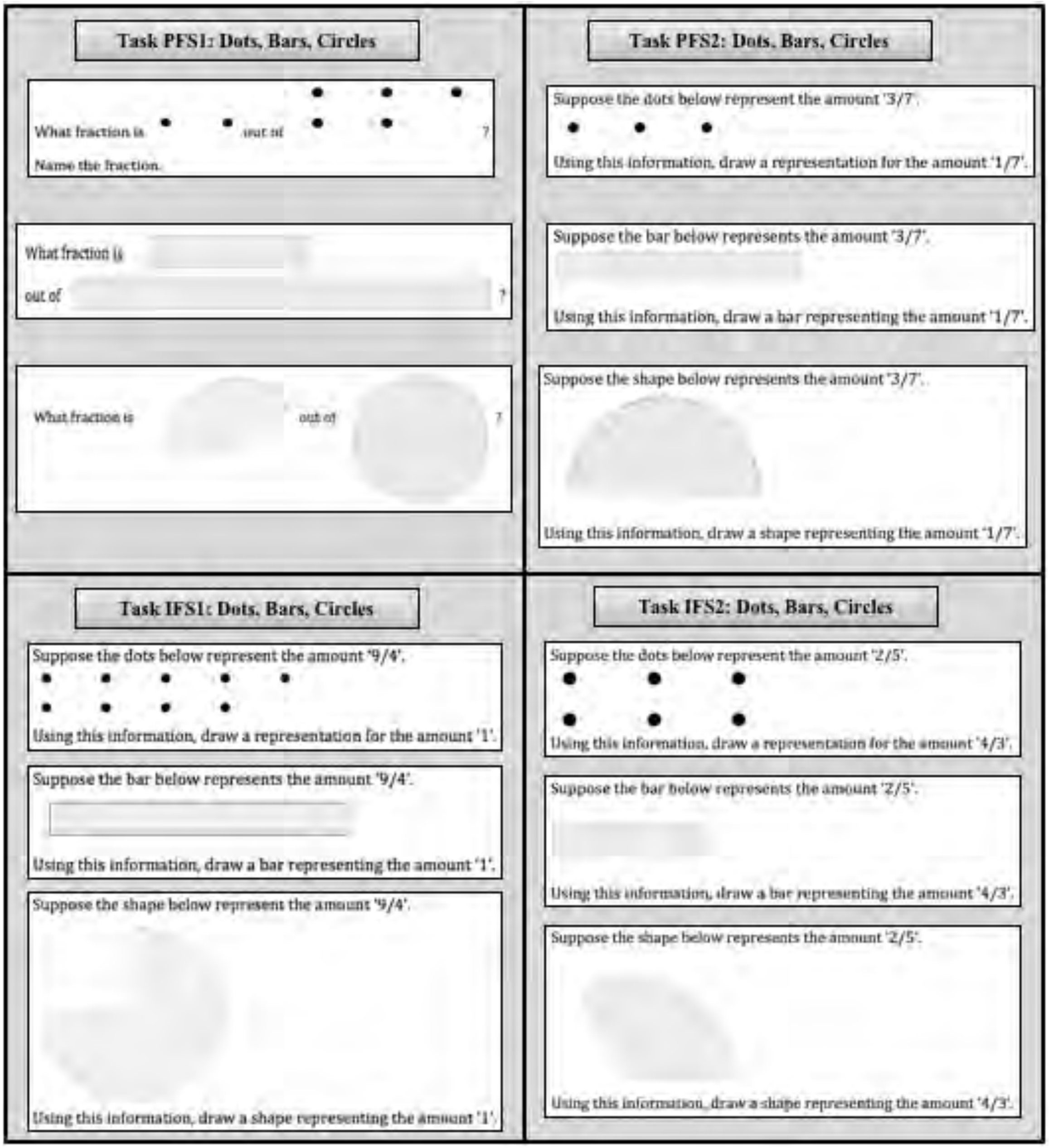

Figure 1. Four fractions task structures with dots, bars, and circles.

Task IFS2 theoretically requires recursive use of an IFS. One could solve the task by first partitioning and iterating to form the size of 1 (using a reversible IFS) and then partitioning and iterating to form an improper fractional size (using an IFS). PTs might instead approach the task by finding a common denominator by which to determine equivalent fractions without forming the size of 1. Lovin et al. (2016) noted that PTs may rely on such procedures when encountering improper fractions because they have yet to coordinate three levels of units: the unit fraction, the improper fraction, and the whole–when attempting to iterate a unit fraction beyond the whole, they lose track of the size of the whole (Steffe & Olive, 2010).

Methods

We administered a written assessment to 76 PTs in an Elementary Math Methods course at a U.S. university, prior to class discussion of fractions. To reduce the length of the assessment and to isolate differences in task representation, each of the PTs completed one of three forms. On the first form, there were four tasks with dots followed by four structurally identical tasks with bars. On the second form, the four dots tasks were followed by four circles tasks, and on the third form, the four bars tasks were followed by four circles tasks (see Figure 1). The four tasks in each representation were consistently given in the sequence: IFS1, PFS2, PFS1, IFS2.

These tasks use the same wording and form of items from Norton and Wilkins (2010), and our process for scoring items also followed their approach. First, a graduate assistant blinded and reorganized scanned pages of the assessments so that the two raters (authors) could not identify PTs' names or assessment forms. If there was strong indication that a PT had constructed a scheme, we scored it '1', and if there was strong counter-indication that a PT had constructed a scheme, we scored it '0'.

Figure 2 displays sample responses scored as '0' or '1' for two items in the bars and circles representations. For instance, for Task PFS2, to assign a score of '1', we were looking for evidence that the individual had partitioned the given size into three equally-sized pieces and drawn one of those pieces. Indication that the individual had instead partitioned the given amount into seven pieces would suggest assigning a score of '0' for Task PFS2. Note that we coded Task IFS2 with '1' if the PT determined the (approximately) correct fractional size by first using a procedure to determine equivalent fractions.

We calibrated our scoring by discussing our inferences and interpretations of ten randomly selected responses to each of the 12 items. As we independently coded the remaining responses, we also assigned '0.6' and '0.4' to indicate "leaning" toward indication or counter-indication, respectively. The (linear) kappa scores for the two raters across the 12 items ranged between .44 and 1, with a mean kappa score of .75, suggesting substantial inter-rater agreement (Landis & Koch, 1977). Agreement was strongest for the dots tasks, for which the PTs' responses were less ambiguous (kappa > .9 for each of the four tasks). The bars and circles tasks had the lower kappa values, as we had to make inferences from the PTs' markings about their intent to create a correct fractional size because they were not provided with a ruler or protractor with which to make exact measurements.

After computing a satisfactory kappa, we reconciled our scores. We assigned a reconciled score of '1' if we had each marked either a '1' or a '0.6', and we assigned a reconciled score of '0' if we had each marked either a '0' or a '0.4'. If one rater had marked '0.4' and the other '0.6', then we assigned a reconciled score of '0.5'. For the remaining responses, we returned to the data to decide on a score of either '0' or '1' for each item. Whereas Lovin et al. (2016) further used the sum of four reconciled scores on similar items to assign an overall '1' or '0' regarding an individual's construction of a fractions scheme, our item scoring remained focused at the item level.

We used the Wilcoxon rank sum (Wilcoxon, 1945) to test whether there were significant pairwise differences in mean scores for each item for this group of PTs. The null hypotheses were that there would not be significant pairwise differences in mean scores across task types or representations. We tested for differences in mean scores across task representations (dots, bars, and circles), controlling for the task types (PFS1, PFS2, IFS1, and IFS2), and we also tested for pairwise differences in mean scores across items within each of the three representations.

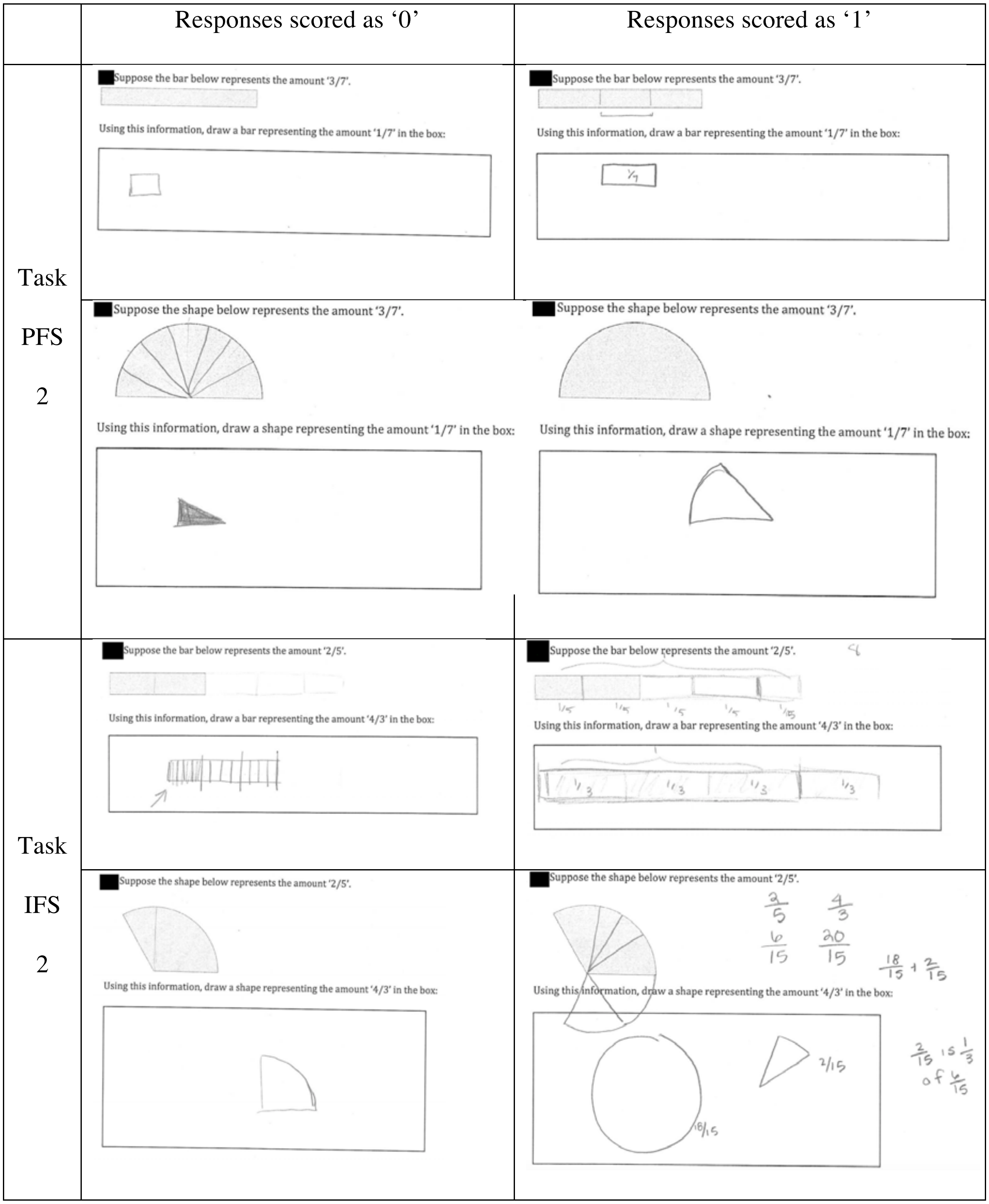

Figure 2. Sample responses coded with '0' or '1' in the bars and circles representations.

PTs' Rankings of Task Demands

Each of the three forms included an identical ninth question, intended to assess PTs experiences of the cognitive demands of their previous eight tasks. The PTs were asked to rank the previous eight tasks they had completed in order from least difficult (1) to most difficult (8) and to then explain their ranking decisions. The instructions emphasized that each of the numbers '1' through '8' were to be used exactly once in the rankings. We discarded PTs' responses to the ninth question in the analysis if they repeated more than one of the numbers 1-8, which resulted in including 71 of the 76 responses. We repeated the Wilcoxon rank sum tests with the PTs' difficulty rankings to test where there were pairwise differences in the PTs' experiences of difficulty across task type or task representation.

Results

Table 1 displays the the mean scores and difficulty rankings for each of the 12 tasks. Within each of the three representation types, the Wilcoxon rank sum test indicated significant differences (at alpha = .05 level) between both the mean performance and difficulty rankings on the two IFS items. Task IFS2 was more difficult than Task IFS1 (and was also more difficult than either PFS1 or PFS2), and this was in concordance with the PTs' rankings of the two items' difficulty. The difference in performance was not significant for the two PFS items across all three representations (p=.500 for dots, p=.355 for bars, and p=.232 for circles). For dots and bars, the ranking of the difficulty of Task PFS2 was significantly greater than Task PFS1; for circles the difference in difficulty was marginally insignificant (p = .012, p=.009, and p=.073, respectively). Across each of the four items, scores on dots tasks were significantly higher than circles tasks. One PT expressed, "The dots make no logical sense to me, and the shapes [bars] are easy until a fraction is more than a whole. What is that supposed to look like?" This response affirms that both task structure and task representation are important considerations for assessing PTs' fractions knowledge.

Table 1: Results Across Task Representation

	PFS1	PFS2	IFS1	IFS2
Dots Mean Score (Mean Dots Difficulty Rank)	.906[a] (2.062)[ab]	.937[ab] (2.917)[b]	.760[b] (3.771)[b]	.344[b] (5.833)[b]
Bars Means Score (Mean Bars Difficulty Rank)	.522[ac] (4.467)[a]	.622[a] (3.333)[c]	.656 (4.467)[c]	.300[c] (6.333)
Circles Mean Score (Mean Circles Difficulty Rank)	.766[c] (4.553)[b]	.681[b] (4.021)[bc]	.457[b] (5.553)[bc]	.128[bc] (6.745)[b]

Notes. Score was from 0 (counter-indication of scheme) to 1 (indication of scheme). Difficulty ranking was from 1 (easiest) to 8 (hardest). Using Wilcoxon signed rank test, with alpha = .05.
[a] Denotes significant difference between dots and bars. [b] Denotes significant difference between dots and circles. [c] Denotes significant difference between bars and circles.

Though Task IFS2 was ranked as the most difficult across all three representations, it was significantly less likely to be answered correctly in the circles representation than in the bars or dots representations. A common explanation for the PTs' assignment of difficulty rankings was that PTs found Task IFS2 "confusing" – particularly in the circles representation. We infer that this was often because the PTs were unfamiliar with a whole circle not representing the size of '1' (see Figure 3 for a sample response).

Supporting this inference, PTs were more likely to correctly answer Task PFS1 (in which the whole circle represented the amount 1) as representing 2/5 in the circles representation than in the bars representation. They more often incorrectly responded with '1/3' or '1/2' in the bars representation. This also explains why the difficulty ranking and score were each lower for Task

PFS2 in the bars representation than in the circles representation. For instance, one PT mentioned that "we didn't really know what a bar represented, so I just had to divide them up equally and go with it."

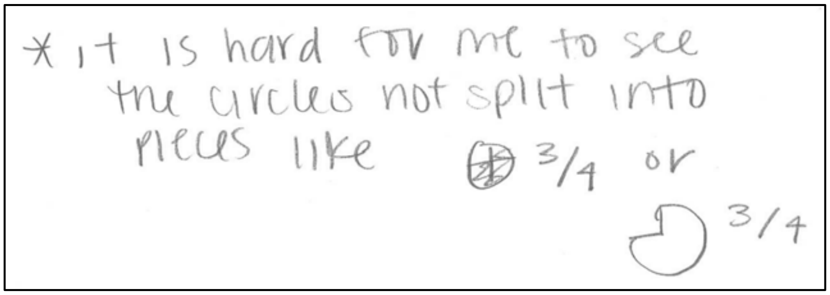

Figure 3. One PT's explanation of her difficulty rankings.

Conclusions

The results of our study generally concur with the learning trajectory described by Norton and Wilkins (2010) and Lovin et al. (2016), in that the PTs in our study were more likely to make sense of PFS tasks than IFS tasks. However, some PTs' familiarity with part-whole interpretations of fractions and proportions resulted in their being able to correctly solve tasks with dots but not structurally identical tasks with bars or circles, even though bars and circles tasks followed the dots tasks in their written assessments. PTs' ranking of task difficulty aligned best with the empirical trajectory of middle grades students' fractions schemes in the circle representation. This suggests that PTs enter their elementary mathematics course well-suited to appreciate the challenges elementary students face in constructing fractions schemes, and that instructors can support PTs by introducing non-standard circular representations of fractions.

We believe more investigation is necessary for the disentanglement of differences between PTs' fractions schemes across representations to inform instructional practice. Anecdotally, we have found in our teaching that focusing on non-standard circular representations early in a course can be frustrating and temporarily reduce some PTs' confidence, but eventually result in their reorganization of their fractions concepts. Future research may investigate the influence of varying the introduction of task structures and representations in elementary math courses. Important considerations include PTs' resilience, motivation, and self-efficacy for teaching mathematics.

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SCHOLARLY PRACTICE AND INQUIRY: DYNAMIC INTERACTIONS IN AN ELEMENTARY MATHEMATICS METHODS COURSE

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This paper represents research that exists at the crossroad of scholarly practice and scholarly inquiry. We share the design, enactment and empirical examination of an elementary methods course activity, Exploring and Supporting Student Thinking (ESST) which engaged 18 prospective teachers in two sessions of one on one problem posing with 3rd grade students. Results mirror outcomes from existing literature on student interviews and letter exchanges.

Keywords: Teacher Education-Preservice, Teacher Knowledge

Research suggests that teachers who understand how students think about particular mathematical ideas will be better positioned to recognize, interpret and support these ideas in their instruction (Brown and Borko, 1992; Fenema and Franke, 1992). Research on Cognitively Guided Instruction (CGI) has demonstrated teacher knowledge of student thinking, reasoning and strategies can lead to gains in student achievement (Carpenter & Fennema, 1992; Carpenter, Fennema, Franke et al., 2000). Ball and her colleagues' work on mathematical knowledge for teaching identified knowledge of content and students as a crucial facet of pedagogical content knowledge necessary for teaching mathematics effectively (e.g. Hill, Ball & Schilling, 2008).

In light of these findings it has become increasingly important for mathematics teacher educators (MTEs) to assist prospective elementary mathematics teachers (PTs) in developing knowledge of children's thinking. Jacobs, Lamb and Phillip's (2010) work on professional noticing of children's mathematics has become a popular framework to explore the ways in which teachers attend to, interpret and respond to students' mathematical thinking. Mathematics methods course activities sometimes provide PTs opportunities to examine and interpret authentic (and/or instructor-generated) samples of student work depicting invented computational strategies or mathematical reasoning as a means to gain experience interpreting and responding to student thinking (e.g. Tyminski, Land, et al., 2014). We term these types of interactions as static, in that there is no student to interact with during the process of interpreting the work, and once PTs have done so, there is no opportunity to respond authentically to students and observe the result. Although we see value in these types of interactions in developing PTs' knowledge of students' mathematical thinking and include examples of them in our methods courses, we sought to design and enact an activity in our early field experience that would foster PTs' understanding of how to elicit and support student's mathematical thinking and which would be dynamic in nature; allowing for a sustained exchange between the PTs and the student.

This paper represents research that exists at the crossroad of scholarly practice and scholarly inquiry. We outline the process in the design, enactment and empirical examination of an elementary methods course activity, Exploring and Supporting Student Thinking (ESST), and answer the question, "What are the experiences of PTs within the ESST activity?"

Literature Review

Scholarly Inquiry and Practice

In methods course activity design and enactment, the authors seek to leverage the interplay between research and practice thorough the processes of scholarly inquiry and scholarly practice (Lee

& Mewborn, 2009). Scholarly inquiry is the exploration of "issues and practices through systematic data collection and analysis that yields theoretically-grounded and empirically-based findings" (p. 3), while scholarly practices are "adapted from empirical studies of the teaching and learning of mathematics and the preparation of mathematics teachers" (Lee & Mewborn, 2009, p. 3). "Scholarly inquiry and practices are interrelated in that MTEs use empirical studies in mathematics education to build practices that are labeled scholarly. In addition, scholarly practices can inform directions for scholarly inquiry regarding PSTs' mathematics teaching and learning" (Kastberg, Tyminski & Sanchez, in press). In order to create the ESST Activity as an example of scholarly practice, we reviewed the literature on existing scholarly inquiry on dynamic interactions in a methods course in order to: 1) synthesize knowledge on the potential impact of these activities on PTs' learning and 2) inform the activity's design by understanding the variation and commonalities of activities described in other researchers' scholarly inquiry.

Dynamic Interactions

Examples of dynamic interaction activities found within the research literature included asynchronous activities such as letter writing exchanges (e.g. Crespo, 2000; 2003; Norton & Kastberg, 2012), as well as face-to-face activities such as interviews with learners (e.g. Ambrose, 2004; Jenkins, 2010), scripted interview protocols (Moyer & Milewicz, 2002), and PTs work with small group of learners (e.g. Nicol, 1998). A brief summary of the activity, context and findings are presented for each example of scholarly inquiry.

In Ambrose (2004), Elementary PSTs worked in pairs to pose open-ended problem solving activities focused on whole number operations and fractions to children. The goal was to impact PSTs' beliefs about teaching, potentially shifting beliefs from teaching as explaining, by leveraging their current beliefs as caregivers. Ambrose concluded PSTs developed new beliefs that were incorporated in existing belief structures.

Jenkins (2010) intervention involved six middle grades PTs working in pairs in alternating roles to pose open-ended tasks focused on patterns and proportions to students. Jenkins searched for evidence of PTs' "interpretive listening skills and awareness of the different ways that middle school students make sense of mathematics" (p. 147). Jenkins reported "the structured interview process fosters an interpretive orientation to listening and initial awareness of the variety of ways that middle school students think about mathematics" (p. 147).

Moyer and Milewicz (2002) engaged 48 PTs in using scripted diagnostic interview protocols focused on rational number tasks to guide their interactions with children. The PTs conducted interviews with children throughout the semester. The final interview was recorded, transcribed, analyzed and reflected upon by the PTs and served as evidence of PTs' experiences and use of questioning. Analysis of the interviews revealed a beginning classification for the types of questioning: 1)"check listing," asking the questions in the protocol with little regard for student responses; 2) "instructing vs. assessing," in which PTs explained mathematics directly to the student with little regard for students' reasoning; and 3) "probing and follow up questions," characterized as PTs genuinely listening to student responses and generating follow-up questions meant to elicit further student thinking.

In Nicol's (1998) activity, 14 PTs were engaged in weekly interactions with small groups of 6th and 7th grade students. The PTs solved problems involving multiplicative reasoning in class and then posed adapted or extended versions of these tasks to students. Nicol examined PTs' abilities to question, listen and respond to students using prospective teachers' journal reflections as sources of evidence for these behaviors. Across the weekly implementations of the activity, PTs began to shift their approaches from those that focused on arriving at a correct answer toward an inquiry-based approach focused on eliciting and understanding student thinking.

Crespo (2000) examined the ways in which elementary PTs listened to the responses of the fourth-grade students in a series of six interactive letter exchanges. PTs' initial interpretations of student work focused on correctness and tended to contain conclusive claims about student understanding based upon small samples of student thinking. Reflective journals were used by PTs to explore their interactions with students. PTs' interpretations began to focus on what the student intended or meant in a solution by the fifth week of the course. Crespo (2003) used the same letter writing activity and data to explore PTs' abilities to pose problems. Initially, PTs attempted to "make their problems less problematic and more attainable to their pupils" (p.251). PTs' questions were worded to avoid student errors or confusion rather than to generate learning opportunities for students or themselves as teachers. Problems included in the last three letters "were puzzle-like and open-ended, encouraged exploration, extended beyond topics of arithmetic, and required more than computational facility" (p. 257). These questions were posed to challenge or extend student thinking and often asked for multiple solutions and explanations.

Methods

Participants and Context

This study examined the experiences of 18 junior-level PTs enrolled in the required elementary mathematics methods course for their university program as they engaged in the ESST Activity. Prior to this course, PTs had completed four mathematics content courses designed for elementary mathematics teachers, including a course on problem solving, and were concurrently enrolled in a fifth content course addressing middle grades mathematics topics. Prior to engaging in this activity, PTs had engaged in activities involving standards documents, CGI problem types and student strategies, responding to students through questioning, number choice and number choice progressions, and opening routines in the grade 2-6 mathematics classroom (Drake, Land, et al., in press).

Exploring and Supporting Student Thinking Activity

The design of the ESST Activity was developed as an example of scholarly practice informed by the literature described above on dynamic interactions. From the literature we identified four contextual factors as potentially supportive in the design and enactment of such an activity: 1) PTs should have opportunities to solve challenging mathematical problems prior to posing them to students; 2) PTs should pose the same problems to students in order to give PTs common experiences to discuss; 3) PTs require opportunities to reflect on their experiences both in a whole group setting as well as through individual, targeted reflection; and 4) MTEs must consistently respond to PTs' reflections.

The ESST activity engaged PTs in solving, planning, and posing a series of 5 tasks. As the instructor of the methods course, the first author provided PTs with 5 tasks designed for use with 3rd grade students. As a class, the PTs and the instructor planned for the enactment of each task using an adaptation of the Thinking Through a Lesson Protocol (Smith, Bill, & Hughes, 2008). During week 5 of our course, PTs visited our partner school where each was paired with a student from a third grade class. During a half-hour session, PTs were asked to pose as many of the five problems as their student could work through, employing extensions and scaffolds as they saw fit. PTs video recorded their session and posted them on the Edthena video tool. PTs were asked to reflect on their own video in terms of their student's solution path and their interactions with the student. They watched and provided feedback for three of their peers, commenting on similar ideas. The instructor also provided feedback on these foci. Following this process, PT synthesized their reflection and feedback into a written plan of ways to improve their facilitation of each task. In week 7 of our course, the PTs returned to our partner school and enacted the same five tasks with a student from a different third grade class. As before, PTs recorded, uploaded, reflected and commented on the videos of this session. The first author provided feedback on these sessions as well. To complete the activity, PTs wrote a reflection paper summarizing their work and what they learned through their enactment and observations. We utilized their reflection papers to make sense of PTs' experience with the activity.

Theoretical Frame

The synthesis of the above examples of scholarly inquiry suggested PTs' dynamic interactions with student thinking can potentially: 1) develop PSTs' knowledge of students' mathematics and strategies (Ambrose, 2004; Crespo, 2000); 2) encourage PSTs to shift their focus in working with students from attaining correct answers to eliciting and understanding student thinking (Crespo, 2000; Jenkins, 2010; Moyer & Milewicz, 2002); and 3) develop PSTs' emerging abilities to use student thinking in crafting responses and posing new problems (Crespo, 2003). We expanded the first code to include not only knowledge of student thinking but any example of what has been defined as knowledge of content and students (Hill, Ball, & Schilling, 2008). We utilized these potential outcomes as our lens as we examined PTs' experiences with the ESST activity, as described in their reflections.

Data Analysis

The authors began the process of data analysis by individually coding the written reflections of the 18 PTs using the three *a priori* codes identified within the theoretical frame. Through several readings of the data and discussion of our existing codes, these three main codes were refined and operationalized using descriptions and sub-codes into our final coding scheme. Excerpts of PTs' written reflection at the conclusion of the activity were taken as the unit of analysis and were coded with both a main code and a sub-code if applicable (Table 1). Inter-rater reliability for the coding was completed demonstrating 74% agreement across 285 units coded. Disagreements were resolved through discussion.

Table 1: Coding Scheme and Number of Coded Units (N = 285)

Knowledge of Content and Students (n = 48)	Attain Correct Answers or Support Student Thinking (n = 120)	Learning to Respond Using Student Thinking (n = 117)
Anticipating possible student solution paths 14	Aware – focused on student obtaining correct answer 13	
Conceptions and misconceptions 27	Unaware – focused on student obtaining correct answer 12	
Task difficulty for students 7	Action -- eliciting and understanding student thinking 42	
	Reflection - the goal of the interaction is to focus on student thinking 53	

Beyond the codes developed from our examination of prior scholarly inquiry, we also employed open and emergent coding techniques (Strauss & Corbin 1998) in order to identify other themes within the data. Two main themes emerged from within the PTs' reflection on their experiences: "the importance of unpacking a task for students" and "PTs' tendency to label or evaluate students based on minimal evidence".

Results

In this section we present our findings related to our five main codes and their applicable sub-codes. We include illustrative examples of each in order to demonstrate our analysis process.

Code 1 - Knowledge of Content and Students (KCS)

As anticipated based upon pour literature search, our analysis of the data revealed a number of instances where PTs identified specific mathematics in their student's work and potentially developed knowledge of content and students from those interactions. There were 48 such examples out of the 285 units. To distinguish among the interactions where PTs seemed to sub-codes were developed and utilized to identify three specific categories of KCS (See Table 1).

Anticipating possible student solution paths was utilized when PTs developed new perspectives on how a student might solve a problem. Susan (all PTs and student names are pseudonyms) provided the following evidence of developing such knowledge while interacting with her first student: "Madison struggled with the coin problem trying to use five coins to make 51 cents. I thought it was very interesting while watching the videos that so many students started with putting out five dimes and a penny. My first reaction is always to make 50 cents with two quarters. Perhaps they all started this way because it would get them closer to five coins total." Of the 48 units identified as Knowledge of Content and Students, 14 involved anticipating possible student solution paths.

Instances where preservice teachers described conceptions or misconceptions of students' mathematical thinking were coded as such. There were 27 units within PTs' reflections identifying their attention to student conceptions or misconceptions. In the following quote from Georgia, we see an example of this code.

> I noticed that my student thought in a different way than I was used to and had some misconceptions about regrouping while doing the standard algorithm for subtraction. In the candy problem, she didn't seem to fully understand how to count up by ones. She was counting by 5's and passed the number she was "counting up to." Then, when she started to count by ones she started at 5 then jumped to 10, 11,12,13.

Anna, as another example, discussed the following misconception when working with her student: "I also found it interesting that once he got a new answer from subtracting, he didn't realize or understand what number he needed to change in the addition problem that was suppose to check his answer."

In 7 instances, PTs demonstrated developing KCS through recognizing, most always in retrospect, the potential difficulty of a task for students. Anna's comment was typical of these responses, "This problem was the hardest for me as a teacher because although he comprehended that each person got 9 brownies, he didn't understand that the last brownie got divided into four pieces and became a fraction".

Code 2 –Obtain Correct Answers or Support Student Thinking

Our examination of the existing literature suggested when working with students can support PTs shift from a focus on students attaining correct answers to eliciting and understanding student thinking. There were 120 units coded as examples of these two mindsets. Within these, we categorized PTs' experiences further using sub-codes. In instances where PTs were focused on their student obtaining a correct answer, we differentiated between PTs who were aware of this focus and those who did not seem to recognize it. For units coded as examples of PTs supporting student thinking, we identified two categories: PTs moves we viewed as supporting student thinking and PTs reflections restating the goal of the activity was to elicit, interpret and support student thinking.

The code "obtain correct answers – aware" was utilized when PTs self-identified their tendency to focus on students arriving at a correct answer. Riley's reflection serves as a typical example of this sub-code. "For the second task, the student struggled in understanding how to approach the problem. I made the mistake of telling him that he should possibly add up. Because of the goal of this assignment, I should not have suggested a strategy to him. Then, I guided him too much through that strategy. Instead I needed to let him approach the problem on his own." Of the 120 units within Code 2, 13 received this code.

The code "obtain correct answers – unaware" was assigned when PTs seemed unaware of their decision or unconscious tendency to lead the student to the correct answer. Marisa's reflection provides an illustrative example, "She had much more trouble with the Brownie Problem and the Coin Problem. I may have prompted her more through these two problems, but she eventually got to the right answer." There were 12 units identified in which PTs were unaware of their actions leading a student to the correct answer.

Action focused on eliciting and understanding student thinking was assigned when PTs' posed tasks or questions intended to help them to better understand the student's thinking. For instance, Heather discussed the following steps she took when interacting with her student: "For example, I created a road map to follow for each problem depending on the child's responses: whether she chose a successful solution path but could not explain the procedure, struggled and needed scaffolding, or completed the problem successfully and explained her thinking. The questions were not random, but rather flowed with her responses." Kim's reflection also exhibited evidence of her actions intended to focus on student thinking. "When Maggie was answering the problems, I would ask her throughout each one what her thinking was, what ideas she was using and how she was sure that her answers to the problems were correct". There were 42 of 120 instances where PTs focused on eliciting or understanding student thinking.

Reflection stating the goal of the interaction is to focus on student thinking was assigned in instances where PTs reflected on the interaction with their student and reminded themselves to keep their focus on the student's thinking. "This was, and still is, an improvement I need to continue to work on so I am able to better explore and support student thinking, assess what strategies a child knows, and determine a child's overall cognitive ability" (Leah). Heather also commented, "Throughout the coin problem, I did not provide enough wait time and found myself explaining too much instead of letting her explore for a longer period of time". There were 53 of 120 instances where preservice teachers reflected on their interaction with the student and stated that the goal was to focus on student thinking.

Code 3 – Learning to Respond Using Student Thinking

We posited our PTs would have the opportunity to develop their abilities in using student thinking to craft responses and pose new problems based on their interactions with students. There were 117 instances where PTs reflected and offered examples of how they might respond if faced with a similar situation in the future. Georgia discussed her interaction with a student: "I also, could have had him do an extension problem with harder numbers to be able to observe his thinking with more difficult numbers." Georgia's thought to pose an extension problem based on the interaction she had with the student is one example of how PTs considered responses as a result of the activity. Landon shared a similar consideration on based on his interaction with a student: "Instead of saying 'take away the smaller number form the bigger number', which is why he put the smaller number on top of the bigger number in his solution path, I could have said 'you have the bigger number and you want to take away the smaller number from it.' This could have prompted him to complete his solution path without any confusion." Landon reflects on his interaction with the student and how he could have responded differently to help the student better understand the problem. Lawson provided an example of responding using student thinking: "Next I decided that in order to help my next student through the problems that I would need to provide my student opportunities to work with 2 digit numbers if they struggled with 3 digit numbers like my first student did. I also would give my student opportunities to work with 1 digit numbers as well if they needed to." Lawson discusses his responses during the interaction with the student and his goal of helping the student when he or she was stuck or struggling.

Code 4 – The Importance of "Unpacking" a Task for Students

Preservice teachers often discussed the concept of unpacking the problem for their student before allowing the student to explore the problem. The unpacking theme was discovered after many preservice teachers reflected on their interactions with the students. For instance, Haley discussed her goal of unpacking the problem: "Unpacking the problem is one thing that I could have done significantly better - making sure the students understood what the story was about and then the method used to solve it. I worked with two students and found that one was on a much higher math level than the other student causing me to work and speak toward the upper level student in the beginning and the lower student as we began working. When I saw that the higher student understood the problem I was quick to move on without checking the lower students understanding. Although I moved on from unpacking the story quickly I think that both could have benefitted from a more in-depth explanation." Eloise provided an example of her goal of unpacking the problem for her student: "Lastly, I want to work on 'unpacking the problem' more. I think it's beneficial for students because it really sets them up correctly for the problem and gives them the most help to complete the problem." There were 58 instances of PT's commenting on unpacking problems for students.

Code 5 - Evaluating or Labeling Students Based on Minimal Evidence

Preservice teachers also had a tendency to evaluate or label students after a minimal time of working with the student. There were several instances where preservice teachers broadly evaluated a student based on minimum experience. For example, Anna wrote, "I would classify Landon as an above average math student who understands most concepts but gets through problems by going through the motions and performing the standard algorithm." Anna had minimal experience working with Landon, but was quick to classify him as an above average math student based off of a small observation. Margaret evaluated her student after working through a few problems: "I think one of the reasons for this was the fact that my student was very smart and she knew how to do all of the problems, and she for the most part solved them all correctly on the first try." There were 16 instances where preservice teachers worked with their student on a problem and then labeled the student based off their ability of a single interaction.

Discussion and Implications

As we examine PTs' experiences with the ESST activity, we can draw several conclusions about the design of the activity and its potential to foster the types of outcomes suggested by the literature. As presently constituted the ESST activity did not seem to afford PTs the opportunity to develop KCS. Perhaps extending the activity beyond two sessions and utilizing a variety of different problems would be necessary in order to support this development. The activity did seem to provide opportunity for PTs to consider their role of listening to and supporting student thinking as well as to provide opportunity for reflection on the ways in which they did and might respond to students. The additional themes of unpacking and labeling students imply these are areas of our course we need to pay explicit attention to prior to our PTs working with students. The continuing cycle of scholarly inquiry and practice will allow us to further refine and empirically examine this activity.

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SEEKING THE INFLUENCE OF COONEY, SHEALY, AND ARVOLD'S (1998) BELIEF STRUCTURES

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Cooney, Shealy, and Arvold (1998) wrote a widely-cited paper that described four belief structures of prospective teachers and argued the structures can aid in describing the ways beliefs change and the influence of authority on the individual. In this paper, we investigate the impact of this manuscript on the field. We first conducted a literature review (n = 48) of journal articles and proceedings published since 1998 covering the same population and goals of Cooney et al. (changing prospective teachers' beliefs). We then conducted an analysis of 106 journal articles citing Cooney et al. to see why the author(s) cited the piece. We conclude the impact of Cooney, et al.'s article differs from that of their results and suggest belief structures should be more carefully investigated by the field.

Keywords: Teacher Beliefs, Affect, Emotion, Beliefs, and Attitudes

There is a long history of studies focusing on the beliefs about mathematics and mathematics teaching and learning of students, prospective teachers, and inservice teachers (Philipp, 2007). These studies have argued that being informed about the beliefs of these populations is helpful in designing appropriate interventions for shifting beliefs (Philipp et al., 2007) and evaluating the success of a teacher preparation program (Charalambous, Panaoura, & Philippou, 2009). Cooney, Shealy, and Arvold (1998) sought to understand the development of and opportunities to change prospective teachers' beliefs and the structure of their beliefs. Cooney et al. (1998) concluded with four belief structures: isolationist, naïve idealist, naïve connectionist, and reflective connectionist. Each of these described the ways the beliefs of the individual were held and could help in explaining the changes in beliefs of the individuals. "We posit the notions of naïve idealist, isolationist, and connectionist with the intent that a description of preservice teachers can enhance our understanding of the ways preservice teachers construct meanings as they progress through their teacher education programs" (Cooney et al., 1998, p. 331).

Although Cooney et al. (1998) is cited frequently in the field (382 times according to Google Scholar as of Jan. 2, 2017), we could not find many studies that used their belief structures as an explanatory tool to shifts in beliefs. Studies have shown the strength of using Cooney el al.'s belief structures: "One of the strengths of the Cooney et al. construct of belief structures is its explanatory power with respect to propensity to change and success in changing" (Conner, Edenfield, Gleason, & Ersoz, 2011, p. 501) We investigated the influence of Cooney et al.'s belief structures by reviewing literature with the same focus, changing beliefs of preservice teachers, and the ways Cooney et al. has been referenced since its publication. We provide a summary of Cooney et al.'s report followed by the methods used to investigate the influence of the article. We then present our results of both investigations and conclude by discussing future research trajectories and issues to be addressed by mathematics education researchers.

Background: Summary of Cooney, Shealy, and Arvold (1998)

Cooney et al. (1998) conceptualized their study through a constructivist perspective influenced by the cognitive and social construction of knowledge. They desired to the students in their secondary mathematics methods course to see themselves as participants in a community of mathematics educators. Cooney et al. used multiple theoretical frameworks to make sense of the beliefs of the prospective teachers. These frameworks helped them to consider the influence of context and reflection (Bauersfeld, 1988; Dewey, 1933), how beliefs are held (Green, 1971), and one's orientation to authority (Belenky, Clinchy, Goldberger, and Tarule, 1986; Perry, 1970). Bauersfeld (1988) provided a way to consider the sociocultural aspects within cognitive interactions with his description of how communities construct knowledge. This was important to Cooney et al.: "Because much of what an individual learns about teaching is through interactions within various communities, it seems reasonable to assume that those contexts are important influencing factors in what is learned" (1998, p. 307). Dewey (1933) aided in operationalizing reflection and how reflection influences the changes of beliefs because reflection is necessary to resolve problematic experiences. Green (1971) described the metaphor of a beliefs systems to emphasize how beliefs are held. He described three characteristics of a belief system: (a) There is a quasi-logical relationship between beliefs; (b) Beliefs are both peripheral or central and derivative or primary; and (c) Beliefs exist within clusters that are isolated from one another (thereby allowing the possibility of an individual having contradictory beliefs). Finally, Belenky et al. (1986) and Perry (1970) were "two schemes that address one's reliance on authority for knowing… These two schemes are similar in that they describe individuals who range from those for whom an authority dictates truth to those for whom truth is seen as contextual" (Cooney et al., 1998, p. 311). To Cooney and colleagues these frameworks collectively provided a way for them to investigate the beliefs about mathematics and mathematics teaching and learning of prospective teachers and gave new insight into the ways prospective teacher hold their beliefs.

Cooney et al. (1998) purposefully selected four participants from a cohort of 15 prospective secondary mathematics teachers. The cohort was in the final year of the teacher preparation program and all students were enrolled in a mathematics methods course. The participants were selected based on survey results as well as observations and assignments completed in the methods course. Each participant was interviewed 5 times during the course and student teaching, and Cooney et al. used a constant comparison procedure (Strauss, 1987) to identify themes that emerged from the data. They reported on four cases.

Based on the four cases, Cooney and colleagues described four possible belief structures: naïve idealist (characterized by uncritical acceptance of ideas presented by authority figures resulting in clusters of contradictory beliefs), isolationist (belief are held strongly and nonevidentially so contradictory ideas are rejected without reflection), and naïve or reflective connectionist (characterized by attempts to incorporate new ideas meaningfully into already present belief structure more or less critically and coherently). Cooney and colleagues end the paper with comments on the importance of considering belief structures in teacher education. "An analysis of belief structures… can provide a forum by which our teacher education programs will be better able to address issues of reform" (Cooney et al., 1998, p. 331). Cooney et al. claimed the goal of teacher education is to develop reflective connectionists, though they admit to having difficulty imagining how to do so with isolationists and naïve idealists. They call for future research to investigate these possible shifts. Based on Cooney et al.'s call to action, we sought to see how researchers had responded after 18 years.

Methodology

We set out to explore the ways Cooney et al. (1998) may have influenced the field. The first exploration entailed conducting a literature review of studies published from 1999 to 2016 with a similar focus as Cooney et al.'s investigation, changing beliefs of prospective teachers. We began by conducting searches in both ERIC and EBSCO using the words beliefs, preservice teacher, change, and mathematics, focusing on peer-reviewed hits. The initial search had over 700 hits. As we began to look through the 700 publications, we realized many of the pieces found did not match our criteria but instead had within the article the searched for words. To aid in focusing specifically on studies about beliefs, we choose to change the search criteria. The second time, we searched both ERIC and EBSCO databases for manuscripts with belief in the title, and including the words preservice, mathematics, and change anywhere in the text from 1999 to 2016. We repeated the searches replacing belief with conceptions and then orientation as we found these words could potentially be synonymous with beliefs. Finally, we repeated the previously mentioned searches in both databases, replacing preservice with prospective. After duplicates were removed, we had 86 peer-reviewed journal articles and proceedings. For this first part the publication was the unit of analysis. In a spreadsheet, we collected information from each publication such as: (a) author; (b) title; (c) abstract; (d) intervention described to change beliefs; (e) beliefs attempting to influence; (f) grade band of prospective teachers; and (g) did they cite Cooney et al. and if yes, then did they use belief structures as an explanatory tool. As we read through the 86 publications we found only 48 pieces fit the search criteria (empirical studies about prospective mathematics teachers' shift in beliefs). Those excluded either focused on the wrong population (e.g. students in mathematics content courses, prospective science teachers, etc.), wrong phenomenon (e.g. content knowledge) or were not empirical.

The second investigation required us to do a citation analysis similar to Leatham and Winiecke's (2014). Using a Google Scholar citation report for Cooney et al. (1998), of the 382 citations listed, we located 106 peer-reviewed English language articles. We then sought out in each article the line of text or statement that was a direct citation to Cooney et al. (1998). Four articles did not have proper citations and were excluded. For this part of the investigation, the unit of analysis was the citation instance in each article (n = 142). We used a constant comparative method (Strauss, 1987) to identify categories of the purposes of citing Cooney et al.

Results

The results of our investigation are reported in two parts. The first part focuses on the 48 manuscripts in which the authors investigated aspects of beliefs similar to Cooney et al.'s (1998) study. We then report on the citation analysis conducted. Overall, Cooney et al.'s construct of belief structures has minimally been used in the field. Two of the 102 manuscripts examined used belief structures as an explanatory mechanism for the change in beliefs of the participants. The citation analysis revealed the purpose of citing Cooney et al. (1998) was to justify claims of the impact of beliefs on teacher perspectives or practices, statements about the influences on beliefs and the difficulty in changing them or to describe how beliefs are held.

Nature of the 48 Publications on Changing Beliefs of Prospective Mathematics Teachers

Cooney et al. (1998) was focused on changing prospective mathematics teachers' beliefs about mathematics and mathematics teaching and learning. The majority of the studies (n = 29) were focused on changing the same beliefs. A number of studies, however, focused on other kinds of influential beliefs, for example, teacher efficacy, epistemological beliefs, the incorporation of a concept, skill, or philosophy (e.g. social justice), and others on specific mathematical constructs (e.g. proof). Table 1 enumerates foci of the beliefs publications. Some publications considered multiple beliefs categories, such as Charalambous et al. (2009) who investigated the change of epistemological and efficacy beliefs of prospective elementary teachers, thus the total number of pieces in the table is greater than 48.

Table 1: Foci of Beliefs Publications

Focus	# of Pieces
Mathematics, Mathematics Teaching and Learning	29
Teacher Efficacy	8
Epistemological Beliefs about Mathematics	4
Technology Use in Mathematics Education	4
Equity/Social Justice Issues in the Mathematics Classroom	3
Proof	2
Confidence in Teaching Mathematics	2
Mathematics Teaching Incorporating Arts	1
STEM Incorporation	1
The Role of Teachers and the Learners	1
Mathematical Knowledge	1

A number of studies used beliefs to evaluate their teacher education programs (e.g. Charalambous et al., 2009). The shift of prospective teachers' beliefs toward more reform-oriented (NCTM, 2014) beliefs provided evidence of success of the content, methods, or general program. This was true for both studies focused on beliefs about mathematics and mathematics teaching and learning, and those focused on teacher efficacy and epistemology. Cooney et al. did not explicitly seek out to evaluate the teacher education program or course in which the prospective teachers were enrolled. Their focus was on the beliefs about mathematics and mathematics teaching and learning. This matched the majority of studies found.

Cooney et al. (1998) did not examine the influence of a particular intervention. Instead, the intervention was more implicit as the catalyst to change beliefs was the teacher education program itself. This can be seen as a macro level intervention because the planned intervention was at the group level. This matched the majority of the studies on prospective teachers. Table 2 lists the categories of interventions used in the 48 publications. Some studies (Philipp et al., 2007) used multiple interventions in different sections of a mathematics methods course for prospective elementary teachers. Each intervention in these cases was counted separately.

Most interventions were more at a macro level or did not intervene at an individual level. The intervention was the teacher education program or course (n = 13), a course or program with a specific philosophy informing the make-up of the course or program (n = 14), or the inclusion of a field component (n = 5). These interventions are not guided towards individuals but instead at the group of individuals. This is different from more micro or individualistic interventions like specific activities conducted in the course (n = 11), or the addition of a technology component (n = 5). These interventions were targeted to individuals to participate in the intervention. A few studies combined both micro and macro level interventions (Philipp et al., 2007).

Finally, Table 3 shows some other characteristics of the 48 publications. Out of the 48 pieces about changing beliefs of prospective mathematics teachers, the majority of the studies focused on elementary teachers (n = 38). Only three studies focused on secondary mathematics teachers, while the remaining seven had a combination of secondary and elementary teachers. The lack of secondary investigations could potentially be due to the methods used to investigate beliefs. Quantitative studies (n = 25) along with heavily quantitative mixed studies (n = 6) represented about 65% of the publications. The use of quantitative methods requires a large number of participants. Secondary mathematics programs typically have fewer students than elementary teacher preparation programs. Additionally, research has shown prospective elementary teacher programs have a high percentage of students with negative dispositions towards mathematics (Szydlik, Szydlik, & Benson, 2003). Therefore, changing beliefs of prospective elementary teachers may seem more relevant to producing reform-oriented mathematics teachers.

Table 2: Interventions Used to Change Prospective Teachers' Beliefs

Type of Intervention (Micro/Macro)	Description of Intervention	# of Pieces
Philosophy Informing Course/Program (Macro)	Describes a theory, construct, or philosophy guiding the structure, activity, and/or goals of the course	14
Teacher Education Program (Macro)	Goal is to evaluate or see the changes incurred by the current teacher education program or course.	13
Specific Activity (Micro)	Describes a specific activity or intervention (observations during field placement, or pedagogical activity) as catalyst for change.	11
Field Component (Macro)	The addition of a field component to course	5
Technology Component (Micro)	The addition of a technological tool to the course (e.g. use of wikis, online discussion boards; online workshops)	5
Program Addition (Micro)	The addition of activities outside of courses (mentoring by experts, monthly seminar, small discussion groups, etc.)	2
Student Teaching (Macro)	Goal is to evaluate or describe the changes incurred by the student teaching experience.	2

Table 3: Characteristics of Manuscripts by Grade Level Focus of Participants

	Elementary	Secondary	Elementary & Middle	Elementary & Secondary	Not Specified
Qualitative	13	2	0	2	0
Quantitative	20	1	1	1	2
Mixed	5	0	0	1	0
Cited Cooney et al. (1998)	7	2	0	0	0
Used Belief Structures for Analysis	1	1	0	0	0

Of the 48 publications, only nine cited Cooney et al. (1998), although all publications were reporting on similar populations and phenomena. Of those nine, two studies (Conner et al., 2011; Mewborn, 2000) used belief structures as described by Cooney et al. This demonstrates the construct of beliefs structures has not been taken up by the field, at least when investigating similar phenomena. Cooney et al.'s study required multiple sources of data and deep qualitative investigation. Most of the studies found were quantitative in nature and therefore would not have the data necessary to use belief structures. If only 9 of the 48 publications cited Cooney et al., then what explains the 382 publications found in the Google Scholar citation report? This finding led us to conduct the citation analysis described below.

Nature of the 102 Publications Citing Cooney et al. (1998)

Following Leatham and Winiecke (2014), we conducted an analysis of the 142 citation instances collected from 102 journal articles. Using a constant comparative method (Strauss, 1987), we found eight categories reflecting the reasons Cooney et al. (1998) has been cited. In this section we briefly describe each category and provide examples. These categories represent the impact Cooney et al. has had on the field. Table 4 summarizes the primary purposes for citing Cooney et al.

Table 4: Summary of the Primary Reasons for Citing Cooney Et Al. (1998)

Reason Cited	Percent (n)
Impact of beliefs on perspective or practice of teachers	25% (n=35)
Influences on beliefs and difficulty in changing them	18% (n=25)
How beliefs are held	13% (n=18)
No mention of beliefs (content knowledge, role of teacher education, etc.)	12% (n=17)
Casual citation (no specific content of article referenced)	11% (n=15)
Potential of reflection	8% (n=12)
Methodology	8% (n=11)
Different types of beliefs exist	6% (n=9)

The most common reason for citing Cooney et al. (1998) (35 or 25%) was to support claims about the ways beliefs influence teachers' perspectives or classroom practice. These included descriptions of the general relationship between beliefs and practice and the way beliefs act like a filter when making sense of a situation. Furthermore, some publications used Cooney et al. to highlight how beliefs influence certain actions in the classroom. For example, Philipp et al. (2007) stated: "Beliefs might be thought of as dispositions toward action, having a motivational force (Cooney et al., 1998…)" (p. 450). Although, Cooney et al. discussed the impact of beliefs on teachers, the focus of their study was not on these relationships. They conjectured how beliefs structures may influence the actions of the prospective teachers, but the relationship between practice and beliefs was part of the rationale for the study.

Researchers also cited Cooney et al. (1998) to describe experiences influencing the development of beliefs of teachers (25 or 18%). Claims included the impact of context on beliefs and how the background of the individual impacts beliefs about mathematics and mathematics teaching and learning. "Beliefs tend to be context specific, arising in situations with specific features (Cooney et al., 1998)" (Philipp et al., 2007, p. 450). As before, Cooney et al. justified their work based on these conceptualizations of beliefs; these statements were not part of their results.

The next largest group of statements described how beliefs are held (18 or 13%). This is the group that is closest to the main focus of Cooney et al.'s (1998) study. Cooney and colleagues were interested in how beliefs are structured and how those structures influence prospective teachers' beliefs about mathematics and mathematics teaching and learning. These statements focused on the use of Green's (1971) metaphor of belief systems, the use of Perry's (1970) discussion about authority and knowing, and in-depth descriptions of belief structures. Discussions of Green and Perry were included in Cooney et al.'s theoretical framework, while in-depth descriptions of belief structures comprised their findings.

The next two groups cited Cooney et al. (1998), but they made no specific link to the conducted research. The category of no mention of beliefs was made up of a collection of statements that cited suggestions or other aspects of teacher education Cooney refers too. For example, Conner et al. (2011) stated: "Cooney et al. (1998) suggest that one goal of teacher education is to help teachers become reflective connectionists" (p. 500). Many of these statements were used to back up claims about the role of teacher education, characteristics of teachers, or prospective teachers' mathematical content knowledge. The collection of casual citations (15 or 11%) cited Cooney et al. in a generic way: "Many studies have been designed to bring about changes in the conceptions of preservice and inservice teachers (e.g. Cooney, Shealy & Arvold, 1998…)" (Steele, 2001, p. 140).

Cooney et al. (1998) emphasized the importance of reflection as a way to change beliefs of prospective teachers, and this was highlighted by a number of publications (12 or 8%). Cooney et al.'s argument about reflection followed their conceptualization of beliefs and how beliefs are held. The claims made focused on the power of reflection to change beliefs or the importance of reflection in teacher education. Philipp et al. (2007) highlighted Cooney et al.'s study to stress the importance of reflection, "Researchers studying teacher education have added to our understanding of the role that reflection plays in teacher education. Cooney et al. (1998) found that…" (p. 471). Many of the researchers built on Cooney et al.'s conceptualization of reflection's role in changing beliefs.

The final two categories cited Cooney et al. (1998), usually in a list of other researchers, either to justify or describe the chosen methods (11 or 8%) or to emphasize the existence of different kinds of beliefs (9 or 6%). Though these are valuable contributions, neither of these reasons for citing Cooney et al. are explicitly about the results of the study.

Discussion and Conclusion

Our results show Cooney et al. (1998), although widely cited, has not significantly impacted research on changing beliefs. The most common reasons to cite Cooney et al. demonstrate the usefulness of Cooney et al.' conceptualization of beliefs and beliefs change. Cooney et al.'s belief structures, however, are minimally considered by those in the field. Only two publications (Conner et al., 2011; Mewborn, 2000) that cited Cooney et al. explicitly used belief structures as an analytical tool. A slightly bigger impact was shown by the 18 citation statements (from 14 publications) that discussed how beliefs are held. We considered statements about how beliefs change to be potentially referencing ideas close to Cooney et al.'s belief structures. The remaining 124 citations cited Cooney et al. without reference to their major results. These researchers cited Cooney et al.'s theories and interpretations of other researchers, often without clarifying their intention, rather than citing the theory resulting from the empirical results of the study. We found this to be problematic. Depending on the citation statement, Cooney et al. could be seen as either a study about the relationship between beliefs and practice, a study about prospective teachers' content knowledge, or a study about the role of teacher education.

Cooney et al.'s conceptualization of belief structures and the studies that explicitly use beliefs structures suggest more focus on beliefs structures could be a powerful direction for future research. Our review of 48 peer-reviewed publications addressing beliefs change in prospective teachers demonstrated that trends in research on changing prospective teachers' beliefs aligns with Cooney et al.'s (1998) chosen intervention and focus on beliefs about mathematics and mathematics teaching and learning. Studies examining beliefs and beliefs change have established that while change is slow, it can happen. However, little is known about why and how beliefs change. That is, researchers have established that reflection and particular interventions within coursework and field experiences can promote change, but little is known about the mechanisms for that change. Cooney et al.'s beliefs structures potentially provide insight into those mechanisms for change. The use of beliefs structures as an analytical tool could move forward beliefs research by providing a deeper understanding of how beliefs shift over time. This will require the development of instruments and possible ways of collecting more pointed data for identifying the belief structures of teachers. Furthermore, specific interventions for different belief structures need to be developed if the goal of teacher education, as stated by Cooney et al., is to develop reflective connectionists.

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SUPPORTING LEARNING TO TEACH IN EARLY FIELD EXPERIENCES: THE UTE MODEL

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Most teacher preparation programs require prospective teachers (PTs) to engage in early field experiences (EFEs) prior to completing required coursework. These EFEs, however, may lack meaningful connections to course content and provide limited opportunities to experience the demands of classroom teaching. In this paper, we share evidence from the implementation of a novel kind of EFE, the "University Teaching Experience" (UTE) model, where secondary mathematics PTs receive mentoring from teacher educators (TEs) as they teach in a undergraduate mathematics course Findings reveal the importance of both guidance from TEs and observations of peer teaching for PTs learning in EFEs.

Keywords: Teacher Education-Preservice

Field experience is an essential part of teacher preparation (Dewey, 1938; Zeichner, 2010). While all certification programs require PTs to complete a capstone field experience (typically called "student teaching"), *early* field experiences (EFEs) emerged around the time of laboratory schools with the premise that PTs should have opportunities to work in K-12 classrooms before and/or during their professional coursework to ground their understanding of pedagogical theory with practice (Dewey, 1938). Typical EFEs involve PTs in observing students in classroom environments during or conducting short episodes of instruction (Cruickshank & Metcalf, 1993).

Recent research indicates that the quality of field experiences matters; having more field experience in a teacher preparation program does not necessarily lead to better prepared PTs (Ronfeldt & Reininger, 2012). Current opportunities for PTs to gain teaching experience each have specific limitations. PTs that only observe in K-12 classrooms for their EFE have limited access to understanding the scope of teaching. Activities like microteaching segments of lessons with peers in a methods course may offer better opportunities for PTs to practice innovative methods than simply observing, as they can receive feedback from a university-based teacher educator. Microteaching is limited, however, in that teaching to one's peers is inherently an artificial instructional situation. School-based EFEs where PTs are engaged in instruction provide authentic opportunities for learning about the complexity of teaching in school settings. However, research documents a disconnect between what PTs see and experience in K-12 classrooms and what they learn about effective teaching in on-campus methods courses (Allsopp, DeMarie, Alvarez-McHatton & Doone, 2006; Zeichner, 2010). Additionally, school-based EFEs can involve some risk for the mentor teacher if the mentor teacher's performance evaluations are based on value-added measures, not to mention risk for students' learning.

In this paper, we report results from the implementation of a novel type of EFE that addresses some of the typical shortcomings of early field experiences in its design. The University Teaching Experience (UTE) model involves an undergraduate remedial, or non-credit, algebra course as a site for an EFE. The UTE model entails four components. One component (inquiry-oriented curriculum and task design) involves mathematics teacher educators (MTEs) as the methods course instructors collaborating with mathematics faculty responsible for the remedial mathematics course curriculum to design curricular sequences for the course that feature tasks (individual or series of problems) requiring a high level of cognitive demand (Stein, Grover & Henningsen, 1996). A second component (plan and implement) features PTs planning and teaching lessons in the remedial mathematics class while enrolled in their initial mathematics pedagogy course (hereafter referred to as the "methods course"). A third component of the UTE is the mentoring that is provided by MTEs during the planning, implementation, and reflection stages of lead PTs' teaching in the developmental mathematics course. The MTEs include the faculty course instructor and graduate assistants with secondary mathematics teaching background. The MTEs also model teaching practices and provide in-the-moment coaching, if needed, during PTs' teaching episodes. Finally, MTEs orchestrate a debrief discussion after the lesson with the lead PTs for each lesson and their peer PT observers to discuss the development of mathematics students' thinking and react to the decision making of the lead PTs. In addition, the debrief discussion offers an opportunity for PTs preparing to teach subsequent lessons to rehearse the beginning, or task set-up, phase of the lesson prior to actual enactment in the developmental mathematics course.

Initial research during the early-stage implementation of the UTE model established the viability of the model for ensuring an effective learning experience for the undergraduate students enrolled in the remedial mathematics course (Bieda, Wolf & McCrory, 2013; Bieda, McCrory & Wolf, 2014). The second phase of the project attended to the viability of this kind of EFE for PTs' learning. In this phase, we analyzed data from several sources to address the following research questions: *To what extent does the UTE support the development of PTs' planning for attending to student thinking as evidenced in their written lesson plans for UTE lessons? To what extent do PTs teach in the remedial math course in ways consistent with the methods and strategies to promote mathematical proficiency, (i.e.,* recognizing and building on students' prior knowledge, anticipating and responding to student thinking, selecting and sequencing students responses to achieve specific mathematical goals, pressing for justification and explanation, and maintaining a high level of cognitive demand during task enactment)? *Finally, how do PTs evaluate the opportunities to learn in the UTE and how do they compare those experiences to their work in a school-based placement during the second semester?*

Theoretical Framework

We use *transformative learning theory* (Mezirow, 1997) as a frame for thinking about how PTs' knowledge about teaching develops through their interactions with activities and experiences in their teacher preparation program. According to Mezirow, transformative learning is the "process of effecting change in a *frame of reference*" (p. 5; italics in original). PTs' frames of reference with regards to teaching practice are composed of both *habits of mind* and a *point of view* (Mezirow, 1997). *Habits of mind* are "broad, abstract, orienting, habitual ways of thinking…" (Mezirow, 1997, p. 5) that are informed by the years of experience PTs have as students in classrooms (Lortie, 1975); by participating in the norms of school as students, PTs have absorbed a "set of codes" (Mezirow, 1997, p.6) that frame their understanding of what teachers do and what they did, as students of mathematics, in response. Similarly, Cuoco, Goldenberg and Mark (1996) talk about *mathematical habits of mind* as the "methods by which mathematics is created and techniques used by [mathematical] researchers" (p. 376) and, as such, are the ways that mathematicians think when solving problems.

Mezirow (1997) argues that points of view are responsive to feedback and shift as we reflect on the outcomes of our actions in the environment. A person's point of view can shift whenever we try to make sense of why something has happened in a way we did not anticipate (Mezirow, 1997). This is precisely the state of novice teaching at the K-12 level; by trying out teaching practices in authentic settings, teachers get feedback in the form of students' responses that they can compare to their assumptions about what they intended to happen. Their reflection upon this experience can change their points of view on what it takes to achieve the kind of learning outcomes they are intending.

Hence, the feedback that PTs receive from a teaching experience - both intrinsically as they react to the setting in the moment and extrinsically as they receive feedback from a mentor or peer – is critical to changing their point of view. Yet the kind of feedback they receive is largely dependent upon the context in which they teach. For example, in a microteaching setting involving teaching to one's peers, PTs are more likely to accurately anticipate the outcome of their teaching moves, and, thus, the intrinsic feedback will be affirmative. Thus, if mathematics teacher educators want to shift PTs' points of view on what it takes to teach in ambitious ways for all learners (Lampert, Boerst & Graziani, 2011), we need to ensure that the context in which they practice ambitious teaching offers an opportunity to get feedback that is representative of the kind of student responses they would receive in K-12 instructional settings.

The emerging research on rehearsals, where PTs rehearse scripted teaching moves in short instructional episodes (Kazemi, Ghousseini, Cunard & Turrou, 2015; Kazemi, Franke & Lampert, 2009; Lampert, Franke, Kazemi, Ghousseini, Turrou, Beasley, Cunard & Crowe, 2013), is moving the field forward in developing teacher education that helps PTs to enact particular teaching practices. Our claim is that the UTE model, like rehearsals, offer PTs an opportunity to ground their learning of how to do particular teaching practices, but in a setting that helps them develop an understanding of what it will take to carry out those practices in live classrooms. A key driver for this situated understanding is the involvement of mathematics teacher educators in providing ongoing instructional support to the PTs in the UTE. Although this support may be more involved than what PTs would normally receive from a mentor teacher in a school-based placement, it is critical support at this stage in their preparation to help them to reflect upon their instructional decision making as they grapple with multiple competing obligations.

Methods

Participants were 19 PTs enrolled in their first semester-long course on mathematics pedagogy (Methods I) in a large teacher preparation program at a Midwestern University. The Methods I course included a three-hour seminar meeting per week, a four-hour school-based placement experience per week, and a two-hour commitment to participating in the UTE per week. Each PT co-taught a lesson in the UTE twice during the course of the semester. Prior to UTE teaching, each pair received a packet with tasks to be completed during the lesson. Then, each pair submitted three drafts of their lesson plan: (1) initial draft completed using the Thinking through a Lesson Protocol (Smith, Bill & Hughes, 2008); (2) revised draft based on feedback from MTE a week before teaching; and (3) revised draft after teaching the lesson in the UTE. To address the research questions, we collected video-recordings of PTs' teaching in the UTE, along with the lesson plan drafts they completed related to their UTE teaching. The results we share in this paper focus on analyses of the first and revised drafts of the lesson plan. We also conducted semi-structured interviews with 11 PTs, who volunteered to be interviewed from the larger sample of 24 PTs, to learn about their perceptions of the value of the UTE for their learning to teach, as well as their reflections on its affordances and constraints as compared to their school-based placement experience. Additional information about the analyses of these data sources will be presented in the Results section.

Results

We will present the results in three parts, with each part corresponding to one of our three research questions: (1) *To what extent does the UTE support the development of PTs' planning for attending to student thinking as evidenced in their written lesson plans for UTE lessons?* (2) *To what* *extent do PTs teach in the remedial math course in ways consistent with the methods and strategies to promote mathematical proficiency?* and (3) *How do PTs evaluate the opportunities to learn in the UTE and how do they compare those experiences to their work in a school-based placement during the second semester?*

Quality of PTs' Teaching in the UTE

To assess the overall quality of PTs' instruction with respect to promoting mathematical proficiency, we rated the video-recorded observations using the Instructional Quality Assessment (IQA, Boston, 2012) across two dimensions: Implementation of the Task and Student Discussion Following the Task. The rating scale for the Implementation of the Task dimension is based on the levels of cognitive demand (Stein, Grover & Henningsen, 1996), with ratings from 0 to 4 where a 4 rating indicates that:

> "Students engaged in exploring and understanding the nature of mathematical concepts, procedures, and/or relationships, such as: Doing mathematics: using complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example); OR Procedures with connections: applying a broad general procedure that remains closely connected to mathematical concepts." (Boston, 2012, pg. 9)

The Student Discussion Following the Task rubric complements the Implementation of the Task rubric by focusing in on the question: "To what extent did students show their work and explain their thinking about the important mathematical content?" (Boston, 2012, p.10). Similarly to the Implementation of the Task rubric, the scale ranges from 0 to 4 with a Level 4 rating indicating:

> "Students show/describe written work for solving a task and/or engage in a discussion of the important mathematical ideas in the task. During the discussion, students provide complete and thorough explanations of why their strategy, idea, or procedure is valid; students explain why their strategy works and/or is appropriate for the problem; students make connections to the underlying mathematical ideas (e.g., "I divided because we needed equal groups"). OR Students show/discuss more than one strategy or representation for solving the task, provide explanations of why the different strategies/representations were used to solve the task, *and/or make connections between strategies or representations."* (Boston, 2012, p. 10)

Raters were trained to use the IQA rubric prior to rating, and achieved an inter-rater reliability in their scoring (within .5 rating points) of 95% on a sample of 5 lessons of 17 total lessons collected. The rating rubric follows a scale from 1-4, without half-point increments. As there were 19 PTs, there were a total of 8 pairs and one team of 3 PTs. Because some pairs were reorganized during the second round of UTE teaching, we selected only the first and second UTE teaching episodes that were taught by the same pairs of students each time. Thus, a total of 6 pairs of teaching episodes, or 12 total lessons, were analyzed for these results.

Table 1: Aggregated Mean IQA Rating

	Pair 1	Pair 2	Pair 3	Pair 4	Pair 5	Pair 6
First UTE	2	3.5	4	2	2	4
Second UTE	4	3	3.5	2.5	2	3

Table 1 provides the aggregated mean scores for each episode, combining ratings for Implementation of the Task and Student Discussion Following the Task. The aggregated score is appropriate as no episode had a difference greater than 1 point in the ratings for each dimension. The table shows two findings of interest. First, for the majority of episodes, PTs' instruction rated at least at a level of 3, indicative of teaching that promotes some level of conceptual understanding. Second, there are no significant patterns in the ratings from the first UTE to the second UTE observation. While two of the pairs improved in their aggregate scores, four pairs either remained the same or received lower ratings.

Growth in PTs' Planning to Attend to Student Thinking

Given that the likelihood of PTs' teaching significantly improving over the course of several weeks during the semester is low, we also analyzed PTs' lesson plan drafts for their first and second UTE teaching to determine whether specific lesson planning practices were improving as a result of the UTE mentoring. We focused our analysis on how PTs planned to attend and respond to student thinking as evidenced in their lesson plan drafts. Table 2 provides the coding scheme we developed from an iterative coding process (Strauss & Corbin, 1998) as well as examples for each of the codes. Two researchers coded a sample of lesson plan drafts for IRR. The agreement of what was to be coded from the lesson plans was 89%, whereas agreement regarding the assignment of the categories reached 65%.

After category codes were assigned to the text, the text was also scored for quality using a 4-point rubric (0 = is not mentioned; 1 = vague/generic, 2 = somewhat specific, and 3 = mathematically specific). We used three values to summarize the data. The *total quantity* was determined by counting the total number of coded instances. The *total quality* was calculated by adding the individual scores of codes. The *quality average* was computed by dividing the quality total for the lesson by the quantity to get an "average" response across all instances. In this paper, we focus on reporting the quality average.

Table 2: Categories Coded in Lesson Plan Drafts

Category	Example text from a PT Lesson Plan
Predictions of students' mathematical thinking	*seeing them write f(x)=???? will be how I know they're putting the pieces together*
Students' mathematical talk	*hearing students talk about inputs of functions and outputs of functions related to my, or their, examples*
Questions students might ask	*students may ask how these extraneous solutions affect the equation's graph*
Student misconception/difficulty	*students may not realize that 'consecutive odd integers' means that the unknown has to be defined as x and x+2*
Student prior knowledge	*students should be familiar and comfortable with the box method for factoring.*
Student learning outcomes	*I want students to walk away from this lesson looking at mathematical functions like they're operations and tasks rather than random grouping of numbers/letters/symbols*

Figure 1 below shows results in the form of average quality scores for the entire sample, across lesson plan drafts and disaggregated by category type. Across nearly all categories, quality scores increased from the first UTE to the second UTE teaching experience. And, not surprisingly, the final drafts for each UTE teaching (Lesson Plan 2 and 4, respectively) had higher quality instances of planning related to attending to student thinking than initial drafts (Lesson Plans 1 and 3). However, it is interesting to note that the quality of evidence linked to predictions of students' mathematical thinking and questions students might ask decreased, somewhat, from Lesson Plan 3 to Lesson Plan 4. This may have happened because, as the math became more challenging in the remedial class, PTs downgraded their expectations in the revised drafts based on experiences with students during the lesson enactment.

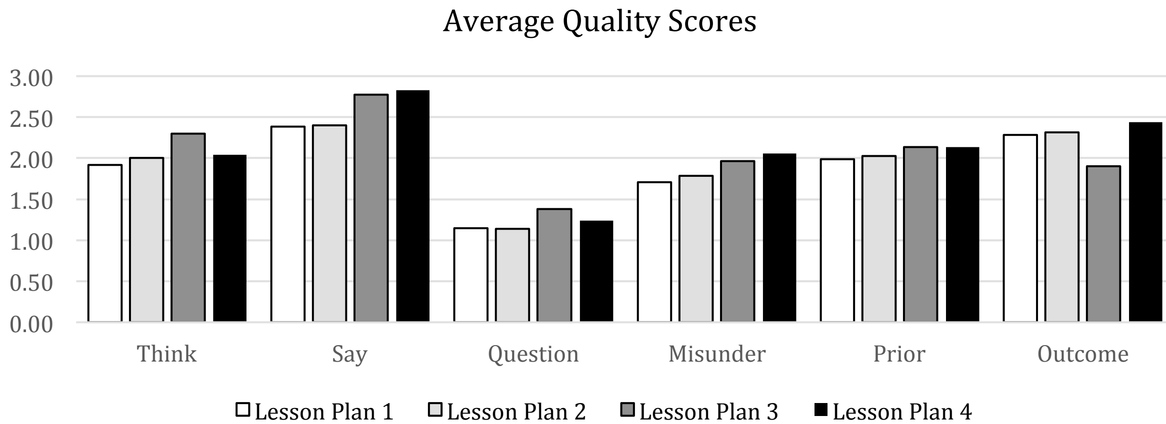

Figure 1. Average quality scores for each category across drafts.

PTs' Perceptions of Learning to Teach in the UTE

Finally, we share results from analyses of interviews with PTs who voluntarily agreed to participate in semi-structured interviews to learn more about their experiences in the UTE. We asked questions such as*: What aspects of the experience did you find useful? What did you learn from observing others teach and taking observation notes? Did the MTL experience influence your work in doing the slices of teaching and lesson studies in your placement classroom?* The interviews were audiorecorded and then transcribed. For select questions, the transcribed responses were coded at the phrase level to capture what participants stated they had learned from participating in the MTL experience (a "what" code) and for how participants stated they had learned these lessons (a "how" code). Using an iterative coding process following methods of grounded theory (Strauss & Corbin, 1998), four codes emerged for "what" was learned (teacher moves, comfort in the classroom, specific discussion strategies, and lesson planning) and four codes emerged for "how" those aspects of teaching were learned (UTE teaching, observing peers in UTE, lesson planning in UTE, peer feedback after UTE).

Not surprisingly, participants most commonly reported that doing teaching in the UTE was the most beneficial aspect for their learning to teach. But, when asked why they felt teaching in the UTE was beneficial, many acknowledged the importance of the support they received while teaching in the UTE. As one participant pointed out, the UTE allowed her to teach "with the guidance of someone there you know well enough to jump in and save you if needed." Another pointed out that working with undergraduate students allowed her to teach "real life students" that "aren't gonna fail if you mess up." However, this pointing to the supportive environment was not universal; other participants stated that they saw any teaching as beneficial to them, and the UTE was just another place to practice teaching, with no special emphasis on the environment. As one participant put it, "the benefit of UTE is getting some experience under you belt, um, kind of getting to know a little bit about yourself as a teacher."

The two most common aspects of teaching the PTs reported learning in UTE were comfort in the classroom and specific discussion strategies. Because teaching in the UTE was among the first teaching experience for most of the PTs, many reported that teaching in the UTE helped them gain some confidence in the classroom. PTs also reported learning how to facilitate whole-class discussions by implementing them while teaching in the UTE. This aspect was discussed as an affordance that the UTE provided that the school-based placement did not. As a PT stated: “in my placement class, um, I don’t think the teacher would have stepped in unless it was a real like, um, train wreck, I guess? Um, meanwhile [Kristen] or the TAs in the UTE would be willing to step in for smaller things, just like, hey, think about this, or whisper in our ears, hey, think about this.” Most of the participants reported similar positive gains in learning to implement discussion-based lessons from the help and support given by instructors during their teaching experiences in the UTE, including the modeling of discussion-based practices by the MTEs early in the semester. This opportunity in implement discussion-based instruction was especially valuable for participants who were later placed in classrooms with teachers who were resistant to using discussion-based instructional practices.

Finally, most PTs reported learning about teaching strategies from watching their peers in UTE, an affordance of the model that school-based placements are unable to offer. Of the 19 instances of participants reporting having learned some sort of teaching strategy, 15 reported learning them from peer feedback or from observation peers. Participants reported favorable on observing and being observed by peers primarily because of the various teaching skills they learned from each other. In looking at the times participants reported learning something from either observing peers or receiving peer feedback, there were only 2 out of the 17 combined instances that participants did not report learning teaching skills.

Concluding Remarks

Taken together, the analyses of data sources suggest that the UTE experience affords PTs with an opportunity to learn about the complexities of teaching in a supportive environment where they can attempt practices such as facilitating whole-class discussions. Findings from our analyses of the quality of PTs’ instruction show that, on average, the quality of instruction is often better than what the literature typically characterizes the nature of teaching in remedial, non-credit, mathematics courses (Larnell, 2016). While the observation ratings show that overall quality does not markedly improve for PTs over the course of the semester, the analysis of lesson planning artifacts reveals that PTs do improve over time in their preparation to attend to student thinking – a high-leverage teaching practice (NCTM, 2014).

Does the UTE model provide better opportunities for PTs to learn from, and within, teaching (Lampert, 2010) than school-based EFEs? Evidence from participants’ reflections about both EFEs in the interviews suggests that the curriculum of the UTE, the structure of the setting, and the mentoring provided by MTEs may provide better access for all PTs to engage in student-centered teaching practices such as leading whole-class discussions. Moreover, the PTs mentioned that opportunities to reflect on their peers’ instruction, which rarely happens in typical EFEs where PTs are placed one-on-one or as a pair with a mentor teacher, was an important aspect of their learning in the UTE. Although the design of this initial study into the effectiveness of the UTE model as a EFE cannot definitively address whether the UTE model provides better opportunities for learning about teaching practice than school-based EFEs, the evidence suggests it is a promising model that would benefit from wider implementation to assess its impact on PTs preparation for the challenges of teaching mathematics in school settings.

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TOWARDS A HYPOTHETICAL LEARNING TRAJECTORY FOR QUESTIONING

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We report on efforts to better understand the questioning practices used by preservice elementary teachers (PSTs), including the range of preferred question types and the values they invoke when evaluating their questions. We sought to determine whether teachers exhibited consistent patterns in selecting questions with certain features, such as funneling students to a particular strategy or eliciting student thinking, across different instructional situations. We found that such patterns did exist; in this paper we use the patterns to propose a trajectory for developing the skill of asking questions that elicit and build on student thinking. The trajectory describes the beliefs, values, and questioning practices associated with PSTs at each stage.

Keywords: Learning Trajectories (or Progressions), Teacher Education-Preservice

Introduction

Asking questions is central to the work of teaching. Yet research (e.g., Franke et al., 2009) has demonstrated that some questions are more likely than others to provide opportunities for students to make their thinking explicit. More research is needed to better understand how teachers might improve the types of questions they ask in the classroom. This study was conducted to learn more about ways to support preservice teachers (PSTs) in developing the skill of asking questions that elicit or build on students' thinking.

Theoretical Framework and Literature Review

We view knowledge for teaching as situated in the context of teaching (Borko et al., 2000), which means that it should be developed through experiences that approximate, to some extent, the practice of teaching (Grossman, Hammerness, & McDonald, 2009). Approximations of practice provide opportunities to learn through the decomposition of teaching into components, which can then be studied and practiced (Baldinger, Selling, & Virmani, 2016). The learning experiences described in this paper use cartoon representations of teaching, developed using the online program Lesson*Sketch*, that provide opportunities for PSTs to choose from specific pedagogical actions (that is, questions), see their (pre-established) impact, and then reflect on those choices (Herbst, Chazan, Chen, Chieu, & Weiss, 2011).

Although this project focuses on developing knowledge about teaching through action and reflection (Ball & Forzani, 2009), we also acknowledge the role of beliefs and values in influencing teachers' practice. In particular, values, which involve teachers' views about what is *important*, have particularly strong impact on teachers' decisions (Bishop, 2012). Efforts to influence the teaching practice of novices must acknowledge and contend with the incoming values of PSTs. In this paper, we explore some interactions between PSTs questioning practices and their stated values with regard to questioning in mathematics teaching.

Features of Questions

The National Council of Teachers of Mathematics' Principles to Actions (2014) advocates teacher questions that "build on, but do not take over or funnel, student thinking," and those that "make mathematical thinking visible" (p. 41). Other productive questioning practices include pressing for mathematical justifications, asking students to make explicit connections between different strategies, and probing errors (Kazemi & Stipek, 2001). These features are in contrast to questions that invalidate students' thinking or impose a way of thinking onto the students. For example, funneling questions are a sequence of closed questions intended to direct students through a series of procedural steps until they obtain the correct answer (Herbel-Eisenmann & Breyfogle, 2005; Wood, 1998). These questions reduce students' opportunities to build on their own understanding because the teacher ends up doing much of the cognitive work and the student merely answers with the expected response (Franke et al., 2009). Although these categories are helpful, more research is needed to articulate how novices improve their questioning practice and what their learning might look like as they transition from asking less productive to more productive questions.

Hypothetical Learning Trajectories

Hypothetical learning trajectories (HLTs) are constructed to represent a possible progression of student learning in a format that is useful for teachers and curriculum designers (Empson, 2011). While some researchers draw solely from existing literature to develop their HLT, others also use insights developed from the analysis of data collected in the first of two research phases (e.g., Meletiou-Mavrotheris & Paparistodemou, 2015). Our study aligns with the latter approach to developing HLTs. Once the HLT is developed, researchers then conduct multiple iterations of their experiment in order to refine their trajectory until it closely mirrors participants' actual progressions of learning (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). Complete HLTs consist of three main elements: "the learning goal, developmental progressions of thinking and learning, and sequence of instructional tasks." (Clements & Sarama, 2004, p. 84). In this study, we examined the patterns in features of questions selected by PSTs in response to student thinking across multiple Lesson*Sketch* experiences and sought to characterize these patterns in terms of a hypothetical learning trajectory. We will focus primarily on the first two elements of the HLT in this paper.

Methods

Participants were 86 elementary preservice teachers (PSTs) in their second of two method courses at a university in the Midwestern United States. Data consisted of PSTs' typed responses to prompts within five online Lesson*Sketch* experiences, two of which we classified as the pre and posttest. In this paper, we will focus on the last three experiences; namely, the Brandon and Cedric experiences and the posttest. In the pre and posttest, PSTs were initially presented with a mathematical task and one simulated student's solution to the provided task. PSTs composed and gave a rationale for a question they would like to ask the student (e.g., Brandon) and then selected all of the questions from a provided list they thought would be good questions to ask the student. In the remaining three Lesson*Sketch* experiences, PSTs were again presented with a mathematical task and one student's solution to the task; this time, they went through two rounds of selecting a question and seeing the student's (pre-determined) response, evaluating the question after each round. At the end of the experience, PSTs decided which of their two selected questions they thought was more effective and explained why. See Table 1 for examples of the questions used in the different experiences, classified according to their question feature. We developed the categories of question types drawing on the literature on effective questions described earlier.

Table 1: Classifications of Question Types

Category	Examples
Suggest a specific alternate strategy (not eliciting)	• (Brandon experience) "Do you know what you need to do to the denominators before you can add fractions?" • (posttest) "Why don't you try dividing each sub into three parts?"
Specific to student's work, but invalidates and funnels	• (Cedric experience) "If it is 4 SQUARE yards, can you just multiply by 3?" • (posttest) "When you are comparing fractions, don't you need to use the same whole?"
Funnels	• (Brandon experience) "In the problem it says that there are three fourths and three sixths. Are fourths the same as sixths?" • (Cedric experience) "Are we talking about one-dimensional units or two-dimensions?" • (posttest) "Is it 1/3 of a sub, or 1/3 of half of a sub?"
Elicits student's thinking	• (Brandon experience) "Can you tell me more about where the fourths, the sixths and the tenths are in your picture?" • (Cedric experience) "Can you show me the square yards in your picture?" • (posttest) "Can you tell me more about how you were thinking about the 1/3?"
Help students build on their own thinking	• (posttest) "Let's look at your pictures for Car A and Car C. Based on the picture, who would get the most?"

Analysis

We began constructing the hypothetical learning trajectory by describing the learning goal, drawing on features of effective questions cited in literature, and hypothesizing which features of questions might be more difficult for students to adopt based on our findings from the Phase 1 data. Next, we looked for patterns among the types of questions PSTs selected and composed in response to student thinking on the posttest and then examined their responses in the earlier experiences to determine whether these groups of PSTs were more likely to select questions with similar features (e.g., funneling or eliciting) in the earlier experiences. As the patterns arose, we identified ways to categorize the different groups of PSTs and adjusted our learning trajectory as needed. During this process, we hypothesized what values and beliefs about teaching might be motivating different types of questions. For example, we conjectured that PSTs would be more likely to ask questions that reference students' work if they valued understanding students' current thinking. In order to gain insight into the PSTs' values, we analyzed their evaluations of selected questions using open codes, which we later condensed into the categories shown below:

- **Building on student thinking:** PST claims that the question provided an opportunity for Brandon to come to a new realization on his own
- **Understanding student thinking:** PST claims that the question helped the teacher to better understand Brandon's thinking or allowed the student to explain his thinking
- **Addressing misconceptions:** PST claims that the question helped the student understand, focused on a misconception, or failed to "fix" a misconception
- **Leading to correct answer:** PST claims that the question helped get the student to the correct procedure or answer

Finally, we investigated the links between PSTs' values (what PSTs believe is important) and practices by examining the relationships between the criteria they used to evaluate questions (i.e., the value codes just described) and the types of questions they tended to prefer (Table 1).

Findings

Patterns of Features of Questions PSTs Selected

After examining the questions PSTs composed and selected on the posttest, we formed groups of participants according to specific features of these questions. We initially classified PSTs as "funnelers" if they a) selected both of the funneling questions **or** b) composed a funneling question. We used similar criteria to create an initial "eliciters" group. These criteria yielded 51 funnelers and 54 eliciters, including 29 PSTs who were listed in both groups. We reclassified these 29 PSTs as "funnelers-eliciters." This resulted in three distinct groups: 22 funnelers, 29 funnelers-eliciters, and 25 eliciters.

A chi squared test of independence showed that in neither the Brandon experience (X^2 (2) = 7.08, p = .029) nor the Cedric experience (X^2 (2) = 5.95, p = .051) were the question types independent of the group. Overall, PSTs tended to select a question that funneled or directed the student in the Brandon experience and a question that elicited or built on student thinking in the Cedric experience (see Figure 1). However, an examination of the standardized residuals revealed that in the Brandon experience, *fewer* funnelers (std. res. = -1.92) and *more* eliciters (std. res. = 1.1) selected the question that elicited or built on student thinking than statistically expected. In the Cedric experience, the inverse was true: namely, *more* funnelers (std. res. = 1.45) and *fewer* eliciters (std. res. = -1.56) selected a question that funneled or directed the student than statistically expected. Figure 1 shows the percentage of PSTs within each group who selected the eliciting and funneling questions in the Brandon and Cedric experiences. Notice that the percent of funneling questions decreases and the percent of eliciting questions increases between each group in both experiences.

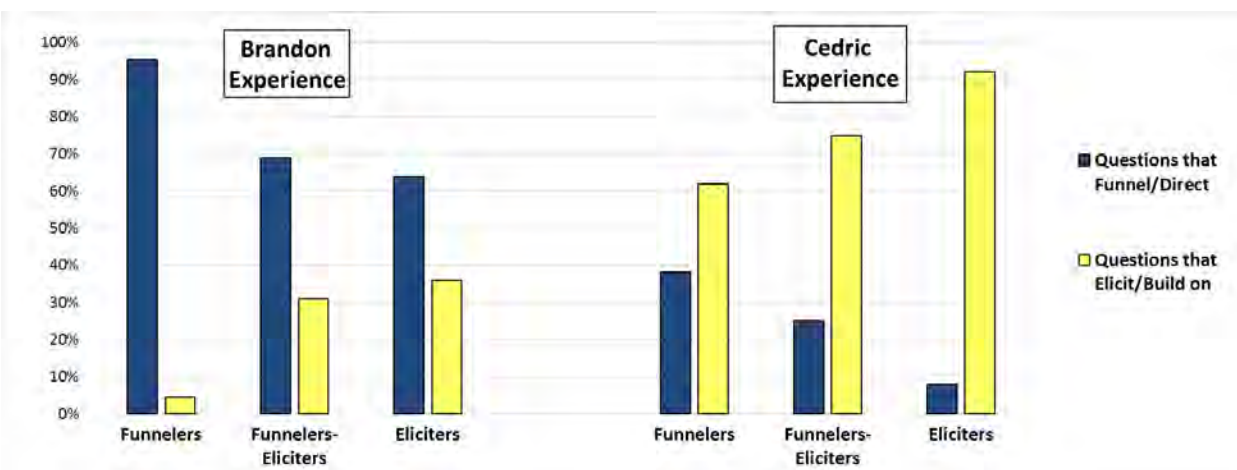

Figure 1. Percent of PSTs in each group who selected either a question that funneled/directed students or elicited/built on student thinking in the Brandon and Cedric experiences.

Criteria for Evaluating Selected Questions

Recall that after viewing the simulated student's responses to two questions they selected, PSTs indicated which question they preferred and why, which we analyzed in order to characterize the PSTs' values underlying their question selection. Over three-fourths of the PSTs in the funneler category gave justifications that focused on whether or not their question resolved Brandon's misconception, compared to less than half of the funneler-eliciters or eliciters (see Figure 2). One PST in the funneler group stated that she preferred the funnel question, “because it made [Brandon] realize that the pieces were not able to be added because they were not the same. He is not realizing it with [the eliciting question], he just keeps labeling his picture and justifying his original answer.” Notice that the PST is evaluating both the funneling and eliciting questions based on whether or not they resolved Brandon’s misconception. Additionally, her negative evaluation of the eliciting question suggests that the PST does not realize that Brandon could have discovered the error on his own in the process of justifying his original answer.

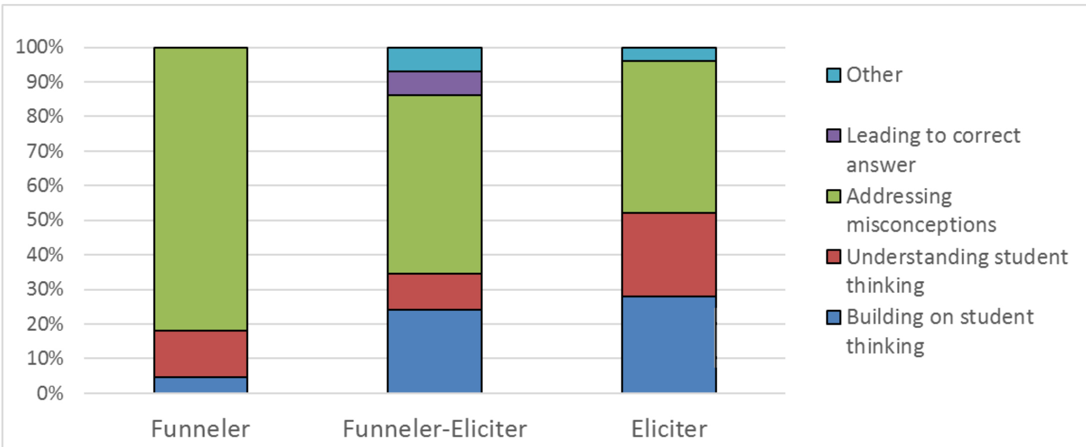

Figure 2. Percent of PSTs who focused on each criterion when evaluating questions in the Brandon experience.

PSTs in the eliciters group were more likely to evaluate their questions based on whether it allowed them to understand or build on the student’s thinking. For example, one eliciter stated that she liked the eliciting question “because we actually get the chance to observe Brandon's thinking and strategies. He is able to explain his thought process for us. The other [funneling] question was more of the teacher telling Brandon what is right and what is wrong.” Here, the PST appears to recognize the value in understanding Brandon’s current thinking before seeking to move his thinking forward. Overall, patterns in the types of evaluations given suggest that not only were PSTs in different groups more likely to select different types of questions, but they also valued different things when asking Brandon a question. We looked for similar patterns in the Cedric experience, but in this case, the differences were not statistically significant. Despite this, the consistency individuals exhibited in the questions they selected and their evaluations in Brandon’s experience lend support to our framework in the next section.

Hypothetical Learning Trajectory

Our trajectory is comprised of three layers – PSTs’ beliefs and knowledge about mathematical understanding, their values about students’ learning, and the features of questions they pose when asking a student about their mathematical work. We propose four main stages that PSTs progress through in the development of asking effective questions. In our descriptions of the stages below, we begin by talking about the features of questions PSTs in the given stage might prefer and then draw connections to the associated values and beliefs/knowledge.

We separated the proposed trajectory for PSTs’ beliefs from values and practice in order to emphasize the distinctions between the components of the trajectory that were based in our data (values and practice) and the components were not explicitly measured, but were emphasized in the methods course and may be connected to the value/practice constructs. The dashed arrows represent the tentative nature of these proposed connections. Although our trajectory depicts four distinct stages, we acknowledge the potential overlap between categories and that some PSTs may not develop understanding in the linear path implied by the figure. Nonetheless, the trajectory serves as a model for understanding a general progression of PSTs' understanding.

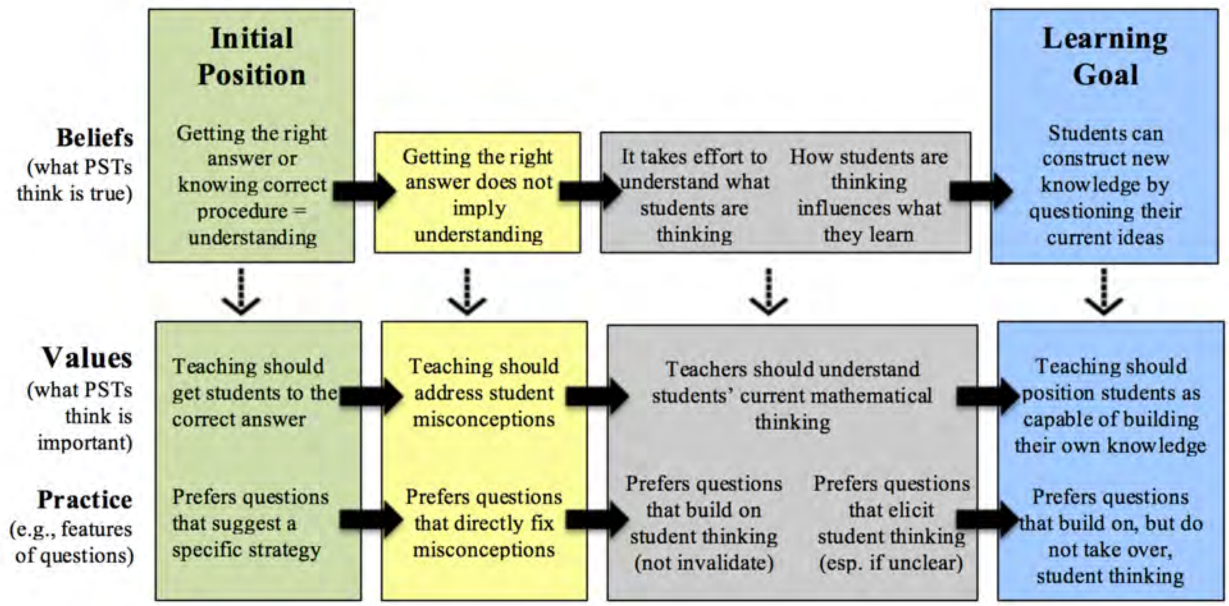

Figure 3. Hypothetical learning trajectory for asking effective questions.

Leading to correct answer (initial position). Based on literature and our prior experiences in methods classes, we began with the assumption that many PSTs enter into undergraduate programs with an unsophisticated view of teaching as telling, and default to using questions as a vehicle for directing students towards specific strategies. For example, one PST explained in the pretest why she thought the question, "why don't you try dividing each sub into 3 parts", was good by saying, "it might make more sense to the student if he divides each piece into thirds and can add them up more easily". Here, the PST projected her own strategy onto the student and assumed that her strategy might make it easier for the student to solve the problem than his current strategy. This reflection lends support to the idea that PSTs who ask questions designed to get students to the correct answer may equate answer-getting with understanding and may have a broad, vague idea of what they think is important for student learning.

Addressing misconceptions (yellow). At this stage, PSTs recognize the importance of asking questions that are specific to the student's work and begin preferring questions that directly confront the student's misconception. Funneling questions, such as "In the problem it says that there are three fourths and three sixths. Are fourths the same as sixths?" fit within this category as the question focuses the student's attention on their misconception without directly telling the student what to do. Of the 49 PSTs who initially selected this question, 34 preferred this question to the eliciting question they viewed subsequently. Nearly all (32/34) of the PSTs who preferred their initial funneling question gave justifications focused on whether they believed the question resolved Brandon's misconception. "My first question was better because he realized that fourths and sixths were not the same. The [eliciting question] just led him to point out where things are in his picture without realizing he was wrong." This response shows a PST who only considers whether the question helped Brandon "realize he was wrong" and failed to consider whether the question revealed more about Brandon's thinking or positioned Brandon as capable of discovering his own error.

Eliciting student thinking (gray). PSTs in the third stage value understanding students' current thinking in addition to helping them develop correct conceptual thinking. As a result, they tend to select questions that elicit the student's thinking about a specific aspect of their work. For example, the question in the Brandon experience, "Can you tell me more about where the fourths, the sixths, and the tenths are in your picture?" focuses the student's attention on how his answer of tenths relates to his drawing of fourths and sixths. This question also elicits his current understanding instead of directly pointing out his error. At the end of the Brandon experience, 19 of the 22 PSTs who preferred this eliciting question provided justifications that highlighted a desire to understand Brandon's thinking or allow him to come to his own understanding. For example, one PST liked the eliciting question because "we actually get the chance to observe Brandon's thinking and strategies. He is able to explain his thought process for us." As PSTs begin to value understanding students' current mathematical thinking, we hypothesize that they begin to realize the effort involved in understanding student thinking and recognize that the student's thinking influences what he/she learns.

Building on student thinking (learning goal). The final box represents the learning goal, where teachers pose questions that aim to build on, but not take over, the student's thinking. Such questions often *do* elicit student thinking, but the description implies that there are additional ways to use questions to help students move forward in their thinking without reducing the cognitive demand or taking over the mathematical work. This corresponds to a value for teaching that not only draws out student thinking, but positions students as capable of developing, questioning, and refining their own ideas. For example, in the posttest, the simulated student Toby determined that sharing two sandwiches with three people equally would result in each person getting 5/6 of a sandwich. The question, "Can you show me the 5/6 of a sub that each person will get?", asks Toby to pictorially represent his solution without indicating that his answer was incorrect. In doing so, Toby would have an opportunity to see that three shares of 5/6 of a sandwich would constitute more than two sandwiches—an unreasonable solution. Although 62 PSTs selected this question on the posttest, only 38 picked a similar question on the other posttest item. The PSTs who selected both of these questions did not exhibit clear patterns in the earlier experiences that suggested a consistent preference for questions that built on, but did not take over, student thinking. We interpret these findings to suggest that our sample did not include a sufficient number of PSTs who were at the final stage in the learning trajectory.

Discussion

Our data suggest that, across different instructional situations, PSTs show consistent patterns in the kinds of questions they select. PSTs who funneled in some situations tended to also funnel in others, and those who chose eliciting questions tended to be consistent in this choice as well. In addition, PSTs who selected both funneling and eliciting questions fell between the two groups in terms of preferences for questions that, on the one hand, take over student thinking, and, on the other, draw out and build on student thinking. These patterns suggest that PSTs were at different stages in their thinking about the questioning practice. We hypothesize that some PSTs (funnelers) prefer questions that lead students to correct answers and do not see value in questions that merely elicit student thinking. Indeed, several PSTs we placed at the beginning of the trajectory expressed dismay when questions seemingly left students still confused, even if those confusions were exactly the ones students needed to work through. At the next stage (funneler-eliciter), PSTs value questions that elicit students' thinking, but they also continue to value questions that lead students to the correct answer or resolve their confusion. Finally, PSTs farther along the trajectory (eliciters) value questions that draw out and build on students thinking, and in this study were also less likely to select questions that imposed a teacher's idea. PSTs at this stage were more likely to criticize questions for being too leading and to value questions that prompted a student to figure something out "on his own."

Establishing these stages is an important first step to designing experiences that support movement along the trajectory. What does it take, for example, for PSTs who value resolving student confusion, to begin noticing and appreciating how questions can help them understand students' current mathematical thinking? Once PSTs value building on student thinking, what kinds of experiences might support them in moving towards a preference for asking eliciting questions? Answering these questions involve consideration of the questioning practices themselves, the PSTs' skills in enacting them, and their values in teaching. In our future work, we hope to further refine this trajectory and continue to develop interventions to help PSTs get closer to enacting the questioning practices in the ambitious goals endorsed by NCTM (2014).

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USING GENERATIVE ROUTINES TO SUPPORT LEARNING OF AMBITIOUS MATHEMATICS TEACHING

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In this paper, we integrate a set of theoretical considerations that together serve as a model for investigating how high-leverage practices could be generative of teacher learning. We use the context of rehearsals to investigate how the use of a specified question sequence aimed at eliciting student mathematical thinking can afford opportunities for novices and instructors to consider goals of ambitious mathematics teaching. In our results, we provide thematic categories for the problems that arose as novices used the sequence of questions, and demonstrate how they afforded the teacher educator opportunities to connect novices' work to goals of ambitious mathematics teaching. In particular, we highlight how these opportunities arose in the midst of modifying to the question sequence and investigating the consequences of its enactment.

Keywords: Teacher Education-Preservice, Rehearsals, Classroom Discourse, Instructional Activities and Practices

Much of the work of teaching is non-routine, requiring a capacity to improvise in the midst of contingent interactions, marshalling knowledge and skill in the service of professional goals (Grossman, Compton, et al., 2009). Mathematics teachers, for example, must make judgments about how to respond to students individually and in groups, drawing on specialized knowledge of both mathematics and student thinking to further instructional objectives. All the while, they must treat all students as sensemakers and provide them with access to cognitively demanding tasks. For mathematics teacher educators, this problem of complexity is associated with another problem, one of enactment. Novice teachers must learn not only to analyze teaching but also to enact it. Some current approaches to teacher education employ "pedagogies of enactment" to engage novices directly in the interactive work of teaching (Grossman & McDonald, 2008). Within these pedagogies, teacher educators are organizing teacher learning around a set of core teaching practices derived from research on student learning and professional standards. These practices include, for example, eliciting and responding to student reasoning, representing student thinking, orienting students to each other's ideas, and attending to students' errors.

To help novice teachers learn to implement these practices, there is also increasing interest in developing enactment tools, such as talk moves or specific activity frameworks. Tools translate abstract conceptual tasks into more concrete steps and objectives (Wertsch, 1998), supporting the user in implementing particular practices toward a goal. There is concern, however, that a focus on enactment tools may reduce teaching to a set of techniques, without attention to important purposes and commitments that guide teachers' practice (Kennedy, 2015). In this study, we conceptualize the idea of "generative routines" as tools that support beginners to enact core teaching practices while simultaneously learning to use goals and professional commitments to guide decision-making. We ground the idea of generative routines in Hatano and Inagaki's (1986) notion of *adaptive expertise*, where they distinguish between routine and adaptive experts and argue that the latter are those for whom performance of procedural skills is enhanced by an understanding of their purposes. Adaptive performance, they argue, requires developing both efficiency in routines and the professional knowledge and judgment to be able to innovate and adapt to new situations. Research on the development of expertise suggests that this balance is achieved through deliberate practice (Ericsson, Krampe, & Tesch-Römer, 1993), which allows the developing practitioner to gradually refine specific aspects of performance through cycles of repetition with feedback.

Our study examines how generative routines can mediate novice teacher learning of ambitious mathematics teaching. We focus on a particular routine, a well-specified question sequence designed to support novices in the beginning work of eliciting and responding to multiple student strategies in mathematics: What did you see (or get)? → Did anyone see (or get) anything different? → How did you see it (or figure it out)? →Did anyone see it (or figure it out) in a different way? As initial prompts, these questions serve a technical purpose by providing the novice with a set of moves to elicit a range of ideas that represent student thinking and their different levels of understanding. As a result, the novice utilizing this tool is confronted with a suite of demands associated with responding to students' contributions that arise in the spaces in between consecutive questions.

Our overarching research question is *How can an enactment tool be generative for mathematics teacher learning?* More specifically, we address this question by investigating (1) the problems of practice that arise for novices in the context of using the question sequence, and (2) how these problems afford opportunities for novices and instructors to connect the sequence to goals of ambitious mathematics teaching. We focus on problems of practice because research suggests that they open spaces that are generative of teacher learning (Horn & Little, 2010).

Theoretical Framework

We conceptualize learning to teach as increased participation in a community of practice where people coordinate their efforts to accomplish culturally-valued activities using tools that mediate goal-directed actions and shared cultural understandings. This sociocultural perspective on learning posits that there is circularity between tool use and the learning it is meant to facilitate (Sfard & McClain, 2002). Cultural tools mediate a learner's participation in a practice while being themselves products of this process. A key aspect of this mediation process is the way tools direct participation toward various goals around which activities are organized. Wertsch (1998) theorizes two complementary ways in which tools mediate activity. First, a tool mediates action by translating what may stand as an abstract conceptual problem for a beginner into a series of concrete operations at which one can become proficient. Thus, learning to use a tool entails developing technical skills. Wertsch also argues that a tool can support enactment through the affordances (and constraints) it contributes to the development of goal-directed activity. In the case of practice routines, affordances arise when the use of a routine towards particular ends opens up "problem spaces," problem solving situations in which the user can work through her understandings of particular concepts (Salomon & Perkins, 1998).

Context and Methods

The context of this study is a mathematics methods course taught by two teacher educators, designed around a summer learning institute (SLI) that provides four weeks of daily remedial instruction in mathematics and language arts for approximately 140 rising third graders of variable mathematics skills. The institute serves as a field setting for twenty-five novice teachers. To prepare for, and subsequently learn from, their work with children, novice teachers participated in daily Cycles of Enactment and Investigation of instructional activities that are common to the elementary mathematics curriculum, designed for novices to work on principled instructional practices and mathematical knowledge in integrated ways (Lampert et al., 2013).

Each cycle begins with the novice teachers observing and analyzing an enactment of an instructional activity (IA) in a classroom context, either live or on video. Following the observation and analysis of the IA, novices next prepare to teach it to the SLI students, rehearsing it first publicly in front of their peers and the teacher educator who participate as students, exhibiting understanding of how children think about disciplinary ideas. The teacher educator acts as coach, enabling both the rehearsing novice and others in the group to study a range of actions a teacher might take in response to particular student performances. All novice teachers then enact the IA with students in elementary school classrooms, video-recording themselves and writing analytic essays on their own performance. Continuing the cycle, the teacher educator guides a collective analysis using records of the enactments.

A Question Sequence as a Generative Routine

In ambitious teaching, student ideas and contributions are the essence of mathematics discourse; thus, a teacher's work to elicit multiple student conjectures is particularly crucial. However, this practice is often at odds with many beginning teachers' instincts to seek correct answers. For this reason, we specified a sequence of four questions to be used routinely across different IAs to provide a beginning structure for some of the initial elicitation work novices must do to facilitate collective problem solving within instructional activities.

The first two questions, "What did you get?" and "Did anyone get a different answer?" (or variations of these questions), enable novice teachers to start gathering a set of possible solutions from multiple students while responding in a non-evaluative manner that positions different answers as conjectures for the group to evaluate. Once a representative set of conjectures has been elicited, the second pair of questions, "How did you figure that out?" and "Did anyone figure it out a different way?" provide novices with initial prompts to begin to elicit students' reasoning about these strategies. These initial prompts serve a technical purpose by providing the novice with a set of moves to elicit a representative range of student thinking. As a result, the novice utilizing this tool is confronted with a suite of demands associated with responding to students' contributions that arise in between the consecutive questions. Demands include pressing students to articulate their reasoning, establishing productive exchanges among students around key mathematical ideas, and representing different contributions clearly for collective consideration (Staples, 2007). These demands constitute a rich problem space associated with responding purposefully to student contributions that can be worked on collectively in rehearsals. In this way using the sequence as a tool creates affordances for novices to experiment with adapting to student performances.

Data Sources and Analytic Procedures

We analyzed 19 video-recorded rehearsals, representing the rehearsals facilitated by one of the teacher educators during the second and fourth weeks of the SLI. These two weeks of rehearsal videos were selected due to the prominence of the focal question sequence in the structure of the IAs being rehearsed.

To analyze the rehearsal videos, we used Studiocode©, a software package that connects analytic codes directly to segments of video. We identified all rehearsal segments in which the question sequence (QS) in its entirety was being rehearsed, hereafter referred to as QS segments, and then narrowed in on portions of these segments where there were pauses in the simulation for exchanges between the teacher educator (TE) and novice teachers (NTs). We refer to these sub-segments as TE/NT exchanges. There were 72 TE/NT exchanges within QS segments in our data set. Guided by our theoretical framework, we characterized the problems of practice that were discussed during these exchanges as a direct result of using the question sequence. To characterize this set of problems, we began with the two broad conceptual categories theorized by Wertsch (1998) and noted earlier: problems related to technical aspects of using the question sequence; and problems afforded through its use, i.e. problems arising as a consequence of asking questions in the sequence and eliciting student responses. Problems associated with aspects of practice unrelated to the question sequence were categorized as "other" and excluded from subsequent analyses. We then followed a process of thematic analysis (Boyatzis, 1998) to identify sub-categories, developing short descriptions of the problems worked on in each exchange and labeling them for themes. We then discussed our emergent thematic categories to refine a final set of inductive sub-categories for each of the two broad categories.

Inside both broad categories of problems, we then used open coding to characterize how participants drew on goals and professional commitments of ambitious teaching in addressing problems of practice. We developed analytic vignettes (Erickson, 1986) for a representative set of the TE/NT exchanges in order to characterize how a teacher educator can leverage work with an enactment tool, like the question sequence, to create opportunities to explicitly connect novice teachers' work on eliciting and responding to a set of professional commitments like treating all students as sensemakers and providing equitable access to content (see Ghousseini, Beasley, & Lord (2015) for more details).

Results

Our analysis demonstrates how the use of the question sequence as an enactment tool, both in its technical aspects and through the problem spaces afforded by its use, brought forward a number of problems of practice that the novice teachers and teacher educator then collaboratively addressed in the rehearsal context. In managing these problems, participants engaged in a form of inquiry during which the teacher educator guided novice teacher participation in considering questions and solutions related to the use of the tool and its consequences. In the process, the teacher educator had repeated opportunities to connect judgments about adapting the question sequence to commitments of ambitious teaching, like providing students equitable access to learning and treating them as sensemakers. The problems of practice related to the technical use of the QS emerged when novices considered both adaptations to the *wording* of specific questions within the sequence and to the *order* of these questions. The problems related to the affordances of using the QS emerged when the NTs had to manage unanticipated student responses. Such responses required the TE and NTs to determine how to respond to student solutions, support their collaboration, and represent their strategies. We share illustrative vignettes from each of these categories of emergent problems.

Problems Related to Technical Aspects of the QS

How to adapt the order of the questions in the QS. During one rehearsal, novice teachers practiced the IA of Quick Images, which focuses on helping students determine the total number of items in two ten-frames that are flashed quickly. Specifically, the novices used the QS to engage students in using the five- or ten-structure of the ten-frames to determine the total quantity. A novice teacher asks, "As we go down the list of questions, if the first student [who was] asked to explain their strategy is understanding the five- and ten- structure, can we just stop at that part, or do we need to [use the other questions in the QS] to ask for different strategies?" In other words, the problem of practice that the novice teacher is considering is "If the teacher hears the correct answer to the problem and the student communicates sound reasoning about it, is it necessary to solicit different answers and strategies?" In her question, the novice mentions the instructional goal of the ten-frame activity as a way of legitimizing the problem of practice that she is bringing forward for everyone's consideration. The TE, in response, provides several reasons why it still makes sense to continue with the next question in the sequence.

> TE: So you want to look for other strategies because you want to find out as much about what students are thinking as you can, because you are still kind of assessing. And it is not about "this is the only strategy that is legitimate." There are other strategies that are legitimate and valid strategies.

NT: But if we identify a student who is understanding the five and ten structure, could we not ask him to explain to the students too?
TE: That would be one move, or you could have other students revoice that strategy and see if they understand it.

This example illustrates the way practicing the question sequence afforded opportunities to attend to novices' personal understandings of the goals of teaching. In this example, the NT seems to be operating with the assumption that teaching is about getting quickly to an explanation of the correct answer. Ambitious teaching rests on a different set of assumptions— like the importance of investigating alternative explanations and incorrect answers— which the TE can negotiate in exchanges like this one. The TE's response stresses that the question sequence serves multiple goals beyond merely identifying the correct answer, and the correct strategy. It also emphasizes some aspects of the commitments of ambitious teaching: the importance of knowing the students as learners (finding out what they know and making instructional decisions accordingly) and treating students as sensemakers (legitimating different ways of reasoning about mathematics). Her response also underscores the importance of allowing students to collaboratively judge what is to be taken as shared. For instance, she suggests that even when a student proposes a correct strategy, the teacher should orient other students to his thinking and give them the space to make sense of it.

Problems Related to the Affordance Aspect of the QS

Representing students' ideas as a result of using the QS. In this example, the novice teacher is faced with a situation that, from an ambitious teaching perspective, demands that she respond to a student contribution in a way that makes their thinking visible to other students and connects it to the mathematical goals of the lesson. This situation occurred during the third rehearsal of a Quick Images activity. The rehearsing novice teacher (R-NT) has flashed a card that showed 12 dots (a full ten-frame on the left side of the card, and another ten-frame with only two dots on the right side). She asks a variation of the question "How did you figure that out?" to elicit a student's strategy for recognizing that a full ten-frame and two more dots was twelve in total: "How did you see 12?" As one student explains that she saw 12 as "the full ten-frame and two more," the R-NT attempts to represent the strategy on the card, roughly pointing with her fingers to the full ten-frame and then the two dots, while saying "10 and 2." The TE deems the R-NT's response appropriate by noting, "What you just did was a good idea, to use your finger [to represent the strategy on the card]." However, the TE points out that the manner in which the R-NT has represented the strategy on the card did not convey meaningful mathematical ideas to the students because she did not deliberately point to where "10" was on the card. One goal of this Quick Images activity is to help all students see that the ten-frame represents 10, which can be done by highlighting that the top and bottom rows each contain 5 dots when they are filled, and together the two rows add up to 10. The TE's comment indirectly underscores a key commitment of ambitious mathematics teaching—to provide equitable access to learning by visually representing student strategies for collective consideration, and to target particular mathematical concepts.

The TE then directs the R-NT to replay her response to the student strategy and practice using her finger more deliberately, tracing with her fingers where the 10 is while revoicing the student's strategy. Before she replays her response, however, the R-NT raises a concern about her own pattern of response in this kind of situation:

R-NT: Should I ask, umm, I feel when I [represent her idea in this way] that I validate her answer by saying it.
NT: Yeah, I was doing that yesterday in my class. I was repeating everybody's answer. I don't think that's what we're supposed to be doing.

R-NT: Like her answer will only be valid if the teacher says it again.
TE: So what would you do instead?

Two problems of practice are identified here by the NTs: one is concerned with how a teacher's revoicing move may be unintentionally interpreted by students to be a form of validation of particular answers; the other relates to the frequency of teacher revoicing of students' answers. Herbel-Eisenmann, Drake, and Cirillo (2009), in fact, documented similar concerns on the part of their in-service middle school teachers. They argued that teachers face dilemmas in using revoicing; they worry that an unintended function of revoicing could be to shift ownership of mathematical ideas from the students to the teacher. As a result, students may stop listening to their classmates and simply wait for the teacher to repeat different ideas. These unintended consequences of revoicing operate counter to a commitment of ambitious teaching: to treat students as sensemakers by giving them ownership of their intellectual work. By voicing their concerns in this example, the NTs seem to be acting on an implicit awareness of this commitment of ambitious teaching. With her question "What would you do instead?" the TE guides the NTs' participation in considering the use of revoicing as a form of representation of student thinking. The R-NT, in response, offers that as the student is explaining a strategy, she could just represent it on the card without doing a lot of talking. The TE directs her to try it; however, as the R-NT replays her response to test it out, she mainly represents the student strategy (of adding ten and two more) by roughly pointing to the ten frames on the card. Her replay of the response opens up an opportunity to investigate its consequences. The investigation starts when the TE intervenes again.

TE: So, the reason why you have to [restate the student strategy] is because what you're doing is you're taking her strategy and making it accessible to all the students by representing it on the card…. If you were just to use your finger, it's not- it's not connecting her strategy to what's on the card. So, you kind of have to say it in a way. Does that make sense?"

What the TE underscores in her intervention is that, in this case, the R-NT's response must support students' understanding of the meaning of "ten" by helping them connect verbal and pictorial representations of it. In this way, the TE connects the work of eliciting and responding to a guiding commitment of ambitious teaching: that giving equitable access to learning requires making explicit the different mathematical ideas that are shared during the lesson explicit.

In another example also taken from a Quick Images activity, a R-NT was standing in front of a white board, flashing different cards to a group of seven students who are sitting in a semi-circle. After flashing a card representing the problem 9 + 4, and asking the first two questions in the elicitation sequence (What did you get? Did anyone get anything different?), the R-NT gets two responses, 13 and 14. In responding to the two different answers, she turns the card face up for students to check their answers. However, as she does that, she orients her body and the card in the direction of one student who had incorrectly seen 14 dots. Given that this student was sitting at one end of the semi-circle, the TE intervenes, noting that by mainly angling the card toward that student, the R-NT was limiting the access of the students sitting at the other end of the semi-circle. She explains,

TE: Remember, this is about everybody processing. So be careful not to walk over and make it about you and this student. So if you could stay [in the center of the semi-circle] and show the face of the card so everyone can see. You can ask her to explain [while you are standing there].

R-NT: They really wanted to—and I made this mistake yesterday—they wanted to come up and show everybody how they did it. And I was letting them, but I think that I would not let them do this anymore.

TE: I would try to cut that out. You can just tell them “explain to me with your words,” and just help them articulate their strategy.

In this instance, the novice gets in a problem space after trying to respond to the two strategies that she elicited with the first two questions of the sequence. The problem of practice that is at play pertains to managing the position of the representation so that students can have access to the mathematical ideas that the group is attempting to address. Positioning the representation in a way that allows every student to see it communicates to them that they are all accountable for judging the reasonableness of answers. Twice in her intervention, the TE reminds the R-NT about this important commitment of ambitious teaching: “Remember, this is about *everybody processing*” and “show the face of the card so *everyone can see it*.” The R-NT’s justification for her move underlines a problematic situation that she was trying to remedy and suggests that she was trying to attend to the goal of students’ joint collaboration: “they wanted to come up and show everybody how they did it.”

Discussion

Our study illustrates the potential of deliberate practice with a generative routine for supporting the learning of adaptive performance. Generative routines, like the question sequence, can function as a stable procedure that can reduce some of the initial complexity of relational practice. At the same time, enacting the procedure opens up a rich problem space for novices, who must confront the contingencies of students’ improvisational responses. As such, generative routines can be more than scripts or processes that scaffold performance. When novices have opportunities to navigate these problem spaces in the company of more experienced others, generative routines can mediate learning about goals of professional practice, including the commitments that enable practitioners’ judgment in situations of uncertainty.

Our analysis reveals the important role played by the teacher educator in connecting the use of the tool to professional commitments of ambitious teaching. She participates in this role through various interpersonal engagements with the novice teachers, allowing them space for practicing the work of teaching while at the same time guiding their participation in it through various forms of interventions aimed at making explicit connections between particular courses of action and commitments of ambitious teaching. Our findings provide evidence of the teacher educator focusing the novices’ attention on the problematic situations that arise in practice and framing considerations for possible solutions around particular commitments of ambitious teaching. Without the teacher educator in this early stage of novice teachers’ experimentation with enactment tools like the question sequence, it may be difficult for them to recognize the problematic nature of situations and to translate their current understanding of the commitments of ambitious teaching into practice.

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"WHEN WILL I USE THIS?" PRESERVICE TEACHERS' BELIEFS AND APPROACHES TO SOLVING MATHEMATICAL TASKS

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The Common Core State Standards Mathematical Practice, Model with mathematics, specifies that students should be able to apply the math they learn in school to "everyday life, society, and the workplace." However, the way that mathematics is traditionally taught in school has led to an ingrained belief about what school math is and how it should be solved which may hinder PSTs from being effective at cultivating modeling practices in their students. This study reveals both roadblocks and openings that can be explored to improve teacher education in the future, specifically regarding transferring mathematical concepts between settings and contexts.

Keywords: Teacher Education-Preservice, Modeling, Instructional Activities and Practices

It is frequently assumed that school mathematics provides students with the preparation they will need for life, however there are multiple barriers in place that prevent this goal from being achieved. Lave (1992) suggests that trying to both provide students with a variety of tools and procedures and preparation for life are "not…compatible values – they are *contradictory.*" Stigler and Hiebert (1998) also reveal how the teaching of mathematics in schools is a cultural practice and that in the United States that practice leads to the belief that "school mathematics is a set of procedures." Simultaneously, the Common Core State Standards for Mathematics (CCSSM) Standards for Mathematical Practice 4, Model with mathematics, emphasizes the need for students to be proficient at applying the mathematics they know to "everyday life, society, and the workplace." Recognizing that Preservice teachers (PSTs) beliefs and understandings about mathematical tasks will influence what they convey to their students, it is important to develop instruction geared towards guiding them towards embracing the CCSSM guidelines which have influence many schools' curricula.

White and Mitchelmore (2010) describe traditional mathematics teaching as the "Abstract Before Concrete" method where procedures are taught with the expectation that students will be able to apply them to other contexts. This has not been shown to be effective and instead resulted in students viewing math as separate from life. A different approach has been met with more success in understanding and application.

The well-known Brazilian street-vendor study by Carraher, Carraher, and Schliemann (1985) revealed that children were able to correctly solve 98% of the given arithmetic problems when given in the context of their street vending responsibilities, 73.7% when given as a traditional word problem, but only 36.8% on a written formal assessment. Other studies have shown similar results comparing the quality of responses and reasoning on tasks that varied in their levels of authenticity (Palm, 2008; Walkington et al, 2013). These outcomes demonstrate the benefit of learning and solving math problems within a context that is relevant or authentic to the students' prior experiences. This is especially important given that the culture of schooling in the United States dictates that math is taught primarily in a classroom setting, and educators need to adapt best practices to the given setting.

Studies done by Cooper and Harries (2002) and Palm (2008) show that students in school settings struggle with problems that are intended to be realistic and relevant to the students' lives. Cooper and Harries suggest that the difficulties arise due to the students reacting to the "ground rules for school mathematics word problems," that allow them to only consider the numbers introduced and the expected operation to complete the problem. By comparing student responses on two sets of story problems, one considered "less authentic," the other considered "more authentic," Palm concluded that increasing the "authenticity" of the tasks will increase the students' ability to create realistic responses that incorporate their background experiences.

We can use information gathered by investigating PSTs' understandings of and approaches to mathematical problem solving to combat the perpetuation of the "mathematics as a set of procedures" mindset (Stigler & Hiebert, 1998). The following research questions were explored to elicit this information: 1. How do the approaches PSTs take to solve two similar sets of mathematical tasks, one presented in a real-world context, and the other presented without context, compare? And 2. Which tasks are preferred by the PSTs and why?

Method

Two sets of mathematical tasks were designed in different contexts. One included a hypothetical financial scenario that was deliberately designed around their intended profession as a teacher. The actual average salary for a fifth-year teacher in the same state as the university was selected for the salary base provided. Also included were common expenses faced by professionals including a cell phone bill, a car payment, rent for housing, and income tax. There was a description of a certificate of deposit (CD) account with compounding interest. The housing costs were based on actual prices of two bedroom apartments in the state while the other bills were taken from statistical averages for the country since more local data was not available. The PSTs were asked to solve tasks related to the scenario. When they asked for clarification, they were encouraged to make their own assumptions about what the question required.

The second set of tasks consisted of one percentage problem that was devoid of any context, and one compound interest problem. All of the necessary information regarding the interest account was provided including rate, initial deposit, length of time and compound frequency. The formula for compound interest was provided as a "hint" but they were not asked specifically to use the formula. This set is referred to as the "abstract" set.

Two first year PSTs, Elizabeth and Rachel, were recruited to complete a videotaped interview. Their teacher preparation program focuses on preparing teachers for grades k-8. They were not given any information regarding the tasks prior to being interviewed.

During the interview, the PSTs were asked to vocalize what they were thinking while solving the tasks. They were given the tasks in opposite orders to examine whether one style of tasks influenced the solution strategies used on the other. They were not allowed to look at the first set of tasks they completed while solving the second set of tasks. After each set of tasks, the PSTs were asked questions relevant to that particular set of tasks. After both sets of tasks had been completed, they were asked more questions about which set they preferred, whether they were now uncertain about other answers that had given in the first set of tasks, and what they found interesting about the two sets while they were solving the tasks. The PSTs had access to a scientific calculator throughout the interview.

The transcripts were coded using a combination of a priori coding along with open coding techniques. Some of the codes included were "numerical calculations": calculations devoid of unit labels, "contextual calculations": units, items, or labels were indicated while performing calculations, "relevance": whether the situation or the tasks currently relate to the PSTs own lives or may be relevant to a situation they find themselves in later, and "preference": indication of a preference of one style of tasks or one method of solving a task.

Results

Two primary findings emerged through the analysis. First, both PSTs expressed a preference for formulas when provided demonstrating a reliance on procedural computations. Second, both PSTs voiced appreciation for and favored the tasks that they considered relevant to their lives.

Both PSTs demonstrated a preference for the formula when solving the tasks related to compound interest even though neither task required use of a formula to determine a correct response. Elizabeth, who had been given the tasks related to the finance scenario first said emphatically, “oh, they give the formula, good!” when presented with the interest task with the formula provided. More importantly, even though she had correctly figured out the interest when first solving the interest task for the set of tasks based in context without the formula, when she was given the opportunity to solve with the formula at the end of the interview, she solved it incorrectly. When asked which method she thought was correct, she said the first way she did it was wrong, even though that was, in fact, the correct solution. “I feel like it should be the same, but they made a formula for it and this is what the formula says. [Elizabeth, 33 min. 10 sec.]” She had more faith in using formula than she did in her own logic and reasoning using the contextual clues given in the scenario.

Rachel, who had been given the abstract tasks first, had similar inclinations regarding her school math experience, “I…remember this formula from previous classes in high school [6 min 30 sec.]” but she would have liked to have been told the meaning of each of the letters in the formula, “saying like, P equals your initial deposit, or like n equals your, I’m not exactly sure, but like r equals rate [16 min 15 sec].” She clearly recognizes this as math she’s seen before in previous high school classes and rather than desiring to understand why the formula works in order to use it more effectively, she preferred to be told where to put the numbers in the given formula. She later says that “I feel like this would work with the formula from the previous task [26 min 15 sec]” when approaching the task in the contextual set of tasks that included compound interest. The formula was not provided on the task paper itself, but she wrote it on her own paper in order to use it. It is important to note that while she showed that she understood how formulas work in general, and what the compound interest formula was used for, she was not able to use it correctly in either context.

In fact, neither PST was able to compute the correct answer using the formula. They particularly struggled with what “n” represented,

> “Now I’m looking at the semi-annually part and seeing if I can do anything with that…Maybe it’s, okay well if I say like twice a year? Then maybe ‘n’ could be maybe I’m either thinking like two or point five [Rachel, 9min 20sec]”

> “…and then ‘n’ I would believe would be two cause it’s twice a year, ... and I have nowhere else to put that number, so I’m just going to go for it and see if this makes sense. [Elizabeth, 27min 45sec].”

If either PST understood the relationships between the variables involved in the formula, they may have had more confidence in determining the value and placement for “n.” Instead, both PSTs frequently mentioned the need to put the numbers in the formula, but neither elaborated on their understanding or interpretation of any of the other variables outside of how they knew which letter the numbers would replace in the formula. Neither attempted alternative methods of solving the problem without the formula while working on that set of tasks.

Both PSTs found the contextual set of tasks more relevant as well as more interesting. Elizabeth likened the bills to her future living arrangements where she will be responsible for paying for her part of a shared apartment. She also said that “it’ll be even more relevant to me in the future [18 min 20 sec].” Rachel admitted that she was not currently responsible for paying any of her own bills, but did acknowledge future relevance, “I should have a car and an apartment and … stuff like that so I should, in the next four years, have to be paying all these sorts of things [42 min 5 sec]." Their mutual preference for the contextual scenario tasks ties into their appreciation of the relevance of the scenario. Rachel articulates this connection well when she said, "You're given this problem where you're directly looking at something you're going to have to do in the future, then it's more interesting because … you will eventually have to take all these things into consideration. So it just makes more sense than just, like, stick this into this formula and find some answer [44 min 0 sec]."

Conclusion

The PSTs interviewed both showed a strong inclination towards the use of procedural methods and formulas over reasoning and alternate solutions. This indicates that both of them have internalized the belief that "math is a set of procedures (Stigler & Hiebert, 1998)" This has the potential to be problematic later as teachers when they try to teach their students proficiency in the use of mathematical skills for "everyday life, society, and the workplace. (NGAC, 2010)." Encouragingly, both PSTs recognized their own interest in solving interesting and relevant tasks which may help them understand the importance of providing similar tasks to their own students.

Many PSTs currently in teacher preparation programs likely did not encounter many modelling opportunities in their own classrooms. For this reason, further investigating their current understandings and beliefs on mathematically similar but contextually different tasks could inform the creation of more effective instruction on the importance of incorporating modelling in their own math classrooms. Smaller scale comparison activities can also be used within teacher preparation programs to demonstrate to PSTs their current biases and the importance to overcome them to potentially make math more inviting to their own students who will then no longer have to wonder, "when will I use this?"

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"WELL, THEY UNDERSTAND THE CONCEPT OF AREA": PRE-SERVICE TEACHERS' RESPONSES TO STUDENT AREA MISCONCEPTIONS

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The purpose of this study was to explore how elementary pre-service teachers responded to student misconceptions about area, within the context of their mathematics content knowledge. We carried out pre-assessments and interviews with 24 pre-service teachers enrolled in a geometry and measurement course. Findings indicated a misattribution of understanding to students, conceptual use of visual representations, and differences in response types depending on initial content knowledge. In many cases, pre-service teachers leveraged their understanding towards productive responses. Recommendations for supporting pre-service teachers in navigating the intersection between content and pedagogical knowledge are discussed.

Keywords: Measurement, Teacher Education-Preservice, Teacher Knowledge

The knowledge required for successful mathematics teaching is complex and made up of several components, including common and specialized subject matter knowledge, as well as knowledge of students and teaching (Ball, Thames, & Phelps, 2008). Specialized knowledge for teaching is significantly related to student achievement gains (Hill, Rowan, & Ball, 2005), highlighting the importance of developing this knowledge in pre-service teachers (PSTs).

Considering the importance of specialized subject matter knowledge, there exist concerns regarding the content knowledge of elementary PSTs. One content area that has drawn attention is area measurement. Past research has found that PSTs struggled with area concepts, exhibiting procedural understanding and misconceptions like those of students (Baturo & Nason, 1996; Livy, Muir, & Maher, 2012; Murphy, 2012). This apparent lack in content knowledge may lead to apprehension about the effectiveness of the PSTs' future area instruction. However, there is neither time to completely readdress the topic, nor would this be useful for all PSTs. Instead, it is fruitful to explore the ways in which PSTs put the area-specific knowledge they *do* possess to use, and the ways in which we can support their own personal transformations of knowledge.

The purpose of this study was to explore how elementary PSTs leveraged their knowledge to help address student misconceptions about area. We were interested in the connections between PSTs' content knowledge, responses to student work, and how PSTs negotiated the intersection between multiple types of knowledge. To this end, the research questions guiding the study were the following: (1) What types of pedagogical strategies do elementary PSTs engage in when responding to student area misconceptions? (2) How do elementary PSTs leverage their own understanding of area concepts into responses to student area misconceptions?

Related Literature

Geometric area has strong connections to experiences with the physical world, as well as other topics in mathematics (Sarama & Clements, 2009). Learning about area marks an early shift for students into two-dimensional mathematics, and has the potential to be used as a tool for estimation, manipulation, and visual representation. Unfortunately, several studies have found that students frequently do not understand these deep concepts, and instead take away from area instruction the formula *length* × *width* (Zacharos, 2006). PSTs have also exhibited the same area misunderstandings as students and may take these forward with them (Livy et al., 2012; Murphy, 2012). Despite some misconceptions, however, PSTs do have valuable knowledge about area measurement. PSTs in past studies have exhibited an understanding of area as two-dimensional space and a concrete measure which can be calculated (Baturo & Nason, 1996; Livy et al., 2012). While perhaps not ideal conceptual understandings of area, this knowledge represents a key starting point in mathematics content courses and for PSTs' future interactions with students.

One approach for understanding how PSTs knowledge may be used in teaching is to analyze their responses to hypothetical student work. This method allows PSTs a longer time for reflection and offers a low-risk setting for exploring pedagogical strategies. At the same time, responding to student work is a central component of mathematics teaching. Analyzing student work in the past has clarified how PSTs respond to errors and student-invented strategies (e.g. Busi & Jacobbe, 2014; Son, 2016). This method can provide insight into how we may support PSTs in developing their content knowledge into relevant and useful pedagogical knowledge.

The framework for this study is that of a mathematical knowledge for teaching composed of multiple components: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2008). We were interested in exploring our PSTs' existing common and specialized content knowledge, and the transformation of this knowledge into mathematical knowledge for teaching.

Methodology

The study took place at a large Midwestern public university. The participants (n=24) were elementary PSTs enrolled in a 15-week geometry and measurement content course. The course is one of several available mathematics courses recommended in the first semester of enrollment. The participants were all female, and had diverse educational and mathematical backgrounds.

Data Collection

Data collected included an area pre-assessment, given prior to the course measurement unit. The assessment consisted of seven questions designed to assess both common and specialized content knowledge, as well as knowledge of students. Four questions addressed conceptual understanding of area (e.g. area as a covering of units). Three questions asked for responses to hypothetical student work. Assessment tasks and student transcripts were adopted from prior studies (e.g. Sarama & Clements, 2009; Zacharos, 2006). A sample task is given in Figure 1.

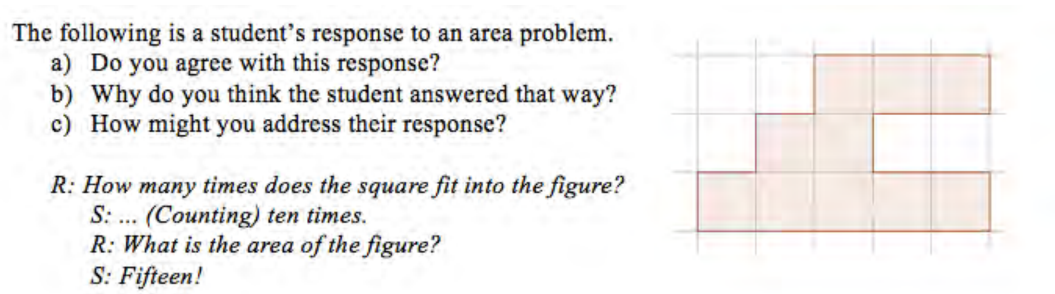

Figure 1: Written pre-assessment sample task.

Following the pre-assessment, interviews were conducted with all participants. Each semi-structured interview was conducted by the first author, and was approximately one-hour long. Participants revisited pre-assessment questions, providing further detail about their responses. Special care was taken to probe responses to student misconceptions, exploring in depth participants' knowledge of content and students as well as knowledge of content and teaching.

Data Analysis

Pre-assessments were examined first to determine participants' procedural and conceptual understanding. Responses were analyzed using a 3-level scale to determine approximate content knowledge, ranging from 0 points for an incorrect response (e.g. suggesting comparison of perimeter when asked for area), to 2 points for correct responses with accurate justification. Participants were divided into three levels according to natural divisions in the range of scores.

Once interviews were complete, these were qualitatively analyzed with pre-assessments and reflective memos. Our analytical framework for pedagogical strategies was borrowed from Son and Sinclair's (2010) framework for analyzing PSTs' responses, and consisted of categories such as conceptual versus procedural focus, show-tell versus give-ask, and pedagogical actions (re-explain, probe thinking, etc.). After an initial coding, the data were revisited to examine the ways in which specific levels of content knowledge interacted with pedagogical strategies. Responses to student misconceptions were viewed within the context of content knowledge, forming a bridge between specialized content knowledge and content-specific knowledge for teaching.

Findings

Analyses of the pre-assessments indicated that most participants possessed a procedural and formula-driven understanding of area. Participants applied the formula *length* × *width* whenever possible, and struggled in the absence of numerical computations. Four PSTs were categorized as high CK, twelve as medium CK, six as low CK, and two were excluded due to missing data.

Choices of pedagogical strategies differed among participants, varying according to content knowledge and personal preference. PSTs' pedagogical strategies were split somewhat evenly between procedural and conceptual responses, but were frequently teacher-focused and centered on *showing* or *telling* the student how to proceed. While most participants could identify key gaps in understanding exhibited by student work, they struggled to supplement those gaps. One frequent suggestion (11/24 participants) for Figure 1 task was for the student to carry out more practice problems, suggesting a high value on procedural fluency. Other recommendations included presenting alternate procedures (18/24 participants) or suggesting the student simply needed a re-explanation of the concept of area (14/24 participants).

Additionally, there was frequently a misattribution of understanding to students. Several participants claimed that the student from the Figure 1 task understood area, and only needed to review correct application of the formula. One participant stated that "They know you multiply base times height for simple shapes…they understand *that*, they just don't understand that you can divide this shape into those shapes that they know how to find the area of and then add them all together" (Participant #12). Through their responses, PSTs relegated errors in student work to procedural mistakes, rather than conceptual misconceptions. This belief then tended to limit the pedagogical strategies employed to a procedural focus.

Several interesting points arose from analyses of the connections between strategies and content knowledge. All four participants with high content knowledge preferred show-and-tell strategies, levering their content knowledge to provide multiple procedural approaches to the problem. These approaches tended to focus on arriving at the correct answer through multiple methods. Meanwhile, participants with low content knowledge were more likely to adopt give-and-ask strategies, offering questions and prompts for students even if they were not necessarily sure of the answers themselves. Participants with medium content knowledge tended to employ all varieties of pedagogical strategies, ranging from telling the student how to carry out a procedure to suggesting cognitive conflict and conceptual probes.

Another finding was the use of visual representations in conceptual responses. This type of response introduced cognitive conflict via visual representations of a "full" rectangle (compared to the partial shape as in Figure 1) and asked the student to note the discrepancy in areas. For some participants, a preference for visual diagrams seemed to stem from an inability to verbalize conceptual ideas, and they struggled in explaining the mathematics behind their diagrams. Despite this, they could still transform their understanding of visual diagrams into conceptual student help, drawing attention to key ideas even if they were unable to verbalize them.

Discussion

Despite exhibiting a lack of deep conceptual knowledge of area, many PSTs in this study were able to identify sources of error in student work and attempted to formulate appropriate responses. While pedagogical actions were frequently teacher-centered, there was a mix of both procedural and conceptual responses, and many participants succeeded in transforming their own understanding into productive advice for students. Most notable is that a high level of content knowledge was *not* required for this process, and many students at low and medium levels of content knowledge could provide conceptual feedback. While there were at times mathematical errors in this feedback, such approaches reflect a positive starting point for teacher educators to build upon. On the other hand, participants with high content knowledge preferred procedural responses, indicating that these PSTs may require an approach focusing on pedagogy rather than content. While a few participants reinforced errors in their responses, these were participants who had particularly weak content knowledge not reflective of the group.

These findings indicate that PSTs at all levels of content knowledge are able to identify and address common area misconceptions among students. While their pedagogical strategies may not be considered ideal, they form a key starting point from which teacher educators can begin to move forward. Encouraging students to not only identify misconceptions, but shift towards addressing them both conceptually and with mathematical accuracy is a key step towards supporting the transition to specialized content knowledge and knowledge about students and teaching. Most importantly, it remains critical to acknowledge the understanding that PSTs bring to the classroom, and to work from it towards common educational goals.

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ADDRESSING MINDFULNESS, MINDSET, CONTENT KNOWLEDGE, AND ANXIETY IN MATHEMATICS FOR PRESERVICE TEACHERS

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Research has demonstrated a connection between teachers' ability to teach and their content knowledge, attitudes, and beliefs (Ball, 1991; Ernest, 1989; Fennema & Franke, 1992; Wilkins, 2008). As such, Wilkins (2008) suggested that teacher education programs address these components alongside content knowledge. To address this, we developed and implemented seminars that incorporated strategies for improving mindfulness, reducing anxiety, improving self-efficacy, and shifting toward a growth mindset for our teacher candidate participants. Our project also included a personalized learning pathway component, designed to develop participants' self-efficacy and content knowledge in mathematics and their knowledge of how to use such pathways in their future classrooms. In this paper, we present preliminary qualitative results from this research.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Teacher Beliefs, Teacher Education-Preservice

Perspectives

Research has consistently shown a link between mathematics teachers' content knowledge, attitudes, and beliefs and their ability to teach effectively (Ball, 1991; Ernest, 1989; Fennema & Franke, 1992; Wilkins, 2008). Similarly, links have been found between elementary teachers' mathematics anxiety and their students' anxiety (Beilock, Gunderson, Ramirez, & Levine, 2010). Likewise, Hadley and Dorward (2011) found that when elementary teachers had lower levels of anxiety about teaching mathematics, their students had increased mathematics achievement scores. This could be explained by the fact that as teachers' mathematics anxiety increases, their self-reported level of efficacy decreases (Swars, Daane, & Giesen, 2006). Unfortunately, even with years of research, students and teachers are still developing anxieties toward mathematics that is affecting students' ability to learn and teachers' efficacy.

Given this issue, Wilkins (2008) noted that teacher education programs must attend to these components in conjunction with content knowledge to improve the value of learning the content. Although these components are certainly necessary as a focus, new research on the ideas of mindfulness and mindsets in the classroom indicate that these might also need to be considered to help the mathematics education community overcome many barriers. In a recent study of fourth and fifth grade mathematics classrooms, students that engaged in mindfulness exercises did 15% better in mathematics than their peers (Schonert-Reichl et al., 2015). This study asserted that mindfulness is also connected to the teacher and subsequently to his or her classroom. Specifically, Schonert-Reichl and colleagues (2015) observed that teachers in the study who participated in mindfulness exercises indicated having less stress. In regard to mindset, Boaler (2016) recently noted the importance of mindset in learning mathematics. She stated, "the fixed mindsets that many people hold about mathematics often combines with other negative beliefs about mathematics, to devastating effect" (p. ix). Hence, the importance of the combined effect of mindset, mindfulness, anxiety, self-efficacy, and content knowledge needs to be explored.

In this project, we developed professional learning community seminars that incorporated strategies for addressing many boundaries within mathematics education including improving mindfulness, reducing anxiety, improving self-efficacy, and shifting toward a growth mindset for our participants, preservice elementary teachers (PSETs) currently enrolled in a mathematics methods course. Our project also addressed issues of access by including a personalized learning pathway component that was designed to develop PSETs' self-efficacy in mathematics, their content knowledge, and their knowledge of how to use and incorporate personalized learning pathways in their future classrooms. In this paper, we present preliminary results from our analysis that is currently underway.

Methods

Our research questions were: How does involvement in the project influence participants' mathematics mindfulness, anxiety, and mindset; and how does involvement in the project influence participants' mathematics self-efficacy? Given that both qualitative and quantitative data were needed to make inferences (Tashakkori & Creswell, 2007), we employed a mixed-methods research design. Specifically, we used a concurrent embedded design so that the quantitative and qualitative data could be triangulated to better understand how PSETs' participation in the project influenced the constructs of interest.

Quantitative data included a pre- and post-survey that assessed participants' mindfulness (Brown & Ryan, 2003), anxiety (Fennema & Sherman, 1978), mindset (Dweck, Chiu, & Hong, 1995), and self-efficacy (Enochs & Riggs, 1990; Enochs, Smith, & Huinker, 2000; Riggs & Enochs, 1990). To compare, the control group also took the surveys but did not engage in the seminars. All participants had taken the PRAXIS by the culmination of the project. This assessment provided an additional reference point that is an accessible reflection of their growth in content knowledge. Finally, we collected results from PSETs' engagement with the personalized learning pathway, Khan Academy. Khan Academy was chosen as the personalized learning platform due to it being free software and because it is frequently used in classrooms. This choice was intended to reveal both the potential usefulness and pitfalls of the platform.

Qualitative data collected from the intervention group included interviews, observation notes from seminars, and response to journal prompts. These sources were analyzed first using an open-coding scheme, and then, using these results to develop themes. All students in the project were enrolled in a mathematics methods course at a large university in the southeast United States. Participants could self-select into either the intervention or control group based upon their availability to participate in the seminars. Participants from the project's Fall 2016 semester included 13 PSETs in the intervention group and 16 in the control group. Participants in the intervention group were expected to participate in three professional learning seminars during the semester, each lasting approximately two hours, as well as engage in a personalized learning pathway during their personal time.

Results

In examining the correlations for both groups, the intervention group showed significant relationships between math anxiety (AMAS) and Personal Mathematics Teaching Efficacy (PMTE) ($r = -.753$), Mindset and teaching efficacy (PMTE) ($r = .451$), AMAS and Mindset ($r = -.496$), Mindset and GPA ($r = .435$), and PRAXIS score and GPA ($r = .462$). Significant relationships were also noted for the control group between PMTE and AMAS ($r = -.60$), Mindset and Mindfulness ($r = .523$), Mindset and GPA ($r = -.487$), PRAXIS scores and GPA ($r = .405$), and PRAXIS scores and the results on the proficiency exam ($r = .540$). Additionally, we used a paired sampled t-test to explore the relationship between variables assessed from pre- to post-test. Results revealed statistically significant changes from pre- to post-test for the intervention group for AMAS ($p = .000$), PMTE ($p = .000$), and Mindset ($p = .025$). Significant differences were not found for Mathematics Teaching Outcome Expectancy (MTOE) and Mindfulness. Statistically significant results were found for the control group for the variables AMAS ($p = .025$) and PMTE ($p = .006$).

Examining the Khan Academy data, we found a strong, positive correlation between the amount of time students practiced skills and the amount of concepts marked as mastered within the platform ($r = .940$). This is not surprising, but can be used to demonstrate to students the importance of practicing certain skills. As part of the participants' journals, they were asked to comment on their use of Khan Academy. These journals revealed that common topics students studied were fractions and topics they were expected to teach in their placement. Additionally, we also found that many enjoyed the personalized pace and focus, however, several did not regularly engage because of a lack of time or because they forgot.

The preliminary analysis of the qualitative journal entries revealed interesting results. In regard to mindfulness, analysis of participant journals indicated that, over the course of the project, several participants developed mindfulness in two areas of their life: their personal life and academic life (i.e., as a teacher candidate). As an example of the former, one participant described being more aware of their surroundings "rather than being on autopilot" (Journal 2, Participant 1). As an example of the latter, another participant claimed becoming more aware of their students so that they "can better teach them core math strategies" (Journal 2, Participant 2).

Similar results were found with respect to participants' math anxiety and mindset. For example, many participants indicated a general awareness of when their internal dialogue aligned with a fixed mindset and tried to change this internal dialogue. We see this in Participant 4's response on the third journal: "I have tried to intentionally speak growth mindset words to myself." Additionally, anxiety toward teaching mathematics seemed to decrease for participants. One participant stated that she "was less afraid [of] teaching [her two] math lessons this semester" (Journal 3, Participant 5). For math anxiety and self-efficacy, the preliminary qualitative results were sorted into three main themes. One theme was that many revealed an increase toward acknowledging their own ability to reduce their anxiety and/or increase their self-efficacy. Another theme was that some connected their anxiety/self-efficacy to events done to them (e.g., timed multiplication tests) or as something done by them (e.g., choosing to avoid "difficult" math classes). The last theme was that many acknowledged the tools that we introduced to them as a way to reduce anxiety and increase self-efficacy. In regard to mindset, awareness of participants' personal mindset and the importance of a growth mindset increased. Second, many saw a disconnect in certain areas where they were fixed in one area and growth in another. Last, many made a direct connection to teaching. After the seminars, they purposefully monitored their body language and speech inside their school placement so as not to indicate their fixed mindedness in a topic in hopes that their students will not have the same mindset that they do.

Conclusion

The "Improving Mindfulness, Anxiety, and Content Knowledge in Mathematics Pre-Service Teachers" project was designed to develop PSETs' self-efficacy in mathematics, their content knowledge, and their knowledge of how to use and incorporate personalized learning pathways in their future classrooms. The results from this study indicated that participants in the project developed a heightened awareness of their mathematics mindset and mindfulness. Participants also demonstrated lower anxiety related to teaching mathematics and increased self-efficacy in mathematics due to engagement with a personalized learning pathway (i.e. Khan Academy). If the analyses of the remaining data indicate similar results, implications for the teacher education community might include addressing how to incorporate similar activities and projects in teacher education programs.

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DEVELOPING CRITERIA TO DESIGN AND ASSESS MATHEMATICAL MODELING PROBLEMS: FROM PROBLEMS TO SOCIAL JUSTICE

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Despite the interest in modeling and the importance of social justice, there has not been much attention to connecting modeling with social justice. To fill this gap, we developed criteria for mathematical modeling problems that embrace the characteristics of problems and social justice through three phases: literature analysis, thematic categories, and piloting. The criteria will help teacher educators when selecting modeling problems to be used in teacher preparation programs and assessing the modeling problems posed by PSTs.

Keywords: Modeling, Equity and Diversity, Teacher Education-Preservice, Problem Solving

The purpose of this study is to develop criteria to design and assess mathematical modeling problems that embrace social justice contexts and to reflect on ways in which the criteria can be used to assess preservice teachers' (PSTs) ability to pose problems. Mathematical modeling has been emphasized since the Common Core State Standards for Mathematics (CCSSM) include model with mathematics as both content and process standards (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). The CCSSM define modeling as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (p. 72). Although the emphasis on mathematical modeling grows in learning and teaching critical mathematics as described in the CCSSM, PSTs are not yet adequately prepared to design and implement effective modeling problems. The characteristics of modeling, especially linking mathematics to real-life situations and involving students into decision-making processes, are shared with social justice problems. However, there is little consideration of social justice in frameworks related to mathematical modeling. For example, frameworks for developing thought-revealing modeling activities (Lesh, Hoover, Hole, Kelly, & Post, 2000) or rubrics developed for evaluating students' processes and solutions to modeling problems (Anhalt & Cortez, 2015) address important real-life contexts in general, rather than focusing on social justice issues. Gutstein (2003), on the other hand, developed mathematics problems with real-life contexts that reveal the injustice world but did not provide criteria for developing such problems. In this paper, we describe how we initiated and piloted the criteria that help teachers pose modeling problems with social justice contexts and use it to assess modeling problems posed by PSTs.

Process of Developing Criteria for Modeling Problems with Social Justice Contexts

Phase 1: Analysis of Prior Research

Lesh and Lehrer (2003) define modeling as a "process" of developing mathematical descriptions for specific purposes in particular situations. In this sense, modeling is placed in a spectrum of problem solving because problem solving is a process that requires solvers to understand a puzzling situation and to find a solution of the situation (Baroody, 1992). Problem solving seems a broader range of mathematical processes than modeling because problem solving do not specify the puzzling situations while modeling is required to involve a real-world situation (Anhalt & Cortez, 2015). A similar relationship appears between problems and modeling problems. Although many teachers mistakenly use the term *problems* for any mathematical tasks, only the tasks satisfy certain criteria

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

can be “problems.” Charles and Lester (1982) argued that problems must not show an obvious way to find a solution, and Van de Walle (2003) agreed problems must not have a predictable solution. When we compared the process of modeling with that of problem solving, more similarities revealed. Pólya (2004) proposed four phases of problem solving as understanding the problem, devising a plan, carrying out the plan, and looking back, which are similar to the mathematical modeling cycle.

The difference between problems and modeling problems is their contexts: the situations of modeling problems are articulated within a specific situation in the real world while problems do not necessarily include a real-world context. Some studies focus on a specific context of modeling problems without using the term, *modeling*. Among these, the problems involving social justice have a unique characteristic because these problems not only use the context of critical parts of the real-world but also encourage students to change their perspectives and take an action to solve the real problem. For social justice, it is critical to help students “understand, formulate, and address questions and develop analyses of their society” (Gutstein, 2003, p. 40).

Phase 2: Thematic Categories to Define Mathematical Modeling with Social Justice

We reconceptualized the relationships among problems, mathematical modeling problems, and social justice problems as shown in Figure 1. Because learners choose appropriate mathematics to analyze empirical situations when working on modeling problems, a specific solution pathway should not be given during this process. Therefore, modeling problems must satisfy the crucial condition of problems and need to be included in the set of problems (Figure 1). In addition, modeling problems are more inclusive than social justice problems.

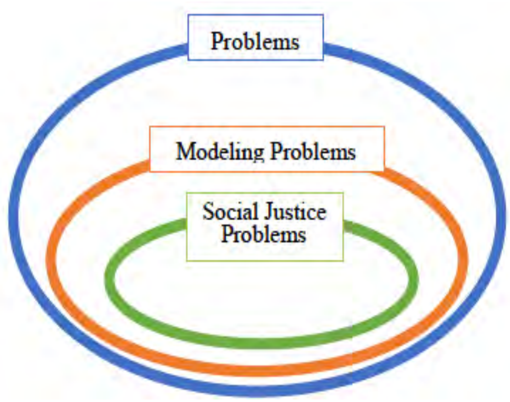

Figure 1. The relationships of problems, modeling problems, and social justice problems.

Although various types of contexts can be integrated into modeling problems, the goals of the problems of social justice is to support students not only to learn mathematics but also to actively develop their capability to read the world and become an agent of change. Thus, we placed the set of social justice problems within the set of modeling problems. The social justice problems are not defined as the tasks that just involve social justice contexts. If a task includes a social justice context only but provides a specific solution pathway, it is not a “social justice problem.” Within this operationalization, we developed a draft of criteria based on the related literature, which was refined through the piloting phase.

Phase 3: Piloting and Finalizing Criteria for Mathematical Modeling with Social Justice

The third phase was to use the criteria to analyze the modeling problems that PSTs developed. We collected data from 30 PSTs in two 4-year university-based teacher preparation programs in the Midwestern United States. Participants were sophomores to seniors enrolled in K-8 teacher preparation programs in each of their programs. The PSTs had some experience of solving modeling problems within their class before assigned the modeling problem development assignment. The criteria draft was not introduced to the PSTs although they learned the definition of modeling through reading relevant articles and discussions in class. The PSTs worked as a group of 3 people for approximately 2-3 weeks to develop their modeling problems. The first and second authors analyzed the collected 10 problems using the initial criteria and managed to reach a consensus through several discussions while finalizing the criteria.

Finalized Criteria for Social Justice Mathematical Modeling

Table 1 is the finalized criteria. The first three columns shown in Figure 1 indicate the subset relationship among problems, modeling problems, and social justice problems.

Table 1. Criteria for Modeling Problems that Address Social Justice Issues

			Criteria	Description
Social Justice Problems			Social Justice Context (Gutstein, 2003)	• The context involves unjust situations of the real world and encourages learners to be an agent of change by identifying mathematical conflicts and resolving the conflicts.
	Modeling Problems		Realistic Context, Problem, and Solution (Lesh et al., 2000; Schukajlow et al., 2012)	• The embedded context must be realistic and familiar to the target students. • The embedded problem(s) requires learners to identify variables and should likely happen in their lives. • The solution(s) must be realistic in the given real-world context.
			Multiple Representations (NGA & CCSSO, 2010)	• Multiple representations (e.g., tables, graphs, symbols, words) can be used to describe the problem situation.
			Generalizable/Transferable Knowledge (Lesh & Lehrer, 2003)	• The problem requires learners to apply their findings to other related problem solving situations. • The problem requires learners to develop mathematical knowledge that can be used in other similar situations.
			Shareable Approach (Lesh & Lehrer, 2003)	• Learners solve problems for a client outside classroom. • The problem-solving process and solutions can be shared with other people for their own use.
		Problems	Focus on Mathematics (Van de Walle et al., 2007)	• The problematic aspect should be due to the mathematics that learners are expected to learn as they solve the task. • Solving the task without using mathematics (e.g., common sense) should not be possible.
			Unpredictable Methods (Baroody, 1992)	• The task does not directly show how to solve it. • The task requires student's own method.

A social justice problem is most specific and must satisfy all seven criteria. The tasks that satisfy all criteria except for *social justice context* are categorized as modeling problems. The tasks satisfying only the last two criteria, *focus on mathematics* and *unpredictable methods*, are problems but neither modeling problems nor social justice problems. Our analysis revealed that most PSTs' modeling problems generally satisfied the criteria of problems but did not sufficiently meet other criteria. Among the criteria shown in Table 1, the most missed one was *social justice context*. None of the problems demonstrated any relevance to social justice. Additionally, the PSTs tended to ignore the characteristics of modeling problems, such as *generalizable/transferable knowledge* and *sharable approach*. Only one of the ten problems included some related features of generalizable or transferable approach.

Discussion and Implications

This study contributes to the extension of literature of mathematical modeling and social justice by demonstrating the process of designing criteria and using them with PSTs. The criteria can be used to develop and assess modeling problems. In our study, most PSTs did not address some of the other components in their problems, such as *social justice contexts*, *generalizable/transferable knowledge*, and *shareable approach*. We realized that these missing components require careful attention when introduced to PSTs. Furthermore, this study provides ideas for future research studies around modeling including social justice contexts. Hernandez, Morales, and Shroyer (2013) present a result that few PSTs identified the role of mathematics teachers as agents of change in society and assumed that one reason might be PSTs' lack of experiences or environments in which they lived and trained. Future studies can focus around changes in PSTs' awareness of social justice issues as they discuss the criteria developed in this study. The criteria have the potential for further investigations and validations in practice, which can initiate discussions among teachers, teacher educators, and researchers as they consider ways to achieve social justice through mathematical modeling.

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DEVELOPING PRESERVICE TEACHERS' UNDERSTANDING OF PRODUCTIVE STRUGGLE

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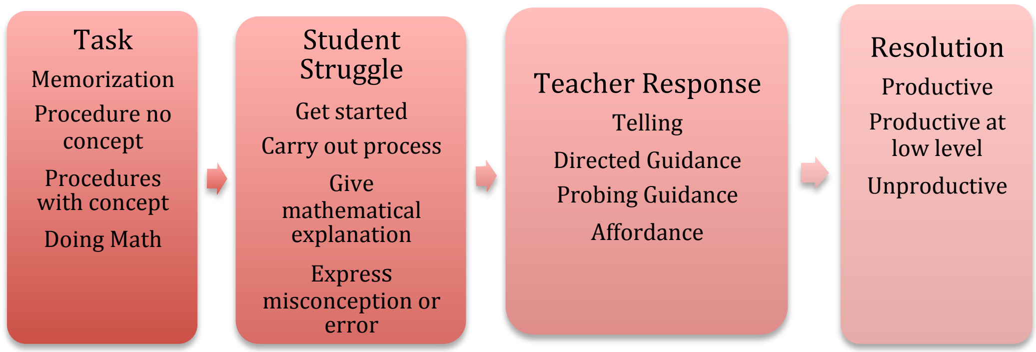
This study examines the development of preservice teachers' (PSTs) understanding of productive struggle in a semester-long mathematics content course. Our qualitative study includes 85 PSTs in four sections of a mathematics content course for prospective elementary and middle school teachers that focused on numbers and operations. Findings suggest that the PSTs develop the ability to attend to and interpret the mathematics underlying the student struggles. They also begin to identify strategies and practices that appear potentially useful for supporting productive struggle.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices, Mathematics Knowledge for Teaching

Introduction

Principles to Actions (NCTM, 2014) calls for teaching practices that support *productive struggle* in learning mathematics. However, struggle is often seen in a negative light (Hiebert & Wearne, 2003) rather than as opportunities for learning. By *productive struggle*, we mean what occurs when, "students expend effort in order to make sense of mathematics, to figure out something that is not immediately apparent" (Hiebert & Grouws, 2007, p. 387). Studies suggest that providing students with opportunities to engage in struggle is an integral part of doing mathematics (Hiebert & Grouws, 2007; Warshauer, 2015). Research also suggests that teachers play an important part in productively supporting student struggles (Warshauer, 2015). Despite recent studies suggesting *productive struggle* as an important component of learning with understanding, teaching practices often attempt to remove the cause of student struggles and rarely to engage students in *productive struggle* with key mathematical ideas (Hiebert & Wearne, 2003; Hiebert & Stigler, 2004).

Our study attempts to address this issue by examining ways to prepare teachers to cultivate a *productive struggle* mindset in a mathematics content course for preservice teachers (PSTs). We introduced PSTs to student struggle episodes in video clip form where the PSTs observed the student and teacher interactions as the student struggled over a task. As a writing assignment, the PSTs were instructed to analyze the episodes aligned with a *productive struggle* framework (Warshauer, 2015) to identify student struggles, teacher responses, and the struggle resolutions. The PSTs were instructed to use the *teacher noticing* framework proposed by Jacobs, Lamb, & Philipp (2004) as a lens to guide them towards *attending* to, *interpreting*, and *responding* to the student struggles in each episode. In addition, they reported on what they noticed of the teacher responses in the struggle episodes that they observed as well as the resolution that was reached in the episodes. The Productive Struggle Framework that we refer to is given below.



This study aims to answer the following research questions:

1. What are the kinds of mathematical interpretations PSTs make about the student struggles they observe in video clips of student/teacher interaction about a task?
2. What teacher responses and actions do PSTs recognize as potentially useful to promote *productive struggle*?

This study seeks to provide insight into what are PSTs understanding of the mathematics that underlie student struggles and the kinds of teacher responses that appear to support the struggles productively. In particular, how are the PSTs coming to understand the role of *productive struggle* in the context of students' understanding of mathematics.

Methodology

Context

Setting and Participants. This study was conducted in 2016 over a 14-week period at a large university in the southern United States. The participants for the study consisted of 85 PSTs, each enrolled in one of four sections of the first of two mathematics content courses for elementary PSTs. Each author was the instructor for one of the four sections. The PSTs experienced three rounds of intervention throughout the semester to introduce them to *productive struggle* and its role in teaching and learning mathematics. The intervention consisted of analysis of video clips and transcripts of a student struggles aimed to expose PSTs to what a student struggle could look like in a classroom and how teachers respond to struggle in a productive manner that supports students learning of mathematics with understanding.

Data Collection. The PSTs completed open-ended pre/post surveys, which included questions about PSTs' familiarity with the concepts of *productive struggle*, accounts of their personal experiences with *productive struggle*, views of struggle as an opportunity for learning, and perceptions of how teachers support student struggles. The PSTs also completed three *productive struggle* writing assignments. The three approximately 3-page writing assignments spaced equally throughout the semester provided PSTs opportunities to witness real instances of struggle with mathematical content that they studied in their course work, examine teacher responses to the struggle, and consider how productive the struggle was for the student.

Data Analysis. The writing assignments and surveys were analyzed using qualitative content analysis. We coded inductively, with the goal of identifying themes and topics of discussion in the PSTs' responses as they aligned with the Productive Struggle Framework (Warshauer, 2015). In general, we examined the attending of student struggle, the interpretations of the mathematics behind the student struggle, description of teacher response, and the productive resolution of the struggle. Prior to coding, the researchers met with a sample of the writing assignments to discuss the coding scheme to promote high interrater reliability.

Findings

We report preliminary findings for our research questions:

Research Question 1: What are the kinds of mathematical interpretations PSTs make about the student struggles they observe in video clips of student/teacher interaction about a task?

In our preliminary analysis, we looked at the data from writing assignments 2 and 3 of the 85 PSTs. We found that the PSTs interpretation of the students' struggle ranged from minimal mathematical basis to those with a detailed mathematical explanation for what could be the source of the struggle. The PSTs identified a variety of mathematical sources to be the cause of the struggle which included: (1) procedural errors and misconceptions; (2) incorrect memorization of mathematical facts; and (3) underdeveloped conceptual understanding. Some PSTs related to the students' struggles in the videos and imposed their source of similar experience and struggle into their interpretation. In both writing assignments 2 and 3, PST generally attended to the students' struggles that they observed in the video clips, identifying the kinds of struggle as getting started, carrying out a procedure, error or misconception, or give a mathematical explanation. Their interpretation of the mathematics behind the student struggles, however, were more limited to errors of procedure or memorization. Interpreting the mathematical concepts often lacked depth and at times were misinterpretations of the mathematics behind the struggle. Table 1 provides some examples.

Table 1: Mathematical Sources of Student Struggles

Procedural Errors and Misconceptions	Incorrect Memorization	Underdeveloped Conceptual Understanding
Example: *"Doesn't understand that you would have to 'borrow' a number from the next column when subtracting with a zero, so that the zero would turn into a ten."*	Example: *"...she needs to remember that the number on the 'bottom' is always subtracted from the number on 'top'"*.	Example: *"She doesn't fully understand the concept of regrouping and base ten."*

Research Question 2: What teacher responses and actions do PSTs recognize as potentially useful to promote productive struggle?

The PSTs were asked to describe the teacher responses they noticed in the videos and identify the types of teacher responses according to the Productive Struggle Framework: (1) telling, (2) direct guidance, (3) probing guidance, and (4) affordance. Through analysis of writing assignments 2 and 3, we found that students considered probing guidance and affordance to promote productive struggle in the students.

Table 2: Teacher Responses

Probing guidance	Affordance
Example: *"She asked questions and used clarification to guide the student...asking him why and reiterating his explanations...allows the student to think for himself."*	Example: *"...teacher responds with patience and allows him the opportunity to once again focus on the problem in solitude and even steps away with the promise of returning on* call..."

It is interesting to note that PSTs changed their opinions of affordance with respect to promoting *productive struggle*. In writing assignment 2, PSTs thought affordance was not productive due to students being unable to answer the problem, but PSTs viewed affordance as a crucial part of writing assignment 3 because the student was given the opportunity to think on his own. In their final survey for the course, some of the PSTs noted teacher's affordances included giving students time and providing encouragement. In their words, let students, "... struggle for a while before returning to provide the students with time to think for themselves."

Conclusion

. As we examined the PSTs' identification of the types of student struggles observed in the videos and their mathematical interpretations, we found that most PSTs could identify when a student was struggling and describe the mathematical sources of student struggles, but many PSTs did not describe clear connections to the mathematical topics behind the struggle. Instead, some focused on describing the struggling student's actions, commenting on their body language and behaviors, and briefly mentioning the underlying content.

However, we found that the PSTs' descriptions of the teacher responses and their considerations of the productiveness of the struggle improved between writing assignments 2 and 3. In writing assignment 3, instead of just describing the teacher actions, the PSTs also included their perceived justifications for the teacher's actions. The PST's explanations of how productive the struggle was also moved away from being focused on whether the student got the right answer by the end of the clip to consideration of how the student could make sense of their mistakes and learn from them. The PSTs also placed emphasis on the teacher being patient and encouraging without giving the student the answer.

Overall, we found that the writing assignments helped the PSTs begin to make sense of recognizing not only when a student is struggling, but identifying what they are struggling with and how to interpret the mathematics behind that struggle. The writing assignments especially helped the PSTs make sense of how teachers respond to student struggles, and why teachers choose certain actions, particularly probing guidance and affordance, to help students through the struggle and build understanding. As PSTs reach the crossroad that connects their mathematical content knowledge to teaching practices, more research is needed into how to help them connect their mathematical content knowledge to interpreting the mathematics underlying student struggles and recognizing teaching practices that support student struggles productively.

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DO THEY BECOME PARALLEL? PRESERVICE TEACHERS' USE OF DYNAMIC DIAGRAMS TO EXPLORE DIVISION BY ZERO

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We conducted an interview study to investigate how pre-service K-8 teachers explored a diagrammatic model of division. To create the diagram, we adapted the classic procedure for constructing the quotient of two segments to a dynamic geometry environment. The diagram had moveable points that allowed participants to set the directed (positive/negative) lengths of segments to be divided and the lengths of these segments determined the directed length of the quotient. Twenty pre-service K-8 teaches were interviewed in pairs. We report episodes of how they explored the diagrammatic representation of division by zero.

Keywords: Number Concepts and Operations, Teacher Education-Preservice

Introduction

Physical manipulatives (Green, Piel, & Flowers, 2008) and intuitive models (Jansen & Hohensee, 2015) can be effective aids that help preservice K-8 teachers hone their understanding of arithmetic. In this study, we adapted the classic procedures for constructing the quotient of two segments (Hilbert, 1902; Mcloughlin & Droujkova, 2013) to a dynamic geometry environment. The diagram had moveable points that allowed participants in the study to set the directed (positive/negative) lengths of segments to be divided and the lengths of these segments determined the directed length of the quotient. The purpose of the study was to investigate how pre-service K-8 teachers related their knowledge of division to a diagrammatic model.

Theoretical Framework

The division diagram is an example of a virtual manipulative (Reimer & Moyer, 2005; Steen, Brooks, & Lyon, 2006). It allowed users to explore ranges of quotients through dynamic dragging (Sinclair, Zazkis, Lilljedahl, 2004). Variations in color, line-thickness, and the gauges of the points in the display (Dimmel & Herbst, 2015) were used to group and emphasize different features of the diagrams: the segments that represented factors were blue and yellow, quotients were green, and the moveable segments of the diagrams were thicker than those that remained fixed (see Figure 1, below). The segments to be divided were perpendicular to each other and aligned with the x and y axes. The quotient of the blue (dividend) and yellow (divisor) segments is the intersection of the y-axis and the line drawn through (0,1) parallel to the segment (not shown) between the yellow and blue points. By dragging either the blue or yellow points, users can modify the diagram to show different quotients.

We focused on a diagrammatic model of division because it has mathematical advantages over physical manipulatives (e.g., base ten blocks, cuisenaire rods) that warranted study. In particular, the diagrammatic model of division provides a visual representation that shows division by 0 to be undefined. When the divisor is 0, the line that defines the quotient (i.e., the *quotient-line*) is parallel to the y-axis and there is no point of intersection. Research has shown that both practicing and preservice K-8 teachers have difficulty explaining why (or even knowing that) division by zero is undefined (Ball, 1990; Crespo & Nicol, 2006; Colleague, Author, & Colleague, date). Given these challenges, a model of division that naturalizes division by zero could be a valuable instructional resource. We asked: How do pre-service K-8 teachers relate their knowledge of division by 0 to a diagrammatic model?

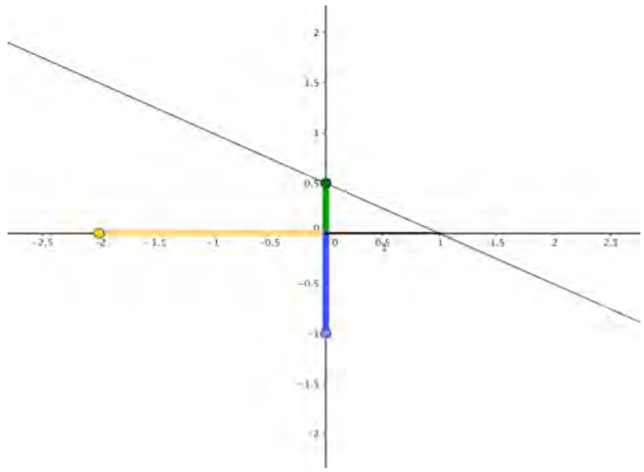

Figure 1. Diagram showing that (-1) / (-2) = (.5). In the diagram, the yellow point and blue point can be slid along the *x* and *y* axes. As they move, they determine the location of the green point.

Method and Participants

We conducted an interview study (Drever, 2003; Zazkis & Hazzan, 1998) of pre-service K-8 teachers. Participants were enrolled an elementary geometry content course at a mid-size university in New England. Twenty (20) students accepted the invitation. Participants were interviewed in pairs. Each of the ten (10) interviews we conducted were approximately 50 minutes in duration. Participants were compensated for their time with $20 Amazon gift cards.

Semi-Structured Interview Protocol

The interviews began with participants exploring the diagrammatic model. The prompts were initially open-ended, e.g.: “Please describe what you see”, and then progressed to more specific prompts used at the discretion of the researchers to guide student interactions with the diagram. The more specific prompts included cues for exploration, such as: “What parts of the diagram can you move ?”, and cues to engage participants’ conceptions of division, such as: “How could you describe the position of the green point in terms of the position of the yellow point and the position of the blue point?”

Data

The interviews were screen and audio recorded. The audio recordings of each interview were transcribed. We prepared multimodal transcripts (Jewitt, 2009) to capture participants’ actions on the diagram. We report below on segments of interviews where participants used dragging to explore division by 0.

Findings

Do They Become Parallel for a Second?

A feature of the diagrammatic model of division is that it visually shows that the quotient (i.e., the length of the green segment) increases without bound as the yellow point (the divisor) approaches 0. As the yellow point approaches 0, passes through 0, and moves away from 0, the quotient segment grows arbitrarily large, disappears for an instant (when the quotient-line is parallel to the *y*-axis), and then reappears the next instant. When it reappears, it is arbitrarily large and pointing in the opposite direction compared to when it disappeared.

Elli and Stella (interview 3, pseudonyms) investigated this switching behavior by moving the yellow point back and forth across the y-axis 11 times (over 3 minutes), at which point one participant asked: “Do they become parallel for a second?” where “they” referred to the quotient-line and the y-axis. The participants continued to explore the behavior of the diagram when the yellow point was at 0 or very near 0, and concluded: “They're not forming the triangle. And then that line that was making the hypotenuse is parallel to [it].” These participants convinced themselves that the quotient-line does become parallel to the y-axis when the yellow point moves to 0, but they did not explicitly relate this to the notion that division by 0 is undefined.

When It’s on 0, the Lines Become Parallel.

Rather than repeatedly dragging the yellow point back and forth across the origin, some participants immediately noticed the special behavior of the diagram when the yellow point passed through 0. Becca and Michelle (interview 9, pseudonyms) initially conjectured that the green line must be “under the blue line” when the divisor goes to zero. Michelle stated that: “Both the yellow and the green are at zero”, and then asked Becca (quietly): “Is that correct?” Becca can be heard saying “Ummmm…” while she considers the question. During this exchange, the divisor is at zero and the quotient-line is parallel to the y-axis. Michelle answers her own question: “I believe”, and as she does so, Becca says: “Ohhh….”. Once Michelle finishes describing the positions of the points, the interviewer asks Becca what she thinks:

Author 1: What do you think Becca?
Becca: Negative 4…you would do like the blue divided by the yellow…so negative 4 divided by 0 is undefined…that’s why the slope is undefined.
Author 1: And so is that the insight that you…it looked like you
Becca: Yeah, the light bulb.
Author 2: And what on the diagram represents undefined?
Becca: Like what line? The intersecting-line…if it’s like, vertical, the slope’s undefined.

Michelle and Becca continue this conversation and conclude that the green is just “not there” or else would be “infinite”. Jenny and Terry (interview 1, pseudonyms) expressed a similar idea. They brought in division by 0 explicitly to make sense of the parallel quotient-line when the yellow point was set to 0. Jenny and Terry interpreted the parallel line to mean that the quotient could be “anywhere or nowhere...because you could subtract 0 from 4 however many times you want, and you are never going to get 0.”

Conclusion and Scholarly Significance

We reported an interview study of pre-service teachers explorations of diagrammatic models of division. We found that participants in the study explored the diagrams systematically and were particularly drawn to the state of the diagram that represented division by 0. That all pairs of participants examined configurations of the division diagram where the yellow point was equal to 0 and that some pairs of participants related this to division by 0 being undefined provides evidence that preservice K-8 teachers could be receptive to the mathematical advantages of these virtual manipulatives.

Elementary mathematics educators are positioned at a crossroads: the basic mathematical ideas they are responsible for teaching rest on deep mathematical footings. How can we best prepare our K-8 mathematics teachers to recognize the intersections of advanced and elementary mathematical ideas they routinely encounter in their classrooms? Our study is significant because a theme in the literature is that elementary teachers have incomplete or incorrect conceptualizations of multiplication and division (Green, Piel, & Flowers, 2008). At the same time, researchers have called for the development of mathematical experiences that can help preservice teachers develop the capacity "for teaching mathematics with understanding" (Crespo & Nicol, 2006, p. 84). This study contributes to the work of creating such experiences by showing the viability of a virtual manipulative that could help preservice K-8 teachers deepen their mathematical knowledge and appreciate the connections between arithmetic and geometry.

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THE FUTURE TEACHER, MULTIPLICATION AND DIVISION OF FRACTIONS

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The present communication is only focused on a few episodes registered in two interviews developed with Tania (an undergraduate teacher of primary education); this methodological tool contributed to Tania's case in the context of a broad research about fractions. The study object considered here was the exploration of multiplication and division of fractions (through problem solving) as well as her identification of the relation between both operations. It also recovered Tania's productions and problems designs for a potential approach to aforesaid operations in primary school classrooms.

Keywords: Teacher Education-Preservice

Introduction

It is necessary to investigate how the future teacher of primary school accomplishes semantic processes immersed in the construction of multiplication and division operations of fractions, considering that the effectiveness of her/his teaching practice in basic education will depend on the strength with which she/he develops these ideas and their corresponding didactic integration.

The previous statement is our research problem, from which we pose the following *questions* and analyze them in the present paper: Which are Tania's thoughts *when she multiplies and divides fractions* within the design and solving problems scope? *How this young future teacher relates both operations* in the semantic field of fractions?

In the next section we present some theoretical categorizations sustained by some authors, in the study of various general forms of thoughts, as well as with respect to the multiplicative thinking linked to fractions.

Theoretical Framework

In the study of thought, Piaget (1986) distinguished the *concept*, which is established as the inclusion of an object in a class as well as one class into the other. It is interesting to highlight that thus *the construction of the concept* is achieved by *multiple modes of abstraction.*

Vygotski (1993), regarding thought, conceived *concept* in terms of a *generalization* that appears in the subject consciousness as a complex act of thought, a process of words significance, instead of assimilation through memory. *Concepts are constructed around pre-concepts,* that is, none conscious content is eliminated only transformed into another of superior level, from a major generalization of the object.

Another point of view related to thought is the one provided by Lonergan (2008), who mentioned that *elaboration of concept* implies *the identification of interrelated aspects in an "object" as well as in the saying*. To say and to conceive are not isolated. *The elaboration of a concept requires a process of selection among data, empirical representation and image.* The concept is what can be expressed beyond empirical.

In contrast to the previous statements, Shön (1996) identified *reflection in action* as a production of a spontaneous and *intuitive thought related to everyday life experience*. He also mentioned that *the teacher can reflect in action* from the own experience and change a routine teaching practice using all her/his thought to act facing a new situation.

Valdemoros, Ramírez, and Lamadrid (2015) propose that in the discourse of a learner may appear *"signification and thinking cores"* full of sense for he/she, this is, *words and phrases that give consistency and clarity to the own praxis*, in an elementary way. Such *"signification and* *thinking cores"* offer cohesion to the cognitive products, despite the fact that the thinking of those who learn could be developed currently at the intuitive level, not having reached even the widespread expressions of the formation of concepts.

Related to the set of rational numbers, Kieren (1988) pointed *in his semantic network* that *multiplicative domains* are *linked to formal construct of equivalence and to multiplicative structure* related to functional and scale relations. Scale relation is assembled through *meanings of fraction* such as *ratio*, which indicate a numeric comparison between two magnitudes and an *operator* (the last construct refers to expand or to contract a whole with reference to another unit).

Fraction is based on two fundamental relations: part-whole relation and part-part relation (Piaget, Inhelder & Szeminska, 1966). The first of these relations is defined as the existing link between whole and a determined number of parts when the latter, continuous or discrete, is divided in equivalent parts. Part-part relation refers to the possibility that all parts can be divisible by themselves and considered as new wholes (Piaget et al., 1966). Such statements were ratified by Kieren (1988).

Conceptual field of multiplicative structures is composed of all *situations that mutually involve multiplication and division,* such as: simple and composed proportions, direct and inverse scales of ratio, fraction, ratio, rational number, among other ideas (Vergnaud, 1988). In particular, the *product of measures* as multiplicative relation shows the difference among diverse types of multiplication problems when, given basic measures, the measure-product is found; concerning division, given basic measures and measure-product, another basic measure should be found (Vergnaud, 1991).

Method

In this report, we show some relevant contents of Tania's multiplicative thought and detailed which are the main mathematical and didactical ideas manifested in her discourse during interviews, which emerged as some *signification and thought cores.*

We decided to interview Tania because she was the most outstanding student among her study group, during the fourth semester of the bachelor's degree in primary education. In this mentioned degree, future teachers are trained for teaching in all subjects that form part of the school curriculum in elementary education (that is the development of these studies in Mexico).

At the time Tania interviews took place, she had already coursed the subject of Arithmetic (third semester in this academic level). Though this research we could not have access -as observers- to Arithmetic and Teaching Practices courses, for this reason the relevance of Tania interviews was ratified for the recognition of some susceptible processes of thought within a possible development in the teacher in training.

We carried out two interviews of didactical cut with Tania because such a methodological proposal from Valdemoros (1997, 1998), and Valdemoros & Ruiz (2008), considered two moments of the development of the interview: a) an exploratory phase in which interviewer only deeply inquire for contents and mathematical processes of the interviewed; and b) a feedback phase, through which interviewer tries to promote a reflection that evidence the contrast between thinking contents and calculus procedures, and thus validate them based on their own production and refine its inconsistencies.

Qualitative validation of these results was obtained by mean of a methodological *triangulation at different moments, during interviews*.

Analysis of Results

Next are some relevant results from Tania's interviews, linked to multiplication and division of fractions, included in the scope of problem solving focused in the teaching of fractions. *Quoted* *expressions correspond to words and phrases* of Tania that we interpret as *signification and thought cores*, linked to the *reflection in action*.

Situation 1

Tania was asked to solve a division problem of fractions (centered on the recognition of the *inverse operation of a product of measures*). Such task was designed for 6th grade of primary school and Tania must communicate her didactic and mathematical reflections about it. The text of the problem requested *to calculate an unknown side of a rectangle board for a bookcase, given area* ($\frac{1}{2}$ *m^2) and a side* ($\frac{2}{3}$ *m)*; this task was complemented with the *graphical representation of the rectangle and the corresponding measures*. Tania thought for a while and then expressed: "you have to solve for the formula of the rectangle area [she holds up] and divide one-half into two-third". The phrase "solve for formula of the rectangle area" represented a heavy semantic load for Tania; it was a clear manifestation of a *signification and thought core.*

Situation 2

We asked Tania to write a problem referred to a division of fractions. She did not provide any oral evidence that she was capable of designing division problems of fractions for the teaching (inside or outside of a geometric field).

Situation 3

Tania continued striving to contrive a multiplicative problem referred to fractions. After proposing an ambiguous elaboration that remained incomplete, she opted to fold a sheet of paper while she was saying "*…this sheet represents a cake that I'm dividing in thirds and quarters, to recognize within it* $\frac{2}{3}$ *and* $\frac{3}{4}$ *fractions"*. Thus, Tania was intuitively leaned on *the relation part-part*, which emerged as a result of the "*paper folding"*. Finally, Tania colored the intersection among both fractions in the piece of paper (appointed the prior as "*the whole"* and "*the unit of reference"*). When we asked Tania which arithmetic operation has employed in both fractions, she could hardly express that she had obtained *"the product* $\frac{32}{3}$ *by* $\frac{3}{4}$*"*.

Overall, although Tania was eloquent during Situations 1 and 3, many of her mathematics and didactic difficulties were evident in the multiplicative field of fractions. Her mathematical thought did not show conceptualized multiplicative proposals, as of progressive *generalizations* and successive *abstractions* (according to those theoretic formulations from Vygotski and from Piaget).

Tania thought was manifested as outstandingly intuitive, partially fragmented, and linked to the own praxis (which promoted plenty of reflections in action, followed by pertinent adjustments to her proposals). For all this, ideas within identified signification and thought cores, provided a kind of strength and a link to such thought. In all this, we do not know in which extent the occasional influence that the Arithmetic course of the Bachelor's degree exerts on Tania.

Conclusions

The aforesaid *signification and thought cores* that Tania expresses in her present discourse, in terms of words and phrases either mathematical and didactical that regulate and define her design activity for the teaching, do not become concepts or pre-concepts yet, but it is possible that such elaborations in the future would constitute roots of thoughts more systematic, general and increasingly better integrated into the multiplicative processes of fractions.

During the interviews, *intuitive thought* immersed in Tania's explicit discourse was sometimes supported by reflections oriented towards teaching practice and a suitable *reflection in action* for those activities that she wants to promote in the classroom.

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EXAMINING PRE-SERVICE TEACHERS' UNDERSTANDING OF THE COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

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This study seeks to probe pre-service teachers' (PSTs') understanding of the Common Core State Standards for Mathematical Practice (SMPs) (NGAC & CCSSO, 2010) by having PSTs identify the potential for SMPs to be practiced and assessed through assessments, curriculum materials, and in practice. The ratings of the PSTs will be compared to those of curriculum authors, as well as to the other PSTs in the sample, in order to determine the amount of agreement based on specific characteristics such as mathematical knowledge for teaching and beliefs about mathematics. After patterns have been identified, PSTs will be strategically chosen for interviews to probe their thinking further. The results of this study will be useful to elementary mathematics teacher educators in thinking about ways to support PSTs in understanding the SMPs as well as helping PSTs to support students' development of the SMPs.

Keywords: Teacher Education-Preservice, Standards, Curriculum, Assessment and Evaluation

Introduction

The Standards for Mathematical Practice (SMPs), part of the Common Core State Standards (CCSSs) *(NGAC & CCSSO, 2010)*, stress the importance of processes and proficiencies that span all grade levels and mathematical content. The work in this study stems from two previous studies (Silver and Mortimer, 2015; Mortimer, 2016) in which the researchers asked what SMPs mathematics teacher educators' and mathematicians' saw potential to be assessed by assessment items. One of the most compelling findings from these studies suggests that, though the SMPs can be seen throughout an international mathematics assessment, experts are not always in agreement about what it looks like to assess the various SMPs. When probed about their differing assignments, experts reported that in many cases the SMPs were vaguely described. In other cases the assessment task could be solved using multiple strategies and thus it was challenging to know exactly what SMP someone would employ. As teachers are expected to be able to support students in learning and assess students' progress in the SMPs, more research is necessary on how to recognize and assess SMPs in assessments and classroom materials.

In this study I will examine the ways that PSTs make sense of the SMPs in the contexts of a lesson from a teacher's manual, a video of teaching and an assessment. Additionally, I will look at the PSTs' mathematics knowledge and beliefs in order to understand how these factors may influence their understanding and interpretation of the SMPs. This study lies at the crossroads between research and teacher education and the results of this study have the potential to provide insight for teacher educators on ways to support PSTs in developing their own understanding of the SMPs as well as support students in practicing and gaining skill in using the SMPs.

Theoretical Justification

Teachers' Mathematical Knowledge

Shulman (1986, 1987) began examining teachers' knowledge and how the ways of knowing content differ between teachers and other professionals. He explained teachers' knowledge as falling into three categories: content knowledge, pedagogical content knowledge, and curriculum knowledge. Building on this work, the research group involved in the Study of Instructional

Improvement (SII) endeavored to build a theory around what teachers need to know to teach mathematics, but began by examining practice and what teachers actually do in classrooms (Ball & Bass, 2003). This group went on to develop measures to assess teachers' mathematical knowledge for teaching (MKT). Studies implemented by this group have gone so far as to attempt to link teacher knowledge to student performance (e.g., Hill, Rowan, and Ball, 2005).

Teachers' Beliefs

In addition to teacher knowledge impacting instruction, McLeod and McLeod (2002) suggest that looking between the cognitive and affective processes, specifically at teacher beliefs, is important to understanding how teachers' approach instruction. Much of the research on teacher beliefs in relation to mathematics education has focused on teachers' beliefs about the nature of mathematics and mathematics teaching and how those beliefs impact their instructional practices (e.g., Thompson 1984).

Research Questions

1. Are PSTs able to identify places in the different domains of practice (e.g., the lesson, video of teaching, and assessment) that they will be expected to use as practicing teachers with the potential to assess or develop students' proficiency in the SMPs? To what degree do they agree with the curriculum authors? To what degree do they agree with each other?
2. How does PSTs' mathematical knowledge influence their ability to identify and make sense of SMPs in different representations of teaching? How does PSTs' mathematical knowledge impact their agreement with the authors of the curriculum materials? Does it differ by task?
3. How do PSTs' beliefs about mathematics influence their ability to identify and make sense of SMPs in different representations of teaching? How does PSTs' mathematical knowledge impact their agreement with the authors of the curriculum materials? Does it differ by task?

Methods

Participants

The twenty-four PSTs in this study are third and fourth year undergraduate students at a large, Midwestern public university earning a Bachelor's degree and teaching certificate, pending successful passing of the state teacher certification assessment. Similar to many pre-service and in-service teachers, these PSTs have had little to no explicit instruction on the SMPs. Though their mathematics methods course includes mathematics practices, as the doing of mathematics always does, the practices specified in the CCSS are not a focus of the course.

Instrumentation

Mathematical Knowledge for Teaching (MKT). Stemming from the work of the Study of Instructional Improvement (SII), The Learning Mathematics for Teaching (LMT) project created an assessment to measure school and classroom processes as well as teachers' facility in using mathematical knowledge in their classroom teaching (Learning Mathematics for Teaching Project, n.d.). The PSTs in this study are given the LMT assessment at two points during their program; the most recent implementation of the assessment will be used as the measure of their MKT.

Mathematics Beliefs Instrument (MBI). In order to measure PSTs' beliefs about mathematics the Mathematics Beliefs Instrument (MBI) will be used (Peterson, Fennema, Carpenter, & Loef, 1989, as modified by the Cognitively Guided Instruction Project). The instrument consists of three subscales: 1) Curriculum: the degree to which one believes that mathematics should be taught in relation to problem solving and understanding rather than focusing on facts and memorization; 2)

Learner: the degree to which one believes that students can construct their own mathematical knowledge; 3) Teacher: the degree to which teachers should organize instruction to facilitate children's construction of knowledge. The instrument is scored on a five-response Likert scale that, when added together, higher scores indicate that the teacher's beliefs are more aligned with cognitive beliefs.

The Tasks

Curriculum materials task. PSTs will be given a lesson from the Everyday Mathematics (EDM) curriculum materials as well as a copy of the SMPs. The curriculum materials will include pages from the Teachers' Manual. Though the authors already included places where they believe students have the opportunity to practice the SMPs, these designations will be removed so that there is no indication of which SMPs the authors intended to be addressed in the lesson. The PSTs, working independently, will examine the SMPs and the lesson. They will go through the lesson and indicate each instance in which they think an SMP is or has the potential to be practiced by students. PSTs will be asked to explain their judgments.

Lesson enactment task. PSTs will view a video of an elementary teacher teaching the mathematics lesson that they examined in the curriculum materials task. In the video the teacher will not explicitly name the SMP that she is intending to teach, but the SMP will be embedded in the work that the teacher and students are doing. Using the online video program Edthena, PSTs will be asked to identify which SMP or SMPs that they see being practiced in the video. They will do this by tagging specific places in the video and explaining their thinking in writing in the tag.

Assessment task. PSTs will look at a formative assessment, the journal pages that correspond to the lesson in the curriculum materials task, and indicate for each problem which of the SMPs have the potential to be assessed. PSTs will be asked to explain their judgments.

Analysis and Expected Results

PST and Expert Rater Comparisons

For each of the three tasks I will compare the ratings given by the authors of the curriculum materials and the ratings given by the PSTs. This comparison will result in descriptive statistics showing the frequency with which all PSTs agreed with the authors, if PSTs were more likely to agree with authors on particular tasks, or if specific characteristics of PSTs made them more likely to agree with the authors (e.g., PSTs with high MKT were more likely to agree with the curriculum authors than those with low MKT). Additionally, I will conduct a comparison among PSTs based on the PSTs' characteristics (i.e., PSTs' MKT and mathematical beliefs).

PST Interviews

I will choose PSTs to interview, selecting representatives of various groups based on the patterns that arise when comparing PSTs' ratings to the authors and other PSTs. These interviews will focus on probing PSTs' ratings in each of the three tasks to understand both why they chose the SMP that they chose in the tasks as well as the processes they used to determine their ratings.

Discussion

As the SMPs are a part of the CCSSs, they are part of the intended curriculum for all states that have adopted the CCSSs. As part of the intended curriculum, teachers are expected to have achieved expertise in the SMPs and feel confident supporting their students in developing the SMPs. Currently many schools of education do not address the SMPs in an intentional way. As discussed, the SMPs are challenging to identify, even for experts, so more support is needed for PSTs in this area. In order for the SMPs to be taught to PSTs in a meaningful way, more understanding of what PSTs already know and understand about the SMPs is needed.

The results from this study will provide the teacher education community with information about PSTs' understanding of the SMPs. The three different tasks may illuminate that PSTs are more skilled at identifying potential for SMPs to be practiced and assessed in different representations of teaching. Additionally we may learn that particular background knowledge and beliefs are important for PSTs in order to complete the tasks successfully. Ultimately the results of this study will support teacher educators in thinking about ways to provide instruction and practice to PSTs so that they enter the teaching field confidently able to support their students in developing their skills in the SMPs.

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EXPLORING AN INTEGRATED NOTICING FRAMEWORK FOR SECONDARY MATHEMATICS TEACHER EDUCATION FIELD EXPERIENCE

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Traditional models for teacher education field experience often perpetuate a split between theory (university) and practice (school classrooms). One way for prospective secondary mathematics teachers to engage with theory at the classroom level is to involve them in a professional learning community during their field experience (internship). This paper shares research on the development of an alternative model for internship and for faculty supervision through the creation of a learning community and an integrated professional development process (incorporating lesson study, video analysis and the discipline of noticing). The research emphasizes the value of community and collaboration when reconceptualizing traditional models for internship, for the role of the faculty advisor, and for being/becoming a mathematics teacher.

Keywords: Teacher Education-Preservice

Research Purpose

Traditional models for teacher education field experience often perpetuate a split between theory (university) and practice (school classrooms). One way for prospective secondary mathematics teachers to engage with theory at the classroom level is to involve them in a professional learning community during their field experience (internship). This paper shares research on the development of a model for internship and for faculty supervision through the creation of a learning community and an integrated professional development process (incorporating lesson study, video analysis and the discipline of noticing). The overall research project has a tri-focal outcome of: (1) disrupting traditional discourses on theory-practice transitions in mathematics teacher education, (2) reconceptualizing the role of the university supervisor (faculty advisor) within the triad of prospective teacher, cooperating teacher and supervisor, and (3) positioning university-school partnerships and collaboration at the centre of professional development approaches in teacher education programs.

Research Context and Modes of Inquiry

My research in the area of secondary mathematics teacher education focuses on the structures and roles of that specific component of teacher education programs referred to as the school practicum or field experience (Nolan, 2012). Specifically, I conduct self-study research into my role as a university supervisor (faculty advisor) during teacher education field experience. In my university's four-year undergraduate teacher education program (located in a Canadian province), the culminating field experience is a four-month internship in schools. Each prospective teacher (intern) is paired with a cooperating (mentor) teacher in the school and assigned a faculty advisor, who is expected to visit, observe, and conference with each intern only 3-5 times during this internship. Over the years, as a faculty advisor for prospective secondary mathematics teachers, my role in this internship model felt superfluous, even token (Nolan, 2015).

To move my role beyond tokenism in the field, I initiated a research project to design and facilitate an alternative internship model for secondary mathematics teacher education field experience. I refer to this model as the Teacher-Intern-Faculty Advisor (TIFA) Internship Learning Community model. During each of four distinct internship semesters (2013-2016), the TIFA learning community consisted of three interns, their cooperating teachers, and me as faculty advisor (and researcher). In the model, TIFA participants engage in a unique professional development process in which I integrate three components: a modified approach to lesson study (Gorman, Mark, & Nikula, 2010), the video recording and editing of classroom teaching episodes, and a video analysis process based in the discipline of noticing (Mason, 2002). I refer to this professional development process as an *Integrated Noticing Framework*.

Theoretical Perspective: Introducing the Integrated Noticing Framework

The *Integrated Noticing Framework* (herein referred to as INF) is based on a belief that "it is critical for teachers to first notice what is significant in a classroom interaction, then interpret that event, and then use those interpretations to inform their pedagogical decisions" (van Es & Sherin, 2008, p. 247). In addition, teachers (both novice and experienced) need to feel safe within a non-judgmental environment in order to further their growth and development as teachers, especially in this age of inquiry (reform-based) mathematics pedagogy. The INF professional development process developed for my internship community model values the experiences and interpretations of teachers, providing an environment for working collaboratively and for sharing experience, expertise, and multiple perspectives on teaching and learning in secondary mathematics classrooms.

The three individual components of the INF involved in my TIFA learning community (lesson study, video analysis, and noticing) are not entirely dissimilar from those described in other research on mathematics teacher education noticing (see, for example, Coles, 2013; Sherin, Jacobs, & Philipp, 2011). What is unique about this research project and professional development initiative lies not so much in WHAT is done but in HOW the process serves to disrupt normalized practices in the education of new mathematics teachers. The INF is conceptualized and enacted in two parts.

Integrated Noticing Framework Part I

In the first part, the TIFA learning community meets together at my university for one full day each month (over the duration of the four-month field experience, or internship) to participate in a *modified lesson study* process. The outcome of this part of the process is the creation of a 'research lesson,' which is then taught by each of the interns in her/his own school classroom setting. The teaching of the lesson is video recorded, then edited by the intern to create a 12-minute video clip, and finally brought back to the next TIFA professional development day at the university. The lesson study process used in part one of the INF is referred to as 'modified' since, as a community, we adopt the position that it makes the most sense for each cooperating teacher and intern pair to further develop the main outcomes and activities of the lesson (as initially planned by the TIFA community) based on their own specific school and classroom contexts, rather than fully developing a uniform and context-free script for all to follow.

Integrated Noticing Framework Part II

During the TIFA learning community meeting, the team engages in video viewing and analysis based in the *discipline of noticing*. As Mason (2011) explains, the *discipline of noticing* is a collection of techniques for preparing to notice in the moment; for reflecting on past events to understand what one wants to, or is sensitized to, notice; and for learning to notice in the moment so as to act freshly rather than habitually (p. 48). As a group, the TIFA community views all three intern teaching videos, using a noticing framework to stimulate discussion and reflection on the videos. My INF noticing process (adapted primarily by drawing on Coles (2013)) consists of four phases: (1) View each video, taking individual 'noticing notes' while viewing; (2) remain in silence for 2 minutes, organizing one's noticing notes to highlight what will be shared with the TIFA community; (3) give an *account of* what was observed in the video (sharing *what was observed directly*, in detail, avoiding all interpretation at this stage); and (4) *account for* what was observed (this is the interpretive stage where possible meanings or explanations for what was observed in the video are presented and discussed). The four phases are enacted, in turn, for each of the three intern videos, being especially careful to separate phases (3) and (4) so as to allow the observations to be voiced and heard, prior to any interpretations, questions, or discussion being introduced into the process.

Data and Discussion of Results

As Coles (2013) notes: "If I am not careful I will only see... that which I already think and believe" (p. 11); this is especially true for mathematics teacher education field experience practice and supervision. In my university's teacher education program, the prospective teachers and school mathematics cooperating teachers already feel that they 'know' what internship is all about and the role of the faculty advisor within. Research indicates that prospective teachers view the practice-based experiences of teacher education as the most important part of their program and the most significant influence on be(com)ing a teacher and shaping a professional identity (Britzman, 2003). There is little surprise, then, that introducing disruptions into the usual, normalized internship experience can be met with skepticism, even frustration. However, my research data indicate that those participating in this community's integrated noticing framework embraced change (the 'disruptions') because of the benefits they saw and experienced firsthand.

Benefits of the TIFA community approach have been noticed on a number of levels, including those beyond what was anticipated in my original goal of disrupting my token role as a supervisor. For instance, there have been noticeable influences on interns' desires and abilities to be reflective on their process of becoming a teacher; they articulate the many benefits of having "more eyes on what you're doing and how you're working with students" (intern, TIFA 2014) through this collaborative community approach to video analysis. Similarly, cooperating teachers have expressed surprise by the ways in which the learning community and INF have had such a major impact on them – not only in their roles as cooperating teachers but as classroom mathematics teachers who, like the interns, are also trying to grow and develop as inquiry-based teachers. Each year of running the TIFA community and collecting research data, cooperating teachers were asked about challenges and rewards of being involved in this enhanced internship project and if the community played any the role in their own professional development as mathematics teachers. One cooperating teacher responded:

> As far as rewards… just seeing what other cooperating teachers noticed, and then getting the opportunity to give feedback on other intern's lessons, and even getting to know other interns and other cooperating teachers… that collaboration piece is nice. (Cooperating teacher, TIFA 2014)

As alluded to in the above quote, the community approach has also had a significant impact on the intern and cooperating teacher's relationship with each other, as well as their relationship with the other intern and cooperating teacher pairs in the learning community. Participants commented on how beneficial it was to plan a lesson together, to video record aspects of that lesson and then engage in the noticing process as a collaborative and supportive group. Additionally, the overall impact of the internship learning community on connecting and building relationships between university teacher education programs and schools cannot be overlooked. School principals and teachers throughout the province have requested involvement in this internship model, recognizing it as a way to enhance the processes of both becoming *and being* a mathematics teacher.

Finally, related to the originally articulated goals of the enhanced internship and INF, my role as a faculty advisor, as explored through self-study approaches, has unquestionably moved my role beyond tokenism in the field. For example, when I asked my interns about my role as a faculty advisor in comparison to what they know about other interns' experiences, one intern responded:

> Just based off of the experiences of my friends who aren't in the TIFA community, the faculty advisor came out once or twice; they didn't really have a relationship established with them. This

> [TIFA community] is good 'cause we were able to establish a relationship and then actually get effective feedback on what we could improve. [Intern, TIFA, 2015]

Scholarly Significance

The complexity of mathematics teacher education field experience means that there is little overall agreement on many issues: the role of the cooperating teacher, the role of supervision, and even the role of practicum in general. Bullough, et al. (2002) suggest that "[t]here is a growing need for experimentation with configurations of field experience and for the generation and study of new models to determine their effectiveness" (p. 69). My professional learning community model, operationalized through the use of the INF, responds to this suggestion, reflecting my efforts as a mathematics teacher educator and supervisor to experiment with and generate new models. In this research, my role as a faculty advisor creates spaces for a collaborative community of teachers to witness the power of transforming theory-practice transitions into engaging and reflective pedagogical practice in mathematics classrooms.

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HOW MUCH DO I KNOW ABOUT MATHEMATICAL MODELING?

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Despite the importance of teachers' conception of mathematical modeling, limited attention is given to this area in the current literature. In this study we examined 78 preservice teachers' (PSTs) views of mathematical modeling and how their conceptions are reflected in their performance of mathematical modeling problems. Analyses of survey responses revealed that our PSTs seem to develop narrow views of mathematical modeling. In addition, although a large portion of PSTs mistook mathematical modeling with mathematical models or with traditional word problems, we found a positive association between PSTs' conceptions of mathematical modeling and their mathematical modeling abilities.

Keywords: Modeling, Teacher Education-Preservice

Introduction

Mathematics education community at large has recognized the importance of mathematical modeling at school level, which concerns how well students are prepared to solve real-world problems that they encounter beyond school, to solve problems in their future professions, as citizens and in further learning (Galbraith & Stillman, 2006). The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) also highlights mathematical modeling as one of the eight Standards for Mathematical Practice for all grades but also as conceptual category in high school. Thus, preparing effective teachers of mathematics who promote students' conceptual understanding and mathematical modeling abilities is one of the most urgent problems facing teacher educators. The purpose of this study is to explore PSTs' conceptions of mathematical modeling and effective modeling instruction and to investigate any relationship that might exist among PSTs' conceptions of effective modeling instruction, mathematical modeling, and their mathematical modeling performance. In exploring the relationship between PSTs' conception of effective mathematical modeling instruction and their mathematical modeling abilities, we specifically focus on two popular modeling problems – (a) Deciding a departing time for airport and (b) finding the best estimate of the total number of people in concert. The research questions that guided this study are: (a) What are the characteristics of PSTs' thinking about effective mathematical modeling instruction and mathematical modeling? and (b) Is there any relationship among PSTs' conceptions of effective mathematical modeling instruction, mathematical modeling, and their mathematical modeling performance?

Theoretical Perspectives

Mathematical modeling is a powerful vehicle for students' mathematical learning. However, the term "mathematical modeling" is easily confused. Research reported several confusions teachers and students have. For example, the term of mathematical modeling is often considered as mathematical models or traditional word problems. Although there exist distinct differences between "modeling as content" and "modeling as vehicle" (Galbraith & Stillman, 2006), teachers tend to "treat [mathematical modeling] more as a venue for learning other mathematics" (Zbiek & Conner, 2006, p. 89). Mathematical modeling involves a cyclical process as shown in Figure 1 in which real-life problems are understood and translated into mathematical language (*formulate*), solved within a symbolic system (*compute*), and the solutions tested back within the real-life system (*interpret, validate, and report*).

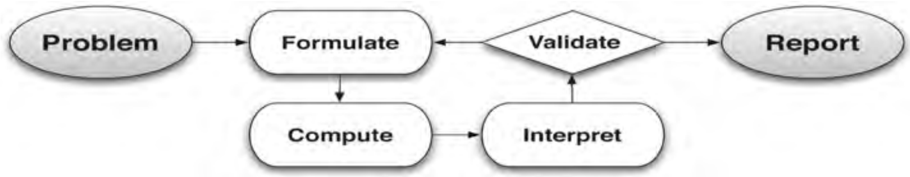

Figure 1. The basic modeling cycle introduced in the CCSSM (NGA & CCSSI, p. 72).

In this study, using the three meanings of mathematical modeling by Stanic and Kilpatrick (1989) and the three teaching approaches to mathematical modeling by Schroeder and Lester (1989), we explored PSTs' conceptions of mathematical modeling and effective mathematical modeling instruction. Building on Schroeder and Lester's framework, the following three perspectives of mathematical modeling instruction can be identified in mathematics classrooms: (1) teaching *for* mathematical modeling, (2) teaching *about* mathematical modeling, and (3) teaching *through* mathematical modeling. In addition, drawn from Stanic and Kilpatrick, we believe that *mathematical modeling as art* should be a goal of effective mathematical modeling instruction. According to them, three different meanings were attributed to the notion of mathematical modeling in mathematics education-- *mathematical modeling as means to a focused end (content), mathematical modeling as a skill,* and *mathematical modeling as art.* In the first perspective, mathematical modeling can be viewed just as *content* to practice skills. Similarly, the second perspective considers mathematical modeling as one skill taught in school mathematics. In contrast, in the third perspective, mathematical modeling should be viewed as an act of discovery through creative use of mathematical thinking.

Methods

78 PSTs from two different university sites were invited for this study. Participants had some experience of solving modeling problems within their class work. In the beginning of the semester, all participants completed the tasks shown in Figure 2. They showed misconceptions on mathematical modeling and did not provide a realistic answer to the problems. Two 3-hour sessions were devoted to help them understand mathematical modeling. By the end of the semester, a written task (see Fig. 2), was used for the study as part of final assessment.

Part 1: Please answer the following questions in as much detail as possible.

1. When people say mathematical modeling, what does the word "mathematical modeling" mean to you?
2. What do you believe constitutes effective mathematical modeling instruction?

Part 2: Solve the following problems.

1. Your best friend is coming to visit. She told you that her bus arrives at 4:00 pm. You live 10 miles away from the bus station. The speed limit is 40 miles per hour. When should you leave your house? Explain your reasoning.
2. A popular band recently came to a music festival. A field of size 100 m by 200 m was reserved for the audience. The concert was completely sold out and the field was packed with all the fans standing. Which one of the following is likely to be the best estimate of the total number of people that attended the concert? 10,000, 20,000, 100,000, 200,000, or 400,000? Explain your reasoning.

Figure 2. Main task of this study.

For the analysis of PSTs' written response to mathematical modeling and effective mathematical modeling instruction, we used an inductive content analysis approach. PSTs' responses to the notions of mathematical modeling and effective mathematical modeling instruction were categorized based on themes emerging as researchers read multiple cases. Then we explored the subcategories under each analytical aspect according to the framework (see Table 1 later). For the modeling task, we first created a rubric based on correctness of PSTs' responses to each item and then assigned a score for each item. To examine relationship among PSTs' conceptions of mathematical modeling, effective mathematical modeling instruction, and their mathematical modeling performance, we run SPSS statistical program (e.g., ANOVAs).

Summary of Selected Findings

Psts' Conceptions of Effective Mathematical Modeling Instruction

To investigate PSTs' conception of effective mathematical modeling (MM) instruction, we reviewed their responses and classified the responses into four aspects based on common themes (see Table 1). Out of the four aspects, the most popular category is *teaching* aspect (i.e., what instructional strategies or teaching practice need for effective MM instruction?), followed by mathematical modeling *steps* aspect (i.e., what step is required for MM lesson?), *problem features* aspect (i.e., what is considered as a good problem for MM instruction?), and *purpose* aspect (i.e., what is a good MM lesson aimed at?).

Table 1: Four Aspects of PSTs' Conception of a Good Mathematical Modeling Lesson and Frequencies

Category	Sub-category	# of PSTs	Relation to 3 MM approaches
1. Purpose aspect (29)	a. To find a realistic solution	10	For
	b. To develop critical/ logical /reflective thinking	14	Through
	c. To develop a good understanding of math	5	Through
2. Problem aspect (39)	a. Real-life problems	14	Through
	b. Problems that requires student prior knowledge	5	Through
	c. Practice problems that use the same technique	4	For
	d. Problems that require multiple solutions	11	Through
	e. Problems that require explanations	5	Through
3. Mathematical modeling steps aspects (42)	a. Structuring a lesson for all five modeling steps	14	About
	b. Identifying problem	12	About
	c. Devising a strategy	3	About
	d. Carrying out	2	About
	e. Interpreting	11	About
	f. Validating	5	About
4. Teaching aspect (78)	a. Emphasizing different ways of solving problems	20	Through
	b. Giving examples about how to solve	10	For
	c. Giving definitions on mathematical modeling	9	For
	d. Giving enough time to work on problems	6	For/Through
	e. Providing a direct, clear direction and structure	25	For/About
	g. Lessons that are interesting to students	8	Through

Note. Majority of PSTs addressed multiple categories. These responses were coded in multiple categories as long as the categories were present in their written responses.

After identifying the four aspects, we collectively considered them to categorize PSTs' conceptions of effective MM instruction into the three groups by referring to Schroeder and Lester's (1989) identification. Out of 78 participants, 42 participants considered effective MM instruction as teaching *about* mathematical modeling, 13 participants as teaching *through* mathematical modeling, and 20 participants as teaching *for* mathematical modeling. This finding indicates that despite the consistent emphasis on teaching *through* mathematical modeling in current mathematics education, a large portion of our PSTs still did not have a clear view of teaching *through* mathematical modeling. In a similar way, we categorized PSTs' conception of MM into three groups by referring to Stanic and Kilpatrick's (1989) identification. Out of 78 PSTs, 42 PSTs considered MM as *content*, 22 PSTs as a *skill*, and 14 PSTs as *art* of discovery.

Relationship between PSTs' Conceptions and Their Modeling Performance

For the first problem that asks students to decide when to leave their house, about 42 % PSTs responded that they would leave their house at times before 3:45 pm to go pick up their friend (correct realistic answer); 58% responded that they would leave their house at 3:45 pm. For the second modeling problem that requires students to determine the best estimate of the total number of people that can attend the concert in the size 100 m by 200 m, 42 % PSTs responded that there were 100,000 fans (correct realistic answer) whereas 38% provided a mathematically correct answer, 20,000. After coding PSTs' written responses and grading mathematics tasks, we quantified the data analysis result to examine relationship among PSTs' conception of mathematical modeling, a good mathematical modeling lesson, and their mathematical modeling performance. A chi-squared test showed that there is a positive relationship between PSTs' conception of mathematical modeling and their conception of effective mathematical modeling instruction, $\chi^2 = 16.888$, $df = 2$, $p = 0.002$**.** In addition, there was a significant difference of mean scores concerning mathematical modeling competence among groups of PSTs who perceived different views on mathematical modeling, $F(2, 73) = 3.292$, $p = .024$. PSTs who perceived mathematical modeling as *art* showed highest mean scores in the mathematical modeling tasks, followed by PSTs who with mathematical modeling as *means* to a focused end.

Discussion and Implications

This study contributes to the current literature on mathematical modeling and the knowledge base of teacher education. The findings of this study suggest that teacher educators need to find a better way to help PSTs perceive mathematical modeling as *art* and effective mathematical modeling instruction as teaching mathematics *through* mathematical modeling (Son, 2016). One approach would be: Have PSTs experience three different perspectives of teaching mathematical modeling and compare affordances and limitations of each approach. Then teacher educators need to give PSTs more opportunities to experience teaching *through* mathematical modeling in their mathematics methods courses where PSTs engage in mathematical modes of thought by analyzing and interpreting the problems. Furthermore, intervention studies that experiment with these suggestions are needed to find a better way to support PSTs' conceptions regarding mathematical modeling, modeling lessons and their mathematical modeling abilities.

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HOW PRESERVICE TEACHERS' CONCEPTIONS OF PROBLEM-POSING RELATE TO THEIR PROBLEM-POSING COMPETENCY WITH FRACTION OPERATIONS

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Despite the importance of teachers' conception of problem-posing and their problem-posing competency, limited attention is given to this area in the current literature. In this study, we examined preservice teachers' conception of problem-posing and their problem-posing performance. We asked preservice teachers to define problem-posing. We then investigated how their conceptions of problem-posing were reflected in their mathematical problem-posing performances in the context of fraction operations. Analyses of 67 preservice teachers' written responses revealed that a large portion of the preservice teachers defined problem-posing as general actions for teaching. However, preservice teachers who viewed problem-posing as a means of improving problem-solving tended to provide correct word problems that include multiple solution paths in a realistic context.

Keywords: Teacher Education-Preservice

Introduction

The purpose of this study is to examine elementary preservice teachers (PSTs)' conception of problem-posing. We also investigate how their conceptions of problem-posing are reflected in mathematical problem-posing performance. With increased interest in curricular and pedagogical innovation in mathematics education, mathematics educators have paid attention to problem posing. According to *Principles and Standards for School Mathematics* (NCTM, 2000), the use of problem-posing activities in the mathematics classroom should be encouraged to improve students' mathematical understanding and creativity. In a similar vein, Kilpatrick (1987) highlighted that "problem posing should be viewed not only as a goal of instruction but also as means of instruction" (p. 123). However, problem posing has been used with multiple meanings that range from "the generation of new problems" or "the re-formulation of the given problem" to "a means to improve students' problem-solving" or "a window into students' mathematical understanding" (Silver, 1994). While the first two meanings of problem posing are related to doing mathematics (Brown & Walter, 2005), the rest two meanings are associated with the features of inquiry-oriented instruction. Thus, what is meant by problem posing is often subject to interpretation in a particular context, which suggests the need for exploring how mathematical problem posing is conceptualized by teachers and whether there exists any relationship between their conceptions of problem posing and their problem-posing abilities. The research questions that guided this study are: (1) how do PSTs view problem posing?; (2) how do PSTs pose word problems related to fraction multiplication and division?; and (3) how is their notion of problem posing reflected in their ability to pose word problems related to fraction operations?

Theoretical Perspectives

Problem posing has long been recognized as a critically important component of the mathematics curriculum, and is considered to be an essential part of doing mathematics (NCTM, 2000). Kilpatrick (1987) conceptualized problem-posing as reformulating an existing problem in order to make it your own. Different from Kilpatrick (1987), Stoyanova and Ellerton (1996) defined problem-posing as "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (p. 518). The subjective nature of this definition—one should decide in which aspect the problem is meaningful and for whom—is apparent. Silver (1994) identified six different perspectives on mathematical problem-posing, which include: (1) problem-posing as a feature of creativity or exceptional mathematical ability; (2) problem-posing as a feature of inquiry-oriented instruction; (3) problem-posing as an important component in the creation of mathematics by professional mathematicians; (4) problem-posing as a means to improve students' problem-solving; (5) problem-posing as a window into students' mathematical understanding; and (6) problem-posing as a means of improving students' disposition toward mathematics. Then, what would be a good conception of problem-posing for teachers? Among Silver's six categories, some are related to doing mathematics whereas other conceptions are considered to be strategies for developing students' mathematical problem-posing abilities. There is no clear distinction in Silver's six categories of problem-posing. Thus, the notion of "problem-posing" is used with a variety of not-always-compatible meanings and is applied to a variety of not-always-compatible teaching/learning situations. In this study, by reconceptualising Silver's six categories, we intended to explore PSTs' conception of problem-posing and its relation to their problem-posing abilities.

Methods

Data for this study came from 67 PSTs at two different university sites. Each PST was enrolled in either an elementary mathematics content course or an elementary mathematics methods course jointly designed and taught by the two authors. A written task was used for the study, which consists of two parts (see Fig. 1). We purposefully selected three fraction multiplication and division problems because these types of problems require deeper understanding of fractions, fraction multiplication and division.

Part 1: Please answer the following questions in as much detail as possible.

1. When people say problem posing, what does the word "problem posing" mean to you?
2. Create a metaphor that describes your conception of problem posing:
3. Why do you think that metaphor is relevant in describing your conception of problem posing?

Part 2: Suppose you are posing a word problem to help students develop a good understanding of fraction multiplication and division. What word problem would you pose teach each fraction operation below? Describe your problem in a way that you pose it to your students.

1. $1/4 \times 2/3 =$
2. $4/3 \times 48 =$
3. $3 \div 1/5 =$

Figure 1. Main task of this study.

In order to answer the first research question, PSTs' responses to the notion of problem-posing were categorized first based on Silver's (1994) six perspectives on problem-posing. As we identified emerging themes from multiple cases, we modified Silver's perspectives on problem-posing into seven categories and then coded the data. For the second research question related to problem-posing task, correctness of PSTs' responses was determined and then various contexts and common misconceptions/errors in the creation of word problems were explored. In answering the third research question that examines the relationship between PSTs' problem-posing performances and their notions of problem-posing, correctness of PSTs' three word problems relating to fraction operations was determined first. After that, we explored patterns that existed between their problem-posing performances and their notions of problem-posing by looking at the number of PSTs with correct word problems, the existence of multiple path ways and problem contexts in the word problems with respect to the different notions of problem-posing.

Summary of Selected Findings

PSTs' Conceptions of Problem-Posing

In order to categorize PSTs' conceptions of problem-posing, we first re-conceptualized Silver (1994)'s six perspectives on mathematical problem-posing based on the following four questions in order: (1) do PSTs define problem-posing from the mathematics perspective of doing mathematics or from the teaching perspective?; (2) if they define from the teaching perspective, do they consider problem-posing as a general action for teaching or an action for creating reform-oriented instruction?; (3) if it is for creating reform-oriented instruction, who is the one posing problems?; and (4) if the main problem poser is the teacher, what is the purpose of the action of problem-posing? We thus sorted PSTs' conception of mathematical problem-posing (PP) into the seven categories:

1. PP as a feature of exceptional mathematical ability or creativity,
2. PP as a necessary component in good problem-solving,
3. PP as essential student actions for inquiry-oriented instruction,
4. PP as essential teacher actions to improve students' problem-solving,
5. PP as a means of *formative assessment* for students' mathematical understanding
6. PP as a means to increase student *interest, disposition, motivation* on doing math
7. PP as general teacher actions of teaching,

Table 1 shows the number of PSTs categorized into each of the seven perspectives of problem-posing. While the first two perspectives (PP1 and PP2) define PP from mathematical action, the remaining perspectives consider PP as a means for instruction. However, depending on the pedagogical purposes for and emphasis on PP, PSTs' conceptions of PP can be further divided into the subcategories ranging PP3 to PP7. About 20% PSTs defined PP only focusing on mathematical action and around 80% considered the purpose and subjects of problem-posing in in defining PP.

Table 1: PSTs' Conception of Problem Posing and Their Frequencies

<table>
<tr><td rowspan="4">Criteria</td><td colspan="2" rowspan="2">Math-focused</td><td colspan="5">Instruction-focused</td></tr>
<tr><td colspan="4">Reform-oriented</td><td>General</td></tr>
<tr><td rowspan="2">Special</td><td rowspan="2">General</td><td>Student Action</td><td colspan="3">Teacher Action</td><td>Teacher Action</td></tr>
<tr><td>Instruction</td><td>Instruction</td><td>Assessment</td><td>Disposition</td><td>Instruction</td></tr>
<tr><td>Conceptions</td><td>PP1</td><td>PP 2</td><td>PP3</td><td>PP4</td><td>PP5</td><td>PP6</td><td>PP7</td></tr>
<tr><td># of PSTs</td><td>2 (3%)</td><td>12 (18%)</td><td>1 (1%)</td><td>10 (15%)</td><td>2 (3%)</td><td>0 (0%)</td><td>38 (57%)</td></tr>
</table>

PSTs' Problem-Posing Competencies in Fraction Multiplication and Division

In order to explore how their conceptions of problem-posing were reflected in their problem-posing actions, we asked PSTs to create a word problem to three fraction number sentences: (1) 4/3 x 48, (2) ¼ x 2/3, and (3) 3÷ ⅕. Among the three problems, we considered the first problem the most challenging (4/3 x 48), followed by the third problem (3÷ ⅕) and the second problem (¼ x 2/3). In the analysis of PSTs' word problems, we considered the following three aspects: (1) whether to focus on mathematics correctly; (2) whether to be realistic; (3) whether to provide multiple solution pathways. We first determined PSTs' word problems based on correctness. As we expected, the first problem, finding 4/3 of 48 was the most challenging to our PSTs; only seven PSTs provided a correct word problem. We further analyzed each PST's word problem with respect to the context and whether multiple solutions/representations are required (Table 2).

Table 2: Frequency of Realistic/Multiple Solutions Among Correct/Partially Correct Responses

Problems	Correct		Partially Correct	
	Realistic	**Single path**	**Realistic**	**Single path**
4/3 x 48 =	7 out of 7	7 out of 7	4 out of 6	6 out of 6
¼ x 2/3	33 out of 33	33 out of 33	5 out of 5	5 out of 5
3 ÷ ⅕ =	20 out of 20	20 out of 20	2 out of 2	2 out of 2

Table 2 presents although our PSTs tended to create realistic contexts, if they were not asked to do, they seem to not consider the tasks that can be solved in different ways by using multiple representations or strategies.

Relationship between the Notion of PP and PP Competency

In order to further understand the relationship between PSTs' conception of problem-posing and their problem-posing competency, we categorized each student based on the correctness of three word problems, which resulted in the following six groups of PSTs: (1) Group 1: students with correct answers to all three problems; (2) Group 2: students with correct answers to #2 and #3; (3) Group 3: students with correct answers to #3 and #1; (4) Group 4: students with correct answers to #1 and #2; (5) Group 5: students with correct answer to #3; (6) Group 6: students with correct answer to #2; (7) Group 7: students with correct answer to # 1; (8) Group 8: students with incorrect answers to all three problems. We found that when PSTs had limited PP competency (Groups 5-8), they tended to view PP as general teacher actions of teaching that focus only on creating the problem or delivering the problem to the students. In contrast, PSTs in Group 2 tended to possess different conceptions of PP, which are related to the features of inquiry-oriented instruction. These findings suggest that we need to consider not only how to develop skills related to PP but also how to change their conceptions of PP.

Discussion and Implications

This study has implications for teacher educators working to design mathematics education courses for PSTs, as well as for researchers interested in better understanding of teacher knowledge, beliefs and strategies. That is, the findings of this study suggest elementary mathematics teacher education programs need to include more problem-posing activities so that PSTs can experience the benefit of problem-posing from a variety of perspectives. This study also suggests the importance for future research to continue to investigate preservice and in-service teachers' conceptions of problem-posing.

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IDENTIFYING CRITICAL TOPICS FOR TEACHING MATHEMATICS FOR SOCIAL JUSTICE IN K-5 SETTINGS: CONNECTIONS AND TENSIONS

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This study examines PSTs selections of critical topics for a social justice project in a K-5 methods course. This study builds on previous work focused on using students' funds of knowledge in instruction. Using the Flint water crisis as an authentic context, I looked for evidence that PSTs selected topics that would allow elementary students to read the world with mathematics. Findings indicated that although PSTs chose topics that "raised awareness," lessons did not fully address issues of power, privilege, and resource inequities. Implications for PST preparation, suggestions for scaffolding, and future research are discussed.

Keywords: Equity and Diversity, Teacher Education-Preservice

Introduction

As a mathematics educator focused on equity and social justice, preparing pre-service teachers (PSTs) to disrupt stereotypes and engage *all* students in meaningful mathematics is a focal point of my practice. After using a funds of knowledge (FOK) approach in a K-5 mathematics methods course for three semesters, findings from my research indicated that what PSTs learned via the FOK project was compartmentalized to that assignment. I also found that PSTs did not use this knowledge when planning additional lessons and small group activities throughout the duration of the course. Additionally, the lessons that PSTs planned were very shallow in their treatment of culture and only included topics such as students' names, foods, favorite toys, and other general interests. In short, their understanding of real world knowledge was shallow and their work lacked substantive evidence of community-based funds of knowledge (Gonzalez, Moll & Amanti, 2005).

Background

For decades, scholars have articulated the need for teachers to teach using culturally relevant practices and build connections between students lived experiences and classroom instruction (Ladson-Billings, 1995; Moll et al, 1992; Gonzalez, Moll & Amanti, 2005). To meet this need, a funds of knowledge framework has been previously implemented in mathematics methods courses (Aguirre et al, 2012). Moll & Greenburg (1990) argued that students' homes and communities are "valuable educational resources for teachers." Moreover, Jiménez and Semingson (2011) argued that funds of knowledge is an additive approach to schooling in which teachers learn to "build on students and families strengths" as opposed to a subtractive model or deficit model (p. 5).

Many students' community experiences and knowledge can be critical depending upon the student's background and current sociopolitical contexts (Gutstein, 2007). Because there can be overlap between community mathematics and critical mathematics, it is necessary to think about the ways in which the constructs work together as well as PSTs beliefs about teaching using such an approach. In a study of 92 PreK-8 PST, Simic-Muller and colleagues found that although PSTs initially had a "narrow" definition of real-world situations, they could articulate more controversial topics when pressed to do so (Simic-Muller, Fernandes, & Felton-Koestler, 2015). They go on to suggest that PSTs could benefit by starting this trajectory with topics that are more widely accessible.

Methods

Context

The study involved 17 PSTs enrolled in my K-5 mathematics methods course in the spring 2016 semester. These PSTs were in a cohort and were simultaneously enrolled in a literacy methods, social studies methods, science methods, and health/PE course. Additionally, PSTs had a 200-hour field component that accompanied their methods block. This course occurred during the first semester of their senior year and preceded their full-time student teaching practicum.

In an effort to move PSTs from focusing on students' interests, I modified the course to include assignments that focused on current events with the ultimate goal of creating "critical tasks." I defined critical mathematics tasks to be tasks that are based on a current societal ill, inequity, or injustice. To introduce this concept to my PSTs we did an in-depth study of the Flint water crisis, what led to this crisis, and how it was being addressed. I chose to study Flint so that I could model an authentic context (Leonard & Evans, 2012) and so that the PSTs could first engage in this pedagogy as a learner before being expected to take it up in their own instruction (Felton-Koestler, 2017).

Before starting this activity, I provided PSTs with a list of articles to read and hash tags to follow on social media for one week prior to starting our discussion. As we unpacked the Flint water crisis, I drew from Gutstein's work in which he defined critical knowledge as, "knowledge about the sociopolitical conditions of one's immediate and broader existence. It includes knowledge about why things are the ways that they are and about the historical, economical, political, and cultural roots of various social phenomena" (2007, p. 110). We then talked about how we could use mathematics to better understand multiple facets of this crisis. PSTs worked in grade-level groups to create lesson plans that were directly connected to the Flint water crisis. After this group project, PSTs were asked to select their own critical context and develop an accompanying mathematics lesson.

Findings

These findings are based on analysis of PSTs group lesson plans, individual lesson plans, and post-project reflection survey. A number of themes emerged among the critical topics selected including: raising awareness, current events (local and national news), student connections, as well as influences from other courses. In the following paragraphs, I share examples from a few of the themes that emerged from PSTs topic selection and justification. I also share tensions that PSTs faced while identifying these topics. A more detailed report of all themes can be found in a forthcoming paper.

Raising Awareness

Some PSTs selected critical topics intended to raise awareness for the students they served. These tasks included world poverty, school poverty, world hunger, and the rice shortage in the Philippines. PSTs explained these selections with the following comments:

> I chose this topic [poverty] because most of the students come from very affluent families. These students pretty much have everything they need to survive. I wanted them to have an opportunity to see that not everyone lives like they do…There are children my students [sic] age that go hungry everyday. Many of these children die of starvation. By showing them with fractions what portion of the world has what they have and what does not, I think the students will come away with a better understanding and be grateful for the things they do have.

Although these examples have the potential to address power, class, and resource inequities, PSTs failed to fully capitalize on that potential in developing their mathematics tasks. For example, the PST indicates that fractions can be used to make comparisons between what some children have compared to others. This lesson could have been enriched by simultaneously discussing issues of privilege with issues of poverty. A subtext to these PSTs explanations is that they wanted to make their students grateful by comparing their current situations to those less fortunate than themselves.

Current Events (Local and National News)

Other PSTs selected topics based on current events. Among these tasks were job creation, flooding in the Midwest, the 2016 election, and school lunch. These task selections were particularly pertinent as one PST suggested that this project caused her to, "pay more attention to the mathematics in current events." Other PSTs indicated that although they did not initially stay abreast of current events, they chose to follow various news sources on their social media outlets. Although each of these examples came from local and national news sources, it is important to note that PSTs justifications for why they were critical varied.

As stated, the 2016 election and the structure of government was selected as a topic. The PST stated that:

> The students in my … classroom are currently studying the three branches of government and how they work together to pass laws and run the country. Also, since the presidential election is coming up, I felt like this was a good task to use to integrate social studies and math using a topic students are interested in and already have background knowledge on. This is a critical topic because it is an issue that people are divided on and that Americans are interested in and playing a role in right now…Students hear about the election and government from their friends, home, and school, and encounter many different opinions that might not be based in fact. If I use this topic in a math task, I am able to give them more information in an unbiased setting that lets them form their own opinion of the critical topic.

A second PST reported on The Frankfurt Company breaking ground in the community surrounding her school. This company was set to invest over $20 million into a local project and create 100 permanent jobs for the area. She commented that The Frankfurt Company, "will employee many families in and around the Bunby County area beginning in 2017 and the years to follow. This is an event that will make an impact on the community, one in which I felt they should be aware of." This topic could have been extended to acknowledge issue of power and class. For example, how can we assume equitable access of job opportunities at the Frankfurt Company? And will there be equal pay for equal work for the given positions?

Tensions

A few tensions arose while analysing this data. First, some PSTs selected topics that they were interested in while others chose topics that they thought their students would be interested in. This shows that PSTs were struggling to balance their interests with students' interests. Some of this tension was because PSTs indicated that they needed to do their own research on a topic prior to developing a task for their classrooms. Another tension that resulted from this analysis was age and grade level tensions. Some PSTs thought that certain critical topics were not appropriate for young students and went on to say that critical topics would have to be adapted to make them "grade appropriate." This tension mirrors findings from other scholars in the field (Simic-Muller, Fernandes, & Felton-Koestler, 2015).

Implications

There are a number of findings that have emerged from this work that demonstrate promise for using this type of pedagogy in a methods course. First, some PSTs were able to use their understanding of the Flint water crisis project to identify other topics and use those topics to develop critical tasks appropriate for their field placement grade levels. Others had difficulty identifying a critical topic. It also clear that PSTs need more information on the historical, political, and social implications of societal issues to help them better frame their understanding of what makes an issue problematic, therefore leading to the development of a critical task. These findings support Bartell's (2013) piece in which she found that in-service teachers had varying definitions of what it meant to teach mathematics for social justice ranging from awareness, to cultural exposure, to student empowerment.

This study highlighted the layers involved in this process, particularly that PSTs may need assistance teasing apart the elements or their critical topic such that any resulting lesson plans fully address issues of social justice. A final implication of this work is that the selection of a critical topic and the development of a task should be an iterative process. PSTs could have benefited from additional conversations that pushed or challenged their thinking even more.

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INVESTIGATING PRACTICE THROUGH REHEARSALS: HOW TEACHER CANDIDATES RESPOND TO STUDENT CONTRIBUTIONS IN WHOLE-CLASS DISCUSSIONS

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Responding to student contributions during whole-class discussions is a complex instructional practice. Coached rehearsals are a way to support teacher candidate (TC) learning. This study investigates TCs' practice of responding to student contributions during rehearsals of leading whole-class discussions. Two key findings emerged through video analysis. The rehearsing TC had general patterns of responding, with some nuanced variation when responding to a contribution that was mathematically problematic. Second, analyzing teacher responses as sets of interactions contingent on student contributions and key mathematical ideas provided greater insight into the practice of responding. These findings have important implications for research on this complex practice.

Keywords: Instructional Activities and Practices, Teacher Education-Preservice; Classroom Discourse

Rehearsals are a valuable pedagogical tool to support teacher candidates (TCs) in learning complex instructional practices, such as leading whole-class discussions (e.g., Boerst, Sleep, Ball, & Bass, 2011). One challenge in leading whole-class discussions is responding to student contributions in ways that position them as resources to advance the class's learning of key mathematical ideas. In this paper, we describe emerging understandings about the practice of responding to student contributions, as well as strategies for analyzing TC practice during rehearsals of leading whole-class discussions.

How teachers respond to students' ideas is consequential for students in shaping their beliefs and mathematical understandings (e.g., Borasi, 1994). There are many different ways that teachers respond to students during whole-class discussions, such as re-voicing what the student said or using other talk moves (Chapin, O'Connor, & Anderson, 2013), steering the discussion toward the mathematical point (Sleep, 2012), building on student thinking (Leatham, Peterson, Stockero, & Van Zoest, 2015), allowing other students to respond, recording the student contribution on the board, and more. Skillfully responding to student contributions rests on noticing and interpreting student's mathematical ideas (e.g., Jacobs, Lamb, & Philipp, 2010) and can be particularly difficult when the contributions have mathematical ambiguities or errors.

Theoretical Framing

We take the view that whole-class discussions are a complex set of interactions among teacher and students. The teacher engages in interrelated moves to elicit student participation, encourage students to respond to one another, and further student understanding of the mathematical concepts being discussed (Boerst et al., 2011). These teacher moves are necessarily situated in the context of the classroom and the type and purpose of the discussion. Developing skill in the practice of leading discussions is challenging for TCs, though they can develop such skill through purposeful opportunities to approximate and reflect on practice (e.g., Grossman et al., 2009), such as rehearsing leading a discussion in a methods class (Lampert et al., 2013). Rehearsals afford opportunities for teacher educators (TEs) to deliberately construct moments for TCs to respond to student thinking; rehearsals also afford opportunities for the TE to provide in-the-moment feedback or engage TCs in collectively reasoning about how to respond (Lampert et al., 2013; Singer-Gabella, Stengel, Shahan, & Kim, 2016). We examine ways rehearsals can support TCs in learning to respond to students' contributions during whole-class discussions. We investigate this through the use of a discussion structure deliberately designed to surface multiple types of student contributions and provide TCs opportunities to respond.

Methods

Our analysis of TC practice takes place in the context of secondary mathematics methods courses at two large, public universities. TCs rehearsed leading sorting discussions (Baldinger, Selling, & Virmani, 2016), where "students" (other TCs) sorted cards with examples and non-examples of linear functions in order to develop and refine their definition of linear function. In each rehearsal, an error was introduced as a way to construct an opportunity for the rehearsing TC to navigate responding to a mathematically problematic student contribution, in addition to correct student contributions. Each rehearsal was video recorded.

The video analysis occurred in two phases. The first phase involved defining the unit of analysis and chunking the videos into segments. The structure of a sorting discussion involved the teacher asking students to share cards that were easy or hard to sort. Based on this structure, one frequent student contribution during the rehearsals was naming a specific example or non-example. For example, in one rehearsal, the rehearsing TC said, "Can you share an example of what you think a linear function is?" The "student" replied, "$y = 3x + 5$." The TC followed up, saying, "Good, $y = 3x + 5$…why did you come up with that?" The student responded, "…because it's in $y = mx + b$ form." In other cases, a student contributed the card and reasoning in a single talk turn. Student contributions also occurred as a card was discussed. These contributions tended to expand on or disagree with what another student said. The teacher's response occurred in the teacher talk immediately following student talk, but also in subsequent interactions. To capture the complex and interactive nature of teacher responses to students, we used the introduction of a new card and all the discussion related to that card as our unit of analysis. We then had to determine what counted as a *student contribution* and what counted as the *response* to that contribution. The initial student contribution was considered to be the card and associated reasoning, and the teacher response was considered to be the set of interactions—including teacher talk, student talk, and recording—that related to that card. This chunking method enabled us to identify patterns that occurred in response to different types of initial contributions. This follows work of others who argue that building on student thinking is more than a single teacher move (Van Zoest, Leatham, Peterson, & Stockero, 2016).

The second phase of analysis involved coding the initial student contributions as having errors (i.e., being mathematically problematic in some way) or not, and identifying the patterns of moves made by the teacher over the course of their response to the initial student contribution. All four researchers independently watched and annotated rehearsal videos to document patterns of responding. The annotations were discussed and synthesized to reach consensus about how to describe the teacher response around each card brought up in the rehearsal. In this paper we present data from one of the six rehearsals.

Preliminary Findings

Our initial analysis of one of the rehearsals revealed two primary findings. First, we documented patterns of practice in how the TC responded to student contributions, and noticed nuanced differences in the nature of the responses based on whether or not the student contribution contained an error. Second, we found that analyzing teacher responses as sets of interactions contingent on student contributions and key mathematical ideas provided greater insight into the practice of responding to students during whole-class discussions.

Patterns of Practice in Responding to Student Contributions

Our analysis revealed patterns of responses across cards. For example, the TC was fairly consistent in her use of a set of talk moves to respond to student contributions. Typically, this TC would elicit student reasoning if it was not initially offered, she would re-voice and often record the student reasoning, and she would orient the other students to the initial contribution by asking other students to elaborate. In some cases, this TC would purposefully steer the conversation toward the definition of linear function, or probe student reasoning. The sequence and frequency with which the TC utilized these moves varied across the six cards discussed during her rehearsal; however, this set of moves was common for every card.

Despite a clear pattern of responding, our analysis also revealed subtle differences in responses when the student contribution was mathematically problematic. For example, the TC often asked questions like, “Can someone else add in why they think that is also an example?” as a way to encourage students to build on the mathematically correct reasoning already shared during the discussion. The TC used a very similar move when a student shared that the card $y = 17 - 5x$ was an example of linear function, but incorrectly reasoned that it was linear because it had a slope of 17 and a y-intercept of 5. In this case, the TC said, “Does anyone want to elaborate on that, or give another opinion, agree or disagree?” In response to initial contributions that are correct or contain errors, this TC used an orienting move. However, in response to the error, the TC explicitly asked about the possibility of disagreement, which was not part of her questioning when there was not an error. Despite using the same general sequence of moves, the TC’s response to the error revealed differences in how those moves were used as compared to responses to contributions without errors. This also raises the question of how different moves interact with one another in the context of different types of student contributions.

Analyzing Teacher Responses as Sets of Interactions

Our approach of analyzing sets of moves that included teacher talk, student talk, and recording had many affordances for capturing the complex practice of leading a mathematical discussion. For example, the orienting moves used by the TC are very similar, but they played out differently. When discussing mathematically correct reasoning, the orienting move resulted in another student adding more reasoning, and the TC incorporated that contribution into the discussion and moved on to the next card. When discussing the incorrect reasoning, right after the orienting move, the TC made an additional re-voicing move, reiterating the part of the student reasoning that was incorrect. The TC used numerous additional re-voicing, probing, and recording moves, and allowed students to respond directly to one another before she brought the discussion of this card to a close. This contrasting use of common teacher moves highlights the importance of considering sets of teacher moves in concert with the student contributions themselves – for instance, whether there is an error or not, or whether the mathematics of the idea is of central importance in that moment. Different moves are more or less effective at different times in the discussion as they are contingent on and interact with student contributions.

Discussion

Rehearsals of sorting discussions provided valuable insight into how TCs respond to student contributions. Through analysis of sets of interactions, we were able to identify common patterns of responding that utilized productive talk moves. At the same time, by considering the complexity of leading whole-class discussions, we identified subtle differences in how these talk moves were used based on the nature of the student contribution. The work of responding to student thinking is contingent on the mathematics being discussed and interactions among the teacher and students (Lampert et al., 2013; Leatham et al., 2015). The particular structure of the sorting discussion allowed for an intentional opportunity to insert incorrect mathematical thinking, and to provide opportunities to respond to different types of student contributions. Using the card under discussion as the unit of analysis potentially makes it easier to see patterns and regularities, and also highlights nuances in teacher practice around responding to student contributions, because the boundaries for different segments of the discussion are clearly marked. This promising analytic technique could also provide a tool for comparing TCs' practice when leading sorting discussions across settings, such as rehearsals and enactments in field placements. On-going work to explore patterns in how TCs respond to student contributions is important for developing more systematic uses of pedagogies of practice for teacher learning.

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MATHEMATICAL MODELING FOR TEACHING: AN EXPLORATORY STUDY

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We traced the impact of a designed unit of instruction on mathematical modeling on Prospective secondary teachers' knowledge about and efficacy towards teaching modeling. Analysis indicate that although teachers maintained modeling to be an important skill to be developed, absence of extensive experiences with mathematical modeling in the course of their own mathematical preparation hindered their ability to access pedagogical actions to be used in teaching.

Keywords: Modeling, Secondary Teacher Preparation, Mathematical Knowledge for Teaching

The demand that mathematics teachers will infuse mathematical modeling in curriculum is now paramount in the educational reform efforts in the US as pioneered by the Common Core State Standards of Mathematics (CCSSM, 2010). There is some evidence that due to absence of information regarding the nature of mathematical modeling misconceptions exist among teachers regarding modeling and its associated pedagogies (Gould, 2013). The international community of researchers on teaching and learning of mathematical modeling have stressed the need to explore models and programs that might assist teachers to meet implementation challenges and to document potential impact of these efforts on teachers (Cai et al. 2014). The current paper addresses these two areas. One question guided our inquiry:

8. What is the impact of a unit of instruction on teaching mathematical modeling on teacher candidates' perceptions of teaching mathematical modeling in schools?

Background

The methods course that served as the site for the current study is the second of a sequence of two courses on methods of teaching high school mathematics. The focus of the first course is on teaching of Algebra, Calculus and number theory concepts. Using Principles of Inquiry based Learning and Teaching (Artigue and Blomhøj, 2013) the course draws attention to connections among student thinking, instruction and assessment. The second course addresses teaching and learning of Geometry, Measurement, Probability, Statistics and Discrete Mathematics. Mathematical modeling strand was to be addressed explicitly during three weeks of instruction.

Methodology

First, a survey of knowledge of mathematical modeling for teaching was designed and administered to collect base line data on the candidates' perceptions about teaching mathematical modeling in schools. Second, in light of survey results and areas that seemingly needed developing, a unit of instruction on mathematical modeling for teaching was conceptualized and implemented during three course sessions (approximately 8 hours). A post implementation survey was then administered on the last session of the academic semester to trace impact of the experiences provided for participants as expressed by them.

Participants

The participants consisted of 11 prospective secondary mathematics teachers. The participants had completed all coursework towards a major in mathematics and were enrolled in the second course of a sequence of two methods courses on teaching in secondary schools.

Survey

The survey consisted of 20 items and addressed four areas. The first part of the survey collected biographical information from the participants, the range of mathematics courses they had completed, and their assessment of courses in which they believed they had gained mathematical modeling experiences as learners.

Four items addressed the participants' claimed comfort level with mathematical modeling, teaching it, and their perception of the importance of modeling for school learners. Additionally, they were asked to identify how frequently they had observed mathematical modeling be implemented in classrooms.

The third part of the survey consisted of open response items that collected data on the participants' description of mathematical modeling, similarities and differences they observed between modeling tasks and other kinds of activities used in classrooms, and processes they associated with mathematical modeling. The participants were asked to provide illustrative examples in each part.

The last portion of the survey aimed to obtain specific data on the participants' ability to identify suitable examples of modeling tasks to be used with middle and high school students as well as how they envisioned gauging learners' progress when engaged in such tasks. We had anticipated that the participants' responses to the last set of questions would allow us to more carefully detail their knowledge related to modeling based curriculum and instruction.

The same survey was administered again during the last session of the academic semester and upon conclusion of the experimental unit of instruction.

Findings

Phase I: What Teachers Felt They Knew

All 11 participants reported having had exposure to mathematical modeling and having gained knowledge of mathematical modeling in their Discrete Mathematics course. A half of the participants reported that they felt they had adequate exposure to modeling experiences in their calculus, differential equations, and probability and statistics courses.

In describing mathematical modeling and its process, 7 (64%) participants characterized it as using mathematics to represent and analyze real world situations. The participants' responses however varied according to the amount of detail they chose to include in outlining their thinking. 3 (27%) participants perceived mathematical modeling as using manipulative, simulations or world problem to demonstrate a mathematical concept.

In explaining specific actions associated with modeling two common themes emerged. One group equated mathematical modeling process with problem solving. The second type of description concerned data modeling with a focus on statistical context. None of the participants referenced defining variables, setting parameters, building a mathematical representation of the situation, and refining the model (Blum and Leiβ, 2007) as part of the modeling process.

On the follow up question that asked the participants to report how they would assess school learners' mathematical modeling progress, all but one participant offered general descriptions that ignored the unique features of mathematical modeling. In order to illustrate differences between modeling tasks and other types of mathematical activities, participants relied on phrases such as "open ended", "multiple approaches", "multiple solutions" to describe their thinking.

Phase II: Course Design

In light of the survey results, we set four goals for our work with the participants so to help them: (1) develop a deeper understanding of the mathematical modeling process and its intricacies, (2) discriminate between mathematical modeling as a process and solving routine application problems,

(3) learn about suitable resources that could be used for simulating modeling tasks, and (4) understand how student learning could be gauged using the modeling cycle as a platform for assessment. The participants were introduced to the modeling cycle on the first day of implementing the modeling unit and revisited it during each session.

Each course session was divided into two parts. During the first part of the session, the participants worked on one or two modeling tasks, compared and constructed their answers, and engaged in refining their solutions. These discussions also granted us the space to introduce how different mathematical tools the participants may not have considered either independently or collectively, could be used to construct more robust models (Approximately 3 hours).

The second part of the session was devoted to deliberations on how the same tasks could be implemented in schools. The participants were introduced to specific resources they could use in designing modeling experiences and available simulations they could utilize in instruction to ground learners' activities. Participants were encouraged to identify the type skills they could address as school learners worked on the specific tasks they examined (approximately 3 hours).

Phase III: Course Outcomes

Descriptions of mathematical modeling and its process. On both surveys the participants were asked to describe mathematical modeling and what they perceived as specific processes involved in this sort of mathematical work. In the post implementation survey, 9 (82 %) participants described mathematical modeling as using mathematics to represent, analyze and solve real-world problems. Comparing these responses to those on pre-implementation survey, their remarks were more reflective of modeling as a process (Blum and Leiß, 2006).

In the post implementation survey, when asked to outline actions that may be involved in mathematical modeling process, 8 (72 %) participants noted specific cognitive actions (making sense of the situation, identifying/defining variables, making assumptions, using mathematics to build a model, interpreting the model, and revisiting the initial model). Compared to their responses to the same question on the pre-implementation survey, these descriptions more closely match stages depicted in the modeling cycle. Two of the participants associated modeling with using manipulative or simulation to demonstrate concepts.

Example generating. In the post implementation survey, when asked to illustrate the differences between modeling activities and other types of mathematical activities, phrases such as "open ended", "multiple representations", "real world connection", "multiple approaches", "a variety of directions" and "multiple entry and exit points" "minimal constraints", "opportunity to define relevant variables", and "revisiting solutions" were referenced by all 11 participants. 10 (91 %) of the participants referenced "minimal constraints", "revisiting solutions", and "define relevant variables" to describe modeling tasks, which were not present in their responses in the pre-implementation survey.

Assessing modeling progress. Compared to pre-implementation survey on which none of the participants appeared to have had a platform for gauging learners' modeling progress on the post implementation survey 7 (64 %) of them offered specific plans relying on the language of modeling cycle for identifying specific behaviors they would seek out.

Discussion

A majority of the participants in our study did perceive mathematical modeling as the process of using mathematics to solve real world based tasks and their understanding of the complexities associated with teaching it increased. The participants also believed modeling cognition to be difficult to nurture. Because of this, compared to teaching other content areas, they felt less efficacious in helping children develop proficiency in the area. Two particular challenges they articulated included how to effectively build on the learners' extra mathematical knowledge when engaging them in model building process as well as managing diverse student backgrounds. These issues have not been adequately addressed in the literature.

Analysis of the post implementation survey data revealed that although course experiences did not have any significant impact on the participants' sense of efficacy towards teaching mathematical modeling, their description of the modeling process, knowledge of task design, available resources, along with ways they could monitor student progress towards establishing more sophisticated mathematical models increased. Participants felt vulnerable towards gauging their own instructional interventions in the course of learners' modeling process. This is not surprising since developing strategic intervention skills (Blum, 2011) has been identified as a particularly complex one to acquire and one that demands time and practice to mature.

Data also indicated that the participants possessed greater control when generating modeling activities suitable for the middle grades learners. Examples of tasks they provided contained detail and tended to precision in describing specific mathematical skills that could be taught or reinforced with them. Lastly, it appeared that the course managed to provide the participants with a language through which they could articulate ideas about mathematical modeling, its form and content. Because of this we argue that while inclusion of experiences we designed appeared to have had been useful in familiarizing the participants with some key issues and methodologies, they were not by any means sufficient to have helped them reach proficiency level. Significant need exists for additional reports by scholars around different models used for assisting prospective teachers develop pedagogical capacities towards implementing modeling based curriculum and instruction.

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METHODS OF ANALYZING PRESERVICE TEACHERS' FACILITATION OF MATHEMATICS DISCUSSIONS

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This study examined how preservice elementary teachers' (PSTs') ability to facilitate productive mathematics discussions on problem-solving tasks developed during a 6-week field experience. Data were collected in 6 weekly cycles of planning (written plans), enactment (video of co-teaching sessions with pairs of children), and reflection (collaborative reflections based on video). Data were analyzed using cognitive demand of a task implementation (Stein, Smith, Henningsen, & Silver, 2000) and math-talk (Hufferd-Ackles, Fuson, & Sherin, 2004). Problematic aspects of data analysis led to the revision of the math-talk framework. I share the process used to modify an existing framework to fit the context of this study and the data analysis used to determine changes in the PSTs' ability to facilitate mathematical discussion.

Keywords: Classroom Discourse, Teacher Education-Preservice, Research Methods

Facilitating a mathematics discussion is a high leverage practice, one that novices need to be able to carry out (Ball & Bass, 2000) and that has a big payoff for student learning (Ball, Sleep, Boerst, & Bass, 2009). Yet, enacting mathematics teaching practice focused on discussion poses several challenges, especially for novices: balancing productive struggle (NCTM, 2014) with student frustration and progress on a task; knowing when and how to intervene (Ball, 1993); and requiring students explain and justify, not memorize and apply, algorithms. Nathan and Knuth (2003) found that teachers "seemed to have in their minds what they expected of their students in terms of dialogue and a solution" (p. 121) and when teacher expectations were unmet, a one-way dialogue resulted. This study focused on preservice elementary teachers' (PSTs) abilities to facilitate mathematics discussions with pairs of children during a field experience. I describe the study design and problematic issues of analysis that led to revising an existing framework.

Existing Frameworks

Several frameworks for studying discourse-based classrooms have been described, but their usefulness for investigating novices who are learning to facilitate mathematical discussions is limited. Knuth and Peressini (2001) categorize discourse as univocal (speaker delivers message with intended meaning to audience) or dialogic (multiple speakers generate meaning of a message), which borders on "stereotyping mathematics class discussions as either good or bad" (Crespo, Oslund, & Parks, 2011, p. 130). Truxaw (2004) adds to dialogic-univocal discourse with categories of inert and generative verbal assessment, but these broad characterizations still only provide snapshots of practice, not development over time. Yackel and Cobb's (1996) sociomathematical norms provide a more detailed picture of mathematics discourse. However, the specific elements of discourse they attended to (what counts as mathematically different and sophisticated solutions and what constitutes an adequate explanation) represent complex issues of practice whose negotiation may not be accessible to novices.

Hufferd-Ackles, Fuson, and Sherin (2004) developed a framework for studying math-talk that addresses some of these problematic issues. Organized hierarchically as a 4x4 matrix, it includes four levels (0-3) and four components: *questioning*, *explaining* mathematical thinking, source of mathematical *ideas*, and *responsibility* for learning. The four components allowed me to zoom in on specific teacher moves and compare them to other features of class discussion. The multi-leveled feature of the framework ranging from a strictly teacher-directed classroom to a description of dialogic discourse allows for a study of changes over time. In this study I used math-talk to examine how teacher moves to facilitate mathematical discussion influenced cognitive demand of a task implementation (Stein et al., 2000).

Methods

This study was conducted via a 6-week field experience integrated within my mathematics teaching methods course. For the field experience PSTs and methods instructors made weekly visits to an elementary school where PSTs worked with children on problem-solving tasks. PSTs were to facilitate mathematics discussions that elicited and explored children's thinking without advancing PSTs' ideas. Classroom teachers and methods instructors had purposefully structured the experience to focus on problem-solving tasks not related to the children's classroom work to relieve PSTs of the need to "cover" material and to give them the freedom to focus on practices coherent with university course work. In methods class, PSTs solved the tasks and hypothesized how students might solve them while I modeled practices for facilitating discussion.

I selected 4 groups of 3 PSTs each from my methods course based on class assignments and observations. Classroom teachers chose 5th graders who they considered to be academically average. In the first 3 weeks, PST groups enacted two high-demand tasks with a different pair of elementary children each week. They repeated this process for the second 3 weeks with two new tasks. Repeating each task three times with new children helped familiarize PSTs with children's approaches to tasks and provided opportunities to refine their responses to children's strategies. Each week was based on a cycle of planning, enactment, and reflection (Kazemi et al., 2010). At the start of each 3-week block PSTs constructed a hypothetical student-teacher dialogue to follow each of several typical (correct and incorrect) solutions for each task (modified from Crespo et al., 2011). Then, using my feedback on the dialogues, PSTs wrote a plan that listed hints and questions they would provide in several scenarios: a child who could not start the task, a child with a specific misconception, a child who was on the right track but had not solved the task, and a child who had solved the task but not connected it to any mathematics concept. I provided feedback again before task implementation. The enactment of the weekly cycle was captured on video by one PST while the other two group members co-facilitated, with recorder role rotating weekly. After each weekly session, each PST group reviewed video to compose a collective written analysis of their and their children's work and to revise their plans based on what they learned from working with children.

To analyze the reflection data I noted excerpts of thoughtful analysis and issues that recurred over several sessions. To analyze planning and video data, I intended to assign a level (memorization-0, procedures without connections-1, procedures with connections-2, and doing mathematics-3) for cognitive demand, and a level (0-3) for each of questioning, explaining, ideas, and responsibility. I hypothesized that a teaching episode could be assigned different levels for each component of math-talk (e.g., Level 2 in questioning, Level 1 in explaining, etc.) which differs from Hufferd-Ackles et al.'s (2004) idea that the four components developed together (e.g., all components should be assigned the same level). However, I encountered several difficulties with the video analysis that led to revisions I discuss in the following section.

Results: Revised Framework

The first problem encountered during data analysis was an inability to cleanly apply levels of cognitive demand and levels of components of math-talk to a single task enactment. The guideline for applying a level was to look at what more than half of the students were doing more than half the time (Stein et al., 2000),but this guideline proved too coarse a grain-size and did not capture interesting conversational turns. What might start as a *doing mathematics* implementation would decline to *memorization* when a PST used leading questions, and later the PST would pull out of a downward trend of low-level questioning when a child began to generate his or her own ideas. Even if I had been able to assign one level from each category to a task implementation as a whole the different contributions to the task enactment made by each participant in the group meant I was not able to attribute the pedagogical decisions and moves to a single individual. Therefore, I decided to parse each task enactment into segments that allowed me to cleanly apply one level for each of the five categories to the pedagogical moves of one PST. To be consistent with plans, which required PSTs to respond to no solution, a correct solution, an incorrect solution, and a partially correct solution, these events also defined the start of segments in the video data where PSTs responded to children's work. When watching the first group's full video data PSTs responded to more than just children's solutions. I identified additional segment markers: responding to an idea (a question or observation about the task) and responding to a strategy (either described by the child or as they executed but had not complete it). A segment started when a PST responded to a child's idea, strategy, solution, or lack of solution and ended when the child offered a new idea, strategy, solution, or when the PST intervened to help a child who stalled out.

The second issue was that the math-talk framework, originally developed from studying one experienced teacher's whole class discourse, did not translate smoothly to the context of two novices working with a pair of children. Within the first group's transcripts I found exemplars for each level of each category and added more specific detail to the descriptions of category levels to help me more easily identify them in my data. I revised the Level 2 descriptions to have the common thread of the teacher prompting student-to-student interactions. Because many of the Level 3 descriptions included activities the teacher "expected" and my participants' intentions and expectations were not discernible from the video data I rewrote Level 3 descriptions to focus on observable data. I then composed a general description of each level so that a feature common to all components within a level distinguished it from previous and subsequent levels. Level 0 is solely teacher directed, in Level 1 the teacher makes attempts to elicit student thinking but is unable to use that thinking to move work on the task forward, in Level 2 the teacher prompted students to talk to one another, and in Level 3 students exercise mathematical authority.

Third, periodically a segment had features of two consecutive levels or fell into a "gap" between the levels. PSTs also made pedagogical changes that improved the discussions, but that did not warrant increasing them to the next level of a math-talk category. Therefore, I concluded mid-levels were needed and searched for examples of them in the data and composed corresponding descriptions. The addition of these levels to the framework occurred in cycles, similar to the process used to develop the original framework (Hufferd-Ackles et al., 2004). When I identified several similar segments that did not fit into the 0 to 3 levels of a component in the framework I coded those segments with the appropriate mid-level, composed a description for that cell of the matrix based on the newly coded segments, and composed a general description of the mid-level. I then recoded any earlier video segments to ensure the assignment of levels for that component was consistent across the data. As I found more examples of mid-levels for a math-talk component I refined the descriptions for that cell. This process continued until I coded all segments of the first group to fit in exactly one level for each component. Using the 7-level revised framework I coded all the other groups' data.

Several types of charts were useful to determine how the PST's pedagogical moves in facilitating mathematics discussions influenced cognitive demand. Plotting in chronological order each PST's segments of all implementations of all tasks along the x-axis and the levels for cognitive demand along the y-axis, I was able to see trends in each PSTs' implementations over the course of the study. To understand how the four components of the math-talk framework were associated with the cognitive demand I plotted in chronological order each PST's segments of all three implementations of each task along the *x*-axis and the levels for each category along the *y*-axis. Plotting segments of particular types of solutions, strategies, or ideas, (none, incorrect, partially correct, correct) on the *x*-axis against the levels for each category along the *y*-axis allowed me to examine how the questioning, explaining, ideas, and responsibility for different types of segments were associated with cognitive demand.

Conclusion

These revisions were necessary to fit the context of this study, and analyzing the video data using two frameworks in the manner described was essential for allowing me to systematically examine the relationship between cognitive demand and quality of the mathematical discussion. This analysis revealed examples of how a teacher might use instances being overly directive to catapult the students' work to a higher cognitive demand and examples of how a teacher's response to student thinking could dramatically shift the cognitive demand and quality of the mathematical discussion. My revisions to the use of the frameworks supported a fine-grained analysis of discussions that highlighted pedagogical moves, key conversational turns within one discussion, and an image of how novices developed and facilitated children's work over a session of problem solving and over multiple sessions.

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NATURE OF MATHEMATICAL MODELING TASKS FOR SECONDARY MATHEMATICS PRESERVICE TEACHERS

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This study investigated the nature of written modeling tasks reported by instructors of required courses in five secondary mathematics teacher education programs. These tasks were analyzed based on a framework addressing potential cognitive orientation (simple procedures, complex procedures, and rich tasks) and purpose (epistemological, educational, contextual, and socio-critical modeling) of the tasks. Our analysis suggests that most tasks included questions of more than one cognitive orientation and more than half of the tasks were coded as contextual modeling. We also found that tasks that were coded as contextual modeling offered opportunities for future teachers to engage with questions at all levels of cognitive orientation. The nature of several modeling tasks, along with the ideas for refining the current frameworks, are presented for future implications of analyzing and developing modeling tasks.

Keywords: Modeling, Teacher Education-Preservice, Algebra and Algebraic Thinking

Connecting educational theory to practice is critical in supporting future secondary mathematics teachers to develop the skills and understanding necessary to enact effective mathematical modeling tasks. The *Common Core State Standards for Mathematics* included modeling as a mathematical practice and content standard; modeling is described as a full iterative process used to solve rich mathematical tasks (NGO & CCSSO, 2010). However, the meaning and purpose of mathematical modeling has been found to vary widely in both theory and practice (e.g., Anhalt & Cortez, 2015; Kaiser & Sriraman, 2006). To connect current theories to practice, we applied two frameworks with distinct perspectives to analyze the nature of modeling tasks. We collected tasks as part of a larger research project Preparing to Teach Algebra (PTA), which investigated opportunities secondary teacher education programs provide for future secondary teachers to learn about mathematical modeling. As we focus on the nature of these tasks, we answer the question, "*What is the nature (i.e., potential modeling purpose and cognitive orientation) of modeling tasks reported by secondary teacher preparation programs?"*

Theoretical Framework

In this secondary analysis of the larger *PTA* study, we use two frameworks proposed to characterize intended modeling purposes (Kaiser & Sriraman, 2006) and cognitive orientations (White & Mesa, 2014) of mathematical tasks. Kaiser and Sriraman (2006) conducted a historical analysis of research papers focused on mathematical modeling and, based on their findings, proposed five categories of purposes of mathematical modeling. Of their five categories, we focus on four we found relevant to our analysis: *epistemological, educational*, *contextual*, and *socio-critical* modeling. Kaiser and Sriraman described *epistemological modeling* as mathematical modeling with the purpose of developing mathematical theory. *Educational modeling* occurs when "real-world examples and their interrelations with mathematics become a central element for the structuring of teaching and learning mathematics" (p. 306). In this type, modeling is used explicitly as a tool for teaching and learning other mathematical content. *Contextual modeling* occurs when the purpose of modeling is to develop further understanding of modeling itself by engaging in the modeling process to solve a task embedded in a real-world context. *Socio-critical modeling* supports "critical thinking about the role

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

of mathematics in society" (p. 306); with this purpose, mathematical modeling is used as a tool to critically investigate and potentially change real-world situations that are relevant to students.

White and Mesa (2014) proposed a framework for differentiating potential cognitive orientations of mathematical tasks. The framework includes three main categories: *simple procedures, complex procedures,* and *rich tasks*. *Simple procedures* are defined as those tasks requiring students to draw on factual or procedural knowledge; they must "remember factual information" or "recall and apply procedures" (p. 14). Students are told which fact or procedures to use, and they must remember them and apply them in the task. *Complex procedures* include tasks requiring students to draw on procedural and conceptual knowledge: to "recognize and apply procedures" (p. 14). In these tasks, students are not told explicitly which procedures to use, but instead are expected to draw on their understanding to choose an appropriate procedure and apply it in the task. Finally, *rich tasks* include any tasks that prompt students to write explanations of procedures, to interpret, compare, or make inferences, or to analyze, evaluate, or create situations or structures. Rich tasks involve high-level mathematical thinking and offer students more opportunities to make their own decisions when solving tasks.

Method

As a part of the larger *PTA* study, we focused on the potential cognitive orientations and purposes of modeling for nine written tasks involving mathematical modeling in instructional materials collected from five universities. Because almost all tasks included multiple subquestions in which each of them varied in terms of richness, we defined our coding unit as a subquestion rather than a task. We coded subquestions in terms of potential cognitive orientation (i.e., simple, complex, rich) and purpose of modeling (i.e., epistemological, educational, contextual, socio-critical). Two researchers each coded the questions independently, and resolved all discrepancies. When we coded a question, we also considered questions prior to the one we were coding. For instance, a question was coded as rich the first time it appeared because it prompted students to analyze. But if similar questions follow in later sections, those later questions might not present new challenges to students. Answering similar questions lowers the cognitive orientation by becoming a routine procedure, thus, we coded such a question as either simple or complex rather than rich.

Results

Cognitive Orientation of Mathematical Tasks

Table 1 presents an analysis of task purposes (first column) and cognitive orientations (remaining columns). If a task is designated in one of the seven categories (e.g., R, SC) in Table 1, it means that its subquestions were coded by the cognitive orientation(s) in that category. For example, the Traffic Flow task is under the category of "Simple & Rich (SR)". This means that all questions in this task were coded as either simple or rich, with at least one in each category.

Overall, three out of nine tasks included subquestions that fell into only one cognitive orientation. For example, all questions from the task Bezier's Curve were coded as simple. One question asks to find $x(0)$ in terms of the constants a_0, a_1 and a_2 if $x(t) = a_0 + a_1t + a_2t^2$. Even though the overall task presented an interesting mathematical problem, each smaller question required using simple procedures (e.g., substituting 0 for t). Only one task (i.e., Traffic Flow) included questions that fell into exactly two cognitive orientations. Most tasks included a variety of questions, with at least one in each of the three cognitive orientations. For instance, the Wooody's Film Frame task focused on using computer animators to tell stories about changes in Wooody's position using linear transformations. One question that was coded as *simple* asked to compute the product of two matrices. Another question stated, "Wooody discovered a pensieve at (1, -2). What is the linear transformation here?" Though this question required students to analyze, students were asked similar questions immediately before, but because the question does not specify a particular procedure, we coded it as complex. At the end of the task, a question asked students to write an ending to the story of Wooody's moving and to describe how to illustrate it. This question involved creating and analyzing a new situation and was coded as rich. We found that most tasks included questions from more than one cognitive orientation and thus provided students with multiple cognitive orientations in modeling tasks.

Table 1: Task Richness According to Questions: Simple (S), Rich (R), Complex (C)

	S	C	R	SC	SR	CR	SCR
Educational Modeling							
Traffic Flow						X	
Heat Transfer		X					
Bezier's Curve	X						
Contextual Modeling							
Wooody's Film Frames							X
Google Page Rank Algorithm							X
Movie Money Making							X
Ferris Wheel Problem							X
Egg Launch Problem							X
Epistemological Modeling							
Quadratic and Its Secondary Difference			X				
Socio-critical Modeling							
(none)							

Modeling Purpose of Mathematical Tasks

Overall, we coded three tasks as educational modeling, five as contextual modeling, and one as epistemological modeling. We saw no socio-critical modeling tasks. Although we were open to coding individual questions within a task with multiple purposes (similar to our cognitive orientation analysis), we did not find any with multiple purposes. Subquestions presented in the Traffic Flow task, which we coded as educational modeling, provided detailed guideline for preservice teachers (PSTs) to solve the problem that embedded specific concepts in linear algebra (e.g., write the augmented matrix). The instructor's main purpose seemed to be for PSTs to practice linear algebra skills rather than building their own models. The Google Page Rank Algorithm task, on the other hand, involved a realistic context and provided PSTs an opportunity to explore different structures of web networks; thus we coded it as contextual modeling. Specifically, this task provided opportunities for PSTs to investigate how Google uses a stochastic matrix along with a Markov chain and its steady-state vector to determine the PageRank of each page on the website. The Quadratic and Its Secondary Difference task was coded as epistemological modeling because of the potential for PSTs to generate relationships between quadratics and second differences through modeling. We found that more than half of the tasks addressed realistic problems and sometimes instructors used modeling mainly to practice the newly learned concepts. Another intriguing point was that tasks coded as Contextual Modeling included subquestions of all types of cognitive orientations.

Discussion and Conclusion

In this study, we described the use of two theoretical frameworks addressing purposes and cognitive orientations of tasks to support the analysis and potential construction of effective modeling tasks. We noticed that tasks' subquestions coded as "rich" can entail varying degrees of richness. We saw three ways to structure a potential rich task in our data. One way was to give an open question (e.g., in the Egg Launch task, one question asks students which of the three teams will win the contest and to explain why) without providing subquestions for scaffolding. The second way is to give a series of questions with each question building on the previous ones. The open question appeared at the end of the task. We argue that even though both tasks included open questions, the first type was richer than the second because the first type required PSTs to create their own process to reach the final goal. The third way is a combination of the first two cases: starting with an open question, providing a series of questions, and restating the same open question at the end of the task. The benefit of the last type is to provide students with differentiated instruction because the instructor can provide the opportunity for students to either explore the task or follow the scaffolding questions depending on students' academic needs.

White and Mesa (2014) argued that the "rich task" category included the subcategories: Understanding, Applying Understanding, Analyzing, Evaluating, and Creating. They mapped these five categories across four types of knowledge: factual, procedural, conceptual, and meta-cognitive. In our analysis, we found that tasks in which students were asked to *create* a mathematical object or process were much richer than tasks where students were asked to simply *explain* a result or process. As we see different levels of cognitive demand within the "rich task" category, we suggest that instructors pay attention to these differences and present various opportunities for students to experience different types of rich tasks.

In terms of our analysis on purposes of modeling, we found no tasks to be socio-critical in nature and would recommend that PSTs are given opportunities to design and modify such tasks. PSTs need to encounter thinking processes, such as those described by Cirillo, Bartell, and Wager (2016) when they converted a modeling task "Dairy Queen" into a task involving social justice. PSTs can also be benefited by posing modeling problems as they lean about characteristics of modeling problems that address social justice issues (I, Jung, & Son, 2017). Opportunities for PSTs to learn about modeling can be enhanced when instructors consider different modeling purposes and cognitive orientation of tasks described in our study.

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NOTICING PRE-SERVICE TEACHERS' ATTITUDES TOWARD MATHEMATICS: COMPARING TRADITIONAL AND TECHNOLOGY-MEDIATED APPROACHES

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This paper describes the implementation of a traditional (face-to-face) and an online module aimed at developing preservice elementary teachers' (PSETs') professional noticing skills and the extent to which participation in these modules affected their attitudes toward mathematics. Using the Attitudes Toward Mathematics Inventory (ATMI, Tapia & Marsh, 2004), statistical analyses revealed significant increases in each of the instrument's four factors (value, enjoyment, motivation, and self-confidence) for those enrolled in a traditional experience, while the online participants experienced significant change in only two factors (enjoyment and self-confidence). Overall, both groups experienced significant improvements in attitudes toward mathematics with no significant differences in the changes between the two groups.

Keywords: Teacher Education-Preservice; Technology; Affect, Emotion, Beliefs, and Attitudes

Introduction

Research exploring mathematics teacher noticing has experienced exponential growth following the work of Sherin, Jacobs, and Philipp (2011). Professional noticing is an ability to recognize and act on key indicators significant to one's profession. In the area of mathematics education, such noticing typically involves the enactment of key skills aimed at facilitating an instructional environment that is responsive to students' mathematical needs and development (Jacobs, Lamb, & Philipp, 2010). Professional noticing has been examined with respect to classroom teaching practices (Sherin & van Es, 2009), preparation and planning (Santagata, 2011), and teacher knowledge (Thomas, Jong, Fisher, & Schack, in press) to identify just a few examples of the breadth of research in this area. Germane to the inquiry described in this paper is the manner in which professional noticing relates to attitudes and if the relationship varies depending upon the delivery mode of the professional noticing instruction. It has long been argued that teachers' attitudes and beliefs are an important part of the way teachers understand mathematics (Jong & Hodges, 2015; McLeod, 1994; Schoenfeld, 2011). As changes to postsecondary systems advance towards experiences that are increasingly mediated by technology, examinations of the relationships between professional noticing experiences and attitudes toward mathematics across varying instructional contexts is warranted. Towards that end, this study addresses the following research questions: (1) To what extent does participation in a professional noticing module influence preservice elementary teachers' (PSETs') attitudes toward mathematics? (2) To what extent do PSETs' attitudes toward mathematics vary in response to participation in a professional noticing module in varied learning environments (face-to-face or online)?

Methodology and Data Sources

Measure Description

PSET attitudes and beliefs toward mathematics were measured using a modified version of the Attitudes Toward Mathematics Inventory (ATMI, Tapia & Marsh, 2004, Schackow, 2005). The

ATMI was selected based on its high level of reliability (α=.97) and ease of administration. This 40-item Likert-scale inventory consists of the following four factors: value, enjoyment, self-confidence, and motivation. It was administered online for both the online and face-to-face participants.

Sites and Participants

The PSETs in this study participated in an instructional module focused on the development of professional noticing and knowledge of an early numeracy trajectory, the Stages of Early Arithmetic Learning (SEAL, Steffe, 1992). All PSETs were enrolled in an elementary mathematics methods or a content and methods blended course at one of five participating public universities in a south central state. The module was a component of the methods or blended course at each institution. PSETs either completed the online module (n = 152) as a component of a face-to-face course or a course delivered via interactive video, or they completed an in-class instructional module (n = 285) as part of their traditional instruction.

Professional Noticing Module Descriptions

Face-to-Face Module

The face-to-face module, four class sessions embedding authentic video vignettes of children engaging in mathematics, used whole-class discussions, small group discussions, homework practice, and a culminating experience in which PSETs completed a video-based diagnostic interview with an elementary student. The ATMI was administered near the beginning and end of the semester in which the module was implemented.

Online Module

The asynchronous online module consisted of four lessons that closely resembled the four days of the face-to-face module. It was embedded in a Learning Management System through which PSETs watched the same video vignettes as the face-to-face participants and attended to their observations through online survey tools. PSETs in the online setting engaged in similar activities as the face-to-face setting, however discussion was asynchronous and a variety of tools were incorporated for small group activities. The pre- and post-tests were completed prior to starting the module and at the conclusion of the modules, typically a two-week period. Students participating in the online module were not required to complete the final assignment of a video-based diagnostic interview that was part of the face-to-face course.

Results

Attitudes and Beliefs in a Face-to-Face Environment

The four factors of the ATMI (value, enjoyment, motivation, and self-confidence) as well as the total scores (sum of each) were analyzed using a repeated measures ANOVA of the pre- and post-assessments. Results from the face-to-face environment (n = 285) revealed a statistically significant increase in all four factors of the ATMI as well as the total score as shown in Table 1.

Table 1: Descriptive Statistics and Repeated Measures ANOVA from Face-To-Face Environment

Variable	Pre Mean	Post Mean	Pre SD	Post SD	F	Sig
Value	44.50	45.07	4.259	4.472	6.978	.009*
Enjoyment	34.51	35.63	9.330	8.986	16.906	<.001*
Self-Confidence	50.92	53.61	14.069	13.742	49.364	<.001*
Motivation	15.27	15.97	4.045	4.121	23.618	<.001*
Total	145.20	150.28	28.354	27.800	49.038	<.001*

*Significant at p = .05

Attitudes and Beliefs in an Online Environment

Initial analyses of the descriptive statistics of the four factors and total scores among the online participants revealed that all four factors of the ATMI, as well as the total, increased between the pre and post-assessments. Repeated measures ANOVA were also conducted (n = 152) to determine if those increases were statistically significant. The enjoyment and self-confidence factors, as well as the total score, were found to have statistically significant increases, while the changes in the value and motivation factors were not significant. The results, along with their descriptive statistics, are shown in Table 2.

Table 2: Descriptive Statistics and Repeated Measures ANOVA from Online Environment

Variable	Pre Mean	Post Mean	Pre SD	Post SD	F	Sig
Value	44.89	45.06	4.30	4.43	.312	.577
Enjoyment	35.40	36.14	8.78	9.43	4.267	.041*
Self-Confidence	50.93	52.41	14.67	14.25	7.674	.006*
Motivation	15.95	16.18	4.04	4.39	.973	.326
Total	147.18	149.8	28.09	28.84	6.949	.009*

*Significant at p = .05

Face-To-Face or Online: Does it Matter?

A one-way ANOVA was conducted to determine if the two participant groups' pre- and post-assessment total scores were significantly different. The results revealed that there was not a statistically significant difference in the pre-test total scores and the post-test total scores between the face-to-face and the online groups ($F_{pre} = .487$, $p = .486$; $F_{post} = .029$, $p = .866$). Significant differences between the online and face-to-face groups could have called into question baseline assumptions between the delivery format of the two groups. The changes (post-pre) in each factor of the ATMI as well as the change in the total scores were analyzed to determine if there were statistically significant differences between the growth of the two groups. The total ATMI score of the face-to-face group was statistically significantly higher than the total ATMI score of the online group, but none of the four factors of the ATMI score were found to be statistically different as shown in Table 3.

Table 3: One-Way ANOVA Results Comparing Mean Change Scores in Face-To-Face and Online Groups

Change Variable	Change Online	Change F2F	F	Sig
Value	.16	.57	1.227	.269
Enjoyment	.74	1.12	.679	.410
Self-Confidence	1.49	2.69	3.397	.066
Motivation	.23	.70	3.262	.072
Total	2.63	5.08	3.986	.047*

*Significant at p = .05

Conclusions

We hypothesized that participation in the professional noticing module, embedding a video-intensive design aimed to capitalize on PSETs' nurturing perspective toward children (Ambrose, 2004), would result in significant changes in PSETs' attitudes toward mathematics. All PSETs, whether participating face-to-face or online, experienced statistically significant changes in total score and two of the four factors, enjoyment and self-confidence. The lack of significance in the motivation and value factors in the online group could be a function of the limited number of items in the ATMI attributable to these factors coupled with the smaller sample size for that group. While both delivery methods revealed statistically significant increases in some factors, indicating that both delivery methods can result in positive attitudinal changes, the significant difference between the change scores of the online and face-to-face groups warrants future research in this area. The results imply that the growth of attitudinal changes in a face-to-face format is significantly higher than those changes in the online format. This can be attributed to the amount of time and intervening events between the pre and post assessments since the face-to-face participants had a longer time period between the two. Or, it could be attributed to the potential lack of depth of online conversations similar to the findings by Wallace (2003). Further study in this area would do much to support or dispel prevailing expectations regarding the design of not only professional noticing instructional materials but also the delivery mode of a mathematics methods course.

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PRE-SERVICE SECONDARY TEACHERS' MEANINGS FOR RADIANS AND DEGREES

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We report the results of a teaching experiment that examined two pre-service secondary mathematics teachers' meanings for angle measure in degrees and radians. Our results suggest that an instructional focus on degrees prior to radians might obstruct pre-service teachers' development of a quantitative way of understanding angle measure.

Keywords: Measurement, Cognition, Teacher Education-Preservice

Introduction

Moore and colleagues (Moore, 2013, Moore et al., 2016) have demonstrated the affordances of leveraging students' *quantitative reasoning* (Thompson, 1990, 2011) to support their understanding of several foundational trigonometry concepts, including angle measure. Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she might assign a numerical value to it in an appropriate unit. *Quantification* refers to the mental actions involved in conceptualizing an appropriate unit of measure as well as a measurement process, and results in an understanding of "what it means to measure a quantity, what one measures to do so, and what a measure means after getting one" (Thompson, 2011, p. 38).

A Quantitative Understanding of Angle Measure

Achieving a quantitative understanding of angle measure involves identifying an attribute of a geometric object to measure and conceptualizing a process by which to measure it. One might conceptualize quantifying the "openness" of an angle as measuring the length of the arc of a circle centered at the angle's vertex that the angle subtends. For the measure of the angle to be independent of the size of this circle, the subtended arc length must be measured in units that covary with the length of the subtended arc so that the ratio of *subtended arc length* to *unit length* is invariant for an angle with a fixed amount of openness. In other words, the unit of measure must be proportional to the subtended arc length and, by extension, the circumference of the circle that contains it. Because the size of the circle centered at the angle's vertex is immaterial to the angle's measure when the subtended arc length is measured in units proportional to the circle's circumference, measuring the length of any single subtended arc amounts to measuring them all. Conceptualizing angle measure in units of *radians* entails understanding that the angle's measure is an equivalence class of arc lengths, each measured in units of the length of the radius of the arc the angle subtends (Thompson, 2008; Moore et al., 2016). Analogously, conceptualizing angle measure in units of degrees involves understanding that the measure of an angle is an equivalence class of arc lengths, each measured in units that are $1/360^{\text{th}}$ of the circumference of the circle, centered at the vertex of the angle, containing the subtended arc.

In summary, achieving a quantitative understanding of angle measure involves: (1) identifying subtended arc length as the attribute being quantified when measuring an angle; (2) recognizing that any unit of angle measure must correspond to a magnitude proportional to the circumference of the circle containing the subtended arc; (3) realizing that radians and degrees satisfy the criterion specified in (2); and (4) understanding the measure of an angle as a measure of a class of subtended arcs.

The process by which students construct a quantitative understanding of angle measure is indeed complex. Identifying a concrete attribute (i.e., a quantity) to measure and conceptualizing units of measure whose magnitudes vary presents unique challenges that make the process of measuring angles considerably more elaborate than other measurement processes. Although researchers (e.g., Moore (2013)) have shed light on the trajectory by which students construct a quantitative understanding of angle measure, much remains to be understood about the conceptual operations involved in developing particular meanings that comprise components of a robust and coherent angle measure scheme. The present study seeks to build upon the work of Moore (2013), Moore et al. (2016), and Thompson (2008) by clarifying the implications of any of the four understandings listed above on students' development of subsequent meanings that are necessary for conceptualizing angle measure quantitatively.

Methodology

We investigated the development of two pre-service secondary mathematics teachers' meanings for angle measure in radians and degrees as they participated in a teaching experiment (Steffe & Thompson, 2000) designed to support their construction of a quantitative understanding of angle measure. In a teaching experiment, the schemes that students construct through spontaneous development are brought forth through exploratory teaching and the interest of the researcher is to discern how students reorganize their cognitive schemes as they experience specific teaching actions.

During the teaching experiment the two participants, Melissa and Kyle, were enrolled in a mathematics content course for pre-service secondary teachers at a large university in the Midwestern United States. Melissa and Kyle participated in four teaching episodes, the first and fourth individually and the second and third together. Each teaching episode lasted between 60 and 75 minutes and all episodes occurred within a span of two weeks. All teaching episodes were video recorded and selectively transcribed. Members of the research team met between teaching episodes to discuss provisional hypotheses about the development and current state of Melissa and Kyle's meanings for angle measure and to modify tasks for subsequent teaching episodes.

We employed grounded theory procedures (Corbin & Strauss, 2008) to analyze the video data. Specifically, we began by performing an iteration of open coding to identify instances in which Melissa or Kyle revealed characteristics of their meanings for angle measure. We then conducted an iteration of axial coding to construct and refine categories of episodes identified with particular codes from the initial open coding. For each category, we articulated the conceptual operations that appeared to inform Melissa and Kyle's language and actions. Finally, we identified shifts in the meanings Melissa and Kyle demonstrated and described the instructional actions that appeared to initiate these shifts.

Results

We limit our discussion of the data to the second and third teaching episodes, and primarily to Melissa's understandings of angle measure.

We began the second teaching episode by supporting Melissa and Kyle's identification of subtended arc length as the quantity one measures when assigning numerical values to the openness of an angle. We accomplished this by presenting Melissa and Kyle with a dynamic animation that showed an angle in standard position with its terminal ray rotating counter-clockwise and asking, "What are some things you notice?" After a few minutes of discussion both Melissa and Kyle abstracted the property that a particular point on the terminal ray traces out a circle centered at the vertex of the angle. They noticed that the openness of the angle covaries with the portion of the circle's circumference traced out by the point on the varying ray of the angle, which allowed them to recognize that the angle's openness is in direct correspondence with the portion of the circle's circumference that the angle subtends. This recognition prompted Melissa and Kyle to conclude that measuring the length of the subtended arc is one way of quantifying the openness of the angle.

After Melissa and Kyle had identified subtended arc length as an attribute one might measure to quantify an angle's openness, the interviewer used the dynamic animation to place the angle's terminal ray in a particular position and asked, "If we wanted to somehow assign a numerical value to the size of the angle, what are some suggestions you have for doing that?" Melissa suggested partitioning the circumference of the circle centered at the angle's vertex into eight equal pieces and then counting the number of these pieces contained in the subtended arc. She drew an image of the angle in standard position on a sheet of paper, sketched a circle centered at the angle's vertex whose circumference was split into eight equal pieces, and approximated that the terminal ray extended between the first and second tick marks on her circle. Melissa then concluded that the angle has a measure of "1.5 out of 8." The interviewer then asked, "What in this picture has a measure of 1.5?" Had Melissa conceptualized angle measure quantitatively, she would have identified a *quantity* (i.e., a measurable attribute) that has a measure of 1.5 and she would have specified a unit of measure. Specifically, she would have explained that the length of the arc the angle subtends has a measure of 1.5 in units of $1/8^{\text{th}}$ of the circle's circumference. Melissa's response, "Just the angle. … Just back to the openness; the openness that appears between the two rays" suggests she was not conceptualizing the value of 1.5 as the measure of a quantity. With prompting she subsequently elaborated: "The angle has a measure of 1.5 and everything else has a measure of a ratio of 1.5 out of 8 … the arc, the area, the angle could all be made as a ratio of 1.5 out of 8." Although Melissa acknowledged that the angle has a measure of 1.5, she did not recognize that this implied that the length of the subtended arc must also have a measure of 1.5 in units of a fractional portion of the circle's circumference. Instead, she claimed that the subtended arc (and the area of the subtended sector) have "a measure of a ratio of 1.5 out of 8."

Melissa reasoned similarly with degrees. She conceptualized an angle's measure in degrees as conveying the fraction of the circle's circumference subtended by the angle. While a useful understanding—and indeed necessary for abstracting the property that any unit of angle measure must satisfy—for her it appeared fundamentally non-quantitative since she did not describe an angle's measure in degrees as resulting from a multiplicative comparison of a measurable attribute and a unit of measure.

The interviewer introduced the idea of radians at the end of the second teaching episode. He showed Melissa and Kyle an image of an angle and stated that its measure is 2.5 radians. The interviewer then asked, "What in this picture has a measure of 2.5?" After a long pause Melissa replied, "The arc length. … That length is 2.5 times the radius." She subsequently explained that to measure an angle in radians, one must divide the length of the subtended arc by the length of the radius of the circle centered at the angle's vertex. Melissa described the value resulting from this division as "the number of radius-lengths in the arc." She also correctly drew an angle with a measure of 3.4 radians and in response to the interviewer's question, "What does it mean to say that an angle has a measure of 3.4 radians?" explained, "The arc length is 3.4 times the length of the radius." These occasions, and others, suggest that Melissa conceptualized angle measure in radians quantitatively—as a measure of subtended arc length in units of the arc's radius.

Melissa continued to demonstrate her quantitative understanding of angle measure in radians at the beginning of the third teaching episode. She accurately approximated the measure of an arbitrary angle in radians and described an angle's measure in radians as resulting from a multiplicative comparison of the subtended arc length and the length of the arc's radius. However, when the interviewer reintroduced degrees a few minutes later by prompting Melissa to describe what it means to say an angle has a measure of 51.7 degrees, she claimed that the angle subtends $51.7/360^{\text{ths}}$ of the circumference of the circle centered at the angle's vertex. After being prompted to identify the length of the arc that an angle with a measure of 51.7 degrees subtends, Melissa attempted to convert the angle's measure to radians so that she would know "how many radius-lengths are in the subtended arc." By failing to recognize that the angle subtends an arc that has a length of 51.7 measured in units of $1/360^{\text{th}}$ of the circumference of the circle centered at the angle's vertex, Melissa again demonstrated a non-quantitative way of understanding angle measure when reasoning about degrees.

Discussion

The findings of this study suggest a different learning trajectory for angle measure than that proposed by Moore (2013). Moore argues for the importance of students recognizing that an angle subtends the same fraction of all circles centered at its vertex. We observed Melissa demonstrating a non-quantitative way of understanding angle measure in degrees that differed from her quantitative understanding of angle measure in radians. We conjecture that her difficulties stemmed from a focus on a multiplicative relationship between a subtended arc length and the circumference of the circle containing it. Our results suggest that introducing radians prior to degrees might support students' construction of quantitative scheme for measuring angles in both radians and degrees.

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PRE-SERVICE TEACHER TASK DESIGN: COLLABORATIONS WITH MASTER TEACHERS

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Pre-service mathematics teachers collaborated with Master Teachers to experience rich problem solving, design of rich tasks, and authentic implementation of rich tasks in high school classrooms. The pre-service teachers made small strides in their own ability to design tasks. There were indications that the pre-service teachers became more open to open-ended tasks and student struggle but their were limitations in their ability to enact these ideas in task design.

Keywords: Teacher Education-Preservice, Teacher Knowledge, High School Education

Introduction

As teacher candidates make the journey to being teachers one of the greatest challenges is designing quality tasks that will engage students in inquiry-based mathematics and be accessible to all students. Adding to the inherent challenge of this task is the fact that typically lesson plans and tasks designed in pre-service contexts, prior to student teaching, are never taught in classrooms making it difficult for teacher candidates to develop and refine their task design. In order to address this challenge we designed the CRAFTeD cycle (Meagher, Edwards & Ozgun-Koca, 2011b), inspired, in part, by Lesson Study research, to give teacher candidates the opportunity to see tasks that they designed implemented in classrooms and, therefore, promote growth in their individual task design.

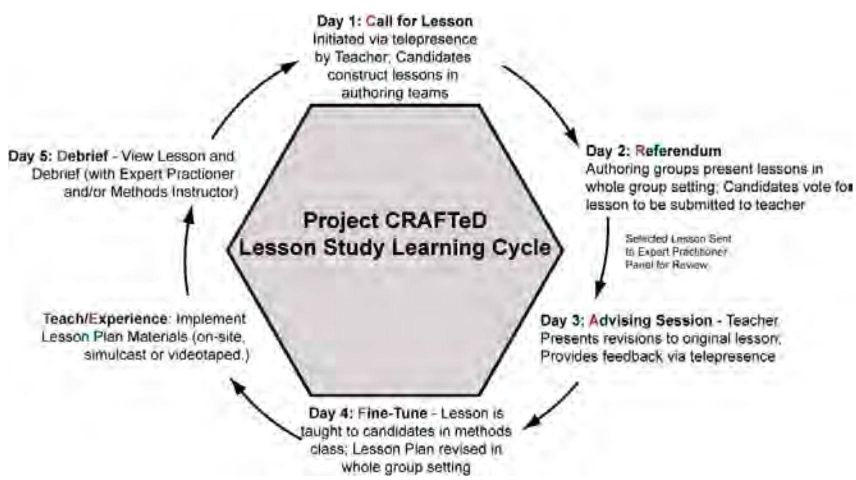

Figure 1. CRAFTeD cycle.

The cycle emerged from our previous work (Meagher, Ozgun Koca, Edwards, 2011a) where the decisive influence of the field placement in terms of exemplars became apparent.

(i) A class of preservice high school teachers will write Lesson Plans on a given topic and then work together to develop improved lessons/short units designed often for technology-rich environments; (ii) experienced inservice teachers will review the lessons/short units and present an initial redesign; (iii) the inservice teachers will teach the lessons, observed by the preservice teachers;

(iv) the preservice teachers and inservice teachers will meet together to reflect on and redesign the lesson based on their experiences in the classroom.

This paper reports the early results of a pilot project designed to measure growth in teacher candidates' ability to design tasks for mathematics classrooms that require "higher level demands" (Stein & Smith, 1998) with the intervention of CRAFTeD cycles.

Literature Review and Relationship to Research

Pre-service teachers early in their program develop lesson plans - typically in isolation - that are never taught in a classroom, creating a disconnect between planning, implementation, and assessment of student learning (Allsopp et al. 2006; Meagher, Edwards & Ozgun-Koca, 2011a). While university methods instructors laud the merits of student-led inquiry, exploration, and discovery-based teaching methods, secondary mathematics teachers in too many schools "set aside" such teaching in favor of instruction directly focused on student preparation for high-stakes, multiple choice state tests (Seeley, 2006). Developing communities of practice (Wenger, 1999) and lesson study groups (Fernandez, 2002) can help candidates and practicing teachers adopt a more research-based focus in their lesson planning and develop a shared repertoire of resources which transcend individual contributions. Providing candidates with opportunities to experience their lessons taught by Master Teachers in authentic classroom settings increases motivation for lesson writing. Preservice teachers demonstrate a trajectory of learning about lesson plans in a cycle of designing a lesson to be taught by a Master Teacher and reflecting critically on the implementation of that lesson (Meagher, Ozgun-Koca & Edwards, 2011b)

Methods and Methodologies

Participants in the study were students in the second semester of a two-semester sequence of methods classes both involving field experience and both prior to student teaching.

At the very beginning of the semester the teacher candidates, working in pairs, revised tasks that were part of the final project of the previous semester (Version 1). The teacher candidates then participated in three CRAFTeD cycles wherein they experimented with different degrees of scaffolding of the tasks. Finally, the teacher candidates, working in their original pairings, revised their tasks from the beginning of the semester (Version 2). Both versions of the tasks were independently scored by two of the research team using a rubric based on Stein & Smith (1999) as measure of growth among the teacher candidates in their ability to design tasks with the potential for higher order mathematics.

We collected (i) Versions 1 and 2 of the Teacher Candidate's lesson plans; (ii) Candidate commentaries on their lesson plans; (iii) Candidate reflections on the semester.

To analyse the data Version 1 and Version 2 tasks were graded independently by two members of the research team. The qualitative data was analysed using the constant comparative method (Glaser & Strauss, 1967) to establish trends in the data.

Results

Quantitative data

Each version of the tasks designed by the students was scored independently but two members of the research team using a rubric based on Stein and Smith (1998) with the scores averaged. Each section of the task was given a score as follows: Memorization: 1 point, Procedures without Connections (2 points), Procedures with Connections (3 points), Doing Mathematics (4 points). The scores were then averaged for the entire task to get an overall score for each task as written by the teacher candidates. The scores for each pair of teacher candidates for each Version of the task are presented in Table 1 below:

Table 1: Scores from Version 1 and Version 2 of the Tasks

	1	2	3	4	5	6	7	8	9	10	11	12	13
Version 1	1.75	1.33	2.57	2.71	1.75	1.80	2.33	2.40	2.50	1.80	1.40	1.67	1.50
Version 2	2.2	1.57	3.33	2.71	1.83	2.00	2.33	2.40	2.50	1.80	1.50	1.67	1.80

As can be seen in the table there was some improvement over the course of the semester (the class average increased from 1.96 to 2.13) with a movement toward "higher order demands" (Stein and Smith, 1998). Furthermore, the number of sections within the tasks scoring 3 or 4 improved slightly from 19 to 22. The improvements were not very large and some possible reasons are explored in the qualitative data analysis.

Qualitative Data

When teacher candidates submitted Version 2 of their tasks they also submitted a commentary on the task in general and responded specifically to a prompt on what they changed from Version 1 to Version 2 and what motivated those changes.

The following significant trends emerged in the data:

(i) A recognition of the need for student struggle with abstract and open-ended tasks. Through experiencing iterations of the CRAFTeD cycle the teacher candidates recognized the need for more open-ended tasks although that recognition did not always result in the design of quality open-ended tasks. The teacher candidates reflected on this with comments such as:

> "In my second revision of the materials, I felt it was important to put more abstract thinking in the problems rather than simple procedures . . . Getting an answer is one thing, but having an understanding of what is actually going on is entirely different, so I added understanding questions to my procedural tasks." (Teacher Candidate 1)
>
> and
>
> [In the revision] the teacher . . . will skip a lecture on the triangle inequality and instead allow the students to discover it on their own." (Teacher Candidate 3)

Other revisions to tasks included moving an open-ended portion to the beginning of the lesson and using a brief open-ended element as a launch in place of the, now widespread, "mini-lesson" of, for example the Santa Cruz New Teacher Project.

While there was a recognition of the need for more open-ended questions the successful design of such questions was not always in place with many of the questions being far too open-ended and therefore unproductive e.g. "With the students around you, see if you can come up with some real-world examples of derivatives and slopes" (Teacher Candidate 1).

(ii) An unsophisticated view of "higher level demands" and a view that differentiation means "easy" and "hard" tasks. There was evidence throughout the teacher candidates' revisions that their notion of "higher level demands" was often tied to more complicated procedural work rather than an engagement with "conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding." Teacher candidate 7 noted that he "ordered the problems on this worksheet in such a way that the least difficult and closed questions are in the beginning" (Teacher Candidate 7). Teacher candidate 9 revised an "angle chase" to involve the students chasing 10 or 12 angles instead of 2 or 3 but without adding any new concepts to the work required to identify the measures of the angles.

(iii) A recognition of the need to give students an opportunity to discuss mathematics. The successive iterations of the CRAFTeD allowed teacher candidates to see their task design in action on classrooms and, in particular, to see students engage with the materials. With this came a recognition that structuring opportunities for students to discuss mathematics is an important element of task design. Teacher candidate 13 noted "math learning is a social activity. When the team communicates using math language, student's memory of the terms learned in this lesson will be deepened. Also, students will be more engaged when they learn together." Another candidate noted that "In this final revision of my lesson plan I chose to add a section where students were asked to discuss what the algebraic properties in the chart really meant. This is something that I had previously ignored but . . . now find important" (Teacher Candidate 12).

(iv) A recognition of the potential of technology to help students engage in mathematics. The instructor of the class and the Master Teachers that were participants in the project emphasized the affordances of technology in designing higher order tasks as well as engaging students. The exemplars experienced by the teacher candidates in the CRAFTeD cycles resulted in changes in the revisions of the tasks. Teacher candidate 14 noted that "Technology seemed to hold students' attention more easily in our field experiences and engage them in an interactive way." Teacher candidate 2 revised her task to allow students to explore limits with technology "When this lesson was originally given, the class had a hard time visualizing the nature of these limits that approach infinity. With the sketch [created in Geogebra], they can explore the infinite limits with more ease" (Teacher candidate 2).

In a similar fashion to the poorly designed open-ended questions discussed in the previous section there was evidence in the teacher candidate's revised tasks that, while they were eager to deploy advanced digital technologies, the tools they provided to students were often unannotated sketches with no instructions as to how they can be used to explore the mathematical content of the task. The potential for students to aimlessly engage with the sketch and discover very little was often quite high.

Conclusion

The pilot study described above was designed to provide pre-service mathematics teachers with experiences in rich problem solving, experiences in designing rich tasks for implementation, and experiences in how those tasks can be authentically implemented in high school classrooms. The pre-service teachers make small strides in their task design but, arguably larger strides in their openness to the importance of open-ended tasks, student struggle and student collaboration in mathematical work. However, there were notable limitations in their ability to enact these ideas in task design.

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PRESERVICE ELEMENTARY TEACHERS' PERCEIVED PREPAREDNESS OF HIGH-LEVERAGE PRACTICES IN MATHEMATICS TEACHING

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This study examined elementary pre-service teachers' (PSTs') perceived preparedness of high-leverage practices (HLPs) in mathematics. Eighty-one elementary PSTs who enrolled in four sections of an elementary mathematics methods course participated in a survey that involved identifying their self-reported confidence and competence levels on HLPs. This study specifically investigated the comparison between PSTs' perceptions of HLPs and the mathematics teacher educators' expectations. Findings showed some glaring differences between the PSTs' perceptions and experts' perceptions in regards to the complexity of some HLPs. This study suggests that initial teacher training programs should include more specific investment in PSTs' insights into details of each teaching practice in mathematics by deliberate decompositions.

Keywords: Instructional Activities and Practices, Teacher Education-Preservice, Teacher Beliefs

Purpose of the Study

The recent recognition of the significant work teachers actually do in classrooms suggests teacher preparation programs offer pre-service teachers (PSTs) more explicit opportunities to be engaged in key teaching practices (Ball & Forzani, 2009; Grossman et al., 2009; NCTM, 2014). An example of these efforts is the establishment of a set of "high-leverage practices" (HLPs), which are considered to be the basic fundamentals of teaching practice (Ball, Sleep, Boerst & Bass, 2009; Davis & Boerst, 2014). This is a critical shift in teacher preparation programs, so it is essential to assess PSTs' current understanding of HLPs as well as to develop activities which promote PSTs' use of HLPs. To do so, teacher educators should first understand how PSTs perceive HLPs in mathematics teaching and learning; however, research regarding PSTs perceptions of HLPs is limited. In response to the need for investigating PSTs' own perceptions, this study intended to accomplish two specific objectives: (a) Identifying PSTs' perceived preparedness for specific HLPs in mathematics and (b) Comparing between PSTs' perceived preparedness and the experts' (mathematics teacher educators') expected learning progressions. Ultimately, this study aimed to gather information on PSTs' conceptualizations of HLPs and provide insight into how to best support the development of practice-based teacher preparation programs.

Theoretical Framework

High-leverage Teaching Practices

There have been continuous efforts towards developing a common set of indicators for the disposition, knowledge, and skill that are required for beginning teachers; however, it is still challenging to define what should be taught in teacher preparation programs across many institutions (Levine, 2006). One of the recent movements in teacher education is focusing on a set of HLPs that support high-quality student learning. By seeing the work of teaching as an 'unnatural' act that should be taught, this view highlights the importance of doing and practicing teaching, rather than simply 'teaching about teaching' (Ball et al., 2009; Grossman & McDonald, 2008). This involves decomposing complex teaching practices into small, teachable HLPs in order for novice teachers to access various components of the work of teaching (Ball & Forzani, 2009; Grossman et al, 2009). This should be done by attending to developing teachers' adaptive expertise to be sure that it does not serve to de-professionalize teaching (Ball & Forzani, 2009; Hamerness, Darling-Hammond, & Bransford, 2005).

Research on Pre-service Teachers' Perceptions of Teaching Practices

Research on PSTs' perceptions of good teaching practices shows mixed results for various aspects of teaching. Some researchers address PSTs who enter teacher education programs with a positive, but generally simplistic view of teaching (e.g., Whitebeck, 2000). Many of them also enter the program with high confidence in terms of their ability to perform well, which may be a display of "unrealistic optimism" (Weinstein, 1988). Some PSTs believe that teaching is easy and that it is about transmitting information (Feiman-Nemser, McDiarmid, Melnick, & Parker, 1989). In contrast, other studies report that PSTs consider a 'teaching personality' more important than content or pedagogical knowledge. This view is in line with the popular myth that some people are 'born teachers.' Whitebeck (2000) suggests that some PSTs enter teacher preparation programs to learn the 'tricks of the trade' and others believe that they are 'born teachers.'

Overall, previous research on PSTs' views of good teaching was very focused on the entry-level PSTs and examined the beliefs, expectations, and perceptions that they bring to the teacher preparation program. It is expected that this study will extend previous research in this area by exploring the perceived self-efficacy of PSTs on specific HLPs and offering suggestions for mathematics teacher educators on how this baseline data can be incorporated when designing PSTs' experiences in mathematics teacher education programs.

Methods

Eighty-one elementary PSTs who enrolled in four sections of a mathematics methods course at a Midwestern university in the United States participated in the study. For most participants, this was one or two semesters prior to the culminating, semester-long, full-time student teaching experience in an actual classroom. At the beginning of the semester, PSTs had a chance to review the descriptions of each HLP (from teachingworks.org). They were asked to select five HLPs they felt they could perform confidently and competently at the time of responding. PSTs completed this listing activity individually followed by a subsequent group debriefing session. For the purpose of this report, we are focusing on reporting the quantitative analysis of PSTs' perceived preparedness as reflected in their lists of HLPs. A pair of mathematics teacher educators (experts) provided their expected learning progressions in teacher preparation programs in order to compare expert judgment to the information coming from the PSTs in this study.

Results

This section reports on three distinct clusters of HLPs from the results of the larger study by highlighting aspects that contrast PSTs' perceived level of preparedness and the researchers' expected learning progressions.

Complexity of Interactive Structure

This study clustered four HLPs together to view PSTs' perceived level of preparedness in facilitating varied interactions with students as well as creating a classroom climate that promotes such interactions: Leading a Group Discussion (LGD), Setting up and Managing Small Group Work (SGW), and Eliciting and Interpreting Individual Students' Thinking (EIIS). Experts' expected learning progression was having PSTs gradually exposed to increasingly complex and challenging situations, starting from working with individual students and small groups to the whole group discussion. However, unlike the experts' expected progressions, PSTs responded that they felt more prepared in LGD and SGW than EIIS (see Table 1).

Table 1: Perceived Preparedness on Complexity of Interactive Structure (N=81)

	Listed as 5 most confident HLPs	Listed as 5 least confident HLPs
LGD	44 (54%)	12 (15%)
SGW	47 (58%)	7 (9%)
EIIS	12 (15%)	27 (33%)

Subject-Matter Knowledge

Another cluster consisted of two HLPs that rely on teachers' subject-matter knowledge and their ability to identify common patterns of student thinking in subject knowledge. This cluster includes Explaining and Modeling Content, Practices, and Strategies (EM) and Diagnosing Particular Common Patterns of Student Thinking and Development in a Subject-matter Domain (DST). Although experts believe these two HLPs should go hand-in-hand, diagnosing students' patterns of thinking in mathematics can provide an effective way of explaining and modeling. PSTs' responses showed different perceptions (see Table 2).

Table 2: Perceived Preparedness on Identifying and Using Subject-Matter Knowledge (N=81)

	Listed as 5 most confident HLPs	Listed as 5 least confident HLPs
EM	18 (22%)	18 (22%)
DST	6 (7%)	44 (54%)

Knowledge of and Relationship with Students

Experts examined several HLPs that rely on teachers' knowledge of and relationship with students: Building Respectful Relationships with Students (BRR), Talking about a Student with Parents or Other Caregivers (TSP), and Learning about Students' Cultural, Religious, Family, Intellectual, and Personal Experiences and Resources for Use in Instruction (LS). Experts expected that LS would provide a good basis for accomplishing BRR and TSP. Table 3 shows PSTs' predominant perceived confidence in BRR. It was the most frequently selected HLP as a confident practice among 19 HLPs.

Table 3: Perceived Preparedness on Knowledge of and Relationship with Students (N=81)

	Listed as 5 most confident HLPs	Listed as 5 least confident HLPs
BRR	66 (81%)	2 (2%)
TSP	23 (28%)	36 (44%)
LS	22 (27%)	19 (23%)

Discussion

These results are based on PSTs' perceived preparedness on HLPs, which do not necessarily represent their actual competencies on HLPs. Regardless of this limitation; however, the data provided valuable information on discrepancies between mathematics teacher educators' and PSTs' perceptions on HLPs. The data also indicated the need for ways to best to support PSTs' learning in teacher preparation programs.

Teaching is complex, and the 19 HLPs provide a new perspective on teacher preparation by decomposing the complex work of teaching (Ball & Forzani, 2009; Grossman, Compton, et al., 2009). However, this study indicates that ambiguity of understanding still exists among PSTs on the meaning and constructs associated with each HLP. For example, the work of leading a group discussion requires multiple practices such as eliciting student thinking, probing, orchestrating, and making contributions (Selling et al., 2015). This contrasts PSTs' high level of confidence in leading a group discussion and low level of confidence in eliciting and interpreting individual students' thinking. This result leaves a question about PSTs' conceptions of "leading discussion," which may rely significantly on their unexamined assumptions. This suggests that there is a need for more deliberate and detailed decomposition of each teaching practice. The practices with a low level of preparedness perceived by PSTs (e.g., Implementing norms and routines for classroom discourse and work, Selecting and designing formal assessments of student learning) also have important implications for PST training. Further research is required to determine the decomposed parts of each HLP and ways to sequence them to establish contextually relevant and responsive teacher preparation programs.

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PRESERVICE TEACHERS' DEVELOPMENT OF KNOWLEDGE OF AUTHENTIC ASSESSMENT MATHEMATICS TASKS

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Mathematics teachers are expected to adopt new approaches to assessment that better address reform goals in school mathematics curriculum. This study is investigating an approach consist-ing of Authentic Assessment Learning Activities [AALA], based on an "authentic intellectual quality framework," to help preservice mathematics teachers to develop expertise in the selection, adaptation, and design of authentic assessment tasks. The approach was piloted with three preservice teachers specializing in mathematics education. Data sources included inter-views, journals, and tasks created by participants. Findings provided initial evidence to support the use of the AALA in teacher education as a potentially effective way of helping preservice teachers to not only develop useful knowledge of authentic assessment, but also mathematics knowledge for teaching associated with the design of authentic assessment tasks in mathematics.

Keywords: Assessment and Evaluation, Teacher Education-Preservice

Mathematics curriculum reforms aimed at mathematical literacy and 21st century competencies (Darling-Hammond & Adamson, 2010; Koh, 2014; Wyatt-Smith & Cumming, 2009), such as critical thinking, reasoning, problem solving, communication, and collaboration, require corresponding reforms in how students' learning and performances are assessed in the mathematics classroom (e.g., National Council of Teachers of Mathematics [NCTM], 2000; Romberg, 1995). This means that teachers implementing such reforms must make changes to their assessment practices to include authentic assessments that enable richer demonstration and more holistic representation of what students know and can perform in mathematical and real-world contexts. However, research indicates that teachers are not well equipped to deal with reform-oriented assessment practices. For example, studies found that a large majority of pre-service teachers in Canada and the United States had low levels of assessment literacy (DeLuca & Klinger, 2010; Mertler, 2009; Volante & Fazio, 2007). Thus, there is a need to support mathematics teachers' development of their assessment literacy in designing and implementing classroom assessments that are well aligned with the objectives of reform-oriented mathematics curriculum. This study is investigating the use of the Authentic Assessment Learning Activities (AALA) to develop preservice mathematics teachers' expertise in the selection, adaptation, and design of authentic assessment mathematics tasks. Authentic tasks replicate real-world problems that require students to demonstrate their mathematical understanding through the application of essential mathematical knowledge and skills in problem solving (Koh, 2014).

Theoretical Framework

The AALA used in this study aim at the development of teachers' understanding of sound principles of classroom assessment and expertise in the selection, adaptation, and design of authentic assessment tasks in mathematics. The AALA include *criteria for authentic intellectual quality* (Koh, 2011a; 2011b), design principles of authentic assessment and associated rubrics, the Structure of Observed Learning Outcome (SOLO) taxonomy (Biggs, & Collis, 1982), and curriculum alignment. The *criteria of authentic intellectual quality,* developed by Koh, consist of the following five criteria and their respective elements: (1) depth of knowledge (factual knowledge; procedural knowledge; advanced concepts), (2) knowledge criticism (presentation of knowledge as a given; comparing and contrasting information; critiquing information), (3) knowledge manipulation (reproduction; organization, interpretation, analysis, evaluation, synthesis of information; application/problem solving; generation/ construction of new knowledge), (4) extended communication, and (5) making connections to the real world beyond the classroom. The mathematics indicators for each of the elements are detailed in Koh (2011b).

The criteria of authentic intellectual quality were adapted from Newmann and Associates' (1996) framework of authentic intellectual work, the revised Bloom's taxonomy of knowledge (Anderson & Krathwohl, 2001), and the different dimensions of learning by Marzano (1992). Newmann et al.'s (1996) "authentic intellectual work" consists of three broad criteria: construction of knowledge, disciplined inquiry, and value beyond the school. Specific standards are embedded within each of the criteria, which "provide a benchmark for teachers to judge whether particular forms of instruction and assessment are likely to help students produce authentic work" (Scheurman & Newmann, 1998, p. 3). The revised Bloom's taxonomy was used to further unpack and define the construction of knowledge and disciplined inquiry criteria.

Method

The AALA are being investigated in a larger funded project with preservice teachers (PSTs). They were successfully used with practicing elementary school teachers in Singapore (Koh, 2011a, 2014) and are now being adapted for use with PSTs. This paper reports on the pilot study with three PSTs specializing in mathematics education; one at the elementary and two at the secondary school levels. The secondary PSTs were in the fourth term of their two-year Bachelor of Education program and had completed an assessment course, while the elementary PST was in the second term of the same program and had not completed the assessment course. All three participated in the AALA during four two-hour sessions led by the researchers at the university during the summer and for homework. They engaged in readings, discussions, and applications of the AALA. Based on the AALA, they individually analyzed actual assessment tasks used in a *unit of work* for a mathematics topic and Grade of their choice and then designed the assessment tasks for a different topic and Grade using the criteria of authentic intellectual quality.

Data sources for the pilot consisted of group interviews with the three participants at the beginning and end of their engagement with the AALA; participants' journals of their thinking and analysis of authentic assessment tasks; researchers' notes and audio-recordings of selected participants' discussions during the sessions with the AALA; and the assessment tasks analyzed and designed by the participants. The interviews explored the PSTs' thinking regarding: their conceptions of the nature and purpose of authentic assessment in mathematics and of the expertise needed in authentic assessment, their learning experiences with the AALA, the authentic assessment task design, their use of the criteria for authentic intellectual quality, the impact of the AALA on their mathematics knowledge for teaching, the use of authentic assessment to promote students' learning and thinking of mathematics, and their plans for using the authentic mathematics tasks they designed in their future teaching. The interviews were audio recorded. All audio-recordings were transcribed. Data analysis included a focus on identifying their conceptions of authentic assessment, what they valued and what was challenging in working with the AALA, and their level of understanding of designing authentic assessment tasks. The data were coded to identify themes in their thinking and experiences with the AALA. For example, coding of interviews to identify their conceptions of authentic assessment was both open-ended and based on the authentic assessment framework. Change/growth in conceptions of authentic assessment was determined by comparison of the pre-and post-AALA engagement coded data. The assessment tasks they developed were scored using the criteria for authentic intellectual quality with 4-point rating scales (ranging from 1 = no requirement/no demonstration to 4 = high requirement/high level) to determine their level of understanding.

Results

Findings of this pilot study indicated that PSTs' engagement in the AALA was effective in helping them to enhance their understanding of authentic assessment tasks in teaching mathematics. Before engaging in the AALA, there was little difference between the two participants who had completed the assessment course in their teacher education program and the one who had not regarding their conceptions of authentic assessment tasks. While the course enabled them to develop initial understanding of forms of assessment *of* and *for* learning, it did not allow them to conceptualize authentic assessment, in general, and authentic assessment tasks in mathematics, in particular, in ways that were meaningful to assess what students know and can perform in mathematical and real-world contexts. They indicated that the AALA were useful in providing them with a systematic way of making sense of selecting, unpacking, adapting, and designing authentic tasks for assessment and helped them to understand what it means to be authentic regarding tasks and process to assess learning in mathematics.

The criteria of authentic intellectual quality, which the PSTs used to guide their analysis of the assessment tasks for the *unit of work* they chose (e.g., slope of a linear function, perimeters of polygons, rates of change), challenged their thinking in evaluating the level to which the tasks required deep understanding and promoted knowledge criticism, higher-order thinking, reasoning skills, and connections to the real world beyond the classroom. But this was central in helping them to understand the strengths and limitations of the tasks they selected from their practicum experiences and to make meaningful suggestions to modify them. The AALA also contributed to their development of mathematics knowledge for teaching as they engaged in identifying the instructional objectives and conceptualizing the authentic tasks. The criteria of authentic intellectual quality challenged their understanding of mathematics concepts involved in the assessment tasks and made them think about them in alternative ways as well as think about alternative ways of engaging students in learning mathematics. As one of the PSTs explained: "I think it's really interesting that our perception of how well we understand a concept is certainly pushed and tested when you're trying to develop a task." Another noted:

> I found it challenging to narrow down the task and imagine what expectations I had for the final project. … As students begin their work on the task, I would like to have exemplars of work for students to gain a better understanding of the expectations.

The final assessment tasks the participants created for *units of work* of their choice (e.g., algebraic expressions (grade 7) and linear relations and functions (grade 10)) indicated a developing practical level of understanding of authentic assessment tasks. Based on the SOLO taxonomy, the tasks covered the assessment of different instructional objectives ranging from a basic understanding of concepts to complex problem-solving. The Grade 10 tasks, for example, consisted of a project designed for students to apply their mathematical concepts and thinking to solve a real-world problem. These tasks included the following expectations:

1. Students will pair up and flashcards with different figures will be distributed (i.e., labeled graphs, images of everyday objects (e.g., basket of fruit, ski hill, thermometer, etc.), table of values, bank statements, and equations. Students will discuss in their groups: *What relationships exist within the photo(s)? How could you represent these relationships?*
2. Data Collection & Explanation. Students will devise a way to relate and graph their collected data. They should be able to interpret and explain the relationships among their data, graphs,

and situations in small collaborative groups. They also should be able to state a reasonable domain and range, and explain any restrictions they set.

3. Data Interpretation & Manipulation. In small groups, students will analyze their graphs and collected data and identify if their relation is linear. They should be able to further explain how they know, orally and through a written piece, which they will keep for their final submission. … At this stage, students should be able to confirm if their relation is a function, through both peer and self-assessment.

The main concern about the AALA was the need for more time and practice with analyzing and creating tasks to allow for deeper engagement with the AALA and group discussions.

Conclusion

The study suggests that the AALA have the potential to help PSTs to develop useful know-ledge of authentic assessment mathematics task design. They also have the potential to support PSTs' development of mathematical content and pedagogical knowledge for teaching associated with the critique, redesign, and design of the tasks. Continued investigation of the AALA in the larger project will involve a larger number of PSTs and aim at deeper understanding of how these activities could support their learning of both authentic assessment mathematics tasks and mathematics knowledge for teaching. They will also be tracked into their teaching as beginning teachers to investigate whether or how they can implement their knowledge in their teaching.

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PRESERVICE TEACHERS' PERSPECTIVES ON AND PREPARATION FOR TEACHING MATHEMATICS EQUITABLY

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This study investigated preservice teachers' perspectives on and preparation for teaching mathematics equitably. Overall, participants indicated that student demographics should influence the way they teach, but this varied by teacher and student group.

Keywords: Teacher Education-Preservice, Elementary School Education, Middle School Education, Equity and Diversity

Purpose of the Study

This study investigated preservice teachers' perspectives on and preparation for teaching mathematics equitably. The research questions were: 1) What attitudes and beliefs do preservice teachers report in relation to teaching mathematics equitably to all students? 2) What preparation do preservice teachers report in relation to teaching mathematics equitably to all students? What preservice teacher preparation do they recommend? 3) What key instructional approaches do preservice teachers suggest for teaching mathematics equitably to all students?

Perspectives

The science, technology, engineering, and mathematics (STEM) disciplines play a key role in societal progress and economic prosperity and possess high status and career promise (Shapiro, Grossman, Carter, Martin, Deyton, & Hammer, 2015). However, some groups are underrepresented in STEM. For example, females tend to show less favorable dispositions toward STEM, which can influence their achievement and participation (Shapiro et al., 2015). Students from underrepresented racial/ethnic groups attain lower achievement test scores on standardized mathematics tests (Snyder & Dillow, 2015), and a sizeable performance gap exists between students who are English language learners (ELLs) and those whose first language is English (Institute of Education Sciences, 2016). Students from lower-income families show weaker self-efficacy, lower school performance, and higher school drop-out rates than students from higher-income families, and the achievement gap between students from lower- and higher-income families has widened in recent decades (e.g., OECD, 2013).

Teachers need to understand the relationship of these types of student background variables to mathematics teaching and learning in order to support all students' needs (Civil, 2014). Teacher education can play an important role in facilitating more favorable teacher dispositions and practices in relation to serving diverse students (Hwang & Evans, 2011).

Methods

Participants

Participants were 203 upper-division undergraduates. Of these, 77.8% were ages 18-25, and 22.2% were ages 25 and older. These prospective teachers were enrolled in elementary education (44.0%), dual elementary/special education or special education (40.5 %), and secondary mathematics education (15.5 %).

Instrumentation and Data Collection

Participants completed an author-constructed survey titled Preservice Teachers' Perspectives on and Preparation for Mathematics Instruction for Diverse Learners, which has nine closed-format items rated on a Likert scale with space to explain ratings and two open-ended questions.

Data Analysis

Quantitative data were analyzed as a whole and disaggregated by age and program of study. The Kruskal-Wallis or Mann-Whitney *U* test was used to test for significant differences between participant subgroups. Written comments were categorized into conceptual categories, or themes.

Results

Beliefs About Planning and Delivering Mathematics Instruction to Diverse Students

Participants tended to agree that student demographics should influence how they teach mathematics. They most strongly agreed that students for whom English is a second language should influence their mathematics instruction and tended to agree that student socioeconomic status should also be considered. Participant responses were fairly equally divided in regard to the role of student race/ethnicity, and participants tended to disagree that gender should be a factor in planning and carrying out mathematics instruction. Participants tended to agree that it is challenging to design and implement fair mathematics instruction according to student race/ethnicity and language and least challenging according to gender. The older age group was more likely to agree that student socioeconomic status should be considered and is challenging in planning and carrying out mathematics instruction. Five dominant themes appeared across participants' written comments, each of which follows with a sample comment:

Students' learning differences. "Students that come from different backgrounds have different ways of learning. Teachers should incorporate these differences into their lesson plans."

Varied teaching methods: "I think that capable teachers are flexible and able to use various methodologies to meet the needs of the array of students they encounter."

Real-world connections: "The way I intend to teach mathematics is to link with the real-world that will benefit all students regardless of demographics."

Students' prior knowledge: "The ideas that students bring into the classroom should help you understand them and teach them better. Gender and race/ethnicity, in some ways, do influence a student's background knowledge."

Resources and supplies: "This is very important in assigning projects or assignments. Teachers have to know what resources are going to be available for all the students."

Preparation for Teaching Mathematics Equitably

Participants tended to agree that they are prepared to support the mathematics learning needs of all students, mainly in relation to gender and socioeconomic status, but they expressed somewhat weak preparation in relation to race/ethnicity and home language. They tended to agree that their strongest preparation for teaching mathematics equitably came from their college coursework. Dual elementary/special education majors were significantly more likely than elementary and secondary majors to report that they can support the mathematics learning needs of all students, especially non-native English speakers.

Written comments show the following themes:

Mathematics as a "weakness." Some participants indicated that they are not well prepared to teach mathematics to diverse students because they consider the subject area their "weakness."

Teaching diverse learners. Some participants reported having a good grasp of mathematics content but not knowing how to teach diverse students. In this regard, one said, "Teaching mathematics is harder than [teaching] other subjects." Some expressed unpreparedness, but others showed confidence that they could find the tools and resources they need.

Teaching non-native English speakers. Participants deem themselves ill prepared to support non-native English speakers in mathematics instruction, explaining, "I believe that non-native English speakers is the only factor that requires different and well prepared instruction."

College coursework, practicum, and personal experiences. Most students said their strongest preparation for teaching mathematics equitably came from college coursework, although some named field experiences and a still lesser number personal work experiences.

Cultural awareness and learning styles. Most participants had limited experience interacting with members of other cultures. They suggested that they be guided in ways to examine the curriculum for cultural inclusiveness and exposed to ways to learn students' background and prior experiences in order to adapt their instructional approaches.

Experience in diverse classrooms. Several participants noted that the program did not provide opportunities to teach mathematics to diverse learners during field experiences.

Hands-on assignments and projects. Participants asserted that they might be prepared to teach mathematics fairly to diverse students if they were taught instructional strategies to draw on real-life situations and use manipulatives and other hands-on materials.

Resources and resourcefulness. Participants identified other people (e.g., special education teachers, aides, counselors, and other teachers) as the most important resource available to them. They also requested assistance locating instructional materials and resources for diverse students.

Perspectives on Appropriate Instructional Skills and Needed Preparation

Participants suggested six main approaches for teaching mathematics equitably to all students:

Group work. Participants stated that when students work collaboratively they are able to tackle and persevere on more conceptually difficult problems.

Mathematical connections. Participants commented that connections between mathematics and other subject areas, one's personal interests, and so forth should be emphasized.

Influential mathematicians and role models. A few participants emphasized the importance of introducing role models from underrepresented groups (e.g., females).

Different teaching strategies. Participants reported that using a variety of modes of instruction can benefit diverse students.

Language support. For example, "We must teach mathematics concepts and vocabulary."

Varied challenges for different groups. Participants voiced most concern about designing mathematics instruction for non-native English speakers and least about gender, such as: "Gender and socioeconomic status shouldn't matter when you teach math, but it might be hard to teach kids that don't speak English and that are from different cultures."

Discussion and Conclusions

This study shows that preservice teachers tend to consider themselves prepared to teach mathematics to diverse students and find this goal important. However, they were least concerned about gender and most concerned about students whose first language is not English. The lack of concern about gender is problematic, given continued issues for females in STEM (Wiest, 2011). Participants' concern about teaching ELLs is worrisome with respect to the need for competence and a sense of self-efficacy in relation to this dimension of teaching. However, it is encouraging that preservice teachers appear to be sensitized to its importance.

The participants in this study consider teacher preparation coursework and field experiences to be important in their preparation to teach mathematics to diverse students. However, they want more field experience and more resources for taking on and continuing this preparation. Some participants reported their own mathematics content weaknesses to be a concern, which is important for teacher education to address (Hourigan & O'Donoghue, 2015).

Older teachers were significantly more likely to consider students' socioeconomic status to be important to consider and to be a challenge in planning and implementing equitable instruction. Perhaps this is due to their extended real-world experience, which might lead to greater recognition of the role of this factor in people's lives. Secondary preservice teachers were also more likely to say they should consider student SES in their instructional efforts. Perhaps SES differences are more obvious in older youth, but more attention to the importance of SES in education appears to be needed. Dual elementary/special education majors were significantly more likely than elementary and secondary majors to consider themselves prepared to support the mathematics learning needs of diverse students, especially non-native English speakers. Participation in a program that includes attention to a student population with specific education needs might raise their sense of self-efficacy for teaching diverse students in general.

Findings such as those reported in this paper can help teacher educators determine how to enhance programs that prepare preservice teachers to teach mathematics effectively to diverse students.

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PROSPECTIVE ELEMENTARY TEACHERS' KNOWLEDGE OF MULTIPLICATIVE STRUCTURE THROUGH CLINICAL INTERVIEWS

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Little is currently known about how students and teachers make sense of multiplicative structure in the context of elementary number theory. This study uses APOS theory to investigate six prospective elementary teachers' developing understanding of multiplicative structure through clinical interviews prior to and following a three-week unit of instruction on number theory. Results reveal the ways in which participants developed more coordinated understandings of multiplicative structure, and suggest benefits for attending to prime factorization in content courses for prospective elementary teachers.

Keywords: Teacher Education-Preservice, Number Concepts and Operations

Background Information

Multiplicative structure is defined by Zazkis and Campbell (1996a) as "conceptual attributes and relations pertaining to and implied by the decomposition of natural numbers as unique products of prime factors" (p. 541). Research has shown that a richer understanding of multiplicative structure can facilitate students' transition from arithmetic to algebra, deepen their understanding of divisibility, and support their work with fractions, decimals, and rational numbers (Campbell, 2006; Brown, Thomas, & Tolias, 2002).

In order for K-8 students to make sense of multiplicative structure, PTs must first develop a deeper understanding of its meaning and applications. However, PTs' knowledge of multiplicative structure reveals an overreliance on computation-heavy procedures (Zazkis & Campbell, 1996a). For example, when asked to determine whether $M = 3^3 \times 5^2 \times 7$ is divisible by 7, many PTs ignored the structure of M, and instead multiplied the prime factors together and carried out long division.

This paper describes the results of clinical interviews were administered as part of a larger study (Author, 2012) examining PTs' understanding of multiplicative structure in the context of five number theory topics: factors, prime factorization, divisibility, greatest common factors (GCF), and least common multiples (LCM). Overall, 59 PTs enrolled in three sections of a mathematics content course at a private university in the northeastern U.S. completed five number theory lessons over a three-week period. Written assessments were administered to all participants prior to and immediately following the three-week instructional unit.

Theoretical Framework

Dubinsky's (1991) Action-Process-Object-Schema (APOS) theory served as the foundation for the 2012 study described above. APOS theory is a constructivist theory of learning that stipulates that individuals construct their own understanding by interacting with their environment. Developing a deeper understanding of a mathematical concept involves constructing and organizing more abstract and connected mental representations. As the individual makes these constructions, he reaches increasingly sophisticated levels of understanding. APOS theory identifies these levels as *actions*, *processes*, *objects*, and *schema*.

Actions are repeatable, algorithmic procedures. An individual who possesses an action-level understanding of a mathematical concept requires step-by-step instructions on how to perform the action. When an individual repeats an action enough times that he can construct an internal representation for that action, the individual can often execute the action mentally without actually performing the individual steps. The action is said to have been interiorized into a *process* (Dubinsky & McDonald, 2001). An individual encapsulates a process into an *object* when he begins to see the process as a single entity consisting of a static structure; the underlying concept exists independent of its associated process. A *schema*, or structured mental network, is formed when processes and objects become connected. Through analysis of the data (as described below), it became necessary to subdivide the *process* code into two sub-codes: *process without coordination* and *process with coordination*. The former reflects an inability to combine, explicitly or implicitly, two or more concepts to make sense of a problem situation; the latter reflects one's ability to make such combinations.

Individual Clinical Interviews

This paper presents the results of six individual clinical interviews administered prior to and immediately following the three-week number theory unit. Six female participants were selected from the larger sample to participate in one hour-long individual clinical pre- and post-interviews. They were selected by random sampling, stratified by prior mathematical achievement levels based on their pre-test scores and recommendations from their math content course instructors.

The purpose of each interview was to identify participants' in-the-moment thinking around multiplicative structure. Using a think-aloud protocol (Patton, 2002), the interviewer asked participants to describe their reasoning out loud as they solved each problem. Both pre- and post-interviews consisted of thirteen math questions assessing participants' understanding of the five number theory topics. Questions were identical across pre- and post-interviews except that the numerical values were changed.

In order to describe participants' understanding of multiplicative structure, interview transcripts were coded using APOS theory's levels of understanding (action=1; process without coordination=2; process with coordination=3; object=4; schema=5). Three phases of coding were undertaken. During phase one, pre-interview data were analyzed by participant and by question. Phase two repeated phase one for all post-interview transcripts. During phase three, participants' pre-interview and post-interview transcripts were compared in order to identify differences in their exhibited levels of understanding.

Findings

Analysis of the six interview participants revealed that all displayed improved understandings, to varying degrees, of topics related to multiplicative structure following number theory instruction. Due to space limitations, only portions of the results of each participant's interviews will be discussed.

Participant #1: Ellen

Compared to her peers in the sample, Ellen was considered a high achieving mathematics student based on her course instructor's recommendation and her performance on the pre-test (57% vs. sample mean of 30%). Her pre-test score was the highest score in the study's sample. During her pre-interview, Ellen showed a tendency to work with procedures (*action* and *process* levels) because she struggled to coordinate prime factors. For example, when asked to determine if $N=2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$ was divisible by 51, Ellen could not coordinate divisibility by 3 and 17 to recognize divisibility by 51.

During her post-interview, however, Ellen revealed a greater ability to make connections across topics (*schema* level) and view concepts flexibly (*object* level):

> If the number can be broken down again into prime factors, I would do that first and then look to see if all of the prime factors needed for the number it's asking for were listed in the number N,

because if the prime factors of – all of the prime factors of the number they're looking for are inside N, then it must be divisible by that number.

Participant #2: Margaret

Based on her course instructor's recommendation and on her initial pre-test score (42%), Margaret was considered a high achieving mathematics student relative to her peers in the sample. During the pre-interview, she worked exclusively at the action and process without coordination levels.

During the post-interview, her understanding fluctuated between action and object levels. While she displayed a deep understanding of the relationship between factors and prime factors on some problems, when a problem had no obvious solution strategy, Margaret reverted back to lower levels of understanding. For example, when asked to identify the factors of $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$ during the post-interview, Margaret could not identify any composite factors that were not already visible in the factorizations.

Participant #3: Amanda

For the purposes of this study, Amanda was considered an average achieving mathematics student based on her course instructor's recommendation and pre-test score (26%). Prior to instruction, Amanda showed evidence of a limited and inconsistent understanding of multiplicative structure. She was able to coordinate prime factors at the process with coordination level, but in certain circumstances required executing action level computations in order to identify divisibility.

Her post-interview work, however, revealed a consistent view of number theory as a connected body of concepts (schema level). Her explanations began to reflect a greater awareness of the close relationships between divisibility, prime factorization, GCF, and LCM concepts. Additionally, Amanda was now able to generalize the relationships between prime factors and divisibility: "I know that any of the numbers, if I can find their prime factorizations within N then N would be divisible by the numbers."

Participant #4: Zoe

Zoe was considered an average achieving mathematics student based on her course instructor's recommendation and her pre-test score (24%). Prior to instruction, Zoe exhibited a limited understanding of how prime factorization can be used to determine divisibility and a procedural understanding of GCM and LCM concepts. Her pre-interview work was most often coded at the process without coordination level for divisibility, prime factorization, and LCM concepts, and at the action level for GCF. For example, she was able to correctly identify that 5 is a factor and 7 is not a factor of $N = 23 \times 3^2 \times 5^2 \times 17^3 \times 31^5$, without doing any division computation, but she had difficulty identifying composite factors and non-factors.

Following instruction, however, Zoe's mental representations of certain procedures strengthened. Her understanding of divisibility and prime factorization improved primarily to the process with coordination level, as she was now generally able to coordinate distinct prime factors in order to identify divisibility. For example, Zoe was able to find almost all composite factors of $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$.

Participant #5: Jane

Based on her poor performance on the pre-test (22%) and her course instructor's recommendation, Jane was considered a low-achieving mathematics student for the purposes of this study. Prior to instruction, Jane's understanding was primarily at the *action* and *process without coordination* levels. Her internalized image for factor was that a factor is a number that is *visible* in the prime factorization of another number.

Jane's understanding of multiplicative structure improved following the three-week unit. Jane's post-interview work was coded almost exclusively at the *object* level, characterized by complete coordination of distinct prime factors and the ability to articulate clear explanations of GCF and LCM concepts. Jane's notion of factor changed from numbers that are *visible* in the prime factorization of a number to coordinated products of distinct prime factors.

Participant #6: Christina

Based on her course instructor's recommendation and her performance on the pre-test (16%), Christina was considered a low-achieving mathematics student. Prior to instruction, Christina's understanding of multiplicative structure was the most procedurally-oriented of all of the interview participants in the study. Her pre-interview work was coded primarily at the *action* and *process without coordination* levels for divisibility topics and exclusively at the *action* level for GCF and LCM topics.

Following instruction, Christina's displayed richer understandings of all of these concepts (process with coordination). Her mental representation of factor became more coordinated and generalizable, so she paid more attention to prime factorization. For example, Christina's revised understanding of factors as combinations of distinct primes allowed her to solve difficult problems, like finding the factors of N = 3 x 4 x 5 and R = 13 x 17 x 19.

Conclusion

The analysis described in this paper provided descriptions of the ways in which six prospective teachers' understanding of multiplicative structure in the context of various number theory topics evolved following instruction. The data reveal that all participants developed more coordinated images for multiplicative structure, though they did not always make even progress across topics. Some made more progress on problems dealing with divisibility but less on those dealing with greatest common factors. Some displayed rich understanding on one question, only to regress back to less efficient strategies on other questions. This study provides evidence that attending to multiplicative structure can support prospective teachers' use of prime factorization and understanding of divisibility.

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PROSPECTIVE TEACHERS' STRATEGIES AND JUSTIFICATION IN THE GENERALIZATION OF FIGURAL PATTERNS

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This study investigated the ways in which prospective teachers (PSTs) solved figural pattern tasks. Specifically, we focused on PSTs' approaches to generalization, the nature of their solutions, and types of justifications. The findings revealed that PSTs employed three approaches to generalization: recursive, recursive-explicit, and explicit. PSTs' explicit generalizations were mostly constructive in nature, as only one PST developed a deconstructive generalization. Additionally, PSTs' justifications fell into two categories: example-based or figure-based. Together, these results suggested that the type of justification PSTs provided related to the type of generalization approach they employed.

Keywords: Algebra and Algebraic Thinking, Teacher Knowledge

Introduction

Substantial research on mathematics thinking and learning has focused on the strategies that students employ as they attempt to generalize mathematical patterns. One vein of this research investigated students' generalization strategies and the types of reasoning and thinking associated with those strategies (e.g., Amit & Neria, 2008). According to Lannin, Barker, and Townsend (2006), "generalizing numeric patterns is viewed as a potential vehicle for transitioning students from numeric to algebraic thinking" (p. 3). However, as school algebra instruction is often criticized for "rushing from words to single letter symbols" (Mason, 1996, p. 75), the importance of deriving explicit generalizations from patterns is often overemphasized. Such an overemphasis can influence students to generalize patterns by manipulating numbers and symbols without meaning, and thus, interferes with the development of their algebraic thinking. For example, Rivera and Becker (2008) found that students who generalize patterns numerically without considering contextual features (e.g., geometric figures) were most likely to attempt to fit an explicit formula onto the numbers they extracted from the figural pattern. This approach, however, excludes an important factor in the development of algebraic thinking and sense-making—connecting the visual images of the pattern and the symbolic generalization.

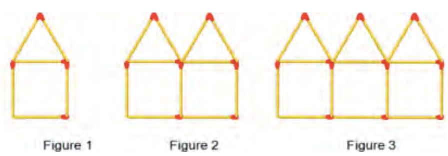
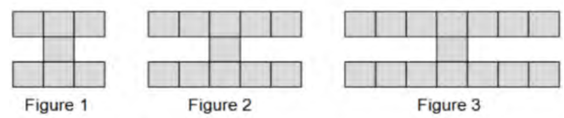
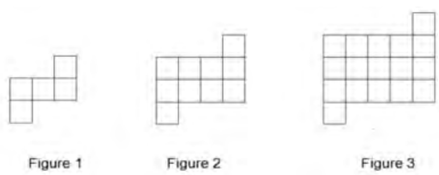
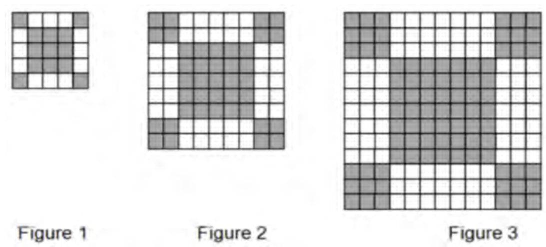
In their study of prospective teachers' (PSTs') generalization of numeric patterns, Zazkis and Liljedahl (2002) found that PSTs typically associated the legitimacy of generalizations with algebraic symbolism. Given that this association might have implications on PSTs' instruction (e.g., how they support their future students in developing algebraic thinking), we were interested in further investigating how PSTs generalize patterns. Additionally, we wondered whether PSTs' association of algebraic symbolism with legitimate generalization in Zazkis and Liljedahl's study was due in part to the fact that the PSTs were asked to generalize numeric patterns. Therefore, in our study we sought to investigate the types of strategies PSTs used to generalize figural patterns. Furthermore, because justifications are related to generalizations (Ellis, 2007), we were also interested in the types of generalizations PSTs derived and the ways in which they provided justification. Thus, in our research, we addressed the following questions: (1) What strategies do PSTs use to generalize linear and quadratic figural patterns?; (2) What types of generalizations do they provide?; and (3) What is the nature of the justifications they provide for their generalizations?

Methods

Eight PSTs (seven females and one male) were selected as participants from a Midwest university in the United States. Of the eight participants, seven were in their early 20s, while one female PST was 37 years old. All participants were in the third year of a traditional four-year teacher education program. The PSTs' foci were evenly split between secondary and elementary mathematics education. Potential differences between these groups of PSTs, however, were not a focus of our study.

The data include audio- and video-recordings of one-on-one clinical interviews with the PSTs and also their written work on four figural pattern tasks (see Table 1). These tasks were designed to afford opportunities for both generalization and justification. They were classified (unbeknownst to the participants) by the type of functions embedded in the pattern (i.e., linear or quadratic). The inclusion of both linear and quadratic tasks was intended to allow for variation in PSTs' approaches. The PSTs were asked to provide a generalization for each pattern and were given a series of prompts depending on their progress. For example, if PSTs employed only a recursive approach, we posed what Stacey (1989) referred to as a "far generalization" task. That is, we asked them to extrapolate the 100th figure to investigate whether they were able to establish an explicit rule for the pattern.

Table 1: Figural Pattern Tasks

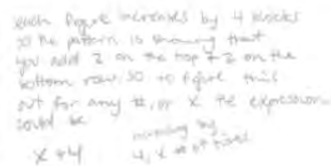
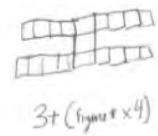
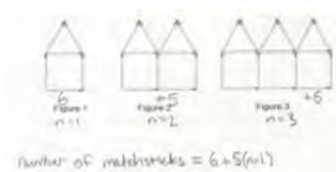
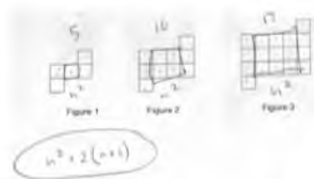
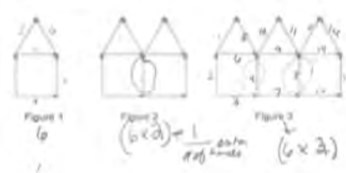
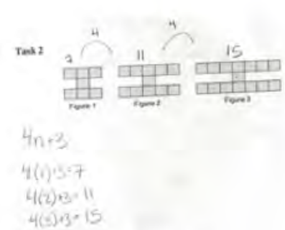
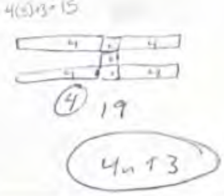
Linear figural patterns	Task 1 	Task 2 
Quadratic figural patterns	Task 3 	Task 4 

After transcribing the interviews and reviewing the written work, we engaged in an open coding process to identify PSTs' strategies, generalizations, and justifications. These three aspects were coded independently (e.g., the approach to generalizing was separated from the type of generalization that resulted), and we generated initial codes individually before meeting as a team to finalize codes. This process yielded the framework in Table 2.

Findings and Discussion

Table 2 includes the number of instances in which each generalization approach, nature of generalization, and type of justification was observed. We found that most PSTs approached generalization by first identifying changes between figures (i.e., a recursive approach). Generalizations derived from this process were most often constructive in nature. This finding is not surprising, as Rivera and Becker (2008) showed that individuals tended to identify additive relations between pattern figures and thus most often developed constructive generalizations. Regarding the types of justifications PSTs provided, the results of our analysis showed that the frequencies of example- and figure-based justifications were roughly similar.

Table 2: Framework for Figural Pattern Generalization and Justification

Component	Description	Example	Number of Instances
Approach to generalization			
Recursive	Participant utilizes patterns that exist between successive cases to determine the next case in a sequence.		3/32 (9.4%)
Explicit	Participant directly relates two (or more) co-varying quantities together, often by a rule or formula.		10/32 (31.3%)
Recursive-explicit	Participant initially identifies the recursive pattern but later attempts to generate an explicit rule.		19/32 (59.4%)
Nature of generalization			
Constructive	Participant recognizes that figures consist of non-overlapping parts.		29/32 (90.6%)
Deconstructive	Participant recognizes overlapping sub-configurations in the pattern structure OR that imaginary parts (that are eventually removed) could be added to the existing figures to create squares or rectangles.		3/32 (9.4%)
Type of justification			
Example-based	Participant substitutes a value in their generalization expression and uses the truth of the statement to determine the validity of the conjectured rule.		15/32 (46.9%)
Figure-based	Participant draws connections between the explicit generalization and the figural pattern.		17/32 (53.1%)

In addition to the results provided in Table 2, we also noticed patterns that spanned across categories. More specifically, there were differences in the ways PSTs employed recursive-explicit approaches and, consequently, in the nature of their justifications. PSTs who used recursive-explicit approaches either extracted the numeric pattern from the figural pattern or focused on both extracted numbers and geometric figures. These "sub-strategies" of the recursive-explicit approach were related to particular types of justification.

PSTs who extracted numeric patterns often abandoned the figures in their reasoning and attempted to fit an explicit formula onto the numeric pattern. Some failed to develop an explicit formula through trial-and-error and accepted their initial recursive generalization as a solution. Other PSTs, however, were successful in fitting an explicit formula to the numeric pattern through trial-and-error. In either case, though, the participant was unable to justify how the explicit generalization related to the figural pattern. Therefore, PSTs who focused on between-figure changes often justified their solutions by providing examples to verify correctness.

On the other hand, PSTs who focused on both the structure of figures and any numbers they extracted from the pattern consistently noticed and utilized the functional relationship between the figure number (i.e., the figure's position in the pattern) and the figural structure in their generalizations. Furthermore, they were always successful in providing both explicit generalizations and figure-based justifications. These results suggest that PSTs who derived generalizations in such a manner were fluent in their transitions between figures and symbols and, therefore, were able to provide valid justifications for their generalizations.

These results have implications for both PST education and future research. As generalization is a central component of algebra, PSTs should be prepared to support their future students in approaching generalization in ways that support the development of algebraic thinking. Provided that algebraic thinking includes the understanding of functional relationships, while generalizing, PSTs should be able identify such relationships within figural patterns. Furthermore, although deriving the symbolic representation of an explicit rule is warranted as a learning goal of generalization in algebra, it should not be overemphasized as it can distract individuals from recognizing important functional relationships (e.g., relationships between algebraic symbols and geometric figures) and developing more sound justifications for their generalizations. Additionally, given that PSTs' prior experiences in generalization and justification likely influence the ways they approach such problems, future research might consider investigating the learning opportunities that might be beneficial for the expansion of PSTs' current conceptions of generalization.

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QUANTITATIVE REASONING AND INVERSE FUNCTION: A MISMATCH

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I extend the body of research investigating students' quantitative reasoning by examining the interplay between a student's meanings developed through her school experiences and her quantitative reasoning in the context of inverse functions. I present one student's activities in a teaching experiment designed to support her in reasoning about a relation and its inverse function as representing an invariant relationship. Although the student engaged in such reasoning, her school mathematics experiences constrained her in relating this reasoning to her inverse function meanings. I conclude with implications and areas for future research.

Keywords: Cognition, Teacher Education-Preservice, Design Experiments

Many researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1996) examining students' inverse function meanings have maintained an emphasis on composition of functions as critical to students developing productive inverse function meanings. However, collectively researchers have found students hold compartmentalized inverse function meanings, typically related to executing specific actions in analytic or graphing situations (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti, Stevens, Hobson, LaForest, & Moore, 2015) and the extent to which students relate these actions to function composition is an open question. For instance, many students rely on a technique of "switching-and-solving" when determining the inverse function of a given function represented analytically (i.e., given $y = x + 1$ they switch x and y then solve for y to obtain $y = x - 1$). Together, these researchers' findings indicate that curricular approaches to inverse function have been ineffective in supporting students in developing productive inverse function meanings. This report, along with Paoletti (2015), which I elaborate on below, begins to address the apparent need to re-conceptualize the teaching and learning of inverse relations and functions.

Theoretical Framing

I examined the possibility of supporting students developing inverse function meanings via their quantitative reasoning (Thompson, 1994, 2011). I conjectured a student could construct a (non-causal) relationship between quantities (e.g., quantities A and B). Once constructed, she could choose to consider one quantity as the input of a relation (e.g. A input, B output) while anticipating the other quantity would be the input of the inverse relation (e.g., B input, A output). With respect to graphing, a student maintaining such understandings can interpret a single graph as simultaneously representing a relation and its inverse relation. Such a student anticipates that the quantity on either axis can serve as the input to a relation; although this reasoning may seem trivial, Moore, Silverman, Paoletti, & LaForest (2014) illustrated that students are often restricted to reasoning about the input quantity exclusively represented on the horizontal axis.

In Paoletti (2015), I presented an undergraduate student's (Arya's) activity as she reorganized her inverse function meanings compatible with this theoretical framing. At the outset of the study, Arya relied on switching-and-solving and understood a function and its inverse function represented different relationships. Arya experienced several prolonged perturbations during the teaching experiment; resolving these perturbations supported her in reorganizing many of her meanings. By the end of the study Arya understood that a relation and its inverse relation represented an invariant relationship and Arya made sense of switching-and-solving by changing the quantitative referent of each variable when switching variables. Arya's inverse function meanings highlight the viability of the ways of thinking I describe above.

Methods and Task Design

I conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with two undergraduate students, Katlyn and Arya (pseudonyms), enrolled in a secondary mathematics teacher education program. I collected data from three interviews and 15 paired teaching episodes. I used the interviews and episodes to investigate Katlyn's mathematical activity, to build models of her mathematics, and to explore the mathematical progress Katlyn made over the semester (Steffe & Thompson, 2000). Data analysis consisted of open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998). Through an iterative process of generating, refining, and adapting hypotheses of Katlyn's mathematics, I was able to characterize her thinking at a specific time and to explain transitions in Katlyn's meanings.

Results

For brevity's sake I do not detail Katlyn's activities throughout much of the teaching experience as they are compatible with Arya's activities reported elsewhere (Paoletti, 2015). Katlyn entered the teaching experiment with her predominate meaning for inverse function involving switching coordinate values graphically and switching-and-solving. She experienced several sustained perturbations that supported her in reorganizing her meaning for graphs (i.e., interpreting the vertical axis as representing a function's input) and her in-the-moment meanings for inverse function (i.e., interpreting a function defining the relationship between volume and side length of a cube and its inverse function as representing the same relationship; she switched the quantitative referents of each variable to make sense of switching-and-solving). Based on my observation of these in-the-moment meanings, I conjectured Katlyn had possibly reorganized her meanings such that she understood a relation and its inverse represented an invariant relationship with the distinction being which quantity she considered the input. I tested this conjecture in a interview two months after the last episode explicitly addressing inverse relationships.

I provided Katlyn with a video of a cylinder with constant radius and a varying height and asked her to determine the relationship between the cylinder's height and surface area. Katlyn determined the analytic rule $SA = 2\pi r^2 + 2\pi rh$ by imagining the net of the cylinder composed of two circles with constant area and a rectangle with varying area (i.e. h varies and r is constant). She drew a linear graph representing the relationship between height and surface area. She stated, "As like the height is increasing, surface area is also increasing." Conjecturing Katlyn might consider surface area as the input, I asked, "Is there another way to read [the graph]?" Katlyn responded, "As surface area increases, height increases.... whatever happens to one is like happening to the other one." Although Katlyn chose to coordinate height first, she understood this was not the only option; from the researcher's perspective Katlyn reasoned about a relation and its inverse relation as she anticipated considering either quantity varying first.

I asked Katlyn to determine the analytic rule of the inverse function conjecturing she would maintain the relationship she had just described. However, Katlyn reverted to switching-and-solving, obtaining $SA = (h - 2\pi r^2)/(2\pi r)$. I asked Katlyn to "talk me through what you did there".

Katlyn: It's funny that you say that 'cause I'm tutoring two girls and we were doing inverses yesterday. And I don't, and I still can't explain why we do this. I was trying to think of a way to explain it to them, and I didn't know the answer. Um [*pause*]. Because that's what I've been told to do for six years…

TP: Okay. So you said you were just tutoring someone on this?

Katlyn: Yeah, and… they were just like, 'well how do I do it?' And so I told them, like you have to make sure the… function is one-to-one so like for every… input there's only one output and for every output there is only one input. All that nonsense that doesn't, I don't really know why we do that. But that's what has to happen before you can switch your input and output and then solve. So, why do we do this? I don't know. But I know this is what the answer is and I. Yeah, I don't know.

TP: Okay and so this is the answer [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?

Katlyn: Yes. Yeah, yeah… I just don't know what it means, like I don't, why do I care about this [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?

TP: So say a little bit more what do you mean you don't know what this [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$] means?

Katlyn: I don't know what it means. I know [$SA = (h - 2\pi r^2)/(2\pi r)$] is the inverse, for surface area of a cylinder. That is all I know. Why is it the surface area? What does it, what does the inverse for surface area mean? I guess I'm thinking like. [*pause*] Okay, it reminds me of that time that we were doing like volume of a cube being like side-squared and then we switched the two and then I was like, okay so now, *s* means volume and *V* means side[length]. So now does here, [*pause*] surface area mean height and height mean surface area? Or did we just not finish the problem in class to conclude about what, I don't, I don't remember. I have no idea why we do this.

TP: So, you're starting to say here [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]. If, if *SA*… represented height, and *h* represented surface area?

Katlyn: Well, it wouldn't make any sense. Because then it would just be the same. Like if you multiplied [$SA = (h - 2\pi r^2)/(2\pi r)$] all back out you would get [$SA = 2\pi r^2 + 2\pi rh$], I guess. And so like I'm attributing [$SA = (h - 2\pi r^2)/(2\pi r)$] to be the same thing where this is now height [*pointing to SA in* $SA = (h - 2\pi r^2)/(2\pi r)$] and this is now surface area [*pointing to h in* $SA = (h - 2\pi r^2)/(2\pi r)$]. That doesn't make any sense. We might as well have kept it that way [*indicating* $SA = 2\pi r^2 + 2\pi rh$]. [*pause*] That's probably not right then cause it has to mean, it has to mean something different.

From the researcher's perspective, Katlyn described the relation and its inverse relation in the moments prior to the term "inverse" being raised. However, when asked to determine the analytic rule representing the inverse function, Katlyn reverted to the activity of switching-and-solving. She used this technique, which she learned as a student and was reinforced as a tutor, despite her being aware that she understood neither *why* she engaged in this activity (e.g., "So, why do we do this? I don't know") or *how to interpret* the results of this activity (e.g., "I just don't know what it means… why do I care about this"). Katlyn recalled the volume-side length situation months earlier and considered switching the quantitative referent of each variable but eventually rejected this concluding a function and its inverse represent different relationships.

Discussion and Concluding Remarks

Katlyn's activity highlights difficulties students can encounter when their quantitative reasoning does not align with their other, possibly non-quantitative, mathematical meanings. Compatible with Arya (Paoletti, 2015), Katlyn reorganized several of her meanings during the teaching experiment (e.g., conceiving that either axis could represent the input quantity). However, Katlyn did not consistently relate these reorganized meanings to her inverse function meanings. One possible explanation for this is that Katlyn was engaging in activity in-the-moment, both in the study and in her tutoring, to alleviate a perturbation without reflecting on how this activity was related to other situations or contexts (e.g., considering that decontextualized and contextualized situations have their own set of rules). Future research examining ways in which to support students in relating their quantitative reasoning with their other, possibly non-quantitative, mathematical meanings for inverse relations as well as other mathematics concepts would benefit the field.

Katlyn's activity highlights how the commonly taught switching-and-solving technique can impede students success making sense of relationships between quantities, interpreting relations and functions in context, and developing connected inverse function meanings. The ways of thinking described in the theoretical framing, as exhibited by Arya (Paoletti, 2015), can provide a way for students to overcome the barriers created when students are taught techniques without understanding why they are engaging in or how to interpret the results of the techniques. Future researchers should continue to explore how a quantitative approach to inverse relations can support undergraduate students, as well as younger students who have not had instruction in function and inverse function, in developing productive relation, function, and inverse function meanings. More broadly, future researchers should continue to reconsider how and why we teach concepts in K-14 school mathematics, like inverse function, in which procedures are emphasized over reasoning about relationships between quantities (Thompson, 2008).

Acknowledgments

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RESPONDING TO STUDENTS DURING WHOLE-CLASS DISCUSSIONS: USING WRITTEN PERFORMANCE TASKS TO ASSESS TEACHER CANDIDATE PRACTICE

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Responding to student reasoning in whole-class discussions and making those contributions central to the mathematical work of the class is especially challenging when students contribute ideas that are incomplete, imprecise, or not yet correct—what we call "errors." We highlight our work supporting and assessing teacher candidates' (TCs') development of skill with the work of responding to student errors. We discuss our use of written performance tasks that call for TCs to play out discussions in response to a classroom scenario. We consider what these written responses reveal about TCs' practice and present findings and examples that have emerged.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Whole-class mathematics discussions support participation in authentic mathematics communities and the development of a broader set of mathematical proficiencies and practices (Kilpatrick, Swafford, & Findell, 2001). Effectively leading discussion entails responding to student reasoning and making student contributions central to the mathematical work of the class (Boerst, Sleep, Ball, & Bass, 2011). It is especially challenging to treat student contributions that are incomplete, imprecise, or not yet correct – what we will call "errors" – as instances of sense-making (Brodie, 2014). The work of responding to errors rests on noticing and interpreting student reasoning (Jacobs, Lamb, & Philipp, 2010) and involves the in-the-moment task of building understanding from errors, in part by positioning others to make sense of the reasoning (Bray, 2011). We are interested in supporting teacher candidates (TCs) in developing skill with leading discussion and the practice of responding to errors. Furthermore, we are interested in ways to assess these developing skills for our work as researchers and as teacher educators.

Theoretical Framework

We take the perspective that TCs can develop skilled practice through purposeful opportunities to approximate and reflect on practice (Grossman et al., 2009). While this serves as a foundation for our use of pedagogies of teacher education such as coached rehearsal (Lampert et al., 2013) and our use of video to better understand TC practice and development, as teacher educators and as researchers we strive to find other ways to approximate teaching practice and to monitor TCs' practice. We have taken up the design and use of written performance tasks (e.g., Bray, 2011) as an additional way to put TCs in the position to make sense of and respond to student reasoning. Through these tasks, TCs demonstrate how they might play out instructional scenarios where they must respond to student errors in the context of whole-class discussions. We highlight our research efforts around the following question: What do TCs' written responses to classroom scenarios reveal about their practice of responding to student contributions?

Designing Performance Assessments

We drew inspiration from the work of Zazkis (2017) and Crespo, Oslund, and Parks (2011) around scripting classroom interactions. In our tasks TCs are presented with a realistic classroom scenario involving a whole-class math discussion and student contributions, including student errors. Respondents are asked to continue the discussion using multiple lines of transcript. TCs are also asked to provide a rationale for why they continued the discussion the way they did, to analyze student reasoning, and articulate how they would want the student's thinking evolve. Performance tasks have the potential to provide more standardized measures of teacher practice across time and contexts. These tasks can also capture TCs' instructional practice, pedagogical reasoning, dispositions about students and teaching, and content knowledge.

We designed two performance tasks for our study. The tasks have parallel structures and differ along two dimensions: (1) mathematical content; and (2) classroom task situation. These differences were built in to diffuse the impact of TCs' specific mathematical knowledge and knowledge of particular tasks and classroom activity structures. For one task, TCs were presented with a scenario centered on the use of a card sorting activity (Baldinger, Selling, & Virmani, 2016) designed to elicit and refine a definition of a polygon. The other task involved a scenario with a task asking students to interpret a position-time graph by writing a story.

Prior to implementing these tasks with TCs, we engaged in two phases of piloting. First, we conducted three cognitive think-aloud interviews with experts in mathematics education or teacher education. This gave us insight into what drew the attention of a reader, what seemed extraneous or distracting, and what seemed unclear. After making revisions, we piloted the tasks with eight first-year teachers or student teachers who had already taken mathematics methods coursework. This process helped us develop a sense of the types of responses we might elicit and led to another round of revisions, specifically around the prompts presented after each scenario.

Methods

We collected responses from 25 secondary mathematics TCs in methods courses at two large, public research institutions. Seventeen participants came from one institution at which they were engaged in a yearlong post-baccalaureate licensure program. The other eight participants from the second institution were enrolled in a shared methods course across multiple licensure programs. Each program's mathematics methods coursework had as a central component a series of practice-focused teacher education pedagogies, such as coached rehearsal.

The performance tasks were administered using Qualtrics in October 2016. TCs completed the tasks individually during the methods class. Response times to complete both tasks together ranged from 11:24 (minutes and seconds) to 42:37, with a median duration of 25:34.

We focus our analysis on the transcripts TCs wrote to continue the discussion in the card sorting scenario. We used a priori and emergent codes to describe TCs' error-handling practices evident in the transcripts. With a more established coding protocol, each transcript was coded individually by two authors. Inter-coder reliability was assessed and any disagreements were resolved through discussion, which allowed for further code refinement. We then engaged in a process of analytic memoing and theme building (Miles, Huberman, & Saldaña, 2014) to capture prevalent, distinct, or novel features of TCs' practice of responding to student errors in whole-class discussions, as represented through their written responses.

Preliminary Findings

From these initial analyses, three themes have emerged that we discuss below, with examples and commentary. These themes also serve as part of our continued analyses. In the card sorting scenario presented to TCs, the classroom discussion began after students were working in small groups with the teacher asking for students to name cards that they knew for sure were polygons. One student, Rosalia, offers Shape Q (see Figure 1) and, after a back-and-forth with the teacher, shares that it is a polygon because "it is a square" and that "all the sides are straight lines." After the teacher asked for another card, Jessie offers Shape J (see Figure 1), stating that it was like Shape Q, emphasizing that it is a square.

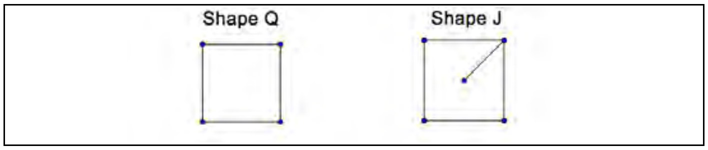

Figure 1. Two cards presented in scenario as examples of polygons.

Bringing Error to Resolution

One feature of TCs' responses was the way in which, within a relatively short dialogue, the error made by the student was resolved or corrected. This occurred in nine of 25 transcripts, and in many cases, the original student corrected their own error, as in the example below:

Teacher: It does look like a square but what is different about Shape J and Shape Q?
Student: There is a line from one corner of the square to the center.
T: Correct, what do you think we can conclude by noticing the line from the corner?
S: We can conclude that this is not a polygon, because they are not all connected.

We found other variations of error resolution, including where another student or the teacher corrects the error. There were five additional transcripts that we considered to be boundary cases of this phenomenon, such as where the last turn of talk has the teacher posing a question that was potentially leading toward resolving or correcting the error in a subsequent student response. In contrast, only one TC used a "tabling" move – explicitly pausing the conversation on a particular idea—with a few other responses including the teacher move of asking for a third card. These findings illustrate that student errors appear to be something that many TCs think could or should be resolved quickly, and often in a one-on-one exchange between teacher and student.

Focus on the Differences Between Cards

As the correction of student errors is a common (though not necessarily productive) response, we went into our analysis of the transcripts with attention to the way errors were resolved (and whether or not they were resolved in the short transcript provided by TCs). From our initial phases of analysis, other codes emerged. One such feature was the way in which transcripts included a teacher move that drew attention to the difference between Shapes J and Q, which occurred in 23 of the 25 transcripts. Furthermore, 17 of the transcripts had such a move as the first teacher move. The prevalence of such a move is notable, though we are not yet sure what this ultimately says about responding to errors. For example, is it a productive move, or is it a move that is explicitly detrimental? While, through further analysis, we have found that the move is often associated with funneling or leading questioning in the response, it is also mixed as to whether or not error resolution is part of the response. We also are continuing to consider what this teacher responding move is, as the move is potentially particular to the circumstances of the scenario—two students, each providing a different example and providing certain reasoning.

Introducing Multiple Students

A final notable finding from our analyses was the way in which some TCs added additional students to the discussion. Nine of the TCs introduced a new student, in some cases creating a new name. Three other TCs reintroduced the other student referenced in the scenario (Rosalia). Other TCs described discussions that included prompts posed to the whole class and, while some of these had unclear direction and purpose, others were more pointed, such as the prompting of students to "turn and talk" with a peer. Overall, we are struck by the way in which some TCs responded to student errors in ways that fostered more discussion and oriented the hypothetical students to one another's ideas. This stands in contrast to the constructed discussions that involved a back-and-forth, teacher-to-student pattern that was present in many other responses.

Discussion and Conclusion

In this report, we have outlined our creation, use, and initial analyses of written performance tasks used to assess TC practice, specifically around the work of responding to student contributions in the context of whole-class mathematics discussions. As part of our full project, the data we highlight in this paper were collected as a pre-assessment. We will be administering these written performance tasks again as a post-assessment, which will allow for further consideration of these tasks and one opportunity to determine changes in TC practice.

We see this work having conceptual, methodological, and pedagogical contributions to the field. Conceptually, this work contributes to the field of literature unpacking the practice of responding to students' contributions. Methodologically, we contribute to the area of research focused on teacher practice through the development of additional tools. Pedagogically, we see the use of performance tasks as not only a data collection and evaluation tool, but also as a formative approximation of practice for TCs.

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STUDYING PRESERVICE TEACHER BELIEFS ABOUT TEACHING MATHEMATICS FOR SOCIAL JUSTICE OVER TIME

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This paper reports on the Teaching Mathematics for Social Justice-Beliefs scale (TMfSJ-B) within the Mathematics Experiences and Conceptions Surveys (MECS). Elementary preservice teacher data were collected across three institutions (n=146) over three time points (i.e. pre-methods, post-methods, and post-student teaching) to examine whether entering beliefs about teaching mathematics for social justice changed over time. Baseline data indicated that homogeneity did not exist between the groups; thus, disaggregate data were used for analyses. One of the two institutions showed positive increase in TMfSJ-B across the three time points. The other two institutions had an increase in TMfSJ-B from the first to the second time point; however, both had a decline from the second to third time point. We explored what factors within the mathematics methods course and student teaching experiences influenced potential changes.

Keywords: Equity and Diversity, Teacher Beliefs, Teacher Education-Preservice

Theory and Objectives

Teaching mathematics for social justice (TMfSJ) is informed by critical theory and critical pedagogy within both teacher education and mathematics education research (Cochran-Smith, 2010; Wager & Stinson, 2012). While TMfSJ is described in a variety of ways to include access and equity to a higher quality mathematics to using social issues to teach mathematics to disrupting school norms (Gates & Jorgenson, 2009; Gutstein, 2006), there is clearly a growing interest in the field of mathematics education to study issues of equity and social justice among preservice teachers. For the purposes of our study, the definition we use is based on the notion that TMfSJ provides opportunities for all students "to learn rigorous mathematics in culturally specific, meaningful ways that seek to improve the economic and social conditions of marginalized individuals and groups, and that work toward[s] reduc[ing] deficit-oriented beliefs about who is or is not 'good' at mathematics" (Leonard & Evans, 2012, p. 100).

While there have been some efforts to integrate TMfSJ theory and pedagogy into teacher education programs (Koestler, 2012; Leonard & Evans, 2012), little is known about how such approaches have influence beliefs and whether beliefs oriented toward TMfSJ are sustained over time. Our study aims to address a gap in the literature by examining elementary preservice teacher (PST) beliefs about TMfSJ from pre- to post-mathematics methods coursework to post student teaching. The research questions explored in this study were:

1. What, if any, changes occur in beliefs about teaching mathematics for social justice over the duration of the mathematics methods course and into student teaching?
2. What factors within the mathematics methods course and student teaching experiences accounted for changes in beliefs, if any?

This focus contributes to the knowledge base on mathematics teacher development by observing both a cognitive aspect of TMfSJ and experiences within a teacher education program. As Philipp (2007, p. 259) explains, beliefs are "psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes."

Methods

The Mathematics Experiences and Conceptions Surveys (MECS; Jong & Hodges, 2015) was administered at three time points: (a) at the beginning of mathematics methods coursework; (b) at the end of mathematics methods coursework; and (c) at the conclusion of the student teaching semester. Participants were undergraduate preservice elementary teachers at three institutions across the United States (n=146). Each version of the MECS included the same TMfSJ-B scale to capture baseline data and changes over time. The MECS also included different experience scales across various iterations (e.g. math methods course experience, field experience, student teaching) to account for factors that might influence beliefs.

Context

Three land grant, research extensive universities in the Southeastern United States participated in the study. A general description of the elementary education program progression and integration of TMfSJ concepts are delineated as follows: University A's elementary teacher education program requires candidates to take a course in social justice and culturally relevant pedagogy upon admission into the professional program (junior year). Candidates take a mathematics methods course that is offered on site at an elementary school, taking part in demonstration lessons by a classroom teacher and working directly with elementary students by making careful observations of children's thinking. The course integrates issues of TMfSJ during these site-based sessions. University B's elementary teacher education program does not require a separate foundational course on diversity, because the goal is to integrate issues of equity throughout methods coursework. The one required mathematics methods course integrates readings and reflections on TMfSJ topics. It is also the case that PSTs are placed in a diverse classroom for at least one field placement. University C, an elementary education program with an explicit focus on STEM content preparation, includes two mathematics methods courses. During the first of the two methods courses, candidates take a co-requisite course in diversity and social justice. Candidates are intentionally placed in diverse school settings during internship.

Data Analysis

To examine changes in TMfSJ beliefs and answer our first research question, we used a Repeated Measures ANOVA (Huck, 2012). To address the second research question, we used multiple regression models to identify factors to explain the TMfSJ beliefs. Prior to conducting our analyses, raw data were converted to logit values using Winsteps (Linacre, 2016). In addition, the TMfSJ-B scales in MECS-2 and MECS-3 were anchored to the MECS-1 scale to be able to make direct comparisons. In Winsteps, the Teaching Mathematics for Social Justice- Beliefs scale had an item reliability of 0.99 and a person reliability ranging from 0.72-0.76 across the three versions of the MECS, indicating that the items had a wide difficulty range and variance but the ability variance of the participants were not as wide in range.

Results

Levine's test of homogeneity indicated that there were statistically significant differences among the baseline TMfSJ beliefs across the institutions; thus, analyses were conducted using the disaggregate data. Mean values for each university across the three iterations of the MECS are presented in Table 1 and Figure 1. Results of the RM-ANOVA analysis indicated only one significant change across all eighteen pairwise comparisons: University B had a significant gain ($p = .002$) for TMfSJ-B from pre- to post-methods.

Multiple regression models were created to examine which factors could help explain TMfSJ beliefs at each time point. At the pre-methods level, we examined whether the PK-12 mathematics experience was influential, but it was not a significant factor across any of the institutions. At the post-methods level, we examined whether the mathematics methods course or field experiences explained the variance in TMfSJ-B, but they were not significant factors. At the post-student teaching level, mathematics methods course and student teaching experiences were entered into the model. The two factors were not significant predictors for Universities A or B. For University C, it was found that the following two factors accounted for 12.6% of the variance in TMfSJ-B post-student teaching ($R^2 = 126$, $F(2, 51) = 3.675$, $p = .032$): the mathematics methods experience and the student teaching scales. In addition, the student teaching pedagogy scale accounted for 12.6% of the variance in TMfSJ-B post-student teaching ($R^2 = 126$, $F(1, 52) = 7.49$, $p = .008$).

Table 1: Mean TMfSJ Scores by Institution

Institution	Pre-methods	Post-methods	Post-Student Teaching
University A (n=38)	1.560	1.813	1.902
University B (n=54)	.877	1.252	1.092
University C (n=54)	.974	1.126	1.038

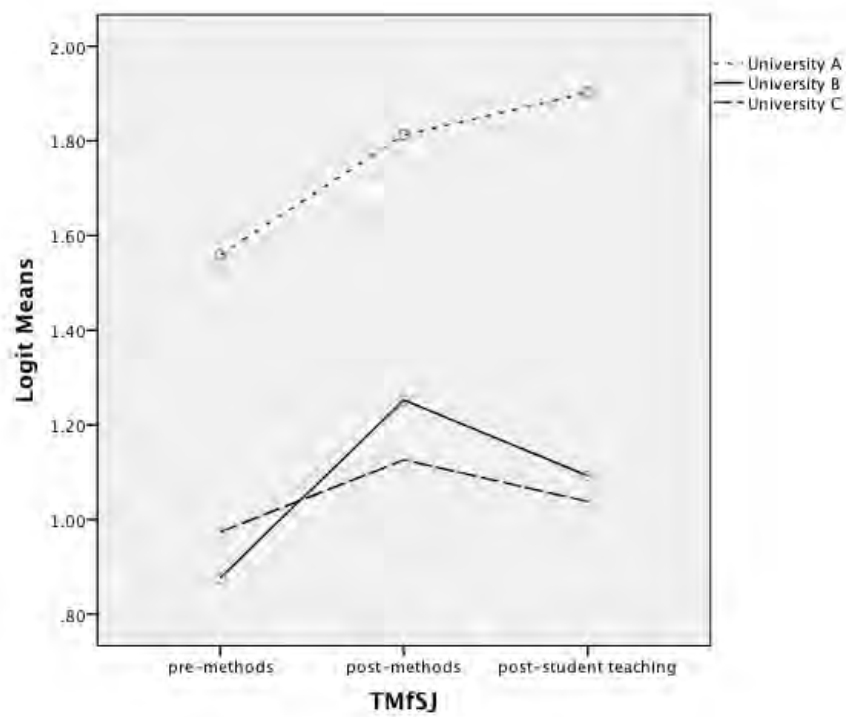

Figure 1. Disaggregate logit means of TMfSJ beliefs over time.

Discussion

Entering TMfSJ-B were somewhat similar for University B and C, while significantly higher for University A. Both University B and C followed similar trajectories – growth in beliefs from pre to post mathematics methods, then a decline in beliefs measured at the end of the student teaching semester. On the other hand, University A saw gains at both the post methods and end of student teaching time points. We were somewhat surprised that a significant change was not found (p=.079) for University A from pre- methods to post-student teaching since positive gained were made at both time points. However, the sample sizes may have been too small at the institutional level. It is also possible that there is a ceiling effect to the beliefs measured at University A, since the baseline beliefs were high.

In exploring the growth trajectories at each university, it might well be that higher entering beliefs among candidates at University A were more sustainable across student teaching in relation to the more tenuous TMfSJ-B found at University B and C. That is, more strongly held beliefs were more entrenched and less susceptible to regression than the beliefs held by candidates at University B and C. As such, foregrounding issues of equity and social justice early in candidates' programs, with the stated goal of developing more productive beliefs, may well lead to sustained teaching that promotes the educational advancement of historically marginalized people.

While our experiences scales did little to explain the variance in TMfSJ-B, there are important contextual differences they may not have been captured in MECS instrumentation. University A candidates have a course with explicit focus on social justice and culturally relevant pedagogy prior to mathematics methods coursework. As such, candidates may be positioned to more thoughtfully consider issues of social justice in mathematics upon entrance to the methods semester. Further University A also includes site-based mathematics methods delivery, where candidates have opportunities to theorize from practice in a diverse school setting (cf. Hodges & Jong, 2015). Given the uneven nature of student teaching experiences, the data presented here suggest engaging in systematic and strategic efforts to increase candidates' capacity to take on productive beliefs about social justice in mathematics early in teacher preparation programs may well lead to beliefs that support more equitable instruction in mathematics classrooms.

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USING NARRATIVES TO ARTICULATE MATHEMATICAL PROBLEM SOLVING AND POSING IN A TECHNOLOGICAL ENVIRONMENT

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In this study, we set out to create an environment for preservice secondary mathematics teachers where we could test the capability of technology to capture episodes of technology-enabled problem solving and answer the question, how do these captured episodes support the articulation of mathematical processes such as problem solving or problem posing? In a collaborative context, screencasting and Interactive Geometry Software were used to create a rich narrative captured in verbalized thinking as well as on-screen activity. The data provided an auditable trail of authentic practice that preservice teachers did not articulate on their own.

Keywords: Teacher Education-Preservice, Problem Solving, Technology

Background and Theoretical Framework

Preservice Secondary Mathematics Teachers' (PSMT) beliefs about technology use in the classroom seem to be drawn from previous vivid episodes or events in their lives (Pajares, 1992), events that likely occurred in the mathematics classroom during their years spent as K-12 students. In order to influence PSMTs' beliefs about the role technology plays in the classroom, we should give them opportunities to experience vivid episodes of mathematical problem solving with technology, and find a way to make these episodes explicit (Cox & Harper, 2016). Equipping PSMTs with accurate records of problem solving practice is necessary for reflection; articulating thinking and reasoning after-the-fact is challenging. Lesh and Zawojewski (2007) call for research into how to document and assess understandings and abilities related to problem solving that relies on data consisting of *auditable trails* of the non-linear activity of problem solvers. We need a way to capture an authentic, accurate and articulated episode to facilitate further reflection on the affordances of technology for doing and learning mathematics. In this study, we set out to create an environment where we could test the capability of technology to capture episodes and *how do these captured episodes support the articulation of mathematical processes such as problem solving or problem posing?*

We choose to use the term narrative as a storytelling term. The narratives that we construct from collected data/auditable trails of documentation (Lesh & Zawojewski, 2007) are mathematical stories about what PSMTs think, say, do, construct, and see while working on a mathematical task. We focused here on the creating and interpreting an auditable trail of problem solving behavior that is specific to a technology-rich context.

Data Collection and Analysis

We assigned 15 PSMTs the task to create a dynamic geometric sketch that embodied a "Kaleidoscope". At the time of this study, the PSMTs were in the seventh week of a course on mathematical problem solving with technology, and were reasonably fluent with Interactive Geometry Software (IGS). We instructed them to engage with the task for approximately 15 minutes in partnerships (and one triad) using a single computer equipped with IGS. The intent was to give PSMTs a task, tools and environment that pushed for and supported verbal communication, negotiation, and collaborative mathematical problem solving.

During the construction phase and class presentations, four sources of data were collected. First, PSMTs submitted the completed IGS sketches of kaleidoscopes. Second, we collected *construction* *screencasts* documenting and linking on-screen activity and verbal communication during the construction phase. We define screencast as a digital recording of computer screen output, also known as a video screen capture, often containing audio narration. Third, we video recorded presentations and discussions of IGS sketches amongst the whole group of 15 PSMTs. Fourth, individual PSMTs recorded *reflection screencasts* where they addressed the problem solving process as well as their intentions behind the completed kaleidoscope.

From each captured episode, we created a rich narrative of problem solving and problem posing activity. In this paper, we focus our attention on the narrative of one partnership: Abby and Olivia. Abby and Olivia are second-year PSMTs who had previous experience working together. This narrative was selected over six others because it demonstrates the potential of technology to both facilitate and articulate mathematical processes, but also for the variety of moments that it provides for analysis. In this sense, this narrative is remarkable in its clarity.

A detailed methodology behind creating the narrative using *thick analysis* can be found elsewhere (Harper & Cox, 2017) thus we will provide only a brief description here in the interest of space. To develop the narrative, we used the creation screencast as the foundation of the analysis while data from the IGS sketch, the recorded class discussion, and the reflection screencasts were used to interpret the creation screencast. Because the IGS sketch was data with which we could interact, it enabled us to form and test our own conjectures about how the sketch worked, giving us insight into the problems Olivia perceived in the behavior of the kaleidoscope model. This is particularly important to the following episode. The data collected during whole class discussion and reflection screencasts gave us descriptions from the PSMTs about the intentions and emotion behind their mathematical work. It was also useful as we compared the way problems were posed by the partnerships during the creation screencast with the way individual PSMTs articulated those problems after-the-fact.

Results

Our narrative of Abby and Olivia's work includes four episodes where problem solving and problem posing can be articulated (Cox, Harper & Edwards, Under Review). For this brief report, we choose to focus on the potential of just the final episode, *A Return to Symmetry.*

Once Abby and Olivia created a first draft of their kaleidoscope sketch, they analyzed their model with a critical eye. With animation features of the sketch engaged, both PSMTs were drawn to the motion of two sets of points: those placed directly on the radii of the circle (*radius points*), and those placed on the circumference (*circumference points*). This visual effect provoked Olivia to pose a question about the model (Figure 1) that articulates a new problem. Olivia's question goes ignored and unanswered as the PSMTs focus on another idea.

After watching the creation screencast we became curious about the problem Olivia was trying to articulate. This curiosity was captured in our researcher notes (Figure 1). Our subsequent discussion produced two questions: 1) what problem was Olivia "seeing" in the motion of the kaleidoscope that caused her to question whether or not the model was valid? Specifically, what points or regions prompted her to ask if "they are allowed to move between the segments;" and, 2) could we use their original IGS sketch to determine how the points were moving and test the mathematical validity of their kaleidoscope model? In essence, we wondered if we could both articulate and solve Olivia's problem.

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Construction Screencast Transcript

(04:10) O: I'm also wondering if they are allowed to move between the segments. If they are allowed to move beyond just…let's stop where it looks like.

[O stops the animation at a strategic, intentional point and gestures to the yellow-circled circumference point as she asks a question. A screenshot of this moment is shown to the right]

(04:23) O: I'm wondering if they are allowed to move beyond, go around the circle past the sector. It's really a mirror so they have to stay within it.

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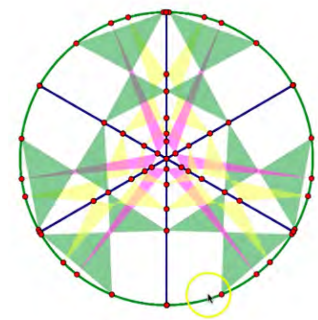
<td></td>
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Researcher Notes

The part where Olivia questions "moving beyond" is really interesting. It seems as if the PSMTs are watching the animated points and imagining that they move from one sector into another. In reality, it's an optical illusion and the animated points, if labeled, would clearly just move within the sector ...Wait? Do the points on the outside of the circle limit themselves to one sector, or do they move out?

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Figure 1. Associated data including transcript, illustration, and research notes.

Since we had access to Abby and Olivia's original IGS file, we were able to conduct a "thought experiment" to find out whether the circumference points were really "moving beyond" the 60-degree arc. This helped us to better understand what Olivia was seeing and describing in the creation screencast. We also wanted to know how the motion of the model was limited in ways by the IGS that would either be removed (solving the problem) or that would prevent it from matching our physical expectations of a kaleidoscope.

To support our investigation, Olivia made herself available for a brief interview after her participation in the study had concluded and during our data analysis phase. In this interview she provided additional insights into the construction of their kaleidoscope. She proudly mentioned that she had recreated a new kaleidoscope on her own, one where the "shapes do not go outside the pie." Considering this comment carefully, we are now confident that Olivia was concentrating on the polygons (and not the circumference points) that spanned multiple sectors of the circle when she asked "if they are allowed to move beyond, go around the circle past the sector." We went back to Abby and Olivia's original creation screencast and captured an image of the initial polygons spanning multiple sectors and since there were no constraints for the circumference points, these points could move "beyond" a 60-degree arc of the circle, yielding polygonal images in the dynamic model that are never seen in a real kaleidoscope.

Discussion and Implications

We have collected an *auditable trail* of problem solving with technology that records the process of problem solving and moments of problem posing. This record of practice is fluid and allows researchers and students to drop into action at any point and explore through thought experimentation "what might have happened if". Specific to this case, it was only because of the auditable record of practice that we were able to see that Olivia had identified a problem in the creation screencast, but had not articulated it clearly to Abby (and perhaps to herself at the time). Interpretation required further articulation and was supported by the IGS data.

Both PSMTs utilized the IGS sketch they created as a source of evidence and illustration in their reflection screencasts, however neither Abby nor Olivia mentioned the symmetry problem. Olivia's interrogation of the polygons in the creation screencast was ignored in both of the reflections produced by the PSMTs. More broadly, Abby ignored the prompt to discuss "insights you had along the way and the impact of those insights on your model" entirely. Olivia went so far as to deny that insight occurred, which stands in contradiction to the data presented here and elsewhere in the narrative. When urged to reconsider her question in the closing interview, Olivia was amazed to hear the insight she had achieved (problem she had posed) and was able to better articulate her concerns as well as a way to improve the original sketch (solve the problem).

Thus, the reflection screencasts provided a limited view of the episode and failed to capture accurate and articulated mathematical practices, in spite of it being an authentic first-hand account. From an analytical standpoint, the PSMTs controlled in their reflections what they presented and in that way limited our view of and understanding of their process to those parts of the story of which they were aware. A comparison to the generated narrative exposed blind spots in their self-analysis. What power we give to our PSMTs when we give them access to an auditable trail with which to narrate a more accurate and articulated account of their insights.

While there is a well-established professional conversation around the role of problem solving and problem posing in mathematics education, more needs to be done to define what experiences PSMTs need with respect to each and conduct research into making these experiences and the supporting mathematical practices more explicit (Cox & Harper, 2016). More can be done to create auditable trails (and from them narratives) to use to prompt deeper reflection about the role technology can play in problem solving. By creating a real-time record of practice, we remove the barrier of having to store within our short-term memory not only the outcomes of our mathematical struggles, but the messiness associated with that work. Mathematical thinking is freed from a chronological and linear path, and we can document the interconnectedness of it as well as the signposts of paths not taken.

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A FRAMEWORK FOR INVESTIGATING NOVICE TEACHERS' PCK

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Our university has developed a one-year master's program with initial teaching licensure for STEM career-changers and individuals with STEM degrees. The assumption underlying such programs is that this population needs only coursework in pedagogy and a teaching practicum (Selke & Fero, 2005). Yet, a focus on only pedagogy without attending to content ignores research indicating the impact of pedagogical content knowledge (PCK) on teacher effectiveness (Ball, Thames, & Phelps, 2008; Goos, 2013). Boundaries between PCK and content knowledge are unclear and no single framework addresses PCK for both mathematics and science teachers. In studying how STEM content backgrounds influence PCK, we developed a framework to help articulate and investigate novice mathematics and science teachers' knowledge.

Participants were 5 mathematics and 8 science teacher candidates in our program. We interviewed them three times throughout their program and first year of teaching. We asked them to describe the impact of and connections between their STEM backgrounds and their teaching practice. Open coding of interview data and existing research on PCK informed our framework.

Our framework focuses on overlapping areas of mathematical knowledge for teaching (Ball et al., 2008) and models for science PCK (Lee & Luft, 2008; Magnusson, Krajcik, & Borko 1999): knowledge of discipline, purposes of teaching, curriculum, students' understanding, and teaching/instructional strategies. *Discipline knowledge* includes content knowledge from all STEM fields. *Knowledge of purposes of teaching* addresses the reasons why certain content is taught. *Knowledge of curriculum* refers to understanding how topics develop and connect over the K–12 curriculum. *Knowledge of student understanding* involves knowing typical student approaches to particular tasks and topics more generally. *Knowledge of teaching/instructional strategies* concerns knowing different ways to teach a particular content.

Our framework has served as a useful guide for distinguishing and analyzing data. We found that participants' *discipline knowledge* did not translate to *knowledge of student understanding*. Participants were challenged by making content accessible to students and "ways to explain what seems elementary to me" (Jennie, 2). However, *discipline knowledge* did support *knowledge of teaching/instructional strategies* by saving them time needed to relearn content and contributing to developing innovative classroom activities that incorporated real-world applications of STEM.

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GENERATING ALGEBRAIC EQUATIONS FOR PROPORTIONAL RELATIONSHIPS

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Keywords: Teacher Knowledge, Teacher Education-Preservice, Algebra and Algebraic Thinking, Number Concepts and Operations

The purpose of this study was to investigate reasoning of 6 future middle grades mathematics teachers on a quantitative definition of multiplication when generating equations for proportional relationships. Past research has consistently reported that proportions involving whole-number multiples is easier to solve than proportions involving fraction multiples (e.g., Kaput & West, 1994; Karplus, Pulos, & Stage, 1983). However, no studies have examined how a definition for multiplication can support and constrain generating equations for proportional relationships.

The theoretical framework of this study is based on a quantitative definition of multiplication articulated by Beckmann and Izsák (2015), as follows: In the equation $M \bullet N = P$, M is the number of equal-size groups, N is the number of units in 1 or each group, and P is the number of units in M groups. This definition of multiplication leads to two solutions using the variable parts perspective, a largely overlooked perspective on proportional relationships (Beckmann & Izsák, 2015). This study was part of a larger, ongoing project on future middle grades (grades 4-8) mathematics teachers' ecology of multiplicative reasoning. One project team member taught a cohort of future teachers for two semesters in 2014-2015. In the first semester, the future teachers received instruction on developing the quantitative definition of multiplication. In the second semester, they reasoned with the variable parts perspective on proportional relationships and developed algebraic equations by reasoning about relationships among quantities. Six future teachers were recruited based on their performance on a fractions survey that focused on multiplication and division with fractions. Data for the present study came from individual cognitive interviews conducted with each future teacher at the end of the second semester.

The main finding we report is that the quantitative definition of multiplication facilitated future teachers' generation and explanation of equations for proportional relationships. Future teachers who did not have this definition of multiplication experienced difficulties in developing appropriate equations. As a future direction, middle grades programs should focus on providing future teachers opportunities to develop capacities for reasoning with the definition of multiplication across problem situations.

Acknowledgements

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ASSIGNING COMPETENCE: HOW CAN WE TEACH IT TO PRESERVICE TEACHERS?

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Mathematics education researchers have begun to leverage practices from complex instruction to address inequities that occur during mathematics instruction (Boaler & Staples, 2008; Featherstone et al., 2011). Featherstone et al. (2011) define assigning competence as "the practice of drawing public attention to a given student's intellectual contribution to a group's problem-solving efforts" (p. 88). While research has provided illustrations of what assigning competence looks like in classrooms, less is known about how to teach the practice to pre-service teachers. This study examines an attempt to teach assigning competence as a practice to pre-service teachers and analyzes how pre-service teachers appeared to take up the practice.

Data Sources and Methodological Approach

Data sources used in this study include lessons plans, notes, and student artifacts (e.g. assignments, assessments, and notes). Given the variety of data sources, we based our analyses on Miles and Huberman (2013) and used triangulation to identify patterns in our data.

Findings

Four themes emerged from our analysis: (1) identifying the practice; (2) connecting the practice to its core aims; (3) developing specific teaching strategies; and (4) conceiving teachers' power and responsibility. First, pre-service teachers grappled with identifying and understanding the meaning of assigning competence, confounding it with praise and making individual students "feel good." Second, we found evidence that pre-service teachers grappled with recognizing the importance and value of assigning competence in mathematics instruction. Third, pre-service teachers had concerns about implementing strategies for assigning competence. Finally, pre-service teachers had concerns about the power and responsibility they could deploy as teachers to disrupt classroom inequities.

Discussion

The work of assigning competence offers a way to bring together core strands of the work of teaching that are high-leverage for beginning teachers: eliciting and interpreting students' strengths, knowing and seeing mathematics in teaching, perceiving status hierarchies and inequities among students, developing specific teaching moves to interrupt those inequities by positioning particular students as competent. However, preservice teachers in our study had difficulty learning to do all of the parts together. In this session, participants will learn about an instructional design for teaching assigning competence to pre-service teachers, details about the design and methodological approach of the study, and consider next steps in developing this important practice with pre-service teachers.

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MATHEMATICAL EPISTEMOLOGY OF PRESERVICE ELEMENTARY TEACHERS

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It is well documented in literature that teacher beliefs influence teaching practices (Stipek, Givvin, Salmon, & MacGyvers, 2001; Wilkins, 2008). However, little research has been done involving preservice elementary teachers' mathematical epistemology. Understanding such beliefs will support improvement in teacher education programs. In this poster, we present our preliminary work to address: What are PSTs' epistemic beliefs regarding mathematics?

We developed a survey on PSTs' epistemic beliefs by combining an open-ended question about learning mathematics together with closed form items measured on a five point Likert scale. The closed form items draw from Wheeler's Epistemological Beliefs Survey for Mathematics (EBSM) and measures beliefs in seven dimensions of epistemology, including: source of knowledge, certainty of knowledge, structure of knowledge, speed of knowledge, personal innate ability, general innate ability, and real-world application (Wheeler, 2007). Each dimension describes a continuum that ranges from non-availing beliefs (having no or negative influence on learning outcomes) to availing beliefs (associated with better learning outcomes). The survey was administered to all PSTs enrolled in a mathematics content sequence for teachers at a large, urban university in the Pacific Northwest. Previous research with the EBSM has not focused on preservice elementary teachers, thus our extension of the tool will provide a new perspective of preservice elementary teachers' mathematical epistemology.

Initial results for the PSTs who responded to our survey (n=53) suggest that certainty of knowledge and speed of knowledge are PSTs most availing dimensions. That is, PSTs perceive mathematical knowledge as evolving rather than absolute and that learning is gradual rather than quick. Conversely, source of knowledge and general ability were found to be the least availing dimensions, suggesting that PSTs view knowledge as external to the learner rather than constructed by the learner and perceive learners to have a general learning skill set that is fixed rather than developed. Additionally, PSTs were fairly homogenous in the belief that learning math depends most on having a good teacher.

Further qualitative analyses of survey responses and follow-up interviews are forthcoming and will serve to identify themes in PSTs' mathematical epistemologies. These findings will help inform preservice elementary teacher education as well as professional development for elementary teachers already in the field.

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AT THE CROSSROADS: INTERSECTING MATHEMATICS EDUCATION WORK OF THE SCHOOL OF EDUCATION AND MATH DEPARTMENT

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With the goal of enriching experiences in mathematics education of elementary and secondary teachers, faculty from School of Education (SOE) and Mathematics Department at University of Indianapolis organized collaborative initiatives. Driving collaboration was the desire to strengthen our education programs by bridging gaps between pre-service teachers' pedagogical knowledge, math content knowledge, and instructional practice. The research questions that drove our efforts were: (1) How do we structure meaningful collaboration? and (2) What specific areas do we focus on to create synergy between departments?

The structuring of our collaboration was grounded in Wenger & Wenger's (2015) community of practice. Such communities are "groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly" (p. 1). We have created a professional learning community (PLC) of mathematics educators at the University. Our PLC consists of two math educators from the SOE along with two math educators and a mathematician from the Mathematics Department. The committee meets regularly to coordinate collaboration on various aspects of mathematics teacher preparation. Through this framework, we problem solve, request information, seek other's experiences, coordinate our efforts, discuss developments, map our knowledge, and identify gaps.

After structuring a PLC, areas of focus were identified. We chose to focus collaborative efforts on: (1) Recruitment and retention: Developing new initiatives to recruit and retain secondary mathematics education majors. (2) Assessment: Although coordinated by the SOE, math department faculty supervise secondary mathematics education student teachers and assist in assessing candidates' mathematics content knowledge. Likewise, mathematics educators from the SOE assist in reviewing candidates' content portfolios required by the math department. (3) Instruction: Mathematics educators collaborate on curriculum and instruction of existing courses in both departments. We further collaborate on programmatic changes for both the elementary and secondary education majors. This promotes greater synergy between different components of the existing curriculum and infuses new teaching methods and content foci while benefitting from the expertise of the faculty. (4) Outreach: Faculty from both departments engage in outreach activities in area schools. These include attendance at PLCs in area schools and possibilities of professional development and the mentoring of practicing teachers.

The structure and focus of our math education PLC has the potential to open doors to explore the collaboration initiative's impact on the University and our students. Our work has just begun.

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MATHEMATICS CONTENT COURSES FOR ELEMENTARY TEACHERS: CURRENT STATE OF PROGRAMS

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The intersection of higher-education practices and elementary education is challenging to navigate. Mathematics teacher educators must consider the trajectory of a preservice elementary teacher throughout her teacher education program and beyond. One way to bring coherence to elementary teacher education programs is to consider and implement the recommendations of organizations with a goal of improving the mathematics instruction and learning for all students.

The Common Core State Standards of Mathematics (National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010) released Standards for Mathematical Practice (SMPs), which span grades K-12 and focus on mathematical practices and processes. After the release of the SMPs, *The Mathematical Education of Teachers II* (MET II) (Conference Board of Mathematical Sciences [CBMS], 2012) responded with updated recommendations for elementary education programs. Included were 12 semester-hours studying mathematics from a teacher's perspective with half in the area of number and operations and the remaining in measurement and data and geometry. There is also a recommendation for some attention to methods of instruction.

A recent trend of addressing the SMPs in mathematics content for elementary teachers textbooks (e.g., Sowder, Sowder, & Nickerson, 2017) attempts to bridge the gap between K-12 schooling and higher education in terms of processes of mathematics. However, we need to know more about what is currently happening in the mathematical content courses to see if these recommendations are being incorporated and to think about where to move forward in these courses to support the mathematical practices of preservice elementary teachers.

This study adds to the work of Masingila, Olanoff, and Kwaka (2012) to gain more information on the current state of mathematics for elementary education courses in the United States (e.g., who teaches them, how many credit hours are required) through a survey of instructors. Forty-four respondents, with an average of nearly 11 years experience instructing mathematics content for elementary teachers courses, described 82 such unique courses. Analysis will include the resources used (e.g., textbook, technology, manipulatives) in these courses as well as the inclusion of and attention paid to the MET II topic recommendations (CBMS, 2012) and SMPs.

This study gives a broader picture of the current state of mathematics content for elementary teachers courses in the United States, taking into consideration multiple recommendations.

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BELLRINGERS: A MEANS OF INTEGRATING MATHEMATICAL CONTENT, PEDAGOGY AND REFLECTION ON PRACTICE IN METHODS COURSES

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Keywords: Teacher Education-Preservice, Teacher Knowledge, Instructional Activities and Practices.

The practice of teaching is complex – involving the coordination of multiple activities and drawing on different types of knowledge related to the learning context (Ball & Forzani, 2009). However there is not time in methods courses to address every aspect of teaching that preservice teachers (PSTs) need to learn. In this study we explore the use of bellringers – brief mathematical tasks implemented as students arrive for class – as a means of addressing multiple instructional goals with preservice teachers.

Seven PSTs enrolled in a middle school mathematics methods course at a large Mid-western university were involved in preparation and enactment of bellringers during the first five to seven minutes of class. To prepare for this we identified characteristics of effective bellringers and used these characteristics to develop a rubric that was used to guide the PSTs in designing high-quality bellringers and to support them to critique each other's bellringer implementations. In preparing and implementing the bellringer, PSTs were expected to apply what they were learning in the course: topics such as task analysis, effective questioning, and the five practices for orchestrating productive mathematics discussions (Smith & Stein, 2011). Implementation was also expected to model instruction based on student thinking – a recurring theme throughout the course. This was followed by a whole-class debriefing session, where they received feedback from their peers and the instructor. The debriefing was used as an opportunity to review important mathematics and as a context to discuss important concepts from the methods course related to pedagogy. All bellringer enactments and debriefing sessions were video taped. After all the PSTs had implemented a bellringer, the PSTs wrote a reflection paper on bellringers.

Analysis of the reflection papers and debriefing conversations showed that PSTs were able to apply what they were learning in the course and deepened their mathematical and pedagogical understandings. The debriefing sessions and writing of the bell ringer reflection further supported the preservice teachers in reflecting on and refining their instructional practice (NCTM, 2000).

The results of this study indicate that the process of bellringer preparation, implementation, debriefing, and subsequent written reflection may be useful as an instructional tool in methods courses to address multiple goals.

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MATHEMATICS TEACHING PRACTICE AT THE CROSSROADS: EFFECTS OF ENGAGING PRESERVICE TEACHERS IN RELATIONAL TEACHING PRACTICES

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Keywords: Teacher Education-Preservice, Design Experiments

The Challenge of Developing Preservice Teachers' Relational Teaching of Mathematics

One of the goals of mathematics teacher educators (MTEs) is to develop activities for mathematics methods courses that support preservice teachers' (PT's) practice for developing relational understanding of mathematics in students. Following Skemp (2006), we define relational understanding as knowing "what to do and why" in contrast to instrumental understanding which only focuses on rules and procedures without reasons (Skemp, 2006, p. 89). Because many PTs come to a mathematics methods course with an instrumental understanding of mathematics, it is common for MTEs to focus on developing PTs' relational understanding of mathematics resulting in less focus on teaching PTs relational teaching methods. This study explores how PTs' practices and views of mathematics learning and teaching evolve and transform from instrumental to relational after the implementation of a relational teaching practice presented in their mathematics methods course.

Response to the Challenge

This impirical study is grounded in the whole-group teaching experiment approach (Heinz, Kinzel, Simon, & Tzur, 2000) and focuses on the PTs' development of teaching practices intended to develop students' relational understanding of a mathematics concept. Prior analysis of a pilot study conducted by the authors produced a three-part framework for use by MTEs to plan and implement practices for developing PTs' understanding of teaching for relational understanding. This poster presents data gathered and analyzed from this more in depth action research study. PTs' self-reflection journals and lesson plan artifacts from three methods courses are analyzed using multiple cycles of In-Vivo and Process coding for emerging categories and themes. Preliminary analyses show that PTs' views of mathematics learning and teaching have begun to evolve from predominantly instrumental to include relational components and thus their teaching practices are evolving to reflect teaching for conceptual understanding. As experience with relational teaching methods deepens, understanding of how students learn mathematics matures.

Implications

This study holds promising positive implications for MTEs' instructional practices. The approach used by these three MTEs presents a clear opportunity to shift the teaching of mathematics from fostering instrumental to relational understanding. Future longitudinal studies will be used to determine if this shift continues into the PTs' full-time mathematics instruction.

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CHANGING THE DESTINATION? ANALYSIS OF A TEACHING ELEMENTARY MATHEMATICS METHODS FOR SOCIAL JUSTICE MODULE

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In an analysis of the final course taken in high school across ethnic/racial groups, Battey (2013) shared disproportionate outcomes among the identified groups, which reveals the road traveled in mathematics education and foreshadows the road ahead…maybe. Gutiérrez (2007) offers another route, a vision for an equitable future, through a definition of equity as "(b)eing unable to predict students' achievement and participation based solely upon characteristics such as race, class, ethnicity, gender, beliefs, and proficiency in the dominant language" (p.41). It is in this intersection of the current destination, implied by Battey, and a potential destination, as implied by Gutiérrez, that we position a social justice mathematics methods learning module for elementary pre-service teachers (PSTs).

The module was iteratively designed over two years to engage PSTs with data from Battey's article through a task that used relevant mathematical content (i.e. proportional reasoning and argumentation) and progressive learning structures (i.e. complex instruction (Cohen, 1994)).

PSTs were asked to represent a future set of data given Gutiérrez's definition of equity. Next, teachers were positioned as critical in determining mathematics outcomes for students through an exploration of literature and examination of PSTs' experiences. Finally, PSTs were exposed to tools, resources, and ways of being that can be used to work toward a more equitable future.

Given 80% of teachers in the U.S. are white (Goldring, et al, 2013), and schools are becoming increasingly diverse, mathematics teacher educators must understand how to develop PST awareness of the inherent power toward academic and economic opportunities that exist within the position of mathematics teacher. The module described above was enacted with PSTs at two large public universities, one within a rural setting in the south with predominantly white students and the other in an urban setting in the northeast with diverse students. One focus of the project is to understand how each context uniquely challenged and supported PSTs. This poster will advance the work in this project and present findings related to the following questions:

1. What was learned from teaching a social justice mathematics module designed explicitly to raise awareness and agency around issues of equity in the mathematics methods courses?
2. How is the nature of what is learned similar and/or different between two distinctive populations of PSTs who engaged with the module?

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PRE-SERVICE MATH TEACHERS' CONCEPTIONS OF CLASSROOM CULTURE

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As pre-service teachers stand at the crossroads between their preparation and their career as practicing teachers, it is imperative that they are prepared to engage in "fundamental element[s] of teaching practice" (Lampert, 2001, p. 53) such as building a classroom culture. This research explores the question: How do pre-service math teachers conceptualize classroom culture?

What Is Classroom Culture?

Classroom culture consists of all activities and expectations that facilitate students' interaction with subject matter content (e.g., Collins & Green, 1990; Lampert, 2001). Teachers' pedagogical practices influence the classroom culture that they co-create with students. More specifically, Bowers, Cobb, and McClain (1999), building on prior work (e.g., Yackel & Cobb, 1996; Yackel, Cobb and Wood, 1991), delineated three aspects of mathematics-related classroom culture: *social norms*, the ways in which students are expected to interact with one another about math; *socio-mathematical norms*, agreed-upon standards for judging mathematical contributions; and *classroom mathematical practices*, knowledge that is taken as shared within a classroom.

Study Design

Five undergraduate seniors at a large Midwestern university participated in the study. These participants were enrolled in their final math methods course before entering their student teaching field placements. Semi-structured interviews, audio-recorded and later transcribed, were the primary means of data collection. Data was coded using the three aspects of mathematics- related classroom culture as a framework, and two additional coding categories were created based on the data. Quotes from each participant's interview responses were identified and coded into mutually exclusive categories. Then, patterns were identified out of this initial coding.

Findings

Social norms were much more a part of participants' responses than socio-mathematical norms; classroom mathematical practices were not addressed at all. Additionally, the participants generally seemed to have a rather broad definition of classroom culture for math classes that went beyond the three aspects laid out by Bowers, Cobb, and McClain (1999). Participants included beliefs about mathematics that they wished to instill in their students as part of their descriptions of classroom culture, as well as specific actions or decisions on the part of the teacher that serve to facilitate classroom culture.

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ENGAGING TEACHER CANDIDATES IN PURPOSEFUL ANALYSIS AND REFLECTION USING VIDEO AND THE LESSONSKETCH PLATFORM

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Preparing teacher candidates (TCs) for the complex and contingent practice of teaching requires TCs to learn to notice (see Sherin & van Es, 2005), analyze, respond, and (later) reflect on what is significant in teaching situations and connect these experiences to broader principles of teaching and learning. By analyzing and reflecting on their own practice, TCs have an opportunity to develop strategies for interpreting what is happening in the classroom (Sherin & van Es, 2005).

This research examined 6 TCs' responses across three video analysis experiences. These experiences are integrated into two secondary mathematics methods courses and consist of group discussions as well as individual analyses. With the second and third iteration there was also a formal write up for the analysis and reflection. Initially, TCs analyzed videos from a public dataset, and in later iterations TCs analyzed videos from their own clinical practice. Performing analysis of video using the LessonSketch platform affords TCs the opportunity to tag, discuss, and reflect on practice in a central location virtually and/or in person. The design of the video analysis experiences includes specific and purposeful structures for analysis and reflection to foster TCs' abilities to improve their practice through noticing.

Preliminary findings show increases in both the level of sophistication and preponderance of evidence employed by TCs to justify claims. For example, in the initial video analysis, TCs' often provided a superficial description of what students and teachers were doing. In contrast, for the third iteration TCs reflected on what students were feeling and thinking, in addition to how and why TCs responded to students. In the third cycle, TCs often tied their analyses to broader teaching principles and practices such as questioning, revoicing, and wait time. Analysis of and reflection on videos of their own teaching provide TCs with a forum to self-report their use of teaching practices and provide justifications. This research confirms video analysis experiences as an intentional tool for developing TCs' abilities to notice while also suggesting potential structures for facilitating this work with TCs. Additionally, this research demonstrates the usefulness of LessonSketch as a workspace for enhanced video analysis experiences.

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PRE-SERVICE TEACHERS' UNDERSTANDING AREA MEASUREMENT: SPATIAL THINKING

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The purpose of this research is unpacking PTs' conceptual understanding of the measurement of area of rectangular and non-rectangular shapes. Area measurement is a significant topic in geometry (CCSSM, 2010; NCTM, 2000) and provides a link between the abstract world of numbers and the concrete world of objects (Hilbert, 1981). In addition, the concept of area can be a model for teaching multiplication of fractions and composite numbers (Hirstein et. al., 1978). Despite the importance of understanding area concept and its measurement and the difficulties pre-service teachers have understanding area concept and its measurement, a few studies has been conducted in this topic.

To address this need, I have chosen to focus on problem solving strategy of PTs in different figures context for determining PTs' understanding of area measurement of rectangular and non-rectangular shapes. My research question will be "how do pre-service elementary mathematics teachers understand the measurement of area of rectangular and non-rectangular figures?"

To explore this understanding, I used theories about understanding and Battista's level of sophistication. My data will be collected using clinical interview methodology (Clements, 2000; Goldin, 2000), which was pioneered by Piaget (1975). I will choose four participants who are PTs enrolled in sections of a course entitled "Mathematics in The Elementary School" at a large public Midwestern university in the U.S. PTs will be interviewed in one-on-one sessions for initial and four explanatory interviews.

Findings determined that PTs' level of understanding area measurement change according to type of figures and numbers. PTs treated the partial units in different ways. In addition, PTs approaches to decimals are dominated by their whole number reasoning. Identifying PTs' understanding provide opportunities to mathematics educators to design more useful and helpful courses for PTs.

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EXPLORING WAYS PROSPECTIVE TEACHERS MAKE COMMENTS AND ASK QUESTIONS IN SMALL GROUPS

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Many mathematics teacher educators agree that teachers learning to implement small groups is important for student learning academically and socially (e.g., Featherstone, Crespo, Jilk, Oslund, Parks, & Wood, 2011). The field of mathematics teacher education, however, knows little of how teacher candidates (TCs) intervene, or make comments and ask questions, in small groups and why TCs intervene in small groups. The lack of understanding how TCs intervene might act as a barrier to helping teacher educators promote the TCs to succeed in implementing small groups in their classrooms. Given that practicing teachers use a range of comments and questions with small groups for different purposes (e.g., Dekker & Elshout-Mohr, 2004; Gillies & Boyle, 2006), examining the types of interventions teachers can make in small groups, and the purposes of those interventions can help teacher educators support TCs to better implement small groups.

This project explores ways TCs intervene in small groups and for what purposes. The research question is, therefore, How do TCs intend to use comments and questions when intervening in small groups in mathematics teaching and why?

This project took place in a mathematics methods course for elementary TCs in their student teaching internship year. Data sources include the responses of 26 TCs to an online survey. The participating TCs were asked to make open-ended responses to survey questions about their perceptions of teachers' roles and their purposes in intervening in small groups. TCs were also asked on the survey to respond to four scenarios describing different small group situations.

Data analysis followed a basic qualitative method. The focus of this analysis was on the range of comments and questions TCs made in response to the scenarios and their intervention purposes. The findings are: (1) TCs tend to make comments and questions that in function extend student's mathematical thinking, elicit students' work, encourage students to work together, and provide content-help; (2) TCs tend to use comments and questions to achieve multiple purposes such as mediating students' thinking and making group-interaction go smooth; and (3) TCs tend to have the same comments and questions but with different purposes.

As a whole, this project will provide more detailed understanding of ways TCs make comments and ask questions to small groups. The understanding will contribute to designing a strategy for teacher educators in an elementary mathematics methods course to support TCs with regard to intervention in small groups.

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PRESERVICE TEACHERS' ANALYSIS OF LEARNING GOALS: THE ROLE OF MATHEMATICS KNOWLEDGE FOR TEACHING

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One ongoing challenge in mathematics teacher education is the difficulty of providing preservice teachers (PTs) with the knowledge, skills, and dispositions required for expert teaching during relatively short preparation programs. Researchers and teacher educators (e.g. Hiebert, Morris, Berk, & Jansen, 2007) have suggested that one fruitful way to navigate this challenge would be to educate PTs to learn from their own teaching over time. Researchers are working to define and operationalize the precise skills needed for teachers to systematically learn from teaching (e.g., Spitzer & Phelps-Gregory, 2017). One identified such skill is the ability to analyze mathematical learning goals. Analyzing learning goals in terms of their conceptually important ideas, or "key concepts," is necessary for teachers to plan instruction around those ideas, collect evidence which illuminates student understanding, and build lasting knowledge (Jansen, Bartell, & Berk, 2009). Previous research indicates that prospective teachers have some skill in analyzing learning goals, but generally only use this skill in supportive contexts and when directly prompted (Morris, Hiebert, & Spitzer, 2010).

Theoretical consideration and emerging empirical evidence indicates that there is a link between teachers' mathematics knowledge for teaching, or MKT (see Ball, Thames, & Phelps, 2008, for details on MKT) and their ability to analyze learning goals. However, much remains to be known about the exact ways in which these two skills support each other. To investigate these links, we asked PTs (n = 53) to decompose a learning goal into its key concepts, using two prompts based on previous research (Morris et al., 2010). We also administered a short assessment of PTs' MKT around the same learning goal (concepts of comparing decimals). Results indicate that there was an association between PTs' scores on the MKT assessment and their ability to identify key concepts of the learning goal. When constructing an ideal student response, PTs with higher MKT scores identified a mean of 0.93 (SD = .95) key concepts of the learning goal, whereas PTs with lower MKT scores identified a mean of only 0.42 (SD = .65) key concepts ($p < .05$). Our poster will expand on this and other findings to illuminate the connection between PTs' MKT and their ability to analyze learning goals.

Implications of this research will inform teacher educators working to prepare teachers who can successfully examine their own instruction, enhancing their teaching at the crossroads of theory and practice.

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FOOD SECURITY: A CONTEXT FOR CONTROVERSIAL TOPICS IN MATHEMATICS EDUCATION

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Keywords: Equity and Diversity, Instructional Activities and Practices, Teacher Education-Preservice

Simic-Muller, Fernandes, and Felton-Koestler (2015) argued the importance of mathematics educators incorporating controversial mathematics tasks into their classrooms. Additionally, mathematics education research suggests that teachers, including prospective teachers, should learn about their students' community funds of knowledge (CFoK) and connect mathematics to that knowledge (Moll, Amanti, Neff, & Gonzalez, 1992). This poster presents the benefit of using food systems and food security as a context that is accessible to real-world mathematics applications and connects to students' CFoK, but can also move into topics that are more controversial. Food offers a progression moving from real-world contexts of food (e.g., how have recipes and portions changed across generations) to controversial topics of food security (e.g., how much more money does healthy food cost?). Food security is defined as "enough food [being] available" from a global to household perspective (Pinstrup-Andersen, 2009, p. 5) or knowing where your next meal is coming from. In regards to mathematical connections, food security is a rich context as it offers tasks for early elementary to high school grades.

In this poster, I will offer specific mathematics tasks for a variety of grade levels that move from real-world contexts to the controversial. For example, in understanding how vast food markets are, second and third graders can calculate the average distance a tomato travels (approximately 2000 miles) as it goes from field to table or geometry students can determine the density of grocery stores as a factor of household income to learn about food deserts--areas with very limited access to fresh food. The context of food security would offer prospective teachers an opportunity to discuss a common everyday concern (eating) while also thinking about how to mathematize the issue. Additionally, the context of food security offers a wide range of ways to connect to CFoK, concerns allowing for real-world situations, controversial issues, and issues of injustice. In summary, food security offers a context to move from real world to topics that are more controversial, pushing prospective teachers to help "mathematics serves as a mirror and a window into people's lives" (Simic-Muller, et al., 2015, pg. 75).

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PROSPECTIVE ELEMENTARY TEACHERS' KNOWLEDGE OF THE ARBITRARY NATURE OF THE FRACTIONAL UNIT

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Keywords: Teacher Knowledge, Rational Numbers

One of the most fundamental ideas in developing fractional thinking is the concept of the unit or whole. Even though there have been research studies conducted to examine PET's knowledge of fractional concepts (e.g., Tobias, 2013), few have focused on teachers' knowledge of the arbitrary nature of the unit. This study analyzes 54 prospective elementary teachers' explicit knowledge of the arbitrary nature of the unit within fractional contexts involving drawn diagrams. The participants were enrolled in two sections of a content course for PETs.

The research subjects completed two questionnaires. The first questionnaire consisted of an open-ended task that asked them to identify and justify what fraction or fractions the shaded diagram depicted in figure 1 could represent. The second questionnaire consisted of six tasks that asked participants to identify and justify whether the same diagram could represent the following fractions: $\frac{3}{4}$, $\frac{3}{5}$, $\frac{3}{10}$, $1\frac{1}{2}$, $\frac{1}{2}$, and 1. After all students completed questionnaire 1, they were given questionnaire 2. By providing a diagram with a shaded portion and asking PETs the fraction or fractions that the shaded part of the figure could represent or whether the shaded diagram could represent certain fractions, there was a demand for an explicit awareness of the arbitrary nature of a unit.

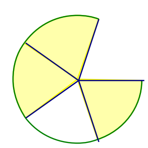

Figure 1. Diagram from first questionnaire.

A content analysis for each student' response was performed. A majority of the students (about 60%) thought that the shaded portion of the diagram could represent (only) 3/4. While other students (about 15%) thought that the diagram could represent 3/5, all students justified their responses based on the idea that the unit or whole needs to be physically present, that is, the circle needs to be completed. It is interesting to report that none of the students was able to think of a situation for which the shaded diagram could represent 3/10. For the picture to represent 3/10, the students needed to physically see 3 parts out of 10. Furthermore, none of the students were able to reconceptualize the three shaded equal parts of the diagram as 1/2 because they did not have physically present the other 2 parts. Other findings and implications of the study will be displayed in the poster during the presentation.

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MATHEMATICS TEACHER EDUCATORS' KNOWLEDGE OF DIVISION OF FRACTIONS AND ITS RELATIONSHIP TO THEIR INSTRUCTIONAL PRACTICE

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Even though some research has been conducted to examine teachers' knowledge of mathematical concepts and procedures, few research studies have examined mathematics teacher educators' knowledge of mathematical concepts and procedures and its impact on their instructional practices. To begin closing this gap, I examined Mr. Frank knowledge of division of fractions and how he used this knowledge when teaching a unit on division of fractions to a group of prospective elementary teachers. Prior to instruction, I asked Mr. Frank to complete several tasks to gain an understanding of his knowledge of division of fractions. The tasks included (a) creating both quotative and partitive word problems whose solutions can be represented by division expressions with and without reminders (e.g., $3\frac{9}{10} \div \frac{3}{5}$ and $4\frac{1}{2} \div \frac{3}{4}$); (b) solving each word problems using diagrams; (c) explaining why each created problem is a division problem; (d) explaining why each problem can be solved using a multiplication expression (i.e., multiplying by the reciprocal of the divisor); and (e) developing and justifying other ways of representing and solving the problems. Considering that Mr. Frank has an undergraduate degree in mathematics and a Masters' degree in mathematics education with over 15 years of teaching experience, it was not surprising that his written responses revealed that he had a profound understanding of division of fractions. He was not only able to create a diversity of word problems and use diagrams to solve them, he was also able to create alternative explanations of why division of fractions can be performed by a multiplication operation.

To examine the impact of Mr. Frank's knowledge of division of fractions on his instructional practices, classroom observations were conducted when he taught the unit on fraction division. It is worth noticing that he developed tasks in which he asked students to solve both quotative and partitive division problem using diagrams and to justify the division of fractions algorithm and interpret the contextual meaning of the reciprocal of the divisor. However, it is also worth mentioning that he did not use other components of his knowledge to provide alternative justifications of why division of fractions can be performed by multiplying by the reciprocal of the divisor because "too many explanations can be overwhelming to students with a weak conceptual understanding of division of fractions."

Evidence of Mr. Frank's knowledge of division of fractions and its impact on his teaching will be displayed during the poster presentation. However, as argued by Castro Superfine and Li (2014), further research is needed to understand the knowledge that mathematics educators need for helping preservice teachers develop their mathematical knowledge.

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THE INTERSECTION OF BELIEFS AND MATHEMATICS ANXIETY IN ELEMENTARY PRESERVICE TEACHERS LEARNING TO TEACH MATHEMATICS

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Theoretical Framework

Mathematics anxiety describes the discomfort or fear that occurs when mathematical tasks are perceived as potential threats to self-esteem (Trujillo & Hadfield, 1999). Negative physiological effects (Luo, Wang, & Luo, 2009), may lead to the disruption of the ability to process information and thus, disrupt learning and performance (Gresham, 2007). Furthermore, mathematics anxiety may shape pre-service teachers' (PSTs) beliefs about doing and teaching mathematics (Stoehr, 2017).

Methods and Results

PSTs were recruited from two larger studies in a large teacher education program (n = 4 narrative participants, n = 53 survey participants). An iterative analysis (Bogdan & Biklen, 2006) was used to demarcate the narratives that pertained specifically to mathematics anxiety and beliefs about teaching mathematics. Narrative themes included: 1) *understanding the mathematics content*, 2) *knowing how to teach a mathematics lesson* and 3) *feeling responsible for student learning*.

An exploratory factor analysis of survey items resulted in four factors being retained based on Eigenvalues > 1.0 and visual analysis of the scree plot. Three factors thematically converged with the qualitative data are described above. Five items (α = .900) explained 32.1% of the variance and converged with the first theme. Four items (α = .886) explained 21.4% of the variance and converged with the second theme. Four items explained 7.7% of the variance (α = .689) and converged with the third theme. The remaining factor assessed *mathematics test anxiety* and explained 9.6% of the variance.

Conclusions

The converging findings in these two studies support that PSTs experience significant anxieties related to learning mathematics content and instructional strategies. PSTs experience additional anxiety when they feel responsible for student achievement. This places PSTs' experience of mathematics anxiety at the crossroads of their own and their students' learning, and suggests future research to explore whether these anxieties are a direct extension of one another.

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NUMBER TALKS WITH PRESERVICE TEACHERS TO DEVELOP THREE LEVELS OF UNIT FOR FRACTIONS

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Teachers' understanding of multiplication of rational numbers including fractions and decimals is often a procedural understanding that lacks conceptual depth (An, Kulm, & Wu, 2004; Armstrong & Bezuk, 1995; Ball, Lubienski, & Mewborn, 2001; Izsák, 2008; Sowder et al., 1998). Furthermore, Izsák (2008) stated that teachers' ability to understand three different levels of unit in rational numbers is a "necessary but not sufficient" (p. 139) condition for them to be able to respond and interpret students' conceptions of multiplication of fractions. In response, we developed a series of number talks for use with pre-service teachers as a way to develop their understanding of three levels of unit within fractions with the anticipated outcome of teachers' having a better understanding of fraction multiplication.

Number talks, or short conversations around mathematical problems that students solve mentally, have been described as useful in K-12 mathematics classrooms because they offer opportunities for students to "reason about quantitative information; utilize number sense; discern whether procedures make sense; identify which procedures are applicable to specific situations; check for reasonableness of solutions and answers; and communicate mathematically to others" (Parrish & Dominick, 2016, pp. 14-15). As such, they can be considered a high-impact practice that has the potential to help students develop a deeper understanding of mathematical ideas. In this study, we utilized number talks in a mathematics content course on rational numbers for pre-service elementary teachers. The participants engaged in carefully designed number talks on the topic of rational numbers at the beginning of each of their class meetings. The number talks were designed to expose participants to three levels of unit within whole numbers and fractions prior to instruction on multiplication of fractions. During this poster session, we will present results from this mixed-methods study including the design of the study, participants' scores on pre- and post-tests for multiplication of fractions, and participants' responses and rationales for number talks. Implications for the teacher education community will be shared.

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LINK BETWEEN MATH EDUCATION AND TEACHING, AN EXERCISE OF DESCRIPTION THROUGH PCK-EC.

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The math education has mainly focused on the problems that arise in teaching and learning mathematics. Thus, it is a priority to highlight the need for an approach between teachers and research in math education. The Seminar Rethink Mathematics (SRM) was born in 2004 looking for this link. The objective of this research is to describe the type of knowledge of the research in Mathematics Didactics that is offered to the teacher through the scientific production of the SRM.

The SRM is an online seminar that is organized into cycles that group sessions. Each session presents the opportunity to interact with the researcher around one of his published results, so each session revolves around a research product (article, report, thesis, book chapter or book). A live dialogue is carried out commonly between two teachers and the researcher, where the reference material is discussed with a view to applying it in one of the dimensions of professional teaching. (Suárez and Ruiz, 2013)

The framework of this study is the Pedagogical Content Knowledge in an Educational Context, PCK-EC model (Pena-Morales, 2016), which characterizes the dimensions of teacher's knowledge in five categories: teachers' attitudes, knowledge of technology, knowledge of learners' cognition, knowledge of the subject matter and knowledge of pedagogy. The database contains the abstracts obtained from the reference document of each session. Here is an example of the result obtained for Session 60: The session "Mathematical visualization, representations and use of technology" is located in the *Knowledge of technology* category, as it analyzes "the problem of using the graphical calculator for the construction of concepts in the mathematics classroom" from the "Constructing Mathematical Concepts" approach. The session looks for "the reflective use of technology in the math classroom" (Hitt, 2003). Based on the results, we can conclude that the approach between teachers and educational mathematics, in terms of scientific production, focuses on two categories that characterize the dimensions of teacher knowledge: almost half (43.6%) of the sessions so far carried out could be classified in the category *Knowledge of pedagogy.* The next highest frequency category (34.5%) is *Knowledge of student cognition*. There are approximately 60% of the sessions classified in one category, 24% in two categories and 16% in three categories.

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PRE-SERVICE SECONDARY MATHEMATICS TEACHERS AS LEARNERS: IMPLICATIONS FOR THEIR TEACHING OF CONCEPTUAL UNDERSTANDING

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Keywords: Teacher Education-Preservice, Teacher Knowledge

The purpose of this study was to describe pre-service secondary teachers' (PST) thinking as they attempted to solve high cognitive demand tasks (Stein & Smith, 2007) by applying the SOLO taxonomy. In addition, we examined the relationship between PST's conceptual knowledge and their problem posing based on videos of their student teaching practice. The following research questions framed this study: (a) What levels of thinking are PSTs' at in the SOLO taxonomy when confronted with problems with high cognitive demand? (b) How might PSTs' level of thinking (in SOLO taxonomy) relate to their problem posing in their student teaching classrooms?

Theoretical Framework

Addressing the multiple components of the research questions posed in this study required a framework with the capacity to examine and crosscut students', teachers', and tasks' "depth of knowledge." The study employed: (1) Skemp's theory of instrumental and relational understanding, (2) the SOLO taxonomy by Biggs and Collis (1982), and (3) the concept of problem posing as it relates to a teacher's capacity to develop rich problems according to Crespo (2003). This latter section described specific teacher roles for problem-posing, the impact it has on students, strategies, and the need for problem posing in teacher education.

Methodology

To answer the research questions, we classified and investigated PSTs' responses to rich mathematical tasks using the SOLO Taxonomy. In addition, we compared their taxonomy levels with the level of problem posing criteria developed by Crespo (2003). The use of video analysis and PSTs reflection statements were analyzed through pattern matching (Yin, 1989) where statements from pre-service teachers were compared with assessments of three mathematics educators and external criteria.

Results and Discussion

Of the 15 students' solutions, eight were classified as multi-structural on the SOLO Taxonomy, meaning that they were able to see many aspects of the problem, yet not comprehend how each piece of information was able to fit together. The remaining students were classified as pre-structural (n=4) or uni-structural (n=3) meaning that they were not able to either grasp the problem or not see the intricacies. The problems posed by PSTs in their videos were transcribed and showed that more than a significant majority emphasized procedural fluency over conceptual understanding. The findings of this study called us to examine the intersection of the qualitative relationship between PST's ability to solve conceptual problems and their problem posing in the classroom. This intersection of problem solving theory and the study of PST's problem posing is an important crossroad in teacher education.

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WHAT COUNTS AS LEARNING: CHALLENGES FOR PREPARING, ENACTING, AND ANALYZING REHEARSALS

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Current reforms have called for a stronger emphasis on teaching and learning mathematics that elicits and uses evidence of students' thinking builds on procedural fluency from conceptual understanding (National Council of Teachers of Mathematics, 2014). Despite the wealth of existing literature on pre-service secondary mathematics teachers' abilities to notice, attend to, and respond to students' mathematical reasoning, we still know little about how these skills related to identifying evidence of students' learning are taught in methods courses. In this poster, we share how three mathematics educators at three different universities are collaborating to study challenges and progress with developing pre-service secondary mathematics teachers' skills with enacting lessons from a worthwhile high school curriculum called Interactive Mathematics Program (IMP). More specifically, this study examines the challenges that the mathematics educators and pre-service secondary mathematics teachers (PSMTs) encountered during the preparation, enactment, and analysis of rehearsals.

This study had 16 PSMTs from the second author's methods course, and each PSMT taught a lesson from the same unit in the IMP curriculum (Fendel, Resek, Alper, & Fraser, 2011) called *Small World, Isn't It?* To prepare PSMTs to learn from their teaching, they were engaged in the pre-rehearsals that included solving tasks, studying enactment of the same tasks, and analyzing the relationship between the instruction and students' learning opportunities. The rehearsals included planning and enacting a task from the IMP, and providing peers with feedback on videotaped lessons. Our data sources included PSMTs' lesson plans with solutions to their tasks, their video recorded lessons, their peers' in-class work, and their teaching reflection papers. To analyze what counts as PSMTs' learning from teaching, we followed the techniques of grounded theory (Strauss & Corbin, 1990) to identify themes to support our understanding about what we believe each PSMT was learning. The results of this study show that the PSMTs' understanding of mathematical concepts within their assigned lessons affected the way they enacted the tasks. For example, if a PSMT solved the task procedurally and anticipated that their classmates will do the same, then they were less likely to have purposeful questions prepared to advance their peers' mathematical reasoning and making sense of mathematical concepts during enactment. These results highlight the need in mathematics teacher education programs to better prepare and support PSMTs to teach mathematics both procedurally and conceptually in their future classrooms.

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PRESERVICE TEACHERS' VIEWS ON SOCIAL JUSTICE TOPICS IN THE CLASSROOM

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Preparing secondary teachers to teach in the 21st century during the Common Core-era of mathematics education requires a changing of perspective from how the preservice teachers (PSTs) learned mathematics themselves (Garmon, 2004). A difference in the PSTs' cultural background and the demographics reflected in urban and inner-city schools could cause disequilibrium and a lack of understanding between the teacher and the students (Gutstein & Peterson, 2005; Sleeter, 2001). Although it can be complicated or hard for some teachers; teaching in this situation requires an alternative view on the preconceived ideas that teachers hold about their students and mathematics (Padron, Waxman, & Rivera, 2002). Teaching from a social justice perspective is a way to effectively cater to an urban population of students (and non-urban as well).

This study addresses issues raised at a large, southern, public predominantly white institution where PSTs have no formal course in preparing them to teach diverse populations in a classroom. PSTs were asked to respond openly and candidly through multiple surveys to express their knowledge and understanding of what it means to teach social justice within a mathematics classroom. Through culturally relevant practices, two instructors led their classes of PSTs in an activity adapted from Gutstein and Peterson (2005) that utilized a social justice lens. The goal was to engage PSTs in an opportunity to experience learning social justice within a mathematics task and to broaden their view of teaching mathematics to include a social justice perspective. We recognize that this is just an introduction to social justice, but we are attempting to integrate this perspective of teaching in a mathematics classroom. The purpose of this research was to primarily use qualitative data to answer the following questions: (a) What are PSTs' initial conceptions of teaching for social justice? (b) How do PSTs view the concept of social justice and its role in mathematics education? (c) After the intervention (activity), what are changes in the initial conceptions of PSTs on teaching for social justice in mathematics?

Using the PSTs' responses, the researchers compared initial conceptions of teaching for social justice to understandings after the activity. This poster will illuminate the responses of the participants in this study and show how students' views shifted.

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WHO TEACHES MATHEMATICS CONTENT COURSES FOR PROSPECTIVE ELEMENTARY TEACHERS? RESULTS OF A SECOND NATIONAL SURVEY

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Keywords: Teacher Education-Preservice

The mathematical experiences that prospective elementary teachers have during their teacher preparation are vitally important. Recommendations from *The Mathematical Education of Teachers II* (Conference Board of the Mathematical Sciences, 2012) call for (a) "mathematics courses that develop a solid understanding of the mathematics they will teach", (b) "coursework that allows time to engage in reasoning, explaining, and making sense of the mathematics they will teach", with "at least 12 semester-hours on fundamental ideas", and (c) "teachers should develop the habits of mind of a mathematical thinker and problem-solver" (pp. 17-19). Additionally, instructors of mathematics content courses for prospective elementary teachers play an important role in helping prospective teachers acquire the knowledge they need for teaching.

In 2010, we conducted a national survey of higher education institutions in the United States to answer the question, "Who teaches mathematics content courses for prospective elementary teachers, and what are these instructors' academic and teaching backgrounds" (Masingila et al., 2012). In 2016, we conducted a second national survey to examine the situation six years later. We made some changes to the survey from 2010 by (a) asking about number of credits instead of number of courses, (b) adding some mathematical content areas when asking what content is included in the courses, (c) asking if a textbook is used and, if so, which one, and (d) changing the questions about academic and teaching background of instructors and supervisors to be more concise. We surveyed 1,740 institutions and a faculty member from each of 413 institutions (23.7%) participated in the survey.

The survey results demonstrate that the majority of institutions are not meeting the recommendations of at least 12 semester-hours of mathematics content. Additionally, most instructors for these courses do not have elementary teaching experience and have likely not had opportunities to think deeply about the important ideas in elementary mathematics, and most institutions do not provide training and/or support for these instructors. If nothing changes with the preparation and professional development of these instructors, the cycle of unprepared prospective teachers whose college experience has little effect on their mathematical understanding (CBMS, 2012) will continue.

We will present all of the findings from this national survey, including (a) the number of credits of mathematics content courses that are offered and required, (b) the content included in these courses, (c) whether a textbook is used, and if so, which one, (d) the academic and teaching background of the course instructors, and (e) whether there is training and/or support for course instructors, and if so, what is the training and/or support.

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PROSPECTIVE TEACHERS' UNDERSTANDINGS OF MATH PRACTICE: MAKE SENSE AND PERSEVERE

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The Common Core State Standards for Mathematical Practice (SMP) build upon efforts to engage students in reasoning and sense making and reduce reliance on memorization of procedures. The first mathematical practice (SMP1) underlies other SMPs with its focus on developing students' thinking strategies and perseverance for problem solving.

Prospective elementary teachers are grappling with learning to teach mathematics in ways that incorporate SMPs. A study by Gerberry & Keazer (2015) found that one third of prospective teachers surveyed recognized that SMP1 was about developing students' thinking processes, while others interpreted the practice as students relying on a procedure or scaffolding techniques to solve a problem. SMP1 may be challenging for prospective teachers to understand because it highlights a shift away from traditional emphases of following procedures, and towards a focus on developing thought processes for solving complex, non-routine problems.

In this study, prospective teachers (n=71) enrolled in four mathematics courses for elementary teachers at two institutions were surveyed about their understanding of SMP1. Specifically, we aimed to answer the research question: *How do prospective elementary teachers conceptualize SMP1, in terms of the tasks they would use to foster SMP1 and how they would assess student engagement?* The survey asked prospective teachers to think about their future classroom where they would be responsible for helping students engage in SMP1. One question was, "Describe a math activity or a math problem that you would pick for your students to give them an opportunity to demonstrate their ability to engage in SMP1. After you describe the math problem or activity, explain how it allows students to demonstrate that they have engaged in the practice." Guided by grounded theory (Glaser & Strauss, 1967), survey responses were reviewed repeatedly, coded, and compared for the nature of the math tasks selected to foster SMP1 and the methods suggested for assessing student engagement.

The most common themes found in prospective teachers' responses were that SMP1 tasks should be context-based problems with multiple ways for students to solve them. In addition, almost half of the respondents indicated that students should generate the solution strategies. A minority of respondents, however, indicated that solution strategies should be provided by the teacher, which contradicts the intention of SMP1. Our findings build on the conference theme *Synergy at the Crossroads* of research and practice because prospective teachers' conceptualizations of SMP1 have significance for informing the design of coursework activities to improve the preparation of prospective teachers.

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THE EVOLUTION OF REPRESENTATIONS AND TALK IN A MIDDLE SCHOOL CONTENT COURSE

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Keywords: Teacher Education-Preservice, Number Concepts and Operations, Classroom Discourse

Departing from the perspective of doing mathematics as an individual's activity, research on doing mathematics within classroom microcultures have been an active research agenda. However, as a field, we know little about the microcultures of mathematics courses for prospective teachers (i.e., content courses). In this poster, I examine two crucial component of the classroom microculture – representations and talk. I take the view of representations as a product (National Council of Teachers of Mathematics, 2000). With this view, representations are encoded objects that result from thinking, maybe for purposes of communicating (Greeno & Hall, 1997; Pimm, 1987). More specifically, I ask how do representations evolve over time? Moreover, I also look at the parallel development of classroom talk with the representations.

To investigate the different ways the talk and representations evolve, I take a cultural-cognition approach (Saxe, 2015). Saxe (2015) posited humans participate in joint problem solving by using collective practices i.e., representing and talking. Moreover, as certain goals of a community shift, the representational forms and functions also shift. For example, as addition problems evolve from adding sets of objects to include fractional amounts, representations and talk evolve. In this poster, I present the evolution of two representations used in a content course for prospective middle school teachers – strip diagrams and double number lines.

Using classroom video data from a semester-long content course, I examine whole-class discussions of student strategies when solving problems involving multiplicative concepts e.g., division, ratio. From the video data, I distill the relevant features of the representations as indicated in student talk. I segment the semester by mathematical topic and examine the features of the representations and talk unique to each segment. The analysis is currently in nascent phases thus, the poster would include a complete set of results I am not ready to report in this proposal. The results section would show examplars indicative of the representations and talk within each segment, the differences between segments. By showing the exemplars, I intend to indicate how the talk and representations evolve over time.

Acknowledgments

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Chapter 9

Statistics and Probability

Research Reports

Brief Research Reports

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Posters

EL RAZONAMIENTO DE PROFESORAS EN FORMACIÓN ACERCA DE LA VARIACIÓN EN SITUACIONES DE RIESGO
PRESERVICE TEACHERS' REASONING ABOUT VARIATION IN RISK SITUATIONS

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En esta investigación exploramos el razonamiento de profesoras en formación acerca de la dispersión de datos (variabilidad o variación) cuando es analizada en problemas donde se comparan conjuntos de datos que involucran situaciones de riesgo como lo son las apuestas en juegos y la duración de vida en tratamientos médicos. En esta comunicación se presentan las respuestas que dieron 97 futuras profesoras de nivel preescolar a dichos problemas. Los problemas fueron resueltos antes de que las estudiantes normalistas iniciaran un curso de procesamiento de información estadística y los resultados mostraron la dificultad que ellas tienen para interpretar la dispersión en este tipo de contextos. Por los resultados obtenidos consideramos importante reflexionar acerca de la instrucción de las estudiantes sobre los significados de medidas de centro y dispersión con la finalidad de contribuir en su formación.

Keywords: Análisis de Datos y Estadística, Conocimiento del Profesor.

Introducción

La variación es la causa subyacente de la existencia de la estadística que está presente en todos lados y por lo tanto en conjuntos de datos (Watson, 2006). Además Moore (1990) enfatiza la importancia de medirla y modelarla, y encontramos que investigadores como Wild y Pfannkuch (1999) incluyen la percepción de la variación como parte de los tipos fundamentales del razonamiento estadístico. Asimismo, Garfield y Ben-Zvi (2008) observan que "la comprensión de las ideas de dispersión y variabilidad en los datos es una componente clave en la comprensión del concepto de distribución y es esencial para hacer inferencias estadísticas" (p. 203). Por su parte, Burrill y Biehler (2011) proponen una lista de siete ideas estadísticas fundamentales en las cuales la variación se ubica en el segundo lugar. Respecto del aspecto escolar, Franklin, Bargagliotti, Case, Kader, Sheaffer y Spangler (2015) mencionan que los profesores deben reconocer las características de la estadística y comunicarlas de manera clara y, particularmente, poner énfasis en la variabilidad y el papel del contexto, y en la descripción de la variabilidad considerar que los datos están constituidos de una estructura (la media o la mediana) alrededor de la cual varían. En relación con ello, Canada y Makar (2006) encontraron que al resolver problemas sobre distribuciones los profesores en formación presentan una percepción intuitiva de la variación, la cual describen con lenguaje informal y que la media es poco utilizada. Otros estudiosos como Mooney, Duni, VanMeenen y Langrall (2014) mencionan que al investigar acerca de la percepción de la variabilidad, en situaciones de azar, los profesores en formación reconocen que debe presentarse cierta cantidad de variabilidad, pero no tienen certeza sobre cuánta. De las investigaciones precedentes se desprende que es necesario proveer a los futuros profesores con experiencias tanto en el análisis de datos como en situaciones de azar donde se desarrollen conceptos como *centro*, *variación*, *distribución*, *valores esperados* y las relaciones entre ellos. Para explorar la comprensión y razonamiento de los alumnos acerca de la percepción, descripción y medición de la variación en los datos se han utilizado diferentes contextos y problemas por ejemplo, variabilidad en el muestreo (Watson & Moritz, 2000), azar (Watson & Kelly, 2004), mediciones repetidas, variación en el crecimiento de plantas (Lehrer & Schauble, 2007; Petrosino, Lehrer & Schauble, 2003), y clima (Reading, 2004). De acuerdo con estos investigadores, las situaciones de riesgo proveen otro escenario para indagar el razonamiento que tienen los estudiantes acerca de la variabilidad (Sánchez & Orta, 2013). Por ello, esta comunicación tiene como objetivo explorar la manera en que futuras profesoras del nivel preescolar interpretan la dispersión de datos en situaciones de riesgo, con la finalidad de que sepan conceptos estadísticos fundamentales con miras a su mejoramiento profesional. Por este motivo es importante incluir en su formación el conocimiento de conceptos estadísticos, que le permitan favorecer la recolección, representación e interpretación de información en el nivel preescolar (SEP, 2011). Además de considerar que estos serán enseñados posteriormente a sus alumnos en otros niveles educativos (Ball, Thames & Phelps, 2008).

Marco de Referencia

Esta exploración se ubica en el área de razonamiento estadístico cuya propuesta es comprender como razonan las personas con ideas estadísticas (Garfield & Ben-Zvi, 2008) y así proponer características para crear escenarios de aprendizaje, puesto que cuando los participantes de una investigación tratan de justificar sus respuestas, muestran los elementos a los que le dan importancia, en particular los datos que eligen, las operaciones que realizan, sus creencias y sus conocimientos. Aunque en ocasiones las respuestas de las personas no son tan explícitas para revelar claramente su razonamiento, de cualquier manera muestran indicios para identificar algunos de sus rasgos. En este estudio identificamos algunas características del razonamiento de profesoras en formación ante situaciones de riesgo. Una parte importante en una investigación en didáctica de las matemáticas son los problemas. Al resolverlos éstos deben promover en las personas la capacidad de pensar y razonar y así proveer al investigador de resultados relevantes que aporten información al área de estudio. Los problemas deben también llamar la atención de quienes los resuelven para que puedan comprometerse con su solución y aumentar las probabilidades de la comprensión del concepto que se quiere estudiar. En la estadística el razonamiento debe articular ideas como media o dispersión, expresadas con números, con situaciones reales basadas en datos, es decir, el razonamiento estadístico está íntimamente relacionado con el contexto, y los números en contexto implican información (Moore, 1990). Los problemas sobre toma de decisiones bajo incertidumbre son comunes en estadística, este tipo de problemas han sido utilizados para promover y analizar características importantes del razonamiento estadístico de las personas. Además, las situaciones que requieren de la comparación de conjuntos de datos son utilizadas frecuentemente para involucrar a los alumnos en el razonamiento con datos (Garfield & Ben-Zvi, 2008). En esta exploración se presentan dos situaciones de toma de decisiones y comparación de conjuntos de datos en los cuales la dispersión es importante, y ésta puede ser asociada con la noción de riesgo, la cual está asociada con la incertidumbre presente en un suceso que implica una amenaza. Estas situaciones aparecen cuando hay resultados no deseados que, como consecuencia, provocan pérdidas o daños. Un problema paradigmático en un escenario de riesgo consiste en elegir entre dos juegos de apuestas de los cuales se muestran pérdidas y ganancias (Kahneman & Tversky, 2000). Considere el siguiente problema:

> Las ganancias observadas de *n* repeticiones de un juego A (x_n) y *m* del juego B (y_m) son:
> Juego A: $x_1, x_2 \dots ; x_n$
> Juego B: $y_1, y_2 \dots ; y_m$
> ¿En cuál de los dos juegos participarías?

Una solución puede ser la siguiente: 1) comparar las medias aritméticas de ambos juegos ($\bar{x}$ y $\bar{y}$); 2) si $\bar{x} \neq \bar{y}$, elegir el juego cuya media es mayor; 3) si $\bar{x} = \bar{y}$, se tienen dos opciones: 3a) elegir cualquier juego; 3b) analizar la dispersión de los datos en cada juego y elegir uno de acuerdo con las preferencias hacia el riesgo. Estas preferencias pueden ser definidas como generalizaciones de las actitudes hacia el riesgo:

En general, la preferencia por un resultado seguro y el rechazo de un juego cuyo resultado tiene un valor esperado igual o mayor a dicha ganancia es llamada aversión al riesgo. Y el rechazo de una ganancia segura y la aceptación de un juego cuyo resultado tiene un valor esperado menor o igual a esa ganancia es llamada propensión al riesgo (Kahneman & Tversky, 2000, p. 2).

En un juego la dispersión de las ganancias (incluidas las pérdidas) puede ser considerada una medida de riesgo: entre mayor dispersión más riesgo. Una persona adversa al riesgo preferirá un conjunto de datos menos disperso en lugar de otro cuyos datos tengan mayor dispersión, mientras que una persona es propensa al riesgo cuando prefiere la opción cuyos datos son más dispersos.

Método

En este estudio participaron 97 profesoras en formación de una escuela normal pública de la Ciudad de México que cursan la Licenciatura en Educación Preescolar (atención a niños de 3-6 años). Para explorar las ideas de las futuras profesoras se utilizó un cuestionario con dos problemas sobre comparaciones de conjuntos de datos (ver Figura 1).

Problema 1. En una feria, se invita a los asistentes a participar en uno de dos juegos. Juan puede participar en un juego, pero no en ambos. Para saber por cuál decidirse observa, anota y ordena los resultados de dos muestras de 10 personas que han participado en cada juego. Las pérdidas (-) o premios (+) en efectivo que han obtenido las 20 personas se muestran en las siguientes listas:

Juego 1:

15 -21 -4 50 -2 11 13 -25 16 -4

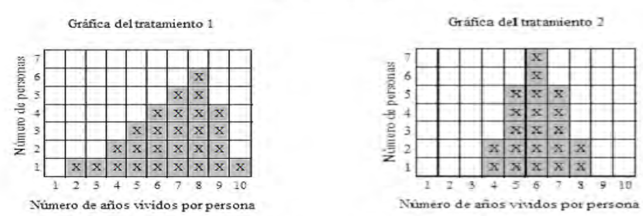
Juego 2:

120 -120 60 -24 -21 133 -81 96 -132 18

a) Si tienes la posibilidad de participar en un solo juego, ¿Cuál juego elegirías? ¿Por qué?

Problema 2. Considera que debes aconsejar a una persona que padece una enfermedad grave, incurable y mortal, pero que es tratable con medicamentos que pueden extender la vida por varios años. Es posible elegir entre dos tratamientos. Las personas tienen diferentes reacciones a las medicinas, para algunas tienen el resultado previsto, mientras que para otras pueden ser más o menos benéficas. A continuación se muestran los años que han vivido varios pacientes tratados con cada una de las opciones mencionadas; cada dato de los que se muestran corresponde al tiempo que ha sobrevivido un paciente con el respectivo tratamiento. Después se muestran las gráficas correspondientes a los tratamientos.

Tratamientos



a) ¿Qué tipo de tratamiento preferirías (1 o 2)? ¿Por qué?

Figura 1. Problemas resueltos por las profesoras.

El cuestionario fue resuelto por las profesoras en formación previo a iniciar el curso de procesamiento de información estadística. Los problemas tenían un inciso donde se plantea una situación de toma de decisiones. En el problema 1 se dan las ganancias y pérdidas de dos juegos y se pide elegir el juego en el que más convendría jugar y en el problema 2, los tiempos de años vividos en forma gráfica de dos grupos de pacientes después de someterse respectivamente a uno de dos tratamientos, y se solicita decidir cuál es el mejor tratamiento. En el primer problema las *medias aritméticas* de los conjuntos de datos son iguales mientras que en el segundo son diferentes. En ambos casos es importante la interpretación de la dispersión asociada con el riesgo para justificar la elección. A continuación se comentan los resultados obtenidos en cada uno de los problemas resueltos por las profesoras, se inicia este apartado presentando las respuestas al problema 1 y después se muestran las correspondientes del problema 2. Para analizar las respuestas en primer lugar

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

se observó la decisión que tomaron, es decir, el conjunto de datos que eligieron y en segundo lugar, se categorizaron las respuestas con base en las estrategias de comparación que describieron en sus justificaciones siguiendo las sugerencias de Birks y Mills (2011).

Resultados

Problema 1

La solución normativa del problema 1 consistiría en comparar las *medias* y posteriormente considerar la dispersión (que a través del rango sería suficiente). En el caso de que se perciba el riesgo en ambos juegos, la opción elegida dependerá de las actitudes del riesgo de quien resuelve el problema: Elegirían el juego 1 aquellos que son adversos al riesgo, mientras que optarían por el juego 2 los propensos al riesgo. Las frecuencias con las que se eligió alguna de las opciones fueron las siguientes: 58 (de 97) futuras profesoras eligieron el juego 1, mientras que 31 de ellas seleccionaron el juego 2, y 5 participantes respondieron que elegirían cualquiera y las 3 restantes no respondieron. Ninguna de las argumentaciones para dichas elecciones siguió el esquema de razonamiento que se describió en el párrafo precedente; aunque algunas se aproximaron. Un procedimiento común en todas las estrategias consistió en sumar las ganancias de cada juego (los valores positivos) y agregar sus pérdidas (valores negativos pero sin considerar el signo), obteniéndose cuatro valores G_1, G_2, P_1, P_2. La forma en que combinaron estos valores produjo las siguientes comparaciones:

Comparación de la diferencia entre ganancias y pérdidas. En 30 casos, la estrategia consistió en encontrar la ganancia global cada uno de los juegos. Esto mediante la comparación de las diferencias entre ganancias y pérdidas: $G_2 - P_2 = G_1 - P_1 = 49$. Este procedimiento prefigura el uso de la *media*. 16 alumnas participantes eligieron el juego 1, 8 normalistas el juego 2 y 6 futuras profesoras mencionaron que cualquiera. Por ejemplo, una estudiante argumenta: “al hacer las operaciones la diferencia de ambos juegos entre perder y ganar es de 49”. Aquí advertimos que no se tiene en cuenta la dispersión de los datos ni consideraciones de riesgo.

Comparación de la suma de pérdidas o ganancias. 19 respuestas se basaron, ya sea en la comparación de la suma de las ganancias (eligieron el juego 2, porque $G_2 > G_1$) o en la comparación de la suma de las pérdidas (eligieron el juego 1 porque $P_1 < P_2$, pues $-P_1 > -P_2$). En algunas de estas respuestas se percibió el riesgo. 9 participantes eligieron el juego 1 y 10 participantes el 2. Por ejemplo una participante eligió el juego 1 y la justificación de su elección fue la siguiente: “Existe la posibilidad de poder obtener una ganancia ya que de acuerdo con los resultados de las muestras del juego 1 las ganancias fueron de 105 y pérdidas de 56, pero en el segundo juego las ganancias fueron de 427 y las pérdidas de 378. En conclusión en el primer juego se perderá menos que en el segundo aunque los premios sean mejores en el segundo”. Mediante esta argumentación consideramos que la justificación de la elección fue con base en la comparación de la suma de las pérdidas, observando que son menores en el juego 1, y descubrimos aversión al riesgo ya que en la justificación se comenta “se perderá menos”.

Comparación de relaciones proporcionales entre ganancias y pérdidas. En 6 casos se compararon relaciones proporcionales entre pérdidas y ganancias, notando que es mayor la del juego 1: $\frac{G_1}{P_1} > \frac{G_2}{P_2}$, por tanto, optaron por este juego. Dado que las *medias* son iguales, la anterior desigualdad se reduce a $P_1 > P_2$, en el fondo esta estrategia consiste en elegir el juego 1 porque se pierde menos. Un ejemplo de este tipo de respuestas fue mencionada por una estudiante, quien mencionó: “porque por los datos se refleja que en este juego hay más probabilidades de salir ganador ya que el número de ganadores casi duplica el de perdedores y aunque fue menos cantidad lo ganado que en el juego 2, en el 1 es más seguro ganar aunque sea poco, y yo no elegiría el juego 2 porque aunque se ganan cantidades más grandes de igual forma se pierde mucho”. En este ejemplo se observa, por un lado el uso de la razón entre las ganancias y las pérdidas y, por otro la aversión al riesgo ya que en parte del argumento se comenta que "es más seguro ganar aunque sea poco".

Comparación de las medias aritméticas. Sólo hubo 2 respuestas en las que el juego elegido fue el 1, donde mostraron el uso de la media aritmética para hacer la comparación entre los juegos, en una de ellas además de presentar la media aritmética (4.9) con la justificación siguiente: "Las cantidades son más bajas, están más cercas de la media". En el ejemplo se advierte, además de la mención de la media, la noción de la dispersión al elegir el juego 1 con valores cercanos al centro del conjunto de datos.

Problema 2

La solución normativa del segundo problema puede reducirse al cálculo de las *medias* de los tiempos de vida de cada tratamiento, notando que los datos del tratamiento 1 tienen mayor *media* (6.7) que los del tratamiento 2 (6). Con lo anterior, también puede elegirse el tratamiento 2, considerando la dispersión mediante el rango e interpretándolo como riesgo. Se creería que el riesgo con el tratamiento 1 (rango = 8) es mayor que el riesgo con el tratamiento 2 (rango = 4) y que la disminución en el riesgo podría compensar la diferencia entre las medias. En este último caso la elección estaría motivada por una aversión al riesgo. En este problema 60/97 profesoras en formación eligieron el tratamiento 1 y 37/97 el tratamiento 2. En los argumentos que justifican las elecciones de las participantes se pueden identificar las siguientes estrategias de comparación:

Comparación de centros. En 31 respuestas se compararon los valores modales observados en cada gráfica. En 19 de ellas se eligió el tratamiento 1, posiblemente comparando las modas de los conjuntos de datos (8 > 6), por ejemplo, una justificación fue: "hay más probabilidades de vivir más años (8 aprox.)". En 12 casos eligieron el tratamiento 2 probablemente con base en la proporción de personas que vivieron seis años, por ejemplo: "aquí me garantizan 7 personas que van a vivir 6 años seguros, sin embargo en el "1" 6 personas viven 8 años, es seguro pero yo voy más por el número de personas que tomaron el tratamiento". En esta estrategia aunque se tienen en cuenta los centros de los conjuntos de datos, se ignora la variación de los datos.

Comparación de valores extremos. 19 participantes eligieron con base en uno de los valores extremos. Cuando el tratamiento elegido fue el 1, en 8 respuestas se argumentó que con ese tratamiento se podrían vivir hasta 10 años; 2 estudiantes justificaron dicha elección indicando que vivirían por lo menos 1 año. En 9 casos se eligió el tratamiento 2, justificando que se vivirían por lo menos 4 años. Un ejemplo de este tipo de respuestas donde se eligió el tratamiento 1 fue: "El tiempo vivido en años por persona es mayor; se puede llegar a vivir 9 o incluso 10 años, lo que en el otro tratamiento no". En las respuestas donde el tratamiento elegido es el 1, es probable que la elección sea motivada por una propensión al riesgo, ya que se menciona que se puede vivir hasta 10 años; mientras que las respuestas donde el tratamiento elegido fue el 2, es probable que sean motivadas por una aversión al riesgo, pues comentan que al menos pueden vivir 4 años.

Comparación de cardinalidad. Doce estudiantes que eligieron el tratamiento 1 basaron su elección en la cardinalidad del conjunto de datos (27 > 21) y entre sus argumentos comentaban que más personas habían vivido con esa opción. Por ejemplo: "porque hay más probabilidad de que funcione ese tratamiento ya que fueron 27 las personas beneficiadas"

Comparación con el rango. En 6 respuestas se hizo alusión al rango. En estas respuestas se ponderó el riesgo, aunque de manera confusa, por ejemplo: "prefiero probar el tratamiento donde hay una mayor probabilidad de obtener un resultado más próximo o al menos que si te asegure o muestre que vivirás mínimo dos años más o máximo 10 años". En 4 respuestas el tratamiento elegido fue el 1, mientras que 2 eligieron el 2 con el argumento del hecho de asegurar un periodo de vida de 4 a 8 años.

Comparación de centro y extremo. En 4 respuestas se eligió el tratamiento 1, en uno de estos casos se expresa que se podían vivir hasta 10 años y en promedio 8 (en realidad es la moda), y en 2 restantes se eligió el tratamiento 2, probablemente, considerando que por lo menos vivirían 4 años y en promedio 6 con la justificación: “porque en el tratamiento 1 la mayoría de personas lograron vivir 8 años y en el tratamiento 2 la mayoría sólo logró vivir 6 años y no hay ni una persona que haya vivido 10 años, al contrario del primer tratamiento que una persona ha logrado vivir 10 años. Por lo tanto si ya una persona logró los 10 años la persona que necesita el consejo puede también vivir 10 años”. En la respuesta es claro que la estudiante observó centros (valor *modal*) y extremos para tomar una decisión, quizás también la preferencia por el riesgo sea la propensión, porque con ese tratamiento “1 persona ha logrado vivir 10 años”.

Comparación del centro y la dispersión. En una respuesta se combinaron centro y dispersión para justificar la elección. El tratamiento elegido fue el 2 y la justificación: “El tratamiento fue más funcional ya que se tiene que las personas viven aproximadamente de 4 a 8 años, con mayor influencia a los seis años pues son años más concretos, donde se puede planear muy bien su vida” .

En los resultados obtenidos 38 de las 97 participantes que seleccionaron el problema 1 y 28 de las 97 que eligieron el problema 2 no mostraron una justificación clara al elegir entre los conjuntos de datos. Por ejemplo en el problema 1seleccionaron el juego 2 y mencionaron “porque en el juego 2 se tienen más probabilidades de ganar más cantidad de objetos que en el juego 1”, sin embargo no encontramos una estrategia evidente que represente lo mencionado, además del uso de la palabra *objetos* que no corresponde con el contexto planteado. En el problema 2, las participantes sólo explicaron su elección diciendo “con alguno de los tratamientos se vive más”, pero sin ofrecer argumentos que muestren cómo se usaron los datos.

Conclusiones

En el problema 1 casi 40% de las respuestas fueron confusas; sin embargo en algunas de éstas se advirtió que las futuras profesoras ponían atención a las diferencias entre los valores de cada conjunto de datos, lo cual es punto de partida en la percepción de la variación. Así, en el problema de los tratamientos médicos se presentan varias respuestas en las que se comparan elementos aislados de cada conjunto (los máximos, los mínimos o las modas), y aunque no siempre fueron combinados de manera adecuada, puede sugerirse una estrategia que incluya todos los datos o combinaciones de éstos. En el problema de apuestas probablemente las respuestas en las que se afirma que se gana más o se pierde menos, la atención de las participantes también se enfocó en elementos aislados (en particular en los valores extremos).

La estrategia de sumar las ganancias y luego las pérdidas tiene la característica de que se consideran todos los datos. En el problema 2 varias futuras docentes usan relaciones proporcionales en las que combinan y hacen uso de más de un valor del conjunto de datos, ya sea operando con ellos o mencionándolos de manera explícita. La comparación de las ganancias totales prefigura la respuesta más sofisticada consistente en comparar las medias, lo que no es una estrategia espontánea ni fácil de elaborar por parte de los alumnos (Gal, Rothschild & Wagner, 1989), y la cual pudo observarse en las futuras docentes. En la enseñanza se insiste en la importancia de la proporcionalidad, y por ello considerarla como estrategia indispensable para resolver cualquier problema, y en nuestra investigación fue empleada en la comparación de las razones en el problema 1. Consideramos que no es una estrategia del todo inadecuada, pues las llevó a elegir el juego en el que la pérdida es menor. Esta manera de resolver es más apropiada en el problema 2, en la que comparan proporciones de los valores modales.

Son pocas las respuestas basadas en la consideración de los rangos (6%) o influenciadas por la percepción del riesgo (problema 2). Lo que mostraron las participantes en relación con los dos problemas es que son muy diferentes, pues ninguna de ellas adaptó en el problema 2 la estrategia seguida en el problema 1; esto nos lleva a suponer que el contexto y el formato de presentación de los datos ejercen mayor efecto que la estructura (oculta) del problema. Asimismo, en el problema 1, en general, hubo revisión de los datos, mientras que en el problema 2, en 20 casos se eligió un tratamiento sin ofrecer ninguna justificación que incluyera un procesamiento de los datos debido, quizá, a la dificultad de extraer los datos numéricos de la gráfica (valores extremos de cada conjunto de datos y aparente ausencia de la lista de datos). En cambio, en este problema 2 las estudiantes ponderaron el riesgo en la forma de elegir el tratamiento 1: "se puede vivir 10 años" o tratamiento 2: "al menos se viven 4 años".

De los resultados mostrados se percibe que al igual que en investigaciones previas (Canada & Makar 2006; Mooney et. al, 2014) es necesario promover en las profesoras en formación, conceptos estadísticos como *centro*, *variación*, *distribución, valores esperados.* Las estrategias a seguir deben ser múltiples (discusiones, talleres, uso de la tecnología, resolución de problemas enmarcados en diferentes situaciones, formatos variados de presentación de los datos, entre otros) para que las futuras profesoras se apropien de estos conceptos. Además, las educadoras mexicanas en servicio deben abordar en el aula la agrupación de objetos según sus atributos, cualitativos o cuantitativos; la recopilación y representación apropiada de datos e información, así como su interpretación (SEP, 2011); los procesos anteriores requieren de la consideración de la variabilidad (Franklin, Kader, Mewborn, Moreno, Peck, Perry & Scheaffer, 2007) por lo que es necesario que desarrollen este concepto. Esta investigación aporta información sobre el conocimiento de las profesores en formación acerca de la variabilidad, concepto estadístico importante (Spangler, 2014), y mostramos, de manera incipiente, que los problemas con los cuales hemos explorado el razonamiento de las profesoras en formación conducen a percibir la variación dándole un significado asociado al riesgo para presentar estrategias que las guiaron a usar la *media* y el *rango* de manera significativa y a motivarlas a planear actividades que propicien dar sentido y significado a los procedimientos basados en el uso de la media y la dispersión.

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Wild, D. J. & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review, 67*(3), 223-265.

The aim of this investigation is to explore the preservice teachers´ reasoning about variation (variability or spread) when they analyze data in situations that involve risk. In particular, in this communication the responses to two problems of a questionnaire administered to 96 preservice teachers are reported. The problems are of comparing groups of data in situations of risk: stakes in games and the life expected after medical treatments. The questionnaire was applied before the preservice teachers began a course of statistical information processing and the results showed the difficulty found by students to interpret variation in this type of contexts. For these results it is necessary to reflect on the instruction of future teachers about the meanings of measures of center dispersion and dispersion to contribute to an improvement in their academic training.

Introduction

Variation is the underlying cause of the existence of statistics and, given its omnipresence, it is also found in data sets (Watson, 2006). Moore (1990) highlights the importance of measuring and modelling variation while Wild and Pfannkuch (1999) include the perception of variation as part of the fundamental types of statistical reasoning. Additionally, Garfield and Ben-Zvi (2008) consider that "Understanding the ideas of spread or variability of data is a key component of understanding the concept of distribution, and is essential for making statistical inferences" (p. 203). For their part, Burrill and Biehler (2011) propose a list of seven fundamental statistical ideas in which variation is placed in the second position. Regarding the school perspective, Franklin, Bargagliotti, Case, Kader, Sheaffer and Spangler (2015) consider that teachers must identify the characteristics of statistics; they must communicate it clearly and, particularly, they should highlight variability and the role of the context. In the description of variability, they have to consider that data are constituted by a structure (mean or median) around which they vary. To that respect, Canada and Makar (2006) found that, when solving problems on distributions, preservice teachers have an intuitive perception of variation. They describe it using informal language while the mean is rarely used. Other researchers, as Mooney, Duni, VanMeenen and Langrall (2014), state that when exploring on the perception of variability in chance situations, preservice teachers identify a certain amount of variability must be present, but have no certainty about how much. From the preceding researches, it is necessary to provide future teachers with experiences in both data analysis and chance situations in which they can develop concepts as: *center, variation, distribution, expected values* and the relations between them.

To explore the students' comprehension and reasoning regarding the perception, description and measurement of data variation, several contexts and problems have been used; among them are: sampling variability (Watson & Moritz, 2000), chance (Watson & Kelly, 2004), repeated measures, variation in growth of plants (Lehrer & Schauble, 2007; Petrosino, Lehrer & Schauble, 2003) and weather (Reading, 2004). According to these researchers, risk situations provide another scenario to explore the students' reasoning on variability (Sánchez & Orta, 2013). Therefore, the aim of this work is to explore the way in which preservice preschool teachers interpret the spread of data in risk situations, so that they know fundamental statistical concepts towards their professional improvement. For this reason, it is of great importance to include the knowledge of statistical concepts in their education; such concepts will allow them to promote collection, representation and interpretation of information at preschool level (SEP, 2011). In addition, we must consider that these concepts will be taught to their students in other educational levels (Ball, Thames & Phelps, 2008).

Reference Framework

This exploration is located within the field of statistical reasoning whose approach is to understand how people reason using statistical ideas (Garfield & Ben-Zvi, 2008). We seek to propose characteristics to create learning scenarios since the participants of an investigation show the elements they consider important—particularly, the chosen data, operations done, beliefs and knowledge—when they try to justify their responses. Although the persons' responses are often not so explicit as to clearly reveal their reasoning, they still show signs to identify some of their features. In this study, we identified some of the characteristics of the preservice teachers' reasoning when they face risk situations.

Problems are a key component in an investigation on mathematics didactics. When solving them, they must promote the ability of thinking and reasoning in people to provide the researcher with relevant results that contribute with information to the field of work. A problem should also attract those who solve it, so that they engage with the solution and increase the probabilities of understanding the studied concept. In statistics, reasoning must articulate ideas, as median or spread, expressing those using numbers; that is, with real situations based on data. Statistical reasoning is closely related to the contexts and numbers in context involve information (Moore, 1990). Problems regarding decision making under uncertainty are common in statistics. This type of problem has been used to promote and analyze relevant characteristics of people's statistical reasoning. In addition, those situations that demand the comparison of data sets are frequently used to involve students in reasoning with data (Garfield & Ben-Zvi, 2008). In this work, we present two situations involving decision making and data set comparison in which spread is important since it might be associated to the notion of risk which, in turn, might be linked to the uncertainty in an event that involves a threat. These situations arise when there are unwanted results that cause, in consequence, losses or damages. A paradigmatic problem in a risk scenario consists of choosing between two gambling games that show losses and gains (Kahneman & Tversky, 2000). Consider the following problem:

The gains observed in n repetitions of a game A (x_n) and m of game B (y_m) are:
Game A: $x_1, x_2 \ldots ; x_n$

Game B: $y_1, y_2 \dots ; y_m$
In which of the two games would you take part?

A solution to the problem might be: 1) comparing the arithmetic means of both games ($\bar{x}$ and $\bar{y}$); 2) if $\bar{x} \neq \bar{y}$, choose the game whose mean is greater; 3) but if $\bar{x} = \bar{y}$, 3a) choose any game or 3b) analyze the spread of data in each game and choose according to the preferences towards risk. These preferences can be defined as generalization of the attitudes towards risk:

> In general, a preference for a sure outcome over a gamble that has higher or equal expectation is called risk aversion, and the rejection of a sure thing in favor of a gamble of lower or equal expectation is called risk seeking (Kahneman & Tversky, 2000, p. 2).

In a gamble, the spread of gains (including losses) can be considered a measure of risk: greater spread, greater risk. A person averse to risk will choose a data set with lower spread instead of one whose data have a greater spread. In contrast, a risk-seeking person will choose a data set with a greater spread.

Method

The participants in the study were 97 preservice teachers from a public teacher training school in Mexico City who study a Bachelor of Preschool Education (care of children aged 3–6 years). A questionnaire including two problems regarding comparison of data sets (see Figure 1) was used to explore the preservice teachers' ideas.

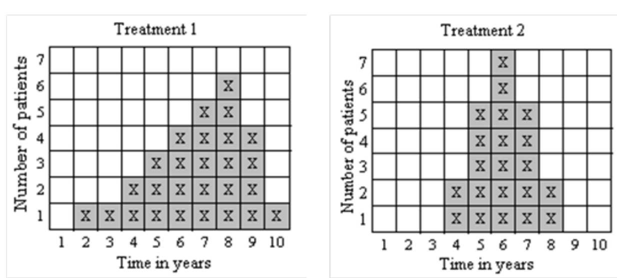
Problem 1.In a fair, the attendees are invited to participate in one of two games, but not in both. In order to know which game to play, John observes, takes note and sorts the results of 10 people playing each game. The cash losses (–) or prizes (+) obtained by the 20 people are shown in the following lists:

Game 1: 15, –21, –4, 50, –2, 11, 13, –25, 16, –4

Game 2: 120, –120, 60, –24, –21, 133, –81, 96, –132, 18

a) If you could play only one of the two games, which one would you choose? Why?

Problem 2. Consider you must advice a person who suffers from a severe, incurable and deathly illness, which may be treated with a drug that may extend the patient's life for several years. It is possible to choose between two different treatments. People show different effects to the medication: while in some cases the drugs have the desired results, in some others the effects may be more favorable or more adverse. The graphs corresponding to the treatments are shown below.



a) Which treatment would you prefer (1, 2 or 3)? Why?

Figure 1. Problems solved by the teachers.

The questionnaire was answered by the preservice teachers before they took the course on statistical information processing. The problems included a section in which a decision-making situation is posed. In problem 1, the gains and losses in two gambles are given and the person is asked to choose the most convenient gamble. In problem 2, the person is asked to graphically choose the times of years lived by two groups of patients after they underwent either of two treatments; the person is asked which the best treatment is. In the first problem, the arithmetic means of the data sets are equal while they are different in the second problem. In both cases, interpreting the spread associated to risk is important to justify the choice.

Below we discuss the responses obtained for each of the problems solved by the teachers. The section starts with the responses to problem 1 and then, those obtained for problem 2. To analyze the responses, firstly we observed the decision the participants made, that is, the data set they chose. Secondly, we categorized the responses based on the strategies of comparison the teachers describe in their justification, as suggested by Birks and Mills (2011).

Results

Problem 1

The normative answer to problem 1 would consist in comparing the means and then, considering the spread (considering it through the range would be enough). In case risk is perceived in both gambles, the option chosen will depend on the risk attitudes of the person solving the problem: those averse to risk would choose gamble 1 while gamble 2 would be chosen by risk-seeking persons. The frequencies corresponding to the options were as follows: 58 (out of 97) preservice teachers chose gamble 1 while only 31 of them chose gamble 2. Only 5 participants responded they would choose any gamble and 3 teachers did not answer. No argumentation for the choices followed the reasoning scheme described in the previous paragraph, even though some arguments came close. A common procedure to all the strategies was adding the gains of each gamble (positive values) as well as the losses (negative values without considering the sign), thus obtaining four values $G1, G2, P1, P2$. The way in which these values were combined produced the following comparisons:

Comparison of the difference between gains and losses. In 30 cases, the strategy consisted in finding the global gain in each of the gambles by comparing the differences between gains and losses: $G2 - P2 = G1 - P1 = 49$. This procedure prefigures the use of the mean. 16 participants chose gamble 1 while 8 preservice teachers chose gamble 2 and 6 answered they would choose any. For example, a student argues: "when doing the operations, the difference in the two gambles between gaining and losing is 49." Here we observe they do not take into account the spread of data nor risk considerations.

Comparison of the sum of gains or losses. 19 teachers based their response whether on the comparison of the sum of the gains (they chose gamble 2 because $G2 > G1$) or on that of the sum of losses (they chose gamble 1 since $P1 < P2$, given that $-P1 > -P2$). Risk was perceived in some of these responses: 9 of the participants chose gamble 1 while 10 chose the second one. One of the participants who chose gamble 1 justified her choice by saying: "There is the possibility of getting a gain since, according to the results of the samples of gamble 1, there were 105 gains and 56 losses, but in the second gamble there were 427 gains and 378 losses. In conclusion, in the first gamble there will be fewer losses than in the second one; although the prizes are better in the second one." Through this argumentation we consider that the choice was based on the comparison of the sum of losses: her perception was that losses are lower in gamble 1. We found risk aversion since, in her justification, the participant says "there will be fewer losses".

Comparison of proportional relationships between gains and losses. In 6 cases, the participants compared proportional relationships between gains and losses, noticing that the one of gamble 1 is greater: $\frac{G_1}{P_1} > \frac{G_2}{P_2}$. That is why they chose this gamble. Given that the means are equal, the previous inequality is reduced to $P1 > P2$. In reality, this strategy aims to choosing gamble 1 because there are fewer losses. As an example, one of the students justified her response by stating: "because from the data, there is a greater probability of winning in this gamble since the number of winners almost doubles the one of the losers and even if the gain was lower than in gamble 2, winning—even if it is a little—is more certain in 1 and I would not choose gamble 2 because although bigger amounts are won, losses are high too." This is an example of both the use of reason between gains and losses and risk aversion, since the argument includes the statement "winning—even if it is a little—is more certain".

Comparison of arithmetic means. Only 2 responses showed the use of arithmetic mean to compare the gambles, and gamble 1 was chosen in both cases. One of the responses showed the arithmetic mean (4.9) and justified the response by saying "the numbers are lower and closer to the mean". In the example we see that besides mentioning the mean, the participant has a notion of spread when she chooses gamble 1, which has values that are closer to the center of the data set.

Problem 2

The normative answer to the second problem can be reduced to the calculation of the means in the life expectancy for each treatment, considering that the data of treatment 1 have a greater mean (6.7) than those of treatment 2 (6). Therefore, treatment 2 could also be chosen if one considers the spread using the range and interprets it as risk. Risk from undergoing treatment 1 (range = 8) might be thought to be greater than that from treatment 2 (range = 4) and that the decrease in the risk might compensate for the difference between the means. In this last case, the choice would be motivated by risk aversion. For this problem, 60/97 of the preservice teachers chose treatment 1 while 37/97 chose treatment 2. In the arguments that justify the participants' choices, we identify the following comparison strategies:

Comparison of centers. In 31 responses, the modal values observed in each graph were compared. In 19 of the responses, the participants chose treatment 1, possibly comparing the modes of the data sets (8 > 6). For instance, one of the teachers justified her response by saying: "there is a greater chance of living longer (8 approx.)". In 12 cases, the teachers chose the second treatment possibly based on the proportion of people who lived six years; for example: "7 persons guarantee that they will live 6 years for sure; however, 6 people live for 8 years in "1". That is for sure but I choose based on the number of persons who were treated". Even though the strategy considers the centers of the data sets, it ignores the data variation.

Comparison of extreme values. In 19 cases, the participants made their choice based on the extreme values. When treatment 1 was chosen, 8 of the responses argued that the treatment would extend the patients' lives for up to 10 years. Two students justified their choice pointing out they would live at least a year. In 9 cases, the students chose the second treatment and they justified their response by saying they would live for four years at least. An example of the type of response in which treatment 1 was chosen was: "The time in years a person lived is longer; a person can live 9 or 10 years; something that does not happen with the other treatment". In the responses where treatment 1 was chosen, the choice might have been motivated by risk seeking since the response states a person can live up to 10 years. In contrast, those responses in which the second treatment was chosen were probably motivated by risk aversion, given that they refer patients live at least for four years.

Comparison of cardinality. Twelve participants who chose treatment 1 based their response on the cardinality of the data set (27 > 21) and, among their arguments, they considered that a higher number of people had lived with that choice. For example: "because there is a greater chance that this treatment works since 27 persons were benefited".

Comparison with range. In six responses, the students referred to range. Although confusingly, they considered the risk; for example: "I prefer to take the treatment that has the higher probability of getting a close result, or at least that ensures you'll live at least two years or a maximum of 10 years". In four responses, the students chose the first treatment while they twice chose the second treatment, arguing that it would ensure a survival period from four to eight years.

Comparison of center and extreme. The students chose the first treatment in four responses. One of these responses expresses a person could live up to 10 years and eight in average (in reality, the response refers to the mode). Treatment 2 was chosen in two responses, probably considering a patient would live four years at least and six years in average. This response was justified as follows: "because in treatment 1, most of the people manage to live eight years and with treatment 2, most only live six years and there is no one who lived 10 years, unlike the first treatment [with which] a person has managed to live for 10 years. So, if a person has already managed for 10 years, the person in need of advice might also live for 10 years". From the response, it is evident that the student observed centers (modal value) and extremes to make a decision. Risk seeking is probably driving the response since the student stated that using the treatment "a person has managed to live for 10 years".

Comparison of center and spread. In a response, the student combined center and spread to justify her choice. The student chose the second treatment and justified her response by stating: "The treatment was more functional since we have that the people live approximately from four to eight years and, more frequently, six years because they are more solid years, where life can be very well planned".

In the results obtained, 38 out of 97 participants who chose problem 1 and 28 out of 97 who chose problem 2, did not show a clear justification when choosing between the data sets. For example, in problem 1, they selected gamble 2 and considered: "because there is a greater possibility of winning a higher number of objects in gamble 2 than in gamble 1." However, we found no evident strategy to represent what was stated. Additionally, the use of the word objects does not correspond to the context laid out. In problem 2, the participants only justified their response by saying "one lives longer with one of the treatments", but provided no arguments to show how the data were used.

Conclusions

In problem 1, nearly 40% of the responses were confusing. However, in some of them we see that the preservice teachers paid attention to the differences between the values in each data set, which is a starting point for the perception of variation. Thus, in the problem of medical treatments, there are several responses which compare isolated element from each set (the maximums, the minimums or the modes) and, although they were not always combined in the correct way, a strategy including all the data or a combination of them might be suggested. In the gambling problem, the attention of the participants was probably focused on isolated elements—on extreme values, particularly—in those responses stating the gains are higher or the losses are lower.

The strategy of adding the gains and then the losses considers all the data. In problem 2, several preservice teachers used proportional relationships in which they combined and used more than one value from the data set by operating with them or mentioning them explicitly.

The comparison of the total gains prefigures the most sophisticated response consistent in comparing means, which is not a spontaneous nor easy strategy to create by the students (Gal, Rothschild & Wagner, 1989), but was observed among the preservice teachers.

When teaching, the importance of proportionality is highlighted; therefore, it is considered a necessary strategy to solve any given problem; in our research, it was used in the comparison of reasons in problem 1. We consider it is not an entirely incorrect strategy because it led the students to choose the gamble in which the loss is lower. This form of solving is more adequate in problem 2, in which the students compare the proportions of the modal values.

There are few responses based on the consideration of the ranges (6%) or influenced by the perception of risk (problem 2). With respect to the problems, the participants showed that they are different from one another, given that no student adapted the strategy followed in problem 1 to problem 2. This leads us to suppose that the context and the format of the presentation of the data have a greater effect than the (hidden) structure of the problem.

In general, the data were reviewed in problem 1 while 20 participants chose a treatment in problem 2 without providing a justification that included an adequate data processing. This situation was likely due to the difficulty of extracting the numerical data from the graph (extreme values from each data set and apparent absence of data list). In contrast, in problem 2 the students considered the risk when choosing treatment 1 "can manage to live for 10 years", or treatment 2, with which patients live for four years at least.

From the results shown, we observe that, as in previous research (Canada & Makar, 2006; Mooney et al., 2014), statistical concepts as center, variation, distribution and expected values in problems, must be promoted in preservice teachers. The strategies to be followed should be multiple (discussions, workshops, use of technology, solving problems with different contexts, and varied ways of presenting data, among others), so that preservice teachers appropriate these concepts. In addition, Mexican preschool teachers in service should deal with grouping objects, according to qualitative and quantitative characteristics, in the classroom. They should also address the collection and the adequate representation and interpretation of data and information (SEP, 2011). These processes demand considering variability (Franklin, Kader, Mewborn, Moreno, Peck, Perry & Scheaffer, 2007); therefore, teachers should develop this concept.

This research contributes with information regarding preservice teachers' knowledge on variability, a relevant statistics concept (Spangler, 2014). We incipiently show that the problems with which we have explored the preservice teachers' reasoning lead to perceive variation. The problems gave variation a meaning associated to risk when the teachers presented strategies that led them to use the mean and the range in a significant way. The problems presented also lead to motivate teachers to plan activities that promote giving sense and meaning to the procedures based on the use of mean and spread.

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EXPLORING HIGH SCHOOL STUDENTS BEGINNING REASONING ABOUT SIGNIFICANCE TESTS WITH TECHNOLOGY

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In the present study we analyze how students reason about or make inferences given a particular hypothesis testing problem (without having studied formal methods of statistical inference) when using Fathom. They use Fathom to create an empirical sampling distribution through computer simulation. It is found that most student´s reasoning rely on data and assimilate natural sampling variation, which are two fundamental ideas of inference. This result represents a significant change in their natural reasoning. An important misconception is believed Fathom simulates samples of the real population instead of a hypothetical one.

Keywords: High School Education, Technology, Informal Education.

Introduction

Literature shows many difficulties on the learning of statistical inference (Batanero, 2000) Castro-Sotos, Vanhoof, Van den Noorgate, & Onghena, 2007). One possible reason is that statistics courses generally focus on teaching procedures and routine concepts and do not offer the opportunity to discuss and understand the fundamental ideas. As a consequence, students reach their first inferential reasoning experience thinking that statistics is only about the computation of numerical values. This has motivated the interest in studying Informal Statistical Inference (IEI) and Informal inferential Reasoning (IIR). Researchers have recently been exploring the idea that if students begin to develop the informal ideas of inference early in a course, they may be better able to learn and reason about formal methods of statistical inference. In this context, the simulation (as opposed to the formal calculation) can be used to begin to teach the process logic and concepts that still need on the contrast of hypotheses (Batanero & Diaz, 2015). However, there are few studies on the RII aimed at students of high school (15-18 years).

Students present a lack of perception of sampling variation (García-Rios & Sánchez, 2014) and a lack of consideration of the data (García-Rios & Sánchez, 2015). This study seeks to show how a sampling distribution simulation activity has the potential to overcome these difficulties. In addition, this proposal presents a simulation by computer that can support the development of inferential reasoning for promoting the understanding of hypothesis tests. Therefore, we are interested in the questions: How students reason on significance testing when they participate in activities using the simulation of Fathom? How would Fathom support the learning of students?

Literature Review

Recently, several studies have focused on the concept of Informal Inferential Reasoning (IIR), as confirmed by the publications of special issues; Statistics Research Journal (Pratt & Ainley, 2008), and Mathematical Thinking and Learning (Makar & Ben-Zvi, 2011). Literature shows two different approaches to study the RII: the first approach focuses on the nature of reasoning about inference given problems and statistical information, while the second approach focuses on the evaluation of the development of the RII as students undergo a course of instruction designed to develop the reasoning. In this paper we focus on the first approach.

Garcia-Rios and Sánchez (2014) show that students have a lack of consideration of data and a low probability language; students often draw inferences based on personal beliefs instead of data and conclusions do not show any degree of uncertainty. In addition, when students based on data,

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they have an inappropriate way to determine if a statistic sample is significant; if the statistic is different from the hypothesis tested then it rejects the hypothesis. A plausible cause of these difficulties is the lack of perception of sampling variation (Garcia-Rios and Sánchez, 2015). These authors also observed that Fisher's test of significance comes more natural to students because they establish a null hypothesis (personal model of the population) to compare the sample and intuitively measure their significance (although inappropriate). It is concluded that in order to develop appropriate inferences before formalizing, it is crucial for students to have an informal method to determine a numerical criterion to know when rejecting or accepting the hypothesis and the simulation seems to be a resource that provides such method.

Rossman (2008) provides a characterization of informal statistical inference and makes a distinction between informal and intuitive, although it does not define it, just exemplified it by establishing some essential features of situations and problems of statistical inference and shows how it can informal methods be used to solve them. Zeiffler, Garfield, delMas, and Reading (2008) proposed a definition of the IIR and exposed three types of activities that should be generated by the tasks to develop it; they also propose a conceptual framework to characterize the RII and develop tasks that allow you to examine the natural IIR of students, as well as the development of such reasoning.

Conceptual Framework

In this work, the conceptual framework is understood as a network of related concepts or categories that together provide a general understanding of the phenomenon of research (Miles & Huberman, 1994).

Significance Tests

There are two different points of view about hypothesis testing: a) significance tests introduced by Fisher and b) testing rules to decide between two hypotheses, which was the opinion of Neyman and Pearson (Batanero, 2000). The approximation of Fisher emphasizes the strength of the evidence provided by the data observed against a null hypothesis. The strength of evidence is captured in the p-value, which measures the likelihood of having obtained an extreme result (or more extreme) if the null hypothesis were true. Under this assumption, the sampling distribution is calculated and from this distribution p-value is estimated; if the retrieved value is very small (statistically significant) the hypothesis is rejected.

Informal Inferential Reasoning

Several papers published in the last few years refer to the concepts of ISI and IIR, however there still no consensus as to what exactly these two concepts mean. In an attempt to combine the different perspectives, Zieffler, et al. (2008) defined the IIR as the way in which students use their informal knowledge of statistics to create arguments based on observed samples that support inferences about a population unknown. To emphasize the importance of informal reasoning we remember the ideas set by Bruner in 1960 (see Heitele, 1975) who believes that it is preferable that student begin to study the subject gradually, although initially only understand it either limited or informal, rather than wait until it matures and can teach directly in more abstract or formal. Teaching is not different in a structural way in the various educational stages, but only of a linguistic form and their level of deepening.

Method

This study is part of a Hypothetical Learning Trajectory (HLT) to develop students reasoning at the high school level. In this proposal, we focus on the reasoning of the students on significance testing with the use of technology while they completed a first task of a series of four, without having studied formal methods of statistical inference.

Participants

Thirty-six 11th grade (16-17 years of age) students, grouped in 18 pairs (referred to as R1 to R18) with a computer per couple, participated in the study. The participants had not studied statistics hence they lacked basic knowledge of statistics and never worked with Fathom. This means that the activities carried out have the objective of emerging student's insights on basic knowledge about significance testing and the use of Fathom.

Instruments

Data collection was conducted through a questionnaire applied in computer, in a two-hour class session. The data set are the answers given by pairs of students in a report about the conclusion of a proportion testing hypothesis problem. This report was written on the computer. The problem says "Coca cola claims that majority (more than 50%) of the population drinking cola prefer Coca rather than Pepsi. To check, an experiment where one gave two glasses of soda (one with Coca and other with Pepsi) to 60 people selected randomly from the population was done and they should decide what liked most. The 60 participants 35 people preferred Coke. Is the Hypothesis 'over the 50% of the population who drink cola in Mexico drink prefers that Pepsi Coca" correct?' Make a report were you: a) explain what your conclusion is: b) details how you came to your conclusion step by step: c) say what so sure of your conclusion are".

Process

Fathom's simulation and the problem were presented to students during the first hour to introduce and operate the software; generate random samples and sampling distributions. In the second hour, pair students were allowed to work freely to make a report of its findings (answers) on the computer; the teacher intervened only to answer small personal questions. When reports were finished students can leave class. Fathom simulates 500 samples (size 60) taken from a hypothetical population, where the parameter can be modified by a slider. Samples are represented in a bar graph and in a table (Figure 1). The simulation is used to generate an empirical sampling distribution and measure the likelihood of the observed data with the empirical method (frequency), i.e. the informal calculation of a p-value using frequencies (Rossman, 2008). The sampling distribution is shown in a table and a graph of points.

Results

Principles and techniques of *Grounded Theory* (Birks & Mills, 2011) were used to categorize students responses. This methodology claims that it is possible to develop emerging categories of data collected and analyzed systematically. The constant comparison of the data favors a full development of the categories and their properties (advanced coding) making it analytically powerful and therefore with the capacity to explain the phenomena under study. The categories of analysis that emerged from the data were: *sample, majority in simulation, mode in simulation and proto-significance test*. Each category reflects the different types of inferential reasoning posted in the student's reports. For analysis, responses were coded with the letter R and a number (table 1).

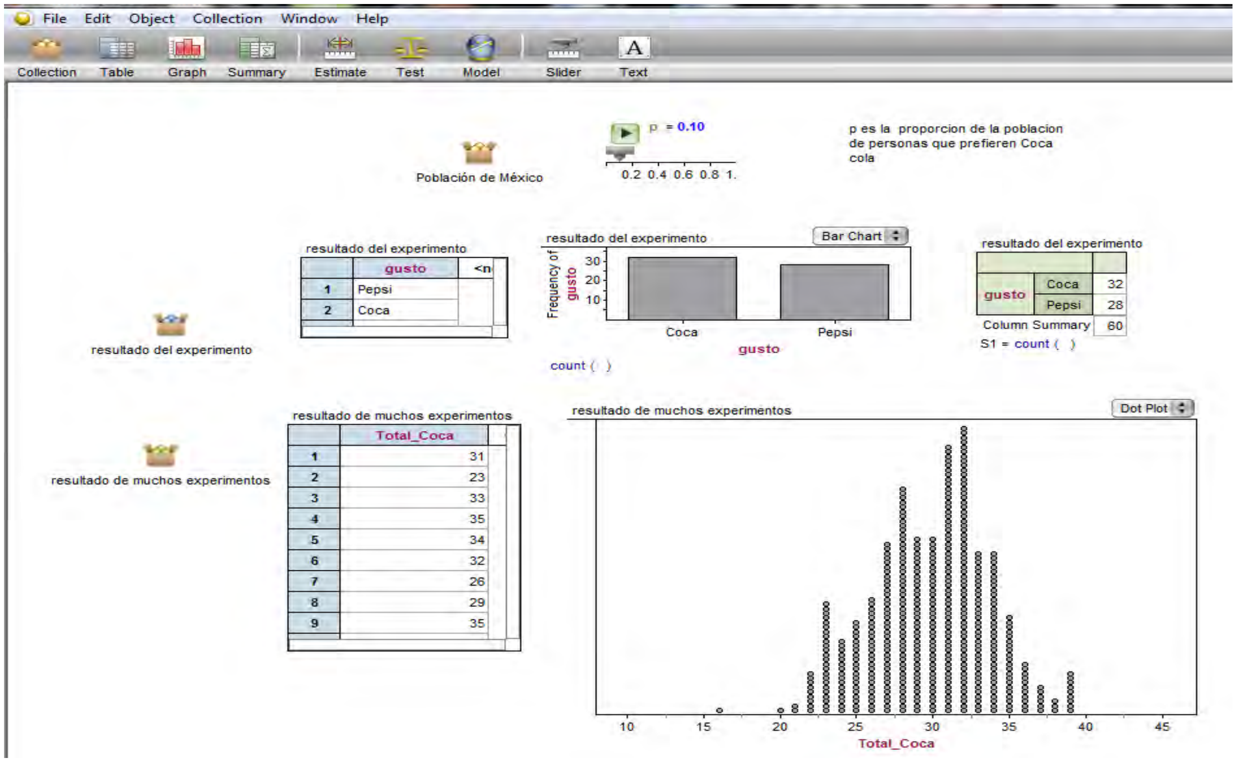

Figure 1. Fathom simulation screen.

The p-value of the statistic (0.58) is 0,098, so the hypothesis P = 0.5 is not rejected at a significance level of 5%. The categories of analysis that emerged from the constant comparison of data and explain the IIR students are: Sample, majority in simulation, mode in simulation and Proto-significance test.

Table 1: Students Reasoning

Category	Pair	Reasoning
Sample (10%)	R1, R3	If sample is bigger than 30 then hypothesis is incorrect
Majority in simulation (69%)	R2, R4, R5, R6, R7, R8, R9, R10, R11, R14, R16, R17	If majority of samples are bigger than 30 then hypothesis is correct
Mode in simulation (16%)	R18, R13, R15	If mode is bigger than 30 then hypothesis is correct
Proto-significance test (5%)	R12	Sample can occur within a population smaller than 0.5 therefore hypothesis can't be prove

Sample

Responses within this category are solely based on sample data and do not consider the simulation results. This reasoning implies an absence of the idea of sampling variation. Therefore, if the proportion of the sample is greater (or smaller) 50%, students conclude that the hypothesis "more than 50% (less than 50%)" is correct. An example of this type of response is the pair R3 who explains in his report: "... from 51% of the population we can say that it is the majority and because the result of the experiment showed that 35 people 60 prefer Coca Cola which is equal to 59% of the total (59% x 60) = 35.4) we can conclude that they are not wrong in what they claim since they are right". In its report, R3 added figure 2.

Figure 2. R3 Report.

Majority in Simulation

In this category the reasoning is to divide the sampling distribution in two regions; samples greater or equal to 30 (50%) represent the region that supports the hypothesis "most prefer Coca Cola" as correct (evidence against the null hypothesis), while less than 30 results represent the region which does not support the hypothesis. Thus, students come to their conclusion determining in which of these two regions are the most of samples. For example, R9 reason "... because if you select the rank of 37 we have 262 surveys... ". These couples add figure 3 and come to the conclusion that the hypothesis is correct. This reasoning suggests that students think that the results of the simulation with Fathom represent the actual population rather than a hypothetical, in addition, assimilate the sampling variation why they resort to determine in which region the most samples are. Some students use parameters greater than 0.5; R11 and R16 used P = 0.51, R4 use P = 0.54, and R10 and R14 use P = 0.6.

Mode in Simulation

The reasoning in this category focused on the mode of the simulated sampling distribution; If mode was greater than 30 (50%) then considered the hypothesis as correct (evidence against the null hypothesis). An example is R13 whom considered that the hypothesis is false, and reason "... our highest value in a survey was 29 people of 60 that liked more Coca-Cola, then from this we see that within that sample less than 50% like Coca-Cola". R13 added Figure 4.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

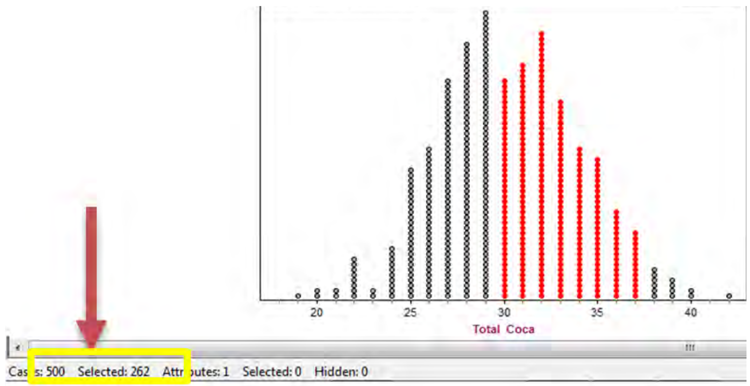

Figure 3. R9 report.

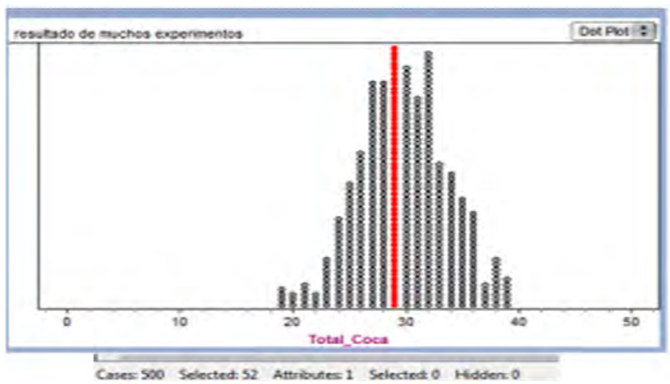

Figure 4. R13 report.

Proto-Significance Test

One interesting answer is the given by R12, whom conclude that it is not possible to test the hypothesis; "we must take a greater percentage of the population in general to the survey so we can conclude that more than 50% of the population actually likes or prefers Coca - Cola, because surveys within a range of 10 values greater and lesser around the expected value must be set" and continue "in the previous survey while the percentage is less than 50% (44%) a value of 26 is expected, I get results up to 16 (being the lowest) and 38 (being the highest) here we see a higher value that that the problem presents, where the majority of the population do not prefer Coca-Cola, so 35 does not ensure that the majority of the population like more Coca Cola". In other words, in a population less than 50% it is possible to obtain the sample; therefore the sample is not sufficient evidence to consider the hypothesis as correct.

Discussion

One of the principles of the constructivist approach applied to teaching is that for any new learning design the knowledge that the student already possesses should be used and articulated. Consequently, if it is to develop the students reasoning, is convenient to have the tools to know what knowledge and reasoning has and what are the false conceptions that limit them or blocked them. The answers to the research questions will give us knowledge to this end.

How students reason on significance testing when they participate in activities using the simulation of Fathom? The first important result is that all the arguments of the students were based on data. This is, no student based his reasoning on personal beliefs, difficulty found in (Garcia-Rios & Sánchez, 2014). The second result to highlight is the assimilation of the idea of sampling variation by the majority of pairs of students (90%). It is considered that reasoning based on simulated sampling distribution have assimilated the sampling variation in some degree, when considering regions (variation of results of samples) and decide to take a statistician; majority or mode. An example of sampling variation assimilation explicit is R14 who wrote "although 50% of the population who likes Coca-Cola has been chosen, there are surveys were Pepsi wins the results". However, this assimilation is not sufficient to choose the outcome of the sample that rejects the null hypothesis (critical value). For a 5% level of significance should be 36, while students consider 30 (50%) or 31 (52%). This difficulty was also found in (Garcia-Rios & Sánchez, 2015).

The difficulties observed in the study are: 1) Sample-based reasoning. (2) A lack of variation to estimate the region that supports the hypothesis. (3) Although students use the simulated sampling distribution they didn't understand their role; responses suggest that students think that fathom simulation represents the actual population rather than a hypothetical.

How would Fathom support the learning of students? The use of absolute values and simulation of surveys made more visible and understandable abstractions such as the sampling distribution and its process. In the traditional teaching of the hypothesis testing a transformation of the statistics (typing or standardization) and the central limit theorem is used to calculate the p-value and determine the critic zone (of rejection) with help of the normal distribution. This is perhaps one of the darker aspects of all the techniques to students. The possibility of putting the notion of sampling distribution in the center of the discussion of a significance test is probably the main contribution of the use of educational software (in this case Fathom). In addition, the students described and explained the observed behavior instead of relying exclusively on theoretical arguments of probability, which often is counterintuitive for students (delMas, Garfield, & Chance 1999). These results show a path to follow for the development of the reasoning in the significance test. First, it must be understood that Fathom not simulates the actual population but a hypothetical; this can help to pass to the next level of reasoning. Secondly, based on the fact that many students assimilated sampling variation, discuss how to choose a sample result (critical value) to reject the hypothesis to verify, this will lead to the idea of p-value.

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MANUFACTURING LICORICE: MODELING WITH DATA IN THIRD GRADE

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This paper reports on a study of 3rd-grade students' modeling with data, which involves comprehensive investigations that draw upon STEM-based concepts, contexts, and questions, and generate products supported by evidence and open to informal inferential thinking. Within a real-world STEM-based context of licorice manufacturing, students experienced the "creation of variation" as they compared and represented the masses of "licorice sticks" they made by hand (using Play-Doh) and those using a Play-Doh extruder. By generating their own statistical measures, students could observe the features of data distributions including center, range, typical, and middle, at a much younger age than usual. They could draw inferences from the models they created, with awareness of how variation limits the certainty with which predictions can be made. The study supports a potential route for advancing early statistical learning.

Keywords: Data Analysis and Statistics, Modeling

Introduction

This paper aligns with the first conference theme of potential routes for mathematics education for the future, specifically, pertaining to the statistical capabilities of young learners. Despite research revealing how young elementary students are more competent in dealing with statistical problems than is acknowledged (e.g., Lehrer & Schauble, 2012; Lehrer & English, in press), many curricula continue to delay core statistical experiences until the middle and early secondary school years (e.g., *Common Core State Standards: Mathematics*; http://www.corestandards.org/Math). Yet young children are exposed to a vast array of statistical information that can, at times, misinform, rather than inform their receptive minds. The ability to reason effectively with data, including entertaining uncertainty and risk, is integral to making meaningful, informed decisions across all spectrums of life. One cannot participate effectively in debates about community issues such as the environment, health care, and education, without this reasoning ability (English & Watson, 2015; Franklin, Kader, Mewborn, Moreno, Peck, Perry, & Scheaffer, 2007). Foundational statistical experiences need to begin early. By undertaking their own investigations, elementary school students can learn to make critical decisions with data, where variation and uncertainty are ever present.

This paper reports on third-grade students' modeling with data, which involves comprehensive investigations that draw upon STEM-based concepts, contexts, and questions, and generate products that are supported by evidence and are open to informal inferential thinking (Lehrer & English, in press). In the present study, students experienced the "creation of variation" as they compared the masses of "licorice sticks" they made by hand (using Play-Doh) with those made using a Play-Doh extruder kit ("factory-made"; adapted from Watson, Skalicky, Fitzallen, & Wright, 2009). Students chose their own forms of representation in displaying their models for the two forms of licorice production, and identified, compared, and explained the features of their data distributions.

Modeling with Data

There are various interpretations of modeling and modeling with data, as reported by English, Arleback, and Mousoulides (2016). As defined here, modeling with data includes: (a) an appreciation of how and why investigative questions are posed and refined within a STEM context; (b) competence in generating, selecting, and measuring attributes; (c) skills in organizing, structuring, and representing data; (d) an ability to interpret evidence-based models including features of data distributions; and (e) making informal inferences while acknowledging variation in the data, and the uncertainty with which any conclusions can be drawn (cf. Makar, Bakker, & Ben-Zvi, 2011; Lehrer & English, in press). Figure 1 displays these core features of modeling with data. Consideration is given to a selection of these modeling components.

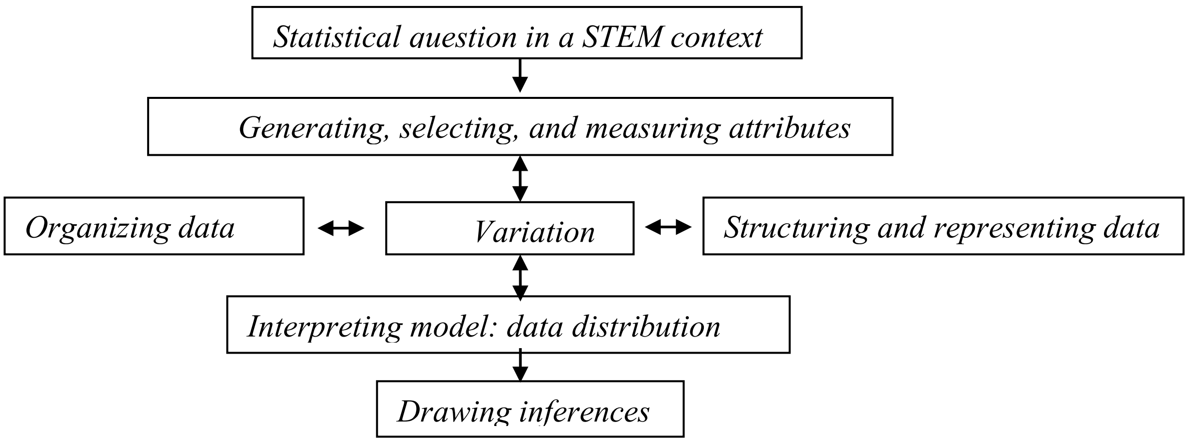

Figure 1. Modeling with data.

STEM Contexts

A statistical question is the starting point for any investigation and immediately raises the issue of problem context and cross-curriculum links. Because data are numbers in context (Moore, 1990), there is no statistics without a problematic situation from another field. Understanding the contextualized nature of data is crucial in developing a facility with statistics. Yet, elementary school curricula tend to give superficial or limited attention to the role of context, especially with respect to whether inferences drawn from the inquiry process align with both the question and context (Lavigne & Lajoie, 2007). With the increased focus on STEM education, including STEM integration (English, 2016) numerous rich contexts arise for undertaking statistical investigations. In the present study, engineering formed the statistical context where students explored the manufacture of licorice and the roles of various engineers (industrial, manufacturing, chemical) in the production process. Such a context highlights the need for quality control in the manufacturing process to reduce product variation.

Variation

Variation is the underlying concept linking all aspects of a statistical investigation; without variation, there would be no need for statistics (Cobb & Moore, 1997; Franklin et al., 2005; Garfield & Ben-Zvi, 2008; Konold & Pollatsek 2002; Moore, 1990; Watson, 2006). In simple terms, variation is “the quality of an entity (a variable) to vary, including variation due to uncertainty” (Makar & Confrey, 2005, p.28). As Watson (2006) highlighted, the reason data are collected and analyzed is to manage variation and draw conclusions and inferences about phenomena that vary. Although there is considerable research on older students’/adults’ awareness of variation there is less so on how this understanding can be developed with young students. This is a major concern especially given that secondary school and university students frequently apply statistical techniques without appreciating or understanding why, when, or how these are applied sensibly to a range of contexts (Garfield & Ben-Zvi, 2008).

Data Distribution

Developing the concept of variation necessitates some understanding of distribution, where patterns in the variability of the data are of interest and are displayed visually (Makar & Confrey, 2005). Exploring learners' concept of variation provides a window into their understanding of distribution. Research has shown that younger students can come to recognize statistics as ways of measuring characteristics of distribution, which guide inferences about the questions posed (Lehrer & English, in press; Makar, 2014). One way to support and advance this early development is to provide children with opportunities to generate statistical measures of center and spread, and to observe how these indicate a distribution's characteristics (Bakker & Gravemeijer, 2004; Konold & Pollatsek, 2002; Lehrer & English, in press). Activities that involve repeated measures yield data distributions that display "signals" (measures of center) and "noise" (measures of variability), which can help students make sense of statistics as measures (Konold & Pollatesk, 2002). It has been argued that when students develop this understanding, they are viewing data through the lens of distribution, in contrast to just a set of data values (Bakker & Gravemeijer, 2004). In today's increasingly data-driven world, young learners deserve access to these core statistical foundations. Future directions in mathematics education need to consider increasing this access in the elementary mathematics curriculum.

Model Representation and Interpretation

Models created through working with data are usually defined as systems of representation, where structuring and displaying data are fundamental; the structure is created, not inherent (Lehrer & Schauble, 2007). Young learners' ability to create and work with a range of representations, including those that extend beyond traditionally accepted formats, is underestimated and needs to be given more recognition and nurturing. In particular, the explicit consideration of variation in relation to representations has not been a key feature of research in the elementary years. Yet, a major foundational component of young students' statistical growth is being able to interpret the meaning, within a given context, of a distribution that displays variation, clusters, modes, and unexpected values; this might not involve conventional text-book types of graphs. Early experiences with a range of representations that effectively display variation in data sets are important but have remained largely neglected in many elementary curricula until recently. Greater insights are needed into how young learners deal with variation in representations that they, themselves, create from their investigations, including how they respond to questions on comparing variation in different data sets, and how they identify and justify the sources of variation that they encounter. With the increasing impact of technology, young students are exposed to more complex and more varied representations that require careful interpretation and critical analysis rather than mere visual inspection.

Drawing Inferences

Informal inference, a precursor to formal inference, has been highlighted as a foundational component that also has not received the required attention especially in the elementary grades. Informal inference is the process of using the evidence provided by data to answer questions beyond the data, acknowledging the uncertainty associated with the conclusion reached (Makar, 2016). Variation is the key to accepting a conclusion with some degree of uncertainty (Franklin et al., 2007). The confidence with which one can form a decision, however, depends on creating a balance between variation and expectation/prediction (Watson, 2006). In the senior secondary courses of study, this balance is expressed in tests of significance or confidence intervals but learning to appreciate variation and its relationship to expectation/prediction needs to begin in the elementary grades with appropriate hands-on experiences and student/teacher questioning.

In addressing these foregoing components of modeling with data, this paper reports on three questions investigated: (1) *How did students represent the models generated for each licorice-*

making method? (2) *How did the students interpret variation and overall data distribution?* and (3) *What was the nature of the informal inferences children drew from their models?*

Methodology

Participants

The activity was implemented in two schools, one a private girls' school, and the other, a co-educational Catholic school. The data in this paper are confined to one class in the former school (mean age of 8.8 years), which was situated in a middle socioeconomic area.

Research Design

The activity was the first that was implemented in a 4-year longitudinal design-based study (Cobb, Jackson, & Dunlap, 2016). This research design caters for complex classroom situations that contain many variables and real-world constraints, supports learning and informs future learning experiences, and facilitates contributions to both theory and practice. Data collection included video-taping of three focus groups as they worked the activity, as well as all class discussions, which were subsequently transcribed for analysis. Focus groups comprised three students of mixed achievement levels selected in consultation with the class teacher.

Activity and Implementation

The activity was created in collaboration with the teacher and formed part of her regular mathematics program in the area of data and probability. The teacher implemented the activity across three school days. The researcher and research assistant were in attendance for the entire activity to observe the students' learning. A detailed lesson description was prepared for the teacher, as was a workbook for students. Students recorded their responses to a number of questions as they worked the activity. Although the students completed the activity in groups, they were to record their own answers and explanations in their workbooks.

The activity comprised several parts including: (a) Reviewing an earlier science activity where students made tubes of lip balm, and discussed variation in their products; (b) Learning about engineers and engineering involved in the manufacture of licorice and foods in general (students viewed a YouTube clip of the American Licorice Co.); (c) Experiencing the notion of variation through exploring packets of manufactured licorice; (d) Investigating questions regarding differences in making licorice sticks by hand (using Play-Doh) and with a Play-Doh extruder ("manufactured"). For each of the hand-made and "manufactured" methods, students identified, measured, compared, and recorded attributes including mass; within-group results were compared; (e) Collating group data on the masses, and representing the group data in a format of choice, for each method; (f) Sharing and interpreting resultant group models from each method with the whole class, including identifying the range and "typical" masses displayed in each group model; (g) Collating all group data and creating a class representation; interpreting the resultant whole-class model of the distribution of the licorice stick masses, for each method.

Data Analysis

For the present paper, data are drawn from the students' workbooks, together with the recorded and transcribed group work and whole class discussions. In conjunction with an experienced research assistant, content analysis (Patton, 2002) was applied in initially identifying, coding, and categorizing the data recorded in the students' workbooks. A further round of refined coding was undertaken to ensure meaningfulness and accuracy. Iterative refinement cycles for video-tape analyses of conceptual change (Lesh & Lehrer, 2000) were applied in reviewing the transcribed focus group and whole class discussions to ascertain the students' learning pertaining to variation, distributions, and inferential reasoning.

Results

This section considers how students represented their models, their interpretations of the variation and data distribution displayed, and the nature of the inferences they drew.

Models Represented

Students represented in various ways their models displaying the licorice-making results. Although forms of bar graphs were most popular, these differed in students' approaches to organizing and structuring their data. For example, many students (78%, N=23) structured their data according to each group member's results (e.g., Monica, Kate, Sarah), while some (13%) ordered the data differently, such as from the "biggest licorice" to "second biggest", to "second smallest", to "smallest licorice" as illustrated in Figure 2. One student displayed each member's heaviest licorice stick only, while another used both tallies and a 3-way table.

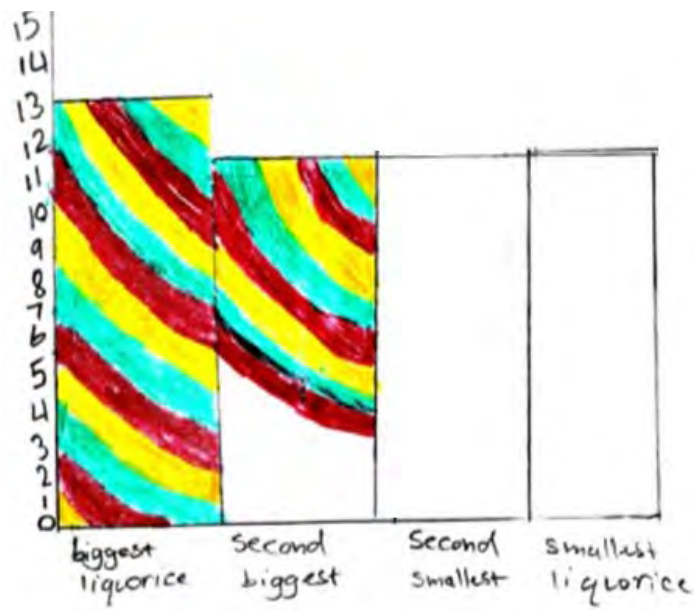

Figure 2. An example of one student's representation for her group's data

Interpreting Variation and Data Distribution

Students were readily able to identify the variation in masses for both the hand-made and factory-made methods, with 83% (N=23) identifying the variation in the former and all students (N=24) for the latter.

Likewise, the students had few difficulties in giving an initial reason for this variation in the hand-made sticks (87%, N=23) with explanations including reference to some sticks being "fatter" or "too thin" or "thicker". They found it more difficult, however, when asked to provide more than one reason.

Sixty-one percent (N=23) were able to offer two acceptable reasons for the hand-made variation, while less than half (48%) were able to give an appropriate third reason (e.g., they simply stated that the sticks "are all different weights"). Over half of the students (63%, N=24) were able to give three appropriate reasons for the factory-made method (58% offering a first reason, 71% a second, and 58% a third).

In collating the group results to form a class plot for each licorice-making method and describing the data distributions, 62% of the students (N=24) offered at least one feature for the hand-made licorice. Of these students, 33% noted multiple characteristics, such as "…lots of spaces and humps and sections and a lot at the start." Those who noted just one aspect gave reasons such as, "very, very lumpy", or "zig-zag." In contrast, all but one student was able to describe the distributional features of the class plot for the second method, with 79% (N=24) describing multiple features, suggesting that their understanding of distribution was developing as they experimented with the licorice-making methods. An example of the class plot for each licorice-making form appears in Figure 3.

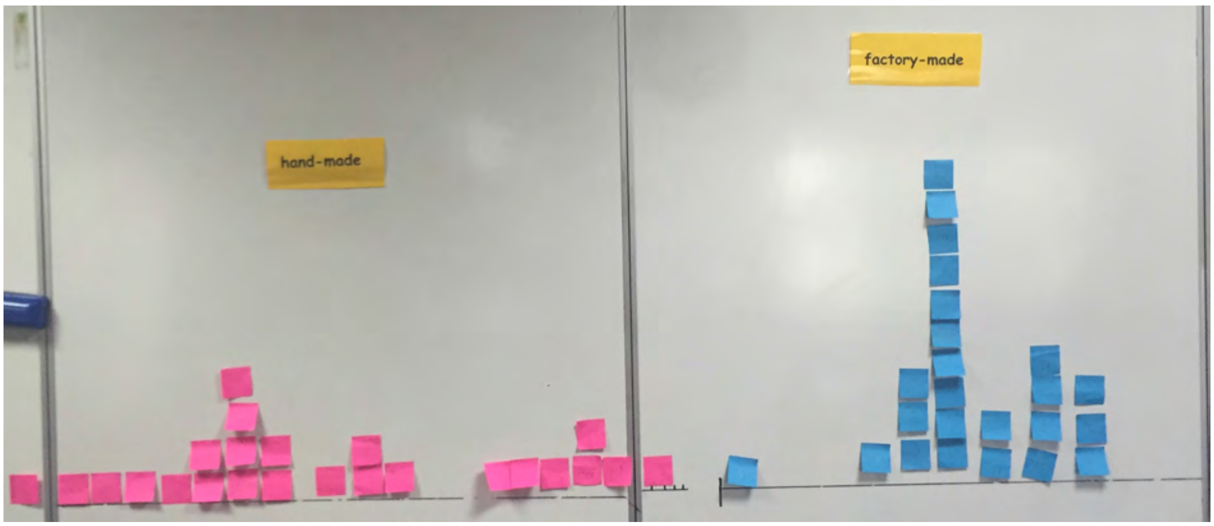

Figure 3. Class plots for each method.

In recording responses to questions about the variation in the masses and the typical mass of the licorice sticks for each class plot, the students again performed better in interpreting the plot for the factory-made licorice. For the hand-made, 58% of students (N=24) could describe the variation in the class plot (e.g., "There's more on 10 and less on 7; there are a lot of people between 8 and 16") and 62% for the factory-made class plot. Although a little over half the students (58%, N=24) could describe the typical mass for the hand-made class plot, 83% (N=24) could do so for the factory-made class plot. The students could readily recognize the difference in the two plots, with 88% (N=24) noting at least one difference. Of these, three students identified multiple differences such as, "Tuesday's (hand-made method) plot had a lot of variation. Thursday's plot (factory-made) had not that much variation. Thursday's plot was a lot taller than Tuesday's plot." The majority of students (71%, N=24) could explain that using the Play-Doh extruder was more accurate in producing sticks of a consistent mass (e.g., "Because it's a machine like, the machine makes them all about the same size and when you're doing them with your hands you can't really tell if they're going to be the same size or not").

Drawing Inferences

On completion of the class models created for each licorice-making method, students were asked, "If you made one more piece of licorice, what do you think (predict) its mass might be? How did you decide?" As part of a follow-up class discussion, the students were also asked, "If another student came into our class and made some licorice, what do you think hers would be (mass of licorice stick)?" Students were readily able to respond to the first question above with 88% identifying an appropriate mass range for the hand-made and 96% for the equipment-made (N=24).

The majority of students could also offer appropriate reasons for each decision, referring to either their own data (42% for hand-made and 33% for factory-made) or the whole class data (29% for hand-made and 46% for factory-made). Their reasons included, “I think because most of mine were around ten and mine were both exactly 1cm wide and 8cm long;” “because it is about the average;” and “I decided because 13g is the typical mass of sticks in the class.”

During class discussions, students frequently referred to chance and uncertainty when explaining what the mass of a licorice stick made by a new student might be. One student explained that, “It might be 13 because most people got … 13 so maybe that’s the typical number.” Another student responded, “I think maybe 12, because if she came in, there’s a chance, because the Fun Factory makes all of them um pretty similar and, and she could make it, but I decided on that [13g] because I think there’s a more likely chance that she would because it won’t always be bigger, she might get it a little smaller than some.” The teacher asked a further question, namely, “Would you expect, say, if we did it again next week and we used the same Play-Doh, and we used the same Fun Factory, would you expect the same plots?” Alesha commented, “I think they might be different because like we could do something, we may have like cut it a bit further or because it’s really hard to get everything exact, so it won’t always be exact.” Monica agreed, “…maybe or maybe not, I sort of agree … you actually don’t know because … when you made three of them like last week they weren’t all the same mass, they weren’t all 15 or they weren’t all 13…”

Discussion

This paper has illustrated how third-grade students can engage in modeling with data involving core statistical concepts and processes, when presented with a motivating context and a meaningful hands-on activity. Using a STEM-based context involving a licorice factory, students were able to explore the important roles of the various engineers responsible for manufacturing high-quality products. The importance of quality control in the real-world provided a valuable context for appreciating how the two forms of licorice making yielded different variations in the sticks produced. By generating and observing “variation in action”, the students could see how variation is an important factor throughout a statistical investigation. They developed an understanding of the reasons behind the greater variation in hand-made sticks and hence, the difference in data distributions of their hand-made and factory-made sticks. Students could identify foundational data distributional features including center, range, typical, and middle, which are usually not introduced until the later grades and then frequently in a computational manner. In line with other research, activities in which students create their own statistical measures of center and spread enable them to observe and understand the features of a data distribution (e.g., Bakker & Gravemeijer, 2004; Konold & Pollatesk, 2002). As Franklin et al. (2007) emphasized, “Statistical education should be viewed as a developmental process” (p. 13) and, as such, these foundational experiences need to commence in the elementary grades.

From the individual and whole-class representations of the models generated, students could draw inferences including predicting the masses of further sticks that might be made. Some awareness of chance and uncertainty was present as the students realized that variation in both licorice-making methods meant that predictions could not be drawn with absolute certainty. Given students’ realization of the uncertainty in drawing conclusions due to variation, the activity can provide foundations for chance explorations. For example, investigations involving the chances of selecting particular candies from factory produced packets can yield unexpected results due to variation in contents. Returning to the conference theme, the present results provide further support for moving mathematics education along a path that capitalizes on elementary students’ early statistical talents. Given the research that has already revealed these talents, greater attention is needed to further advance the field.

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' STATISTICAL KNOWLEDGE: A SNAPSHOT OF STRENGTHS AND WEAKNESSES

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Amid the implementation of new curriculum standard regarding statistics and new recommendations for preservice secondary mathematics teachers [PSMTs] to teach statistics, there is a need to examine the current state of PSMTs' common statistical knowledge. This study reports on the statistical knowledge 217 PSMTs from a purposeful sample of 18 universities across the United States. The results show that PSMTs may not have strong common statistical knowledge that is needed to teach statistics to high school students. PSMTs' strengths include identifying appropriate measures of center, while weaknesses involve issues with variability, sampling distributions, p-values, and confidence intervals.

Keywords: Teacher Education-Preservice, Teacher Knowledge, Data Analysis and Statistics

Many have argued the need to increase students' understanding of statistics (Shaughnessy, 2007). Accordingly, there has been recent increased emphasis on statistics content in secondary curricula standards in the U.S., informed by recommendations from the National Council of Teachers of Mathematics (2000) and the Common Core Standards for Mathematics (National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010). However, a recent study of 1,249 high school students in the U.S. suggests that students are not developing a conceptual understanding of statistics (Jacobbe, Foti, Case, & Whitaker, 2014). Since many teachers, including preservice secondary mathematics teachers (PSMTs), have likely had minimal experience with statistics in their own K-12 education, they also may not have had many opportunities to develop strong statistical understandings.

The Conference Board of the Mathematical Sciences (2001, 2012) as well as the American Statistical Association (ASA, Franklin et al., 2015), present recommendations for developing statistical knowledge and pedagogy needed by preservice mathematics teachers to teach statistics. However, the lack of research focusing on the statistical knowledge of PSMTs was highlighted and called for in the 2011 International Congress of Mathematics Education Topical Study (Batanero, Burrill, & Reading, 2011). The majority of research on preservice teachers' statistical knowledge has focused on elementary teachers (e.g. Browning, Gross, & Smith, 2014; Hu, 2015; Leavy & O'Loughlin, 2006)). The limited research conducted on PSMTs' statistical knowledge has been small-scale studies, from a small number of institutions on specific statistical content (e.g., Doerr & Jacob, 2011; Lesser, Wagler, & Abormegah, 2014). While some smaller studies have suggested that PSMTs may struggle with statistics (e.g. Casey & Wasserman, 2015), there are no large-scale studies that describe the current state of new teachers' statistical knowledge. This study examines the statistical knowledge of a large cross-institutional sample of PSMTs as they enter student teaching to answer the question: *What are the strengths and weaknesses of PSMTs' knowledge of the statistical content they will be expected to teach*?

Framework

Groth (2013) developed a hypothetical framework for Statistical Knowledge for Teaching consisting of two domains of knowledge teachers need to develop: subject matter knowledge and pedagogical content knowledge. Developing subject matter knowledge and key developmental understandings of statistics is foundational to be able to develop pedagogical statistical knowledge.

Within subject matter knowledge there are three types: common content knowledge, specialized content knowledge and horizon knowledge. Common content knowledge refers to knowledge gained through statistics taught in school and is considered common because it refers to knowledge for daily literacy or in any profession that uses statistics. This study examines the common statistical knowledge of PSMTs since they will soon be expected to teach these common statistical ideas as part of curricula for high school students.

The *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A Pre-K-12 Curriculum Framework* (GAISE, Franklin et al., 2007) describes statistical reasoning students should develop in K-12 and suggests this reasoning develops across three levels A, B, and C. Although there are not explicit definitions given for statistical reasoning in each level, the levels increase in statistical sophistication and become more abstract. The content in Level A represents topics for early or novice learners of statistics (elementary and middle school), Level B represents slightly more advanced statistical content (middle school or early high school), and Level C represents even more advanced content (high school or introductory college courses) (Franklin et al., 2007). The GAISE report recommends that students learn statistical topics through engaging in a statistical investigative cycle consisting of: posing questions, collecting data, analyzing data, and interpreting results. Therefore, when examining PSMTs' common statistical knowledge, it is useful to consider their understandings across these cycle phases and all three GAISE levels.

Methodology

Participating Institutions

This study focuses on PSMTs prepared through university-based teacher preparation programs in the US. Since a random sample of all mathematics teacher preparation programs was unavailable, this study began with a purposeful narrowing on PSMTs who attend institutions in which some faculty have participated in the last 13 years in particular National Science Foundation (NSF)-funded or ASA-funded programs to increase the emphasis of statistics education at that institution. Faculty from 57 institutions participated in the NSF-funded program, Preparing to Teach Mathematics with Technology (PTMT, ptmt.fi.ncsu.edu), and/or the ASA-funded Math/Stat Teacher Education: Assessment, Methods, and Strategies (TEAMS, www.amstat.org/sections/educ/newsletter/v9n1/TEAMS.html) conference between 2002-2014. These institutions were chosen since faculty members received professional development focusing on explicit content and strategies for preparing PSMTs to teach statistics. Our assumption was that PSMTs from these institutions may have had opportunities to engage in statistics content and pedagogy activities in their coursework.

The sample was obtained by contacting all 57 institutions through their undergraduate program coordinator for mathematics education to inquire if the program was interested in participating. Twenty-four programs expressed interest, and 18 participated. The coordinator identified the last mathematics teaching methods course PSMTs take before student teaching, which would constitute the data collection point in either fall 2014 or spring 2015. Of the 18 institutions, all but one were public institutions. The majority of institutions (61.1%) had an Carnegie Classification™ (Carnegie Foundation for the Advancement of Teaching, 2011) enrollment profile of high undergraduate. Approximately 84% of participants attended institutions with a basic classification of Research Universities/Very High, Research University/High or a Master's college and university with a larger program.

Participants

Across 18 institutions, there were 221 PSMTs recruited by their mathematics teaching methods instructor to take the assessment of their statistical understanding, described in the next section, as an assignment as part of the course. Those who took exceptionally less time (10 minutes) than recommended by authors of the assessment were eliminated (Jacobbe, personal communication). This resulted in a sample size of 217 PSMTs. The PSMTs were undergraduate juniors and seniors, or graduate students earning initial licensure; all were enrolled in their last mathematics education course prior to student teaching. The number of PSMTs participating from each institution ranged from 2 to 31, with a mean of 12. Fourteen institutions had 100% participation of PSMTs who were eligible to participate, with the remaining four institutions having between one and four students who did not complete the assignment. The majority of PSMTs were female (71%), and 88% were Caucasian. Almost all (93.4%) reported they had taken at least one statistics course at their institution or had completed Advanced Placement Statistics in high school.

Data Collection and Analysis

To examine PSMTs' common statistical knowledge, the Levels of Conceptual Understanding of Statistics (LOCUS) assessment (Jacobbe, Case, Whitaker, & Foti, 2014) was administered online (locus.statisticseducation.org). The LOCUS instrument assesses understanding of statistics across the three GAISE levels of development and also assesses understanding within each phase of an investigative cycle: formulating questions, collecting data, analyzing data, and interpreting results. Participants took the 30 multiple choice Intermediate/Advanced Statistical Literacy version of the assessment, which was designed for students in grades 10 – 12. The test consists of two level A questions, 11 level B questions, and 17 level C questions. This version has been validated and reliable with students in grades 6-12 to assess statistical knowledge across levels B and C and the four phases of the investigative cycle (Jacobbe, personal communication); while this instrument is not intended as a high stakes assessment of knowledge, it does represent the statistics content PSMTs are expected to teach their students in the near future. Thus, teachers are expected to score fairly high on the assessment. While actual test items cannot be released due to test security, sample items for the four categories at different levels are available on the LOCUS website (locus.statisticseducation.org/professional-development). Each test-taker receives an overall score (percent correct), as well as sub-scores for Level B, Level C, Formulating Questions, Collecting Data, Analyzing Data, and Interpreting Results.

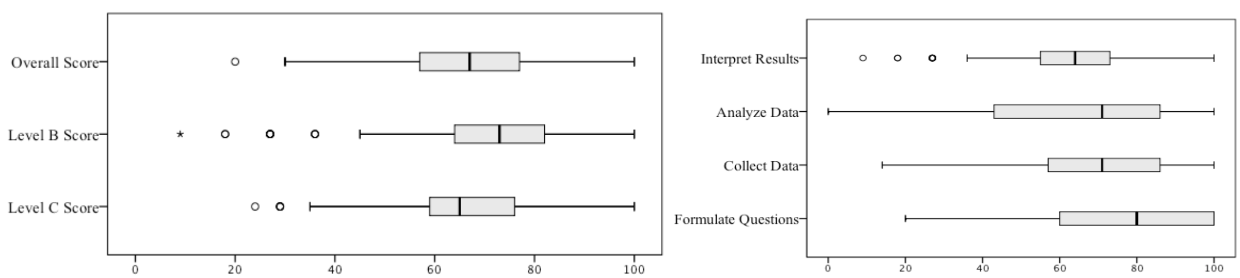
To examine the statistical knowledge demonstrated by PSMTs, descriptive statistics were computed for the overall score and each subscore. Paired samples t-tests were used to test for significance of PSMTs' statistical knowledge between GAISE Levels B and C and a repeated measures ANOVA used to test for significant differences in PSMTs' statistical knowledge between the four phases of a statistical investigation. An item analysis was conducted to closely examine PSMTs' strengths and weaknesses.

Results

Trends in scores on the LOCUS test can help in describing what PSMTs from these 18 universities currently understand about the statistics content they will soon be responsible for teaching. The summary statistics for PSMTs' scores are reported in table 1. With a mean overall score of 69%, and a standard deviation of 14.06, PSMTs do not seem to demonstrate a conceptual understanding of the statistical content they will teach high school students. PSMTs scored, on average, significantly higher on Level B questions than on Level C questions ($t=5.772$, $p<0.001$), demonstrating that their statistical knowledge is weaker as items increase in sophistication. The distribution of PSMTs' scores is shown in figure 1. The boxplots show that for the overall scores and subscores, there are at least some PSMTs who scored between 90-100% correct, indicating that they likely have strong common statistical knowledge of topics they will soon be responsible to teach. However, there is a concern since only one-quarter of PSMTs scored overall above 77%, and a quarter scored below 57% overall. The variation in scores seems somewhat similar for Level C scores. However, higher standard deviation in Level B scores is likely due to the increased quantity of low scoring individuals, indicated as outliers in figure 1.

Table 1: PSMTs' Percent Correct on LOCUS Instrument

	Number of items	Mean	SD
Overall Score	30	68.61	14.06
GAISE Levels			
Level B Score	11	70.85	17.69
Level C Score	17	64.87	14.16
Phases of Statistical Investigative			
Cycle	5	80.37	21.51
Formulating Questions	7	70.40	19.70
Collect Data	7	63.34	22.22
Analyze Data	11	60.48	16.25
Interpret Results			



Figures 1 and 2. Distribution of PSMTs' LOCUS scores

Examining subscores by phases in the statistical investigative cycle, PSMTs scored higher on average on Formulating Questions and lower as the cycle progresses, scoring lowest on Interpreting Results items (Table 1). A repeated measures ANOVA determined that mean scores differed significantly between scores for the four phases [$F(3,648)=64.73$, $p<0.001$]. Post hoc tests using a Bonferroni correction revealed that PSMTs scored significantly lower as the cycle progressed ($p<0.001$). However, there was only a slight difference between mean scores for Analyze Data and Interpret Results ($p=0.32$). The distribution of scores across the four phases is shown in figure 2. The boxplots show that for all four phases, there are again some PSMTs who scored between 90-100%, indicating that those PSMTs likely have the common content knowledge that will be needed when teaching that phase of the investigative cycle. On Formulating Questions items, at least half of PSMTs scored 80% or higher, and a quarter of those scored 100%, indicating stronger understanding for these PSMTs about Formulating Questions. However, half of PSMTs scored below 71% on Collecting Data and Analyzing Data items, and half scored below 64% on Interpreting Results items. Even being conservative, this result is convincing that the majority of these PSMTs do not have the common statistical knowledge that can provide a foundation for teaching students key concepts related to Collecting Data, Analyzing Data, and Interpreting Results.

Item Analysis

Upon further analysis of individual items classified by the statistical investigative cycle, themes emerged concerning PSMTs' strengths and weakness. As previously mentioned, PSMTs scored the highest on average for Formulating Questions items, with no common misunderstanding identified. As an example of their strength in understanding this phase, PSMTs were able to read a description of a study and measurements taken to identify an appropriate statistical question of interest.

Collecting Data. On average, PSMTs scored the next highest on Collecting Data items. PSMTs were able to identify ways to improve a study design given a study and measurements, identify which study design would be best based on a question of interest, and identify a data collection plan based on a study description. Thus, these PSMTs seem to have strong common content knowledge related to the design of a statistical study.

Even though PSMTs were able to develop a data collection plan, they struggled more when asked to identify how to choose a sample to minimize bias. Only 64.5% were able to choose a correct sampling method; instead, 30% chose a convenience sample or a stratified sample that seemed complicated but was not random. Thus, they do not seem to have a strong understanding of the role of an appropriate sampling method within the design of a study. Another common misunderstanding of PSMTs was the conclusion that could be drawn from a specific study design. Figure 3 is a similar item to the one PSMTs were asked on the assessment. Over 58% of PSMTs chose an answer similar to answers (A) and (C) that allowed a researcher to generalize results to an entire population based on a sample of volunteers. These findings highlight PSMTs' need for a deeper understanding related to ways in which study designs and data collection processes impact the conclusions that can be drawn.

Each member of a random sample of 1,000 adult males from the United States was asked a number of questions, including questions about height and annual income. When the responses were analyzed, it was determined that taller men had greater incomes than shorter men, on average, and the difference was statistically significant. Which of the following conclusions would be most appropriate based on these results?

(A) The study establishes that being tall causes men to have greater incomes, on average, and this conclusion can be generalized to all men in the United States.
(B) The study establishes that being tall causes men to have greater incomes, on average, but this result only applies to the men in the sample.
(C) The study establishes that being tall causes men to have greater incomes, on average, than shorter men, and this conclusion can be generalized to all men in the United States.
(D) The study establishes that being tall causes men to have greater incomes, on average, than shorter men, but this result only applies to the men in the sample.

Figure 3. Sample Collect Data item from locus.statisticseducation.org.

Analyzing Data. PSMTs' average scores for Analyzing Data items were the second lowest among the phases, and had the highest variability. PSMTs demonstrated that they understand which measure of center is appropriate for a given context, how measures of center and variation change when data values are changed, and a justification of an association from a two-way table. However, PSMTs demonstrated more difficulty with Analyze Data items that involved understanding of variation in data. Only 43% of PSMTs could identify a histogram containing data that varied the least from its mean. Instead 30% of PSMTs chose a uniform distribution and about 20% thought variability from the mean was the same for all three distributions. PSMTs demonstrated another misunderstanding related to expected variation in sample means when repeatedly sampling from a population. When given the distribution of a population and population mean, 36% of PSMTs could not identify the distribution of sample means. Instead they chose distributions that resembled the general shape of population distribution. These results point to PSMTs' need for more common content knowledge in regards to variation, sample distributions, and distribution of sample statistics.

Interpret Results. PSMTs scored the lowest on average on Interpret Results items. However, on five of the eleven Interpret Results items, 84% or more of PSMTs answered the items correctly. PSMTs were able to compare distributions in a context using the center and spread, demonstrate an understanding of the effect of sample size on a sample mean, and interpret survey results with a given margin of error. These are important concepts often taught in middle and high school curricula. On the other six Interpret Results items, the percentage of PSMTs responding correctly to these items ranged from 21% to 48%, and their misunderstandings were related to ideas of formal inference. PSMTs struggled most with statistical significance, identifying and interpreting a p-value, and explaining confidence intervals. About half (48%) of PSMTs were able to correctly interpret results given a large p-value and fail to reject the null hypothesis (Figure 4). However, 40% of PSMTs chose a conclusion that a large p-value meant they could reject the null hypothesis.

Mario wondered if students who graduated from a nearby college with an engineering degree would have higher yearly salaries, on average, than students graduating with a degree in mathematics. To investigate, he randomly selected ten engineering graduates and eight mathematics graduates. He calculated the mean yearly salary to be $58,421 for the ten engineering graduates and $58,402 for the eight mathematics students. The difference in the sample means of $19 per year resulted in a p-value of 0.478. Based on this information, which of the following statements is correct?

(A) The difference in means is both statistically significant and practically significant.
(B) The difference in means is neither statistically significant nor practically significant.
(C) The difference in means is practically significant, but not statistically significant.
(D) The difference in means is statistically significant but not practically significant.

Figure 4. Sample Interpret Results item from locus.statisticseducation.org.

On another item regarding p-value, PSMTs were asked to reason if a p-value would be large or small for comparing means of two distributions given data on a dotplot. Only 35% of PSMTs were able to correctly identify that the p-value would be small due to the large gap between distributions. Almost 47% incorrectly answered that the p-value would be large due to a large gap between the distributions. These findings demonstrate that PSMTs on average do not have an understanding of what it means to be statistically significant and what a p-value represents, aspects of common content knowledge expected in statistics, and included in many high school curricula.

The item PSMTs had the most difficulty with in Interpreting Results asked the test taker to explain the meaning of a 95% confidence interval for a mean. Approximately one-fifth chose the correct response that a 95% confidence interval represents that 95% of confidence intervals constructed from random samples would capture the true mean. Almost half of PSMTs chose the response that there was a 95% probability that the mean was in between the lower and upper limits of the confidence interval. These misunderstandings highlight the need for PSMTs to have more experiences with interpreting and understanding confidence intervals.

Discussion and Conclusion

Our study was situated within a purposeful sample of PSMTs enrolled in teacher education programs where a faculty member had participated in professional development projects that promoted increasing attention to statistics in secondary mathematics education courses. It is not known exactly how those teacher education programs currently include an emphasis on statistics, nor exactly what these PSMTs experienced at all 18 institutions. Nonetheless, there are several findings of this study that are significant to consider. Our results provide empirical evidence that PSMTs in this study generally do not exhibit a strong common content knowledge of many aspects of statistics needed for teaching high school students, and in particular they struggle more with the later phases of a statistical investigation. Previous research has shown a similar trend with inservice teachers and students measured by LOCUS (Jacobbe, 2015; Jacobbe, Foti, et al., 2014). Thus, PSMTs need more experiences in collecting data, analyzing data and interpreting results to develop a deeper understanding of all aspects of the statistical investigative cycle and to develop common statistical knowledge needed for teaching.

PSMTs exhibit some similar strengths and weaknesses with concepts that high school and introductory college students develop. An important strength that PSMTs demonstrated is that they are proficient at identifying an appropriate measure of center for a given context. PSMTs' strength in understanding measures of center suggests they should be well equipped to assist their future students develop stronger conceptions. PSMTs' weaknesses involve issues with variability, sampling distributions, p-values, and confidence intervals. Many researchers have identified that these topics are also often misunderstood by many students in undergraduate statistics courses (e.g., Aquilonius & Brenner, 2015; Castro Sotos, Vanhoof, Van de Noortgate, & Onghena, 2007; delMas, Garfield, Ooms, & Chance, 2007); thus, PSMTs' common statistical knowledge may be no better than those of other college students not preparing for teaching.

These findings, even though from a purposeful sample, suggest there is a critical need for mathematics teacher education programs to reevaluate the opportunities PSMTs' have to increase their common statistical knowledge. Our results specifically indicate that effort should focus on developing PSMTs' knowledge of variability, sampling distributions, and formal inference, particularly as they are applied in the analyzing data and interpreting results phases of an investigative cycle. While this study only reports on one aspect of PSMTs' statistical knowledge for teaching, the larger study (Lovett & Lee, 2017) provides more details about PSMTs' confidence to teach and the experiences they perceived had contributed to their confidence and understandings in statistics. Additional large-scale studies are needed on all aspects of PSMTs' statistical knowledge for teaching and the impact that teacher education programs have on PSMTs' preparedness to teach statistics.

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PRE-SERVICE TEACHERS' DEVELOPMENT OF STATISTICS AND PROBABILITY KNOWLEDGE IN A TECHNOLOGICAL COLLABORATIVE ENVIRONMENT

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This study investigates the development of pre-service teachers' (PSTs) probability and statistics knowledge in a technological collaborative environment. The teachers collaborated synchronously in an online environment to solve a probability task that involves investigating the fairness of different dice. Teachers used simulations to roll six dice and collect data about them with different sample sizes. The simulations allow users to roll each die up to 1,000 rolls and represent the outcomes in a frequency table, bar graph, and a pie chart. While investigating the fairness of the dice, teachers engaged important statistical and probabilistic concepts to reason about data. Results showed that interactions between experimental and theoretical probability helped teachers further their understanding of distribution, data dispersion, and the Law of Large Numbers. This informs supporting PSTs learning of probability and statistics.

Keywords: Probability, Data Analysis and Statistics, Technology, Teacher Knowledge

Introduction

Today's society relies heavily on technological innovations that make collecting, analyzing, and presenting quantitative data easily accessible. This stresses the importance of preparing students for such reality. As a consequence, mathematics educators are attending more to teaching and learning statistics and probability. The National Council of Teachers of Mathematics (NCTM) suggests that students, starting from elementary grades, should learn how to collect and present data and make decisions based on them (National Council of Teachers of Mathematics, 2000). Students need to develop a meaningful understanding of data variability and to develop an understanding for randomness of different events that can occur in nature, technology, or society.

Different conceptions of probability make learning and teaching of probability a challenge for mathematics teachers. Interpretations of chance and randomness and their relationships with the subjective, theoretical, and experimental conceptions of probability contribute to the challenge of teaching and learning probability (Batanero, Henry, & Parzysz, 2005). Responding to this challenge, mathematics educators turn to technology and simulations to support teaching and learning of statistics and probability. There are different technologies and educational software such as graphing calculators, Spreadsheets, and TinkerPlots that can be used to help students develop important statistics and probability concepts. However, there is a need for studies that provide more insights into how tasks in technological environments can support teachers' and students' learning of statistics and probability (Biehler, Ben-Zvi, Bakker, & Makar, 2013; Garfield et al., 2008). As a response to this need, we investigate the development of PSTs knowledge of probability and statistics while working collaboratively in an online environment to solve probability problem. The PSTs used interactive simulations with multiple representations to determine the fairness of different dice. Our study responds to this question: Using simulations in a collaborative environment, how do PSTs engage theoretical and experimental probability to develop different statistical and probabilistic concepts?

Related Studies and Theoretical Framework

Surveys of the literature of teachers' knowledge of statistics and probability show that in-service and pre-service teachers lack deep understanding of graphical representation of data and important statistical concepts (Eichler & Zapata-Cardona, 2016). In addition, teachers lack theoretical and empirical probabilistic knowledge in general (Batanero, Chernoff, Engel, Lee, & Sánchez, 2016). Professional development programs usually focus on content knowledge. There is a need to help teachers develop flexible understanding of statics in contexts and build connections between theoretical and experimental probability (Batanero et al., 2016; Eichler & Zapata-Cardona, 2016).

Knowledge of statistics and probability includes many important concepts. Statistics deals with aspects related to producing data, either through observation or experimentation, and analyses that aim to find patterns as well as test hypotheses (Cobb & Moore, 1997). On the other hand, probability involves ideas such as randomness, events and sample space, combinatorics, independence (conditional probability), Law of Large Numbers, sampling and sampling distribution, and simulation (Batanero et al., 2016). In general, probability deals with two aspects of phenomena, empirical and theoretical. Empirical probability focuses on data generated from experiments where theoretical probability focuses on expectations beyond empirical data (Batanero et al., 2016).

To support students' learning of statistics and probability, teachers need to acquire deep content and pedagogical knowledge of the subject. Understanding how teachers develop statistical and probabilistic concepts informs the design of tasks in which pre-service and in-service teachers can engage and build their knowledge of statistics and probability.

Methodology and Data Collection

In our study, 12 PSTs interacted synchronously in an online environment to solve a probability task. The online environment, Virtual Math Teams with GeoGebra (VMTwG) integrates GeoGebra with a white board and a chat panel for synchronous discussion. The probability task involves investigating fairness of six different dice. The task has three main components: online collaboration using simulations, watching middle school students' discussing the same task, and analyzing artifacts of middle school students' solutions of the same task. The data for this paper focus on the first two components of the task. Data consist of online interactions of PSTs to solve the probability task, which included their chat logs, GeoGebra activities, and their chat logs reflecting on the middle school students' arguments for the same task. We were interested in studying how the PSTs, using simulations, moved back and forth between theoretical and experimental probability and engaged different ideas to judge the fairness of multiple dice.

The PSTs were provided with simulations for six dice that were weighted differently. They could roll each dice between 1 and 1,000 times to determine whether the dice are fair. The task provided them with three different representations of the data; a frequency table, a pie chart, and a bar graph. As a group, the PSTs were to make judgments about the fairness of each die and provide evidence to support their arguments. The video they watched contained a group of eight seventh graders engaged in a debate about how many times one would have to roll the dice to determine if the dice were fair or not. For analysis, two researchers openly coded the actions of the teachers inside of the tool and their chat logs discussing the fairness of each die and middle school students' arguments. Altogether, four groups of three PSTs were analyzed.

Results

While solving the probability task, PSTs engaged aspects of theoretical and experimental probability in their discussions. Even though the four groups of PSTs had different prior conceptualizations of theoretical and experimental probability, they used similar strategies to response to the task. Our analysis of PSTs actions and chat logs revealed that they relied mainly on three statistical and probabilistic concepts to investigate the fairness of dice. All groups focused on data distribution, data dispersion, and aspects of Law of Large Numbers (LLN). These concepts are related to each other in multiple ways. Because of these relations, PSTs discussions of these concepts were intertwined at moments. The following sections present how PSTs engage each concept while solving the task.

Distribution

The bar graph of the discrete probability distribution of each die allowed PSTs to discuss the experimental and theoretical distribution of data. Some students started with only theoretical understanding of the probability for six-sided dice. They expressed difficulty conceptualizing the notion of investigating the fairness of a die. To them, dice, by definition, have six faces with equal chance of selection. At the beginning of working with the task, a member of the first group asked "what does 'poor quality dice' even mean?" This indicates that this student has a theoretical base for understanding the probability of dice. All groups noticed the normality of data distributions generated from rolling the first die. They described these distributions as "bell curve" and "pretty normal". The normality of the distribution of the first die's data drew students' attention to the notion of distribution and allowed them to reflect on what a fair die's distribution should look like. In addition to bell curve or normal distribution, PSTs described different data distributions with terms such as "even" and "inconsistent". Three groups used "even" or "equal" to describe approximately uniform distributions. These distributions were of data generated from the only fair die in the task.

Dispersion

The second statistical notion to which students attended was dispersion of data. The PSTs attended to extreme values at first and moved to using range as a measure of dispersion. The second die has a low probability for number five. This allowed PSTs to talk about the lowest and highest values in their data. This attention to high and low values was used in investigating dice that followed the second die. After using extreme values to discuss the fairness of dice, PSTs used range to describe the die's performance. They attended to the highest and lowest value and reported the difference between them. While attending the spread of data, the third group demonstrated interactions between experimental and theoretical probability clearly when discussing fairness of the fourth die. A student stated that "at 500, fair dice would be in the range of 83 for each… but 1 is way below that range. i do not think its [*sic*] fair". Starting with an understanding of the theoretical probability of a fair die, the students divided 500 by six to estimate the outcomes of each face of the die. The spread of data is an important concept and relates closely to data distribution and, in our task, the Law of Large Numbers.

Law of Large Numbers

In our task, PSTs were able to change the number of rolls for each die to test its fairness from 1-1000. The simulation used a pseudorandom process to generate data based on certain probability we specified for each face of the die. The groups of PSTs demonstrated different initial and final understanding of LLN. The first three groups started with limited understanding of LLN. That was evident through their choices of how many rolls to use. The first group did not attend to this issue where the second group discussed using a small number of rolls that is a multiple of six. This indicates limited understanding of LLN but could show an understanding of the theoretical chance for each side being one-sixth. One student of the second group commented about the outcomes of each side saying, "obviously it is not going to be perfectly even", which shows that the student has an understanding of experimental probability but did not connect it to the LLN. Without any justification, the third group started with investigating three cases: 100, 500, and 1000 rolls. After testing these cases, a student questioned: "it should be the more you roll, the more equal the graph should look right?" Other members agreed with her. This group is starting to conceptualize the importance of LLN. The last group demonstrated understanding of the importance of LLN and decided to roll the dice 1000 times from the beginning.

Discussion

This study reports on four groups of PSTs interactions in an online collaborative environment to solve a probability task. The task asked PSTs to investigate the fairness of six differently weighted dice through using an interactive simulation for each die. PSTs collaborated synchronously to discuss the fairness of each die. They had the freedom to roll each die between 1-1000 times. At the end of their problem-solving session, PSTs watched a video of middle school students reasoning about the importance of using a large number of rolls in the same task. PSTs' discussions showed significant interactions between empirical and theoretical probability that supported their reasoning of data and investigation of dice fairness. To judge the fairness of dice, they engaged three main ideas: data distribution, dispersion of data, and the Law of Large Numbers. Three groups of the PSTs demonstrated stronger theoretical understanding of probability which align with findings in the literature (Batanero et al., 2016). These groups were able to demonstrate understanding of experimental probability by the end of working on the task.

Except for the last group, the PSTs worked on our task without formal introduction to statistical or populistic concepts related to the tasks. This did not limit them from engaging important concepts that are critical for this task. Their discussions were informal and used non-standard vocabulary. Additionally, we did not ask PSTs to discuss certain ideas when investigating the fairness of the dice. It was interesting to see how all the groups develop similar strategies. This provides important implications for designing tasks that aim to help PSTs develop their knowledge of probability and statistics.

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF BINOMIAL DISTRIBUTION

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The binomial distribution is important for its connections to probability, sampling proportion, and inference. This research describes preservice secondary mathematics teachers' understanding of the binomial distribution in the context of an investigative task. A three-tiered framework was developed and used to classify preservice teachers' understanding, with implications for further research on the same or various statistical concepts.

Keywords: Data Analysis and Statistics, Teacher Education-Preservice

Introduction

Statistics education research has made great strides in recognizing common misconceptions and working on best practices to promote students' statistical thinking. To add to research on student thinking in statistics, this research study investigates teachers' understanding of a particular concept, the binomial distribution. The binomial setting is important for its stand-alone use, but also for the opportunity it provides to connect ideas related to probability, sampling proportions, and inference (Kazak, 2010). Thus, the research question is: "How do preservice secondary mathematics teachers (PSTs) understand the binomial distribution?" The frameworks used to analyze PSTs' understanding are the developmental levels described in the GAISE Report (Franklin et al., 2007) and the statistical habits of mind in the Framework for Supporting Students' Approaches to Statistical Investigations (SASI, Lee & Tran, 2015).

Literature Review

The main focus of the task used in this study is to introduce students to the binomial distribution and to encourage conceptual thinking about the probabilities of certain outcomes in that setting. Students have shown evidence of struggling with probability (Batanero, Henry, & Parzysz, 2013), inference (Harradine, Batanero, & Rossman, 2011), and repeated sampling (Harradine et al., 2011); the concrete binomial setting examples can help bridge these gaps. Students will be able to see what constitutes a "weird" or unlikely result by comparing it to an intuitively calculated expected value. For example, students will see that they are unlikely to get six or more questions correct when randomly guessing on a ten-question quiz. The binomial setting in the task also allows PSTs to see how sample size affects experimental probability and its relationship with theoretical probability. Combining ideas of data and probability before approaching formal inference can be useful for students (Kazak, 2010).

Students have intuitions about combinatorics that sometimes conflict with mathematics. For example, they may not attend to order when it is actually important for the question (Abrahamson, 2009; Kazak, 2010). They generally believe that the probability of flipping three heads and then one tail is greater than the probability of flipping four heads in a row, even though these are equally probable when you consider order (Abrahamson, 2009).

The simulation is helpful in allowing students to actually see different "quiz attempts" and the results happen in real-time. It also helps them see the probabilities at work. They would be very shocked to see all ten of the answers be correct, and much less shocked to see none of them correct. The fact that students first "take the quiz" by randomly guessing and then use the computer to simulate many more repetitions follows the recommendation of doing a tactile simulation before using technology (Chance & Rossman, 2006). The TinkerPlots simulator specifically is used in this study because it is possible to actually see the questions being randomly answered, avoiding the "black-box" technology problem (Chance & Rossman, 2006, p. 5). Overall, students should have a concrete idea of the binomial setting after completing the task. Students will develop an informal sense of how to combine data and probability to predict certain results. The level of their understanding is the main focus of this research project.

Framework

The theoretical framework that forms the basis of this research is the SASI framework, which is an extension of the framework in the GAISE Report (Franklin et al., 2007). The GAISE framework combines the investigative cycle and the different levels of student understanding during each phase of the cycle. For example, for the first phase, Formulate Question, the framework describes student thinking at each level, with Level C being the most sophisticated statistical thinking. The levels are not associated with an age or grade level, but rather assess students' current level of thinking. The SASI framework identifies specific statistical concepts, called statistical habits of mind, to which students should attend at different phases of the investigative cycle (Lee & Tran, 2015). The SASI framework was used to pinpoint concepts related to the binomial setting to which students should be attending.

Methods

Data were collected from 12 female PSTs who were enrolled in a 400-level course on teaching mathematics with technology offered at a large public university. They were given a task that asked them to go through the investigative cycle of a statistics problem in a binomial setting. They used the software program TinkerPlots to help them visualize the simulation. The original task (taken from: http://apstatsmonkey.com/StatsMonkey/Statsmonkey.html) was modified to encourage Level B and Level C thinking according to the GAISE framework (Franklin et al., 2007). Written responses to the task were used to assess their level of understanding according to the A, B, C levels of the GAISE framework (Franklin et al., 2007).

Results

The results come from the analyzed data collected in the form of the written responses given by the students on the last section of the task. The GAISE (Franklin et al., 2007) and SASI (Lee & Tran, 2015) frameworks provide a pathway for analysis that involves categorizing PSTs' responses according to Levels A, B, and C, with C being the deepest level of understanding. The GAISE framework focuses on understanding at the different stages of the investigative cycle, rather than on specific statistical topics. For this reason, the framework's structure was used to develop similar levels of understanding specifically for the binomial distribution context. The "statistical habits of mind" (Lee & Tran, 2015) were used to identify concepts on which students should be focused for each level of understanding.

PSTs' responses that indicated a firm understanding of the binomial setting had some common characteristics, and attended to related statistical habits of mind. Those with correct responses 1) include a *context* where it is possible to know that every event has the same probability and that all events are independent, 2) attend to *variability* when predicting likely/unlikely values, and 3) pose *statistics* questions appropriate for a binomial setting. PSTs' responses with all three characteristics were considered to have a Level C understanding (n=4). PSTs with two of the three were considered to have a Level B understanding (n=3). Responses that included none or only one of the three, but were still in the realm of the binomial setting (e.g. outcomes are success or fail, probability is involved, expected value is used) were indicative of a PST with a Level A understanding (n=5). Sample responses are shown in Table 1.

Table 1: Task Questions and Sample Responses from each Level of Understanding

Questions from Task	Level C Response	Level B Response	Level A Response
1. What is another binomial setting you can think of that fits these characteristics? (i.e., the four conditions of the binomial setting)	1. Flipping a coin a set number of times (ex: 30) success (head) failure (tails) fixed observations (30) observations are independent, P(correct)=.5	1. Rolling a die 15 times to see if "1" appears. (1/6 chance each time & each is independent).	1. Cards-standard deck
2. In the setting you have chosen, pose an interesting question you can answer using experimental and theoretical probabilities	2. John and his brother are betting that whoever's end of a coin appears most, has to do the others dishes for the week. Who will have to do the dishes if John chooses heads and his brother chooses tails and they repeat the process 30 times?	2. Does one number appear more often, or is there a higher probability for 1 # than for others.	2. What is the probability of pulling a heart suited card in 10 pulls from a deck.
3. If you define the number of successes to be a random variable Y, what values of Y do you expect to be most likely? Why?	3. y=15, the numbers closest to 15 because as you repeat the task many times, the totals should be close to the actual probability. (.5)(30)=15	3. Y=# successes. 2 or 3 because there's a 1/6 chance & 15 rolls --> 2.5 (1/6)^2=.0278 --> low but possible. lower y more likely to occur.	3. between 2-3 because the probability is 25% (shows work for 52/4=13)
4. What values of Y do you expect to be least likely? Why?	4. The numbers further away from 15, on both ends (+ or - 15) because you are straying away from numbers that are reasonable from the probability	4. Numbers from 5-6 because the probability of rolling a 1 5-6 is (1/6)^5=.000129 which is extremely low. The higher y is, the less likely it will happen.	4. 5-10 would be least likely for the same reasons as other example

It is not enough to judge that a PST is able to state that all observations in a binomial setting have the same probability of success. PSTs should also be able to produce or identify a context in which it is possible to know that is the case. In the Level C example given to item 1, this PST was able to produce an example on her own that satisfies the conditions. In contrast, the Level A example gives a situation with too little detail to infer that she has produced a satisfactory binomial setting. This is also true for the condition that all observations are independent. PSTs should be able to produce or identify a context in which that is true.

PSTs with a deep understanding should also have an awareness of the likelihood of specific outcomes. This includes attending to variability when predicting likely or unlikely values. Although the binomial setting allows for theoretical probability calculations, PSTs should understand that data sampled from simulations or real-world contexts will not match theoretical probabilities. When PSTs were asked to predict values, many calculated the expected value. For example, the Level C example answer to item 3 provides the "actual probability" (when she has calculated expected value) and suggests that likely outcomes will lie around that value. She has attended to variability, but not combinatorics. The Level B example provides a rare appeal to theoretical probability, but leaves out the combinatoric component and independent probabilities of failures. The sample Level C answer to item 4 shows consideration for variability, using language like "the numbers closest to" and "the numbers further away." In contrast, the Level B PST chooses two numbers deemed likely and two numbers unlikely.

Finally, PSTs should be able to produce an investigative question that requires a statistical, not mathematical, answer, appropriate for a binomial setting. For example, the item 2 Level C Example provides an appropriate statistical question in her binomial setting that can be answered using theoretical probability calculations, or simulations. In contrast, the Level A Example provides a basic probability question that does not depend on the binomial setting.

Discussion

Without overgeneralizing from this small sample, the results allow deeper insight into the types of understanding involving the binomial setting, especially in the context of an investigative task. In general, the binomial distribution has not been a focus of much research, even though it is a foundation for the concept of sampling proportions. The framework used for this study can be used to inform future research on the topic, and serve as a model for similar frameworks for a variety of statistical concepts.

The PSTs' responses to questions on the binomial setting after the investigative task show a majority of Level A and Level C understanding, with fewer in the middle. The tactile and electronic simulations, and the real-world context of the task, may have aided the strong responses. It was beyond the scope of this research to investigate PSTs' exposure to binomial distribution prior to the lesson; follow-up research should consider this aspect. In general, PSTs did exhibit struggles with applying the four conditions of the binomial setting to novel situations, attending to variability when predicting outcomes, and producing statistical questions that went beyond basic probability. This echoes research on student misconceptions.

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THE MIDDLE GRADES SETS INSTRUMENT: PSYCHOMETRIC COMPARISON OF MIDDLE AND HIGH SCHOOL PRE-SERVICE MATHEMATICS TEACHERS

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This study reports psychometric evidence for the use of the Middle Grades Self-Efficacy to Teach Statistics (SETS-MS) instrument for both middle and high school pre-service mathematics teachers. Results indicate the reliability from scores is similar for both groups, and there were no statistical differences in the subscale means between the groups. This evidence suggests the instrument is appropriate for use with either population of pre-service mathematics teachers. Suggested SETS-MS uses include measuring self-efficacy after the implementation of recommendations from the ASA's SET report (Franklin et al., 2015). The impacts of these results are discussed within the paper.

Keywords: Data Analysis and Statistics, Teacher Beliefs, Middle School Education, Teacher Education-Preservice

While the National Council of Teachers of Mathematics has long advocated for the inclusion of statistics and probability in high school mathematics curricula (National Council of Teachers of Mathematics, 2000), the adoption of the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010) has increased the emphasis on statistics in these grade levels. Mathematics teacher education programs are faced with the challenge of preparing pre-service teachers (PSTs) to teach statistics. With the release of the American Statistical Association (ASA)'s *Statistical Education of Teachers* report (SET; Franklin et al., 2015), we are at a crossroads in mathematics teacher preparation in both research and practice. While the recommendations for appropriate content and supports for each certification/licensure level in the *SET* report are an initial start, researchers and mathematics educators still need to evaluate the practices in teacher education programs that increase teacher efficacy to teach statistics and develop teachers' statistical habits of mind using psychometrically-sound measures.

The two grade-level-specific versions of the Self-Efficacy to Teach Statistics (SETS) measure teacher efficacy to teach middle and secondary students the skills to conduct specific statistical tasks, based on the *Guidelines for Assessment and Instruction in Statistics Education* (GAISE) *Pre-K-12 Report* (Franklin et al., 2007) and the CCSSM. The Middle Grades version (SETS-MS; Harrell-Williams, Sorto, Pierce, Lesser, & Murphy, 2014) aligns with the CCSSM standards for grades 6-8. The SETS-MS instrument was validated with PSTs seeking licensure that included some portion of grades 5-8. However, the grade levels covered by some states' secondary licensure/certification include some middle grades in addition to high school, so the SET-MS instrument needs to be evaluated using high school PSTs as well.

Thus, the purpose of this study was to compare psychometric performance of the SETS-MS instrument using two samples of PSTs, one that is seeking licensure/certification that includes at least one of grades 5-8 (referred to as "middle grades" in this paper) and one seeking high school licensure/certification that focuses on grades 9-12 (referred to as "high school"). Specifically, the study sought to answer the following research questions about the differences between middle grade and high school PSTs: Do the scores and reliability of the scores from the two SETS-MS subscales differ? Does the distribution of item response category usage differ? Additionally, is there evidence of measurement invariance?

Methods

Participants

The participants came from two separate studies using the SETS instruments. The first study included a convenience sample of 309 PSTs whose intended licensure/certification was referred to as the "Middle Grades" PSTs. Data were collected across four different large public institutions of higher education in four states (IN, TX, OK, KY) in the US. Approximately 78% of the participants were female and predominately self-identified as Caucasian (88%).

The second study included 290 PSTs from 20 universities in the United States with secondary mathematics teacher education programs, referred to as the "High School" PSTs. While two universities were selected from convenience, the other 18 institutions spanning 14 states in the U.S. were selected because at least one faculty member participated during 2002-2014 in a program (funded by either ASA or NSF) to increase the emphasis of statistics education of teachers at that institution. Similar to the first study, the participants were predominately female (70.3%) and approximately 82% self-identified as Caucasian. While these percentages seem high, they follow national trends and are similar in demographics to pre-service teachers in other studies on teacher efficacy (Duffin, French, & Patrick, 2012; Knoblauch & Woolfolk Hoy, 2008).

Instrument

The SETS-MS has 26 items using a 6-point Likert scale, with 1 = *not at all confident* and 6 = *completely confident*. Harrell-Williams et al. (2014) provides information regarding instrument development and evidence for reporting two subscales for the SETS-MS instrument, identified as "Reading the Data - Level A" and "Reading Between the Data - Level B".

Analyses

The analyses in this paper fall under two categories: those done at the subscale level and those done at the item level. Cronbach's alpha was calculated as reliability estimates for scores from the two aforementioned subscales comprised of the 26 middle grades SETS items. MANOVA was used to determine if the means of the two correlated subscales scores differed for middle grades and high school PSTs.

Three item-level analyses were completed. Item means, classical test theory measures of item difficulty, were obtained for each PST sample. Chi-squared tests of homogeneity of proportions compared the distribution of response categories percentages across the groups for each item. The Benjamini–Hochberg procedure (1995) controlled the false discovery rate for the 26 Chi-squared tests, using a false discovery rate of 0.05. Lastly, differential item functioning (DIF) was assessed across the pre-service teacher groups using the Wald test method in IRTPRO (Cai, du Toit, & Thissen, 2011), with item estimation occurring within each subscale. In general, DIF was assumed to exist if any of the Wald test p-values were smaller than 0.05. The Benjamini–Hochberg procedure was also employed in the DIF analysis to minimize the false discovery rate.

Results

Subscale Analyses

The Cronbach's alpha reliability estimates for the subscales exceeded 0.90 for both the middle grades and high school PSTs (see Table 1), indicating very little measurement error in the subscale scores and almost no difference in the reliability across groups. According to the MANOVA results, the mean subscale scores for Level A and Level B were not significantly different across the two groups, $F(2,596) = 0.037$, $p = 0.964$.

Table 1: Subscale-Level Results

		Middle Grades PSTs			High School PSTs		
Subscale	Number of Items	Cronbach's Alpha	Mean	SD	Cronbach's Alpha	Mean	SD
Reading the Data (Level A)	11	.92	4.54	.91	.93	4.55	.81
Reading Between the Data (Level B)	15	.94	4.11	.98	.93	4.10	.84

Item-Level Analyses

For the Level A items (items 1–11), the item means ranged from 4.26 to 4.86 for the high school PSTs and from 4.28 to 5.01 for the middle grades PSTs. For Level B, the item means ranged from 3.45 to 4.72 for the high school PSTs and from 3.62 to 4.65 for the middle grade PSTs. The observed difference in item means indicates the two groups were likely responding differently to some items. The chi-squared tests of homogeneity of proportions in response category usage revealed that middle grade PSTs responded differently than high school PSTs on 13 of the 26 items. Five of these items were from Level A (items 4-7, 10), while eight were Level B items (items 12, 14-19, 22). The majority of these items addressed creating or using specific visual displays to summarize, describe or compare distributions (dotplot, histogram, boxplot). In most cases, the middle grades PSTs were using the higher response categories (5 or 6) with higher frequency than the high school PSTs, for whom 5 was the category with the highest frequency. All *p*-values for the Wald tests for evaluating differential item functioning across the two groups of PSTs were 0.82 or greater, indicating that the items did not perform differently across the groups when participant self-efficacy levels were taken into account.

Discussion

The current SETS-MS instrument seems appropriate for use in a mixed licensure/PST audience. The results showed no statistical differences in mean scores or reliability at the subscale level between middle and high school licensure/certification candidates. The differential item functioning analysis indicated when participants' level of self-efficacy was taken into account, the items performed in the same manner. There were slight differences in response category usage at the item level with middle grades PSTs using 5s and 6s with more frequency, while high school PSTs used 5s. However, there was not enough difference across all PSTs to come through as a difference in means in the scores of either the "Reading the Data (Level A)" or the "Reading Between the Data (Level B)" subscales. This suggests that middle school PSTs feel more confident in teaching graphical representations than high school PSTs. Since reading and interpreting graphical representations are present in middle grades and high school CCSSM, to increase their confidence, high school PSTs need more opportunities in their coursework to engage with graphical representations.

A limitation of this study is that both samples were not a random sample of PSTs at either licensure/certification level or across the entire US. However, a complete list of all institutions that prepare middle and high school PSTs was not available. Most studies in mathematics education have been conducted at a small number of institutions, while our data included 23 institutions from 16 different states, collectively. Furthermore, the demographics of our participants reasonably mirror national demographics in race and gender for PSTs.

Conclusion

As suggested changes to preparation of middle and high school mathematics teachers, as described in the ASA's *SET* report (Franklin et al., 2015) are implemented nationally, the hope is that PSTs will be more prepared and confident to teach statistics. The evidence provided in this study indicates that the SETS-MS instrument is a psychometrically-sound instrument for mathematics educators to use in measuring changes in their PSTs, as teacher preparation programs implement recommendations in the *SET* report. For example, when using the SETS-HS instrument to evaluate high school PSTs statistics teaching efficacy, Lovett (2016) found that taking Advanced Placement Statistics in high school had a positive influence on their statistics teaching efficacy. As a future direction of research, similar studies can be carried out psychometrically comparing PSTs to inservice teachers' scores on the SETS-HS.

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WHAT CONTEXTUALIZED SITUATIONS ARE MADE AVAILABLE TO STUDENTS USE STATISTICS IN MATHEMATICS TEXTS?

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In our modern information age societies that are drenched in data, statistical literacy is a crucial literacy for citizens to be able to actively engage in their society and government. Statistical literacy is intended to be developed through the instruction of statistics concepts in school mathematics classrooms. However many mathematics teachers have little to no prior experience with statistics, which means many teacher need to draw upon resources to help teach statistics concepts one of which is their textbooks. The objective of this study is to investigate what contextualized situations are available for students for the use of statistics by two major high school mathematics textbook series.

Keywords: Data Analysis and Statistics, Curriculum Analysis, High School Education

Problem Statement

Statistics and data based arguments are common in people's everyday lives today and are used in a number of powerful ways including to guide government policy, business decisions, influence public opinion, and to influence consumers in making purchases. Societies today are drenched in data (Steen, 2001) and this trend is only increasing in our current information age. Statistical literacy has become a crucial literacy for citizens today (Franklin et al., 2007).

In the context of schooling in the United States, the development of statistical literacy is supposed to come from the teaching of statistics concepts in middle and high school mathematics classes (Franklin et al., 2007; National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010). Unfortunately, most mathematics teachers have had little to no prior experience with statistics (Shaughnessy, 2007). So how then do teachers with little experience in statistics teach statistics concepts to their students? One very likely resource for teachers to rely on is their textbooks. Scholars have reported on the influence of textbooks on shaping classroom instruction and students learning (Fan, Zhu, & Miao, 2013). In line with the conference theme of crossroads, the broad objective of this study is to consider what kind of contextualized situations high school mathematics textbooks give students access to explore with statistics and to consider a possible change in route of the types of contextualized situations used in mathematics textbooks for the use of statistics.

Background

In statistics, "data are not just numbers they are numbers with a context" (Cobb & Moore, 1997, p. 801). This makes the consideration of context central to statistical enquiry (Wild & Pfannkuch, 1999). This is also a departure from the mathematics commonly taught in school where numbers are frequently presented and used in their abstract form without any connection to context (Gattuso & Ottaviani, 2011). In statistics the analysis of data cannot be considered without thinking about the context of the data (Cobb & Moore, 1997; Franklin et al., 2007; Wild & Pfannkuch, 1999). Context determines how and what data to collect, as well as how to analyze the data and interpret the results. This results in a constant interplay between considering a statistical problem and the context of the problem (Wild & Pfannkuch, 1999). For an instructor to teach statistics concepts well they must know more than just the relevant theory, but they must also have a vast supply of relevant contexts with which to see and apply the theory too. In mathematics on the other hand a knowledge of theory is generally enough to create problems and examples on the spot (Cobb & Moore, 1997). Due to such disciplinary differences the consideration of context in school mathematics classrooms is important.

In spite of the influence of textbooks on the enacted curriculum of the classroom there is surprisingly little work specifically focusing on investigating the statistics content of school mathematics texts in the context of the U.S. Much of the previous research has been focused on analyzing the proportion of middle and elementary school texts that focus on statistics content and their alignment to the GAISE Framework (Bargagliotti, 2012; Jones et al., 2015; Pickle, 2012). Only one study focused on the statistics content of high school mathematics texts, which was a dissertation looking at the learning trajectories related to bivariate data in a single text (Tran, 2013). Based on this review of the literature there seems to be a significant lack of research on the statistics content of high school mathematics textbooks

Methodology

Research Question

The specific research question to be considered in this study is: *how are the contextualized situations appropriate for the use of statistics formed by the statements of two major high school mathematics textbook series?*

Selected Texts

In a survey of a nationally representative sample of U. S. mathematics teachers, Banilower et al. (2013) found the top two companies making up the market share of high school mathematics texts are Houghton Mifflin Harcourt with 35% (SE=1.6), Pearson with 30% (SE=2.0). The data for this survey was collected during the initial transition to the CCSSM (NGA Center & CCSSO, 2010). Therefore, I chose to extrapolate the survey results and select Houghton Mifflin Harcourt and Pearson's most recent and CCSSM aligned textbook series to study predicting that they would still make up a majority of the market share. For Pearson I analyzed their Algebra 1, Geometry, Algebra 2 Common Core curriculum (Randall et al., 2015) and for Houghton Mifflin Harcourt I analyzed their Algebra 1, Geometry, and Algebra 2 curriculum (Kanold, Burger, Dixon, Larson, & Lienwand, 2015). All lessons with statistics standards as the explicit focus were included as well as lessons on modeling that included modeling data sets with variation. Counting rules and the development of theorems on the mathematics of probability were excluded, consistent with the GAISE framework (Franklin et al., 2007) recommendations.

Archeology

For this study I drew upon Foucault's (1972) methodology of archaeology, which is focused on the study of discourse to interrogate the "regimes of truth" or knowledge, constituted by the rules or regularities of statements in discourse. From this perspective a discourse consists of a regulated set of rules that are generally taken for granted, and also constrain and "specify what is possible to speak, do, and even think, at a particular time" (Walshaw, 2007, p. 19). Statements in discourse operate in different ways one of which is, "materiality (which is not only the substance or support of the articulation, but a status, rules of transcription, possibilities of use and re-use)" (Foucault, 1972, p. 115). For this study the operation of materiality was considered in terms of the contextual situations in which the use and re-use of statistics is formed as appropriate or normal. From this perspective regularities in the contextualized situations (e. g. rolling dice, test scores, profit, personal characteristics, science, etc.) presented in high school mathematics textbooks influence and regulate in what contextualized situations that teachers and students use statistics in the future.

The focus in this analysis was on the regularities in the types of contexts that were presented in the text and in what form data based information was provided for each (if at all). The forms in which the data were presented that were considered included the types of variables described in the contextualize situations (quantitative or categorical), what sample size was described (if any), whether raw data, summary statistics, or representations were provided with the contextualized situation, and finally whether or not real data was provided with the contextualized situation. I operationalized real data as data that was provided with a citation of its source. The analysis of the texts was done through an iterative process of multiple readings of the data.

Findings

In analyzing the two textbook series a number of strong regularities came out in the types of contextualized situations that were presented for investigating with statistics, which I used to create contextual categories (see Table 1). There was little variation from Algebra I to Algebra II texts within both series and there was also a significant amount of overlap in the categories between the two series.

Table 1: Contextual Categories

Pearson Contextual Categories	HMH Contextual Categories
Entertainment/Sports/Exercise	Entertainment/Sports/Exercise
School/Testing	School/Testing
Science/Weather	Science/Weather
Personal Characteristics	Personal Characteristics
Voting/Personal Preferences	Voting/Personal Preferences
Transportation/Travel	Census/Population data
Business/Economy/Sales	Transportation
Manufacturing/Product Quality	Business/Sales
Food/Farming/Agriculture	Manufacturing/Product Quality
Rolling Dice/Random Draw	Food
Making Decisions/Fair Decisions (Only Algebra II)	Pets/Animals
	Work/Salary/Savings
	Random Draw/Flip coins/Roll dice (Only Algebra II)
	Health/Medical (Only Algebra II)
	Fair Divisions/Outcomes (Only Algebra II)

In considering the forms in which the data based information for each task was presented, contexts were presented in a wide variety of forms. Some statements included no data-based information, while others only provided descriptive statistics such as the measures of center or spread for a situation. Some statements presented data in representations, which included histograms, bar graphs, line graphs, scatterplots, box and whisker plots, data tables, frequency tables, and two-way tables. Some statements presented ordered or raw data. A mixture of any number of these forms was also observed. No statements from the Houghton Mifflin Harcourt text contained real data, while Pearson did present some statements with real data. In the statements only one or two variables were presented with any regularity, if any were at all. For those statements that did present data, the mean sample size was 11(7.4) ranging from 3 to 50 in the Pearson textbooks and the mean sample size was 12 (10.5) ranging from 3 to 100.

Discussion

It is promising that so many categories of contextualized situations were present in the textbooks as it has been pointed out in the past that statistics is often taught abstractly and focused on decontextualized calculations in mathematics classes (Gattuso & Ottaviani, 2011). However, the promise ends there as the contexts in which the texts construct as appropriate for the use of statistics generally go no further than those typical of small talk, such as the weather, sports, or personal preference, or related to work or business. The types of contextual situations presented in the texts are predominantly neutral with almost no controversial issues like those that citizen's face in their daily lives being presented. Some of the most prevalent issue societies are facing today including immigration, race, gender identity, women's rights, climate change, water access and quality, gun rights, gentrification, urbanization, wealth distribution, poverty, and government spending are not at all present. Essentially these texts are not constructing statistics as useful for one to make sense of the world around them. This means there is serious need for curriculum to be created that fosters students to make sense of the world around them through statistics.

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ELEMENTARY PRESERVICE TEACHERS' UNDERSTANDING OF VARIABILITY AND USE OF DYNAMICAL STATISTICAL SOFTWARE

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Keywords: Data Analysis and Statistics, Elementary School Education, Middle School Education, Teacher Education-Preservice, Technology

A major component of the statistical thinking involves understanding and describing variability in data. Moore (1997) points out the importance of both variability and the measuring and modeling of variability in statistics learning. However, Reading and Shaughnessy (2004) claims that there was a lack of attention to the measurement of variability, which results in a gap in students' concepts of variability. Moreover, the Guidelines for Assessment and Instruction in Statistics Education (GAISE) (2007) has suggested that the use of technology will help students develop a better sense and understanding of variability through more engaged data analysis tasks in upper elementary and middle school grades. However, little research has been done regarding how K-8 pre-service teachers (PSTs) have developed an understanding for variability for themselves in preparation of teaching the topic to children. What experiences are being developed that prepare K-8 PSTs to understand variability while exploring data with technological tools such as dynamic statistical software?

This study analyzed tasks used in a statistics course designed for elementary/middle school PSTs that focused on developing PSTs' understanding of measures of variability. A case study methodology was used with data collected from the statistics content course in Western Michigan University, where dynamic statistical software, TinkerPlots, was used almost daily in class. PSTs' class works and classroom observation notes were analyzed and coded based on Garfield and Ben-Zvi's (2005) framework of seven components that comprise a deep understanding of variability. Based upon the preliminary findings from tasks and class sessions examining variability in data, specifically that of making sense of the mean absolute and standard deviation measures, we see that PSTs are beginning to develop a more meaningful understanding of the measures of variability with the help of dynamical statistical software such as TinkerPlots. TinkerPlots provided several means for students dynamically engaging with the data while thinking about spread. Such work is needed in order to help our pre-service teachers be current in their understanding of the use of technology tools in the teaching and learning of statistics.

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UNDERSTANDING AND APPLICATION OF SLOPE BY EIGHTH GRADE STUDENTS DURING A STEM LESSON

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A major challenge for students is to apply what they have learned in mathematics classrooms in other contexts, and it is a task for their teachers to prepare them to do so. In this poster we describe the ways in which students applied their knowledge of the concept of slope and how that knowledge evolved within a science lesson on density. The lesson was part of an integrated science, technology, engineering, and mathematics (STEM) unit where students studied the physical properties of minerals as they worked toward designing a process for sorting minerals using those physical properties. In the lesson studied, students compared the densities of two minerals by examining the scatterplots of mass vs. volume for several samples of two different types of minerals. The unit and lesson were specifically designed to support students in developing meaningful understanding of both science concepts such as density and mathematics concepts such as proportional relationships and slope (Moore et al., 2014).

The participants in this study were four groups (13 total) of eighth grade and two groups (six total) of sixth grade students at schools in two different mid-western states. Students were interviewed about their understanding of linear relationships in data and the specifics of slope both before and after the observed lesson. These task-based interviews presented students with a scatterplot of data and a graph showing linear relationships in context and asked them to interpret and answer questions about each. Additionally, audio and video recordings were captured during their group work sessions, and the lessons were observed by the researchers.

Students began the unit with a variety of understandings about slope and linear relationships in data. Some students began with very little knowledge, while other students were able to demonstrate fairly sophisticated understanding of the concepts. After the lesson on density and slope, most students demonstrated a better understanding of the concepts or were able to articulate what they had already demonstrated more clearly. Additionally, several of the students were able to point directly to things that they had done during the lesson which helped them to better understand the interview tasks. However, not all students were able to apply what they had learned or demonstrated in one context in the other contexts. This was true both for students who were able to demonstrate knowledge during the interviews but not during the lesson and visa versa. This observation is consistent with research arguing for the situated nature of learning (e.g. Lesh, 2010; Cobb, 1999). Implications for teaching and curriculum design are addressed.

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CONTEXT AND VARIABILITY: HOW DATA CONTEXT SHAPES PRESERVICE TEACHERS CONCEPTIONS OF VARIABILITY

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Keywords: Data Analysis and Statistics, Teacher Education-Preservice, Teacher Knowledge

This project builds on by research done by Thanheiser et al. (2011), which investigated elementary preservice teachers' (PSTs) statistical thinking. In one of the tasks given to PSTs, they were asked to provide prices of seven potato chip bags with a specified average. The PSTs tended to provide data close to the average; however, the authors noted that the potato chip prices context may have influenced responses to be close to the expected value, as prices are often the same across stores. The authors asked "would PSTs construct data sets with a wider variability if the context was not restrictive?" (Thanheiser et al., 2011, p. 545) We hope to not only answer that question, but also provide further insight into how PSTs perceive variability of different contexts. In practice, educators should choose a data context in a purposeful way, and this research aims to improve on this practice by improving the education of PSTs in relation to statistics and the data context. Wild and Pfannkuch's (1999) framework on statistican inquiry highlights both the omnipresence of variability in statistics while also noting that statistical reasoning requires shuffling between the statistical and contextual realms of thought.

Our research question is "How does data context affect how preservice teachers think about variability?" To answer this question, we designed a mixed-methods study including a survey and interviews with a subset of the survey respondents. We adapted the task from the previous study into three tasks, all which prompted students to come up with seven data points from a population that has an average of 23, but each of the tasks was placed in one of three data contexts: hourly wages, college t-shirt prices, and no context. The survey also included questions to probe PSTs for their reasoning. Our sample was 66 students enrolled in math courses for PSTs at a Pacific Northwest university, and we randomly assigned each PST to one of the three versions of the survey.

Survey responses indicated that the wage data context elicited students to provide data with more variability than the t-shirt context. One student in the hourly wage context group writing "I thought about just writing numbers 20-27 in the boxes… this would be easy, but I didn't think this was a realistic salary range. So I changed it to counting by 2's to widen the range ever so slightly." However, our initial review of interview responses indicate that there are several wrinkles in this conclusion, and we will present further results from the interviews to better explain the between group differences noticed in the surveys.

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SURVEYS OF ATTITUDES ABOUT STATISTICS: AN ANALYSIS OF ITEMS

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Keywords: Data Analysis and Statistics; Affect, Emotions, and Attitudes; Teacher Beliefs

Attitudes toward statistics are considered important outcomes of introductory statistics courses because they have been linked with student achievement and because they inform people's lasting impressions of the discipline (Gal, Ginsburg, & Schau, 1997; Ramirez, Schau, & Emmioğlu, 2012). These attitudes have been studied since the 1950s, and dozens of instruments have been created during this time (Nolan, Beran, & Hecker, 2012; Ramirez et al., 2012). Over this 60-year period, there has been substantial evolution in the discipline's understanding and conception of attitudes. These surveys are generally described and reported on in isolation, but few studies have made comparisons across instruments. One notable exception is Nolan, Beran, and Hecker's (2012) meta-analysis that examined the reliability and validity of 15 instruments designed to measure the attitudes of post-secondary statistics students. However, this meta-analysis focused on the surveys and their subscales without examining the original items used in the surveys. Statistics education is at a crossroads with an increase in research with populations other than undergraduate students, i.e. teachers and pre-service teachers. To chart a route forward, this study examines the original items used on survey instruments from the past 37 years to 1) provide a more robust understanding of attitudes about statistics as enacted in the literature and to 2) clarify how constructs that have used common labels (e.g. "values," attitudes," and affect") have been operationalized by different surveys.

The data for this study were the individual items included on the operational or final versions of surveys of statistics attitudes (broadly conceptualized). The final or operational versions of 17 surveys were found by searching the literature, requesting original dissertations, and contacting authors. The data analysis will proceed in two phases. First, the items will be analyzed using an inductive process as described by Hatch (2002); the result of this iterative phase will be a set of codes representing constructs (and subconstructs) measured by the items from the surveys. Then, each of the survey items will be categorized using these proposed codes by the research team. Inter-rater reliability will be assessed, and the team will meet to discuss and resolve disagreements. The development and description of constructs will address the first goal of this study, and the categorization of items using the codes will address the second.

With an increasing focus on statistics education research with new populations, a rich description of attitude constructs and subconstructs that have been valued and assessed in the literature will inform future instrument-development projects. Describing how different surveys assess nominally-similar constructs will also benefit researchers seeking to use or interpret surveys with groups other than undergraduates, such as K-12 students and in-service teachers.

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Chapter 10

Student Learning and Related Factors

Brief Research Reports

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Posters

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BARRIERS TO ENGAGEMENT IN MATHEMATICAL DISCOURSE: MALIGNANT POSITIONING IN THE SECONDARY MATHEMATICS CLASSROOM

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In this study, we demonstrate how a teacher draws on validating storylines as support for the malignant positioning of students within classroom interactions, which reflects broader storylines about honors versus non-honors students, mathematical proficiency, and differential access to powerful mathematics.

Keywords: Classroom Discourse, High School Education, Research Methods

Objectives and Purpose of the Study

In this paper, we develop an integrated theoretical approach to examine mathematical discourse in a high school classroom. Drawing on theories of positioning and the social forces that shape interactions, we discuss the *storylines*, or patterns of interaction based on commonly shared narrative conventions (Davies & Harré, 1990), at play within the moment-to-moment interactions between a secondary mathematics teacher and her students. The goal of this work is two-fold: (1) to demonstrate how validating storylines serve as support for enacted subject positions; and (2) to illustrate how malignant positioning can limit students' engagement in mathematical discourse.

Theoretical Framework

In this study, we draw on positioning theory to uncover the ways in which teachers and students make sense of one another's actions within the classroom, something we argue is an integral part of mathematical meaning-making. As Davies and Harré (1990) describe, positioning is "the discursive process whereby selves are located in conversations as observably and subjectively coherent participants in jointly produced story lines" (p. 48). Taken together, storylines and positions are used to interpret actions within discourse. The way in which an action gets interpreted by others depends on the storylines (and related subject positions) that others believe to be in play. One of the key theoretical assumptions of our study is the belief that discourse shapes the positions that are available to participants, even as these positions shape the discourse.

In order to more accurately describe the discursive work that teachers and students do in the mathematics classroom, we distinguish between several different types, or what van Langenhove and Harré (1999) called "modes," of positioning, including intentional and malignant positioning. For example, on the first day of school, a teacher may clearly outline her expectations for the year in order to intentionally position students as having a right to succeed but a duty to work hard. Such intentional positioning can be considered part of the discursive work people do every day in order to create identities, relationships, and practices, and even to privilege one way of knowing over another (Gee, 2014).

The term "malignant positioning" has been used to describe the way in which Alzheimer's patients can be stripped of their social persona and rights in order to be characterized as having less ability and status (Kitwood, 1997; Sabat, 2003). As Harré and Moghaddam (2003) explain:

> To say that someone cannot quickly bring to mind certain words is to predicate a certain psychological attribute to that person. However, to slide from that attribution to positioning someone as no longer having the right to make life decisions for him or herself is an act of positioning. (p. 8)

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We define malignant positioning as acts of positioning that limit the rights and duties of individuals or groups. In the mathematics classroom, for instance, to say that a student cannot quickly bring to mind the solution to a problem is to predicate a certain attribute to that student regarding their mathematical competence. However, shifting from that attribution to positioning the student as no longer having the right to engage in powerful mathematics, or mathematics with "clout" (Bruner, 1986), is an act of malignant positioning. Sometimes, the attribution in question is institutionalized (e.g., tracking students into non-honors and honors courses). As our findings show, when teachers draw upon "validating storylines" (Sabat, 2003) to justify acts of malignant positioning, students become entrenched in positions with limited rights to powerful mathematics and minimal duties (expectations) as students of mathematics.

Methods of Inquiry

This study was conducted in Ms. Mason's Algebra II classroom, which was located within a large suburban high school that enrolled approximately 1888 students at the time of this study. According to information provided by the school district, 22.89% of the school's students qualified for free and reduced lunch and the teacher to student ratio was approximately 26:1. Ms. Mason, who has 7 years of teaching experience, was chosen due to her willingness to provide access to her classroom over an extended period of time, which was crucial to gain insight into how teachers and students position themselves and one another within mathematical discourse.

Data Collection and Analysis

Our goal in analyzing classroom discourse was to examine teacher and student interactions from multiple and complementary perspectives. Sources of data included video recordings of 11 classroom sessions (approximately 90 minutes each), interviews with the teacher and focus students, and mathematical artifacts created by the teacher and students. Data analysis proceeded in three overlapping phases: (1) initial, collaborative coding of all classroom videos; (2) micro-analysis of focal interactions; and (3) triangulation across data sources. The first phase of data analysis involved viewing and coding classroom video from all 11 class sessions. Using qualitative coding procedures informed by grounded theory (Strauss & Corbin, 1990), we employed open coding to identify concepts and themes for further analysis and axial coding to organize and integrate categories.

The initial coding process (Phase 1) allowed us to identify focal interactions for more detailed micro-analysis (Phase 2). Using a scale (Herbel-Eisenmann et al., 2015) as the unit of analysis, we focused on particular utterances and exchanges at the levels of 10^0-10^2, words and interactions among people (Lemke, 2000). In order to examine the ways that participants drew on storylines to validate positioning acts in moment-to-moment interactions, we foreground these communication acts while backgrounding attention to storylines. The third phase of analysis involved triangulating our analysis across data sources. In addition to checking and confirming findings and interpretations across class sessions, we analyzed and coded video interviews and artifactual evidence. By looking across multiple scales, including lessons (10^3), lesson sequences (10^4), and units (10^6) (Lemke, 2000), we were able to examine the relationships between positions and storylines (Herbel-Eisenmann et al., 2015).

Results

To illustrate instances of malignant positioning and the effects of this positioning on students' engagement in mathematical discourse, we focus on sample interactions between Ms. Mason and her students at the 10^2 level, words and interactions among people (Lemke, 2000). In the following exchange, Ms. Mason stands at the board, introducing a lesson on how to analyze the behavior of a rational function near a vertical asymptote.

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Ms. Mason: If this was an honors class, what I would have you do is pick x points between -4 and 4 and plug them into the equation. Then, I'd have you pick x points below negative 4 and above negative 4 and plug them into the equation=

Student: (shift forward, hands coming up in questioning gesture to shoulders): So, why wouldn't we do=

Ms. Mason: =So now what we're going to do, we're going to look at the calculator. [Pause]. We've got enough to get us started, because I need at least these things labeled on the graph.

The preceding example illustrates how positions and storylines are mutually constituted within the everyday workings of the classroom. In Line 1, as Ms. Mason explains her process for graphing the local behavior of rational functions using the "honors vs. non-honors" storyline, she concurrently introduces an associated set of subject positions that are now made available to others within the conversation. As she positions the students in the room as "non-honors," not capable of achieving the same level of rigor that she would demand of more capable "honors" students, she narrows what is *logically possible* (students algebraically determining local behavior of function near vertical asymptotes) into what is *socially possible* (the calculator doing the work of determining local behavior) (Harré & Moghaddam, 2003). When a student attempts to challenge this position in Line 2, asking "Why wouldn't we do," Ms. Mason cuts her off and instead proposes an alternative: "So now what we're going to do, we're going to look at the calculator." The discussion continues:

Ms. Mason: I'm not going to accept this. I'm not going to accept this. You're going to have to be more specific. Some of you, on the cumulative test, when you had a parabola, you did this (*shakes student's paper*). Nothing labeled. I will not accept that. You have to give me something. I will not accept random swooshes (*puts paper down on desk*). This isn't Nike. I need specifics.

Student: Why are they=

Ms. Mason: =Get your calculators out. All right. You need to write this in your notes, because this is extremely important when you work with rational functions.

Student: Why are they not there?

Ms. Mason: They're not actually lines, they're boundaries. Kind of like a fence. Keeps you from crossing.

Student: What did you say this was for?

Ms. Mason: So, this is rational functions. So, here's what you need to star. This is really crucial—put all of numerator in parentheses and put all of denominator in parentheses. If you don't do that, you are not going to get the right graph. All right?

In the preceding exchange, Ms. Mason draws on a validating storyline (honors vs. non-honors) in an act of malignant positioning that limits the rights and duties of these "non-honors" students. Instead of investigating mathematical understandings through the use of multiple representations (i.e., numerical or algebraic) as honors students might, non-honors students rely heavily on calculators and follow explicit instructions to get the "right" graph. In this case, the larger educational storyline about differential access to powerful mathematics validates the positions at play in the moment-to-moment interactions between Ms. Mason and her students.

Discussion

In a recent commentary, the NCTM Research Committee (2016) suggested that education researchers have a right and duty to "intervene to shift these storylines and positionings and to have greater impact on policy, practice, and public perception" (p. 103). We agree that researchers can play a key role in transforming conversations about the teaching and learning of mathematics. However, we also believe that it is equally important to examine the ways in which storylines unfold

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in the moment-to-moment interactions between teachers and students in classrooms. In these spaces, more than mathematics is being created—students create positions and are positioned by others as certain kinds of people: a good student, a bad student, a class clown, an honors student, or a non-honors student. As Morgan (2012) argues, "the challenge is to connect such classroom-level analyses to a developed understanding of the broader context" (p. 192). In this study, we demonstrate how malignant positioning emerges within the mathematics classroom and reflects broader storylines about honors versus non-honors students, mathematical proficiency, and differential access to powerful mathematics. Moreover, we highlight the teacher's role in the ongoing negotiation of such positionings, providing insight for those seeking to remove barriers to mathematical engagement within their own classrooms.

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CHANGES IN STUDENT PERSPECTIVES: WHAT IT MEANS TO BE "GOOD AT MATH"

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This paper describes how a cohort of sixth-grade mathematics students shifted their collective understanding of being "good at math" from passive recipiency to active agency.

Keywords: Equity and Diversity; Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs

For many students, school in general, and mathematics classes in particular, are places to learn that they are not smart. In some cases, student interests and abilities are discouraged as counter to the classroom model of good student. Cobb, Gresalfi and Hodge's "normative identity" accounts for the "general and specifically mathematical obligations that delineate the role of an effective mathematics student in that classroom" (2009, p. 58). Thus, normative identity can be thought of as the set of markers by which a person could identify accomplished learners—if you asked individual students what it means to be good at math, what is the consensus? Traditionally, the normative identity of "good at math" has been limited to being fast and accurate (Horn, 2007). Classrooms that are student- and inquiry-centered often value not only accuracy, but also: bravery in taking risks; embracing uncertainty; collaboration; communication; using misconceptions as springboards for discussion; the use of multiple representations; and being helpful (Boaler, 2015; Ruef, 2016). There is ample evidence telling us that teachers' beliefs about "good at math" matter (e.g. Gresalfi & Cobb, 2011). But students also need and benefit from a more open vision of mathematical success (Cohen & Lotan, 2014). When there are more ways to be successful, more students succeed. This paper describes the shifts in normative identity for 62 middle school students, finding that perceptions of what it meant to be "good at math" shifted from the restrictive normative identity of smart, fast, and correct, to the more inclusive identity of brave, mistake-making, and able to share one's thinking.

The school in this study embraces public sensemaking—the communal act of sharing individual student thinking in a public manner, with the goal of better understanding and refining a mathematical argument. Research has shown that public sensemaking has the potential to powerfully impact student learning (Cohen & Lotan, 1997; Stein & Smith, 2011). Risk-taking is related to "growth mindset"—the belief that one gets better at mathematics through effort—as opposed to remaining stuck a "fixed" innate level of ability (Dweck, 2006; Ruef, 2016).

Related to growth mindset, Boaler links "closed" vs. "open" mathematics to student opportunities to see themselves in the mathematics they do, and mathematics as relevant to their lives (Boaler, 1998). Closed mathematics refers to tightly procedure-bound and rule-based mathematics, which supports a culture of quick and accurate calculation. Open mathematics refers to opportunities to explore, create, collaborate, and invent new methods to make sense of mathematics. Sun (2015) found that teachers' implicit messages of what it means to do mathematics are more impactful than explicit messages about having a growth mindset. In other words, practicing what one preaches matters—it does not work to tell students to have a growth mindset when the teaching and learning environment is infused with closed mathematics.

Methods

This report is a subsection of a larger study, which followed a cohort of 62 sixth-grade mathematics students across most of the 2015-2016 academic year. The students were divided into three classes, and all shared the same mathematics teacher, Ms. Mayen. The students identified as: 76% Latinx, 15% African American, 8% Asian, and 1% Filipinx. They attended City School, a San

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Francisco Bay Area middle school that serves a diverse student body drawn from a large urban setting. According to the school's website, more than 91 percent of students qualified for free lunch. Ms. Mayen identifies as Latina, trained at a prestigious teacher preparation program, and was in her second year of teaching at the time of the study. All names in this study are pseudonyms.

Data included pre- and post-surveys constructed from 26-item Likert-scale and three short answer items (Table 1); field notes; analytic memos; classroom video; interviews with students and Ms. Mayen; and three sets of exit tickets. The post-survey was administered on March 18, 2016, to accommodate the schedules of both the participants and the researcher. Analysis of interviews, video, exit tickets, and field notes included coding for a-priori and emergent codes. The analysis was iterative, with emergent themes tested for counter examples, and refined through comparison to additional data (Charmaz, 1995). Coding was validated by comparison with trained raters, resulting in an average inter-rater reliability ratio of 87 percent.

Analysis and Findings

"Good at Math" in August

On the first day of school, students reported that someone who is good at math is compliant; smart; helpful; gets good grades; gets answers quickly; and is focused or pays attention in class (Table 1). This normative identity paints a portrait of "passive recipiency" with markers of success that map to a traditional classroom, including paying careful attention and following rules, and getting answers quickly and accurately (Boaler, 1998; Ruef, 2013; Horn, 2007). Asked to describe someone who is "good at math," Jazmin wrote "They follow directions; do their homework. Teacher's favorite student." Mia shared "They finish their work faster than other people do." These responses were coded as "compliant" and "speed/fast." Several students valued peers who help others with their mathematical work. This maps to reports from September student interviews wherein several students referenced the role of helpers in the classroom, from supporting presenters at the board to collaborating on seatwork. Krystal shared that "A person who is good at math is really smart. They are good at math and help others at math like teachers. They do hard problems." Notice that she sees the role of a smart and helpful student as being similar to that of a teacher. This was a common theme among responses—"smart" is a resource to be shared by helping others. While helping is positive, this early version was laced with the notion that smart and helpful people show how to complete procedures, which reinforced the idea that "smart" meant "knowing the rules."

"Good at Math" in March

This "good at math" portrait included being focused; making mistakes; having a growth mindset; being compliant; explaining well; being smart; presenting one's thinking, and being helpful (Table 1). There is evidence of a shift toward a normative identity of "active agent." The shift makes sense when compared to the norms and culture the classes had co-established with Ms. Mayen (Ruef, 2016). Of particular note is the importance of making mistakes and explaining well. The reason for the growth in valuing mistakes may be attributed to Ms. Mayen consistently valuing them as important for learning and understanding mathematics, which showed up in the student responses on exit tickets and survey responses. Barney wrote "How you know someone is good at Math is if they help and they make a lot of mistakes." Ms. Mayen and her students worked hard to establish a culture centered on efforts to share and understand one another's thinking (Ruef, 2016). It follows that the students valued presenting one's thinking publicly, helping, and explaining well.

Not one student referenced grades on the post-survey, and while students answered the pre-survey with rigidity in following the "rules" of filling in dots, students modified the Likert-scale portion of the post-survey. Some added doodles, one sketched a block letter **YET** (classroom shorthand for growth mindset), and some circled rather than shading in dots. This may be an

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indication that students felt greater agency to exercise creativity in answering the surveys. Montse invented a new word, “mistakealable,” to describe math—for her, math is a place where making mistakes is an important part of the learning process. The biggest shifts apparent in the Likert-scale items illuminate changes in the students’ collective vision of doing mathematics (Table 2). Based on results from the paired t-tests of the pre- and post-surveys, students came to value open mathematics, creativity, and the power of consensus to determine the validity of an answer. They disagreed most strongly that “in math, the most important thing is to get the correct answer.”

Discussion

It is clear that Ms. Mayen’s students shifted their beliefs about what it means to be “good at math,” but how and why? These students came to see being brave in presenting at the board as a necessary precursor to sharing their thinking. Relatedly, sharing their thinking supported deep understanding of each other’s perspectives. Perhaps most importantly, making mistakes was central to the work of being a good at math. This short report focuses on the shifts that occurred rather than the mechanisms that drove those changes. I have reported elsewhere on the careful work done in concert between Ms. Mayen and her students to establish the norms and culture of the classroom, which were essential to developing safe ground for the risk-taking that public sensemaking demands (Ruef, 2016). Ms. Mayen inherited, sought out, and augmented a mathematics curriculum that was made up primarily of open mathematics tasks. Further, she negotiated and reinforced norms that supported public sensemaking. It is therefore not surprising that collectively, her students took up the view that math class was about open mathematics and growth mindset, and that they would be successful if they were active agents (Ruef, 2016; Sun, 2015).

In closing, I share two collective stories drawn from these normative identity composite sketches. In the beginning of the year, Ms. Mayen’s students seemed to believe this narrative: Mistakes are bad. If I make a mistake, I am bad at math. By March that story had change to this: Mistakes are good. If I make a mistake, I am good at math. Ms. Mayen’s students re-engineered their understanding of what it means to be smart, and in so doing, opened more doors for themselves and others to identify as successful mathematics students.

Table 1: Percentage of Respondents who Identified Traits of Being “Good at Math”

A person who is “good at math…”	8/25/15 Survey	2/22/16 Exit Ticket	3/18/16 Survey
Is compliant in traditional ways (takes notes, studies hard, behaves well, is quiet).	38%	9%	16%
Is smart.	28	4	12
Is helpful.	20	4	9
Gets good grades.	10	8	0
Gets answers fast.	7	4	2
Explains well.	0	21	12
Has a growth mindset.	2	9	16
Makes mistakes.	0	19	21
Presents their thinking.	2	9	11
Is focused or pays attention.	20	6	25

pre-survey (8/25/15, n = 60); exit ticket (2/22/16, n = 53); post-survey (3/18/16, n = 57)

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Table 2: Statistically Significant Shifts from pre- to post Survey. (n = 55) Italics Indicate Items that Grew in Strength from pre- to post Survey.

Question[1]	$\bar{X}$ pre	$\bar{X}$ post	$\bar{x}_{post} - \bar{x}_{pre}$	*t*-value	*p*-value
Positive Identification with Mathematics					
I like math.	4.44	4.89	.45	-2.51	**.015***
Public Sensemaking					
The best way to learn math is it talk about it with other people.	4.89	4.36	-.53	2.61	**.012***
Open Mathematics					
In math, you can be creative.	4.49	5.07	.58	-2.95	**.005***
In math, you can invent your own ways of doing things.	4.44	5.07	.64	-3.60	**.001****
Math affects people in the world outside of math class.	3.51	4.26	.75	-3.30	**.002***
Fixed Mindset					
You can't solve math problems if you don't know the right formulas.	4.15	3.43	-.71	3.12	**.003***
Some people just aren't good at math.	2.93	2.50	-.43	2.11	**.040***
Closed Mathematics					
Mathematics involves mostly facts and procedures that have to be learned.	4.66	4.03	-.64	3.30	**.002***
In math class, it is important to get answers quickly.	2.67	2.13	-.55	3.21	**.002***
It is important to avoid mistakes in math class.	3.31	2.45	-.86	2.67	**.010***
In math class, the most important thing is getting a correct answer.	2.82	1.85	-.97	4.56	**.000****
The best way to learn math is to pay attention to the teacher. (Passive Recipient)	5.02	4.41	-.61	3.25	**.002***
In math class, only the teacher should decide if an answer is correct. (Authority)	3.31	2.44	-.87	3.36	**.001****

[1] 6-point scale: 1 = Strongly Disagree; 2 = Disagree; 3 = Slightly Disagree; 4 = Slightly Agree; 5 = Agree; and 6 = Strongly Agree. * $p < .05$ ** $p < .001$

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COLLEGE STUDENTS' GENDER AND RACIAL STEREOTYPES OF MATHEMATICIANS

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Many individuals are negatively affected by stereotypes of mathematics and mathematicians, which further prevent their interest in pursuing mathematical careers. This paper assesses college students' stereotypes of mathematicians with regards to race and gender. In this study, we asked 179 college students to "Draw a Mathematician." We also conducted four focus group interviews with a total of 12 participants, in which we asked them to view photos of individuals and determine which they believed to be a mathematician. Through our analysis of the data, we have found that many college students do have certain stereotypes of mathematicians.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity and Diversity; Gender; Post-Secondary Education

Stereotypes about mathematics and mathematicians affect how individuals view those who enjoy mathematics and those who enter mathematical careers. As a result, these stereotypes can also influence certain students' mathematics performance, perseverance, and career choice in the field. It is likely that negative stereotypes more greatly affect women and certain minorities, who are already underrepresented in the field of mathematics. Therefore, it is important to know what stereotypes of mathematicians currently exist and how they are viewed by society.

Scholars began studying stereotypes of scientists as early as the 1950's (Mead & Metraux, 1957) and have continued to do so in more recent years (Thomas et al., 2006). Around the turn of the century, scholars also began studying students' stereotypes of mathematicians. For example, Rock and Shaw (2000) collected drawings of "mathematicians at work" from kindergarten through fourth-grade students and found that while the majority of kindergarteners and first grade students drew female figures, the second through fourth grade students drew almost an equal number of male and female mathematicians. In a similar study with middle school students in five different countries, Picker and Berry (2000) also had students "draw a mathematician at work" (p. 70). Picker and Berry found many common themes amongst the drawings. In particular, they noted that many students drew white, male mathematicians. These mathematicians were often wearing glasses, had facial hair, were either balding or had unruly hair, and were dressed in unfashionable clothing.

While some work has been done on children's stereotypes of mathematicians, we know much less about college students' stereotypes of mathematicians. Given that it is during the college years when many individuals are making choices directly relevant to their future careers, we find it critical to know more about college students' stereotypes of mathematicians, especially those stereotypes related to gender and race.

Methods

This research study was conducted in two phases. During the first phase of the study, 179 college students completed a survey in which they were asked to "draw a mathematician." Colored pencils were provided to the students for use in their drawings. Of the 179 participants, 66 (37%) identified as male, 112 (63%) identified as female, and 1 identified as neither. The majority of the participants identified as Caucasian (79%). More than half (58%) of the participants were college freshmen, with another 28% as sophomores. A large variety of majors were represented, with majors from every

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college at the university, and the participants had completed anywhere from zero to six mathematics classes at the collegiate level.

During phase two, we conducted four focus group interviews with a total of 12 volunteers who had participated in the first part of the study. For the focus group interviews, we presented the participants with 16 photos of individuals and asked the participants to determine which individuals they believed were mathematicians and which were not, and to explain their reasoning. The participants were initially asked to do this independently. After each participant recorded their decisions about each of the photos, we led a group discussion about what they had determined. The focus group interviews were audio and video recorded and later transcribed.

Results from the Survey

Of the 179 participants who completed the survey, 87 (49%) drew a male mathematician, 37 (21%) drew a female mathematician, 6 (3%) made sure that both genders were represented, 43 (24%) drew a figure that had an indeterminable gender (such as a stick figure with no hair or clothes), and 6 (3%) drew something other than a person or left the page blank. What we found even more interesting, however, are the results when we analyze our data by the gender of the participant. Therefore, Table 1 contains the data for the male participants and Table 2 contains the data for the female participants. For the one individual who identified as neither male nor female, the drawing of the mathematician was a male.

As can be seen by Tables 1 and 2, both genders drew male mathematicians more often than any other category, however, male students were substantially more likely to draw a male mathematician than female students were. Furthermore, only 3% of male students drew a female mathematician, while 31.5% of female students drew a female mathematician. Similarly, only one male student ensured that both genders were represented while 5 female students made sure that both genders were represented in their drawings.

Table 1: Male Participant Drawings **Table 2: Female Participant Drawings**

Male Participant Drawings			Female Participant Drawings		
Gender of drawings	**N**	**%**	**Gender of drawings**	**N**	**%**
Male	37	56.1	Male	49	43.8
Female	2	3.0	Female	35	31.5
Both genders represented	1	1.5	Both genders represented	5	4.5
Indeterminable gender	24	36.4	Indeterminable gender	19	20.0
No person drawn/blank	2	3.0	No person drawn/blank	4	3.6

We also investigated the implied race of the mathematicians in the drawings, which we compared to the stated race of the participants. To determine implied race of the drawings, we used the colored pencil shading of the face to ensure consistency. For shading options, we developed four categories, which included: no shading, brown, yellow/orange, and other. Of the 173 surveys that had drawings of people, 155 (90%) did not shade their mathematician, 7 (4%) shaded their mathematician brown, 10 (6%) shaded their mathematician yellow/orange, and 1 drawing we coded as "other" because the participant had drawn multiple stick figures with each figure being a different color (red, orange, green, purple, etc).

We further analyzed these results by race of the participants. Table 3 presents the results for Caucasian participants and Table 4 presents the results for the participants who identified as a racial minority or as multi-racial. In our interpretation, no shading could be classified as Caucasian and brown shading could be classified as African American or multi-racial. It is hard to determine what yellow/orange shading is meant to represent, but given that mostly Caucasian participants chose to shade their mathematician as yellow/orange, it is possible that this, too, was intended to represent a

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Caucasian mathematician. If that were the case, then 97% (134 of 138) of Caucasian participants who drew a person, drew a Caucasian mathematician.

Table 3: Caucasian Participants Drawings **Table 4: Minority Participants Drawings**

Caucasian Participant Drawings			Minority Participant Drawings		
Shading of Face	**N**	**%**	**Shading of Face**	**N**	**%**
No Shading	125	90.6	No Shading	30	85.7
Brown	3	2.2	Brown	4	11.4
Yellow/Orange	9	6.5	Yellow/Orange	1	2.9
Other	1	0.7	Other	0	0.0

Results from the Focus Groups

During phase two of the study, we held focus groups with two to four participants at a time, asking them to look at actual photographs of individuals and determine whether or not they think each individual was a mathematician. We had a total of 12 participants in our focus groups, three men and nine women.

We selected 16 images to use for the focus group interviews. We purposely chose images that encompassed many different stereotypes of mathematicians. Of these 16 images, eight were male and eight were female. Four images were of racial minorities. We had an image of an Asian male, an Asian female, an Indian male, and an African American female.

Each image was selected to be a mathematician by at least one participant during the focus group interviews. No image was selected by more than nine participants. In our analysis, we classify whether or not an image was considered to be a mathematician by the participants if more than 50% of the participants (at least seven out of 12) chose it to be a mathematician. Based on this criterion, eight of the 16 images were selected to be mathematicians. Of the eight selected as mathematicians, four were men and four were women. Therefore, male images and female images were equally selected to be mathematicians by our participants. When considering race, the images of the Asian male, Asian female, and African American female were all selected to be a mathematician. The image of the Indian male was not.

During the interview discussions, participants rarely brought up the topic of gender on their own and when we specifically asked about gender, most participants stated that gender did not play a role in their decisions. However, a few participants did admit to using different criteria for men as they did for women. While it was not completely clear from their comments how they used the criteria differently, it is clear that they recognized that they were using different criteria for different genders.

Although discussion on gender had to be solicited by the interviewers, many participants brought up the topic of race on their own. For example, some participants wrote comments such as “racial stereotyping” or “he looks Indian and smart” as their reason for selecting certain images to be of mathematicians. This occurred for the images of both the Asian male and Asian female, and the Indian male. No comments about race were written for the image of the African American female, though she was selected to be a mathematician by eight of the 12 participants. One thing we found interesting was that the Indian male was only selected to be a mathematician by three of the 12 participants. One reason for this might be because we specifically chose a photo of an Indian male who was playing the drums, to put into conflict the racial stereotype of individuals from India being good at mathematics with the stereotype that mathematicians do not have hobbies or interests outside of mathematics (Piatek-Jimenez, 2008). While many participants said they considered that he might be a mathematician, they also stated that because he was playing the drums in the photo was why they determined he must be a musician instead.

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While gender and race seemed to play some role in these participants' decisions about the photos, there were three other criteria that appeared more compelling to them. First, in order to be considered a mathematician, the individual needed to be "dressed professionally" but not be "too dressed up." If the individual was wearing a t-shirt or polo shirt, it was assumed that they were not a mathematician. Yet, if they were wearing a formal dress or a sports coat, that also led our participants to assume that they were not a mathematician. Secondly, the setting of the images influenced our participants' decisions. For example, the woman who was camping and the man playing the drums were rarely selected to be a mathematician, with the backdrop of the image often being provided as the reason. Finally, one of the most cited reasons that participants gave for their decision was whether or not the individual in the image reminded them of someone they know. If the image reminded them of a past mathematics teacher, then they assumed the image was of a mathematician. If the image reminded them of someone else they knew, a past history teacher for example, then they determined the image was not of a mathematician.

Discussion

During phase one of our study, when we asked college students to "draw a mathematician" we found that they mostly drew images of white males. This was true amongst our female participants as well. However, during the focus group interviews, exactly half of the male images were selected to be mathematicians and exactly half of the female images were selected to be mathematicians. In other words, gender appeared to play a much smaller role when the college students considered actual photos of real people. These results suggest that while a male may be the first image that comes to many people's minds when they hear the word "mathematician," they recognize that both men and women become mathematicians and use additional criteria when making assumptions.

While differentiated races were almost nonexistent amongst our participants' drawings of mathematicians, race played a more prevalent role during the focus group interviews. Our participants were not shy about admitting their stereotypes that Asians and Indians are smart, and therefore more likely to become mathematicians. We found, however, that the stereotype that mathematicians do not have hobbies outside of mathematics appears to be stronger than the racial ones, at least for many of our participants. Our participants appeared to feel more strongly that a mathematician would not play the drums than they did about someone from India having to be a mathematician.

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FROM TRAJECTORIES, DEFICIT, AND DIFFERENCES TO NEURODIVERSITY: THE CASE OF JIM

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Cognitive differences intrinsic to children with learning disabilities (LDs) have historically led to deficit assumptions concerning the mathematical experiences these children "need" or can access. We argue that the problem can be located not within children but instead as a mismatch between instruction and children's unique abilities. To illustrate this possibility, we present the case of "Jim," a fifth-grader with perceptual-motor LDs. Our ongoing analysis of Jim's fractional reasoning in seven equal sharing based tutoring sessions suggests that Jim leveraged his knowledge of number facts and alternative representations to advance his reasoning.

Keywords: Cognition, Elementary School Education, Equity and Diversity

Over their school-age years, many children tend to experience difficulties with fractions. For children with learning disabilities (LD), the difficulties can become persistent and grow into unique learning challenges. As a result, researchers continue to focus on ways these children develop *understandings* of fractions as quantities (Hunt, Westenskow, Silva, & Welch-Ptak, 2016) to provide evidence of and access to the potentially rich mathematics in which these children *can* engage.

Unfortunately, these illustrations stand in sharp contrast to the current literature base and policy recommendations for mathematics instruction for children with LDs. Previous research clearly illustrates instruction for these children have been dominated by basic concepts (Kurz, Elliott, Wehby, & Smithson, 2010). A recent review (Lambert & Tan, 2016) of articles researching the mathematics learning of Kindergarten through 12th grade students found significant differences between the mathematical teaching practices used with children with and without disabilities. Mathematical teaching and learning were informed largely by constructivist and sociocultural perspectives with children *without* disabilities. For children *with* disabilities, mathematical teaching and learning were informed primarily by medical and behavioral perspectives. The distinction is concerning as it suggests two categories of mathematics learners who "need" different kinds of mathematics.

Our work both builds on and critiques work on learning progressions (or learning trajectories). Original discussions of trajectories (Martin, 1995) stressed their hypothetical nature and did not separate actualized trajectories from the teachers and children involved in specific instances of learning. Currently, the term suggests an expected course of development or simplified learning path in a concept area, such as early fractions. When considering children with LDs, who may use different or more informal ways of reasoning than educators might expect, we are concerned that educators might use progressions to direct children to move across the levels or stages of a progression without paying attention to the reasoning that children employ and work to support children to explore, revise, and advance that reasoning.

From Explicit and/or Leveled Instruction to Neurodiversity

Attempts to remediate, or "fix," children through procedural training or steering thinking through predetermined pathways or conceptual steps seems problematic if educators wish to provide access to and support reasoning that children with LDs *do* possess and build from it (Hunt et al., 2016). In fact, we contend that these kinds of approaches to remediation may work in part to disable these children more so than their learning differences. Disability Studies (DS) recognizes that although individuals

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have natural biological variations, it is the social effects of difference that disable rather than the impairments themselves (Siebers, 2008). From the DS perspective, the behavioral/directive tradition apparent in much of the instruction these children experience portray learning differences as deficits *within individuals* that results in viewing difference as something to be fixed as opposed to a natural strength that can be leveraged in instruction. Neurodiversity (Robertson & Ne'eman, 2008) positions cognitive differences as not only natural biological differences, but as potential strengths. For example, individuals with LD demonstrate cognitive strengths in three-dimensional reasoning and creative problem solving (Eide & Eide, 2012). Using this lens, we sought to understand the fractional reasoning of one fifth grade children with perceptual-motor LDs as he worked in fractional tasks meant to support two ways of reasoning fundamental to early fraction knowledge: partitioning and iterating (Steffe & Olive, 2010). We utilized equal sharing (i.e., equally sharing an object or objects among a number of people, where the result is a fractional quantity) because it invites multiple means of reasoning and representation. The research question was, "What ways of partitioning and iterating does a fifth-grade child with LDs display in equal sharing tasks?"

Method

"Jim"

"Jim" (age = 12 years) attended elementary school in the Northwestern United States. He was identified by his school system as having a learning disability in mathematics. Jim's performance on the mandated standardized state measure of math performance was at a failing level in 3rd, 4th, and 5th grade, which suggests sustained low achievement in mathematics. His reading scores were at average levels. Jim had received over two years of additional support in fraction concepts and operations that included shading pre-partitioned models and procedures for operations. Finally, Jim evidenced significant difficulties with visual motor integration (i.e., coordination of visual perception and motor skills at 2nd percentile).

Teaching Experiment

Data collection was collected in seven sessions of a teaching experiment (Steffe & Thompson, 2000). Sessions took place during school hours and were in addition to the child's regular math class time. The first and second author attended all tutoring sessions and collaborated throughout the ongoing analysis of teaching episodes. The first author was the researcher-teacher. The second author acted as a witness (e.g., took extensive field notes, observed the interactions to provide an outsider's perspective during on-going analysis). All authors are engaged in retrospective analysis of the data (described below). Researchers collected three sources of data: transcribed video recordings, written work, and field notes.

Tasks, teaching moves, and representations. We prepared problem tasks, representations, and possible teaching moves based on previous evidence of how children with LD might reason in equal sharing tasks (e.g., 3 people share 4 items, Author). Tasks were planned to be dynamic (i.e., adaptable to the child's current conceptions) and presented to Jim in realistic contexts that we changed according to his preference. In each task, the number of sharers ranged from two to ten and the number of objects shared ranged from three to 13. The problem-solving tasks were designed so that Jim could use a variety of strategies and representations (e.g., drawings, Cuisinaire rods) to reason about the mathematics. Teaching moves were broadly defined as responsive to the child's thinking (e.g., extending, supporting).

Data Analysis

Ongoing analysis of critical events (Powell, Francisco, & Maher, 2003) in the child's thinking and learning were noted and discussed before and after each session. The focus was on generating

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(and documenting) initial hypotheses as to what conceptions could underlie the child's apparent problem solving strategies during these critical events. These hypotheses led to planning the following teaching episode. We are currently using retrospective analysis to delineate Jim's informal conceptions of fractional quantities, how his reasoning shifts during each tutoring session, and what his conceptions were during the final session. We are also currently working to identify possible indicators of Jim's conceptual growth using the constant comparison approach (Leech & Onwuegbuzie, 2007). Reported results are tentative.

Preliminary Results

Initial Reasoning: Session One

Jim had just solved several tasks involving whole number partitive division. Excerpt a begins with an extension of the partitive division situations.

Excerpt a: Share 7 granola bars between 3 friends

J: [gives two whole items to each person] And of course there would be one left over. [draws a long bar; carefully marks a small dot at the top middle of the bar and draws a line straight down; then uses the same mechanism to mark each of the two parts into two more parts]. Ok, so they each get a slice of the one that's left. And there's another piece left [begins to partition the fourth part into four parts using the same mechanism].

T: Oh. So [labels each part], so if this is my part, and this is yours, and this is Nita's, you say you have this piece left. And we cut it up again. Any way to do it, so we don't have to keep cutting?

J: [attempts to partition two additional times by spinning the paper and draw a line from one of the corners] I really don't know of any other way to do it.

T: Ok. Any way to know what to call that parts you made?

J: [pauses for 5 seconds] I'm not sure.

In the first session, Jim evidences what we call a *midpoint partitioning strategy*. His partitioning seems to be supported by a careful identification of the midpoint of the length and a unilateral partition. The strategy does not yet seem to be linked with the number of sharers in the situation, a consideration of the magnitude of the parts created, or an iterative consideration of the parts to the whole. In other problems in the session, Jim continues to use the midpoint strategy regardless of the number of sharers. It is unclear whether Jim was conflating partitions with parts (i.e., three lines as opposed three parts), yet Jim's alternate strategies provide counter evidence of this possibility. It is interesting to us that, throughout the session, Jim seems to view the midpoint as the only valid partition at this point. In later sessions, Jim continued to evidence this strategy in his work in equal sharing tasks, regardless of the number of items or the number of sharers.

Session Four

In session four, Jim began to connect his number knowledge to his midpoint partitioning strategy to bring about an early iterative reasoning. Excerpt b shows Jim's reasoning in a task involving five sandwiches and four sharers.

Excerpt b: Share 5 sandwiches between 4 people

J: [draws 5 boxes on the paper; makes a lengthwise and widthwise partition to create four parts in each box. Numbers and names each part].

T: Tell me about your drawing.

J: Well, I made lots of tiny pieces.

T: Oh. How many?

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J: [pauses for 5 seconds and looks at his drawing] Well, there are five boxes and four tiny pieces in each. Four times five is 20 and 4 + 4 + 4 + 4 + 4 is 20 [points to each box as he counts].
T: Ok [points to drawing]. How much of a sandwich would I get, do you think?
J: [Looks at drawing] Well, you get five tiny pieces out of all of the 20 pieces of sandwiches.
T: Oh ok. How about for one sandwich?
J: Hmm. Well, [mutters 'four times one is four'] four of the tiny pieces is a whole sandwich because four times one is 4, and one part and one part and one part and one part is a sandwich [points to one part; taps table four times]. And then one more. So a whole sandwich and a piece.

Jim shows three subtle shifts in his reasoning. First, his way of partitioning seems to have changed to a repeated halving, perhaps due to his self-prompted change of representation from a bar to a square representation. Second, he partitions *each* square into the number of sharers (as opposed only the last item). Partitioning each item also seems to support the final shift which involves a nascent iterative consideration of the parts to the whole. This reasoning seems further supported by Jim's leveraging of his whole number fact recall to support his reasoning of each person's share as first a share of a subset of the total number of 'tiny pieces' he creates and then as a rudimentary coordination or iteration of the part to the whole. We are further examining how Jim's use of multiple modalities (i.e., changing representations, verbal number facts, gesturing) support his partitioning and iterative reasoning in later sessions, especially in tasks where partitioning proves difficult (e.g., requests to share between 3 or 5 shares).

Discussion

Jim's significant initial misunderstandings about fractions would typically be addressed by behavioral interventions in special education focused on memorizing procedures. In this study, we explore how close analysis of previous understandings based on research in fraction learning can be the framework of an intervention, particularly when the student is understood not as deficient, but as always already having knowledge of the mathematical topic. These two excerpts highlight one of the multiple shifts we document in Jim's understanding of fractions. We argue further that these shifts in understanding were supported by 1) problem solving in contexts, 2) access to multiple modalities, and 3) instruction that builds from careful attention to previous understandings.

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HOW NEURODIVERSITY CAN SHAPE RESEARCH; RETHINKING ENGAGEMENT THROUGH THE PARTICIPATION OF STUDENTS WITH AUTISM

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Mathematics education must include students with disabilities in research, not simply adding these learners to already developed studies, but by engaging with new perspectives that such students bring. Part of a larger study, this paper documents two dilemmas that the authors were faced with when investigating the mathematical engagement of two fifth graders with autism in a standards-based mathematics classroom. Using neurodiversity, we analyze how the unique engagement of these two students challenges preconceived notions of students with autism in mathematics, as well as how we conduct mathematical research.

Keywords: Equity and Diversity; Affect, Emotions, Beliefs and Attitudes; Research Methods

Introduction

Mathematics educational research rarely includes students with disabilities (Lambert & Tan, 2017). In this paper, we focus on two students with autism learning mathematics, and we draw on a political and theoretical movement initiated by people with autism called *neurodiversity*. Instead of understanding autism as a deficit, neurodiversity understands the cognitive differences of autism as a natural and beneficial aspect of biological diversity (Robertson & Ne'eman, 2008). From this perspective, we assert that increasing the participation of students with autism in mathematics will not only help the students, but will help mathematics. Furthermore, we argue that including students with disabilities such as autism in mathematics education research will help educational research in mathematics by expanding both concepts and methods. This paper presents a theoretical argument for including students with autism in mathematics educational research by analyzing how including two students with autism has expanded both theoretical and methodological practices in mathematics research.

Conceptual Framework

This brief research report is based on emerging data from a larger project arguing for a shift from designing intervention around content to intervention in participation (Lambert & Sugita, 2016). Building proficiency in mathematics for <u>all</u> learners means sustained and deep participation in practices such as problem solving, reasoning, and critique, otherwise known as the Standards for Mathematical Practice in the Common Core State Standards for Mathematics (CCSS-M, 2010). Yet, very little is known about how students with disabilities, particularly with autism, engage in the Standards for Mathematical Practice, as the topic is significantly underrepresented in special education research (Maccini, Miller, & Toronto, 2013).

Taking a situated view of learning, we use the terms participation and engagement interchangeably and define participation as the actions of individuals in particular activity systems, which are one or more individuals interacting with each other and with particular sets of material and ideological resources (Greeno & Gresalfi, 2008). Activity systems are understood through

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documentation of the norms of participation that construct the taken-for-granted activities and assumptions of the cultural space. Additional information on the activity system are obtained from interviews with students and teachers.

The majority of research on autism is not focused on academics (Gevarter et al., 2016). Oswald et al., (2015) argue that the lack of research in mathematics that includes learners with Autism Spectrum Disorder (ASD) stems from assumptions that individuals with ASD are higher performing in mathematics than their non-disabled peers. Some research has supported this connection between autism and mathematical talent (Baron-Cohen, Wheelwright, Burtenshaw, & Hobson, 2007). Yet Wei and colleagues (2013) found that while a larger percentage of individuals with ASD in college major in STEM fields (34%) than non-ASD students, individuals with ASD are far less likely to attend college, when compared to other disability categories. From a neurodiversity standpoint, we recognize that individuals with autism have unique ways of processing the world that may provide particular strengths in mathematics. However, we also assert that explanations for persistent low achievement for individuals with autism in mathematics (Wei et al. 2013) must explore the relationship between the student and the mathematics classroom.

Methods

We have situated this paper in a larger study of a fifth-grade classroom that includes four students who receive special education services. The larger study investigates shifts in student participation in the mathematical practices, particularly MP1 (problem solving) and MP3 (discourse) over the course of one academic year. In this short paper, we explore emerging data, focusing on the initial mathematical understandings and participation of two students with autism. Using *neurodiversity* as a theoretical tool, we explore several issues:

1. What does it mean to include neurodiverse individuals in mathematics educational research? What challenges are presented? What can be gained from such work?
2. How do current conceptions of participation privilege neurotypical learners? How can we broaden these notions?

Findings

In this section, we will present two dilemmas that we faced in our initial work on this project, particularly around the two students with autism. We will introduce our two participants, then present the dilemmas, and explore what the findings might mean for broadening mathematics education to include all learners. Andrea and John are the two students of focus in this study. Andrea is a 5th grade girl with an Individual Educational Plan for autism. She identifies as white and Asian. John is a 5th grade boy with an Individual Educational Plan for autism. He identifies as Latino.

Dilemma 1: Interviews

This research included student interviews. In our initial plan, we assumed these would be individual. However, after consulting with the teacher, we were concerned that both Andrea and John would have significant difficulties with this participation structure. In the lead author's first visits to the classroom, Andrea was both very excited about the research, and also clearly uncomfortable with the new adult and the idea of being filmed. John had a history of being very reserved at school, taking months to speak to previous classroom teachers. After discussing this with the teacher, we decided to interview all students in pairs, created by the teacher. We felt that this would make these students more comfortable in the interview. In addition, we planned to allot extra time with each of these interviews. We did not video-record Andrea's interview, and we began that interview by providing a comic book related to her interests. The comic book seemed to help Andrea relax and answer the interviewer's questions.

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We were sensitive to the reality that our interviews might feel like evaluative assessments to students, which might trigger negative emotions because of past experiences being taken into rooms with unfamiliar adults and being asked questions. Children with disabilities in schools are routinely assessed by unfamiliar adults, and these events are often confusing, leaving them with memories of failure and stress (Connor, 2005). Even as we planned to ask questions that we felt were not stressful, we recognized that the situation might mimic other evaluative experiences. We changed our research plan to adapt to this need. The lead author also used play as a relationship building device, engaging in talk around the comic book with Andrea in order to help her transition into the interview and feel comfortable. We wonder how much these shifts in methods would assist other students.

While these two individuals with autism prompted our switch to paired interviews, we wonder if this new structure made other students more comfortable as well. Or, alternatively, did paired interviews introduce additional layers of stress for students, as we found in one interview in which a student seemed to ask for permission to speak, not from the interviewer, but from the other student. We believe that when we worked to make the interview more inclusive of the two students with autism, we looked far more carefully at the emotional and social implications of interviews, assessing the dialogical discursive implications of the triad (Riessman, 2007). Including the students with autism made us question the interview structures that we had taken for granted.

Dilemma 2: What counts as participation?

In our first day of video recording, Andrea and John were seated right next to each other, right in the front of the classroom next to the teacher's whiteboard. The teacher first led the class in a numeracy routine called a number string, and then asked students to solve a Cognitively Guided Instruction (CGI) story problem on equal sharing (Empson & Levi, 2011).

Their participation in this mathematics lesson was quite dissimilar. During the number string, Andrea raised her hand for each problem, and more than once shouted out an answer. When called on, she shared her strategy, explaining how she broke apart a multiplication problem into parts (using the distributive property). Each of her answers were correct. For the CGI equal sharing problem, her first exposure to that problem type, Andrea drew out the sharers and the brownies that were to be shared, partitioned the final brownie into fourths, and accurately identified all fractional parts. Her strategy was non-anticipatory, with coordination at the end resulting in a accurate answer. She solved this very quickly, within 3-4 minutes, and then began clearing out her desk, and then transitioned to reading a comic book. She did not speak in any small group situation, nor did she share a strategy for the CGI problem. Andrea demonstrated understanding of the targeted mathematical concepts of the lesson, and was able to engage in whole group strategy sharing, although she did not participate in small group discussion.

John did not verbally participate in the number string. However, throughout the number string, his attention seemed rapt towards his teacher and the board on which she represented student strategies. Like Andrea, this was his first exposure to an equal sharing problem. Also like Andrea, John used a non-anticipatory strategy with coordination at the end. While Andrea solved the CGI problem quickly, John worked slowly, carefully drawing out each portion as he portioned, twice representing all the shared pieces and the sharers. His partitioning was accurate but he did not accurately name the quantity in fractional terms. When solving the CGI math problem, he did not speak to any other students, nor did he during opportunities to speak to a partner during a share. During the share that followed the CGI story problem, he did raise his hand very slightly in response to a question about which strategy he used. At all times, however, his attention seemed wholly focused on his teacher and her representations on the white board. Similarly, when he solved his CGI problem, he focused completely on his work, drawing and redrawing his solution for the entire work time. He worked longer on his paper than any other student in the room.

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Conclusion

What do we learn from these differences in participation? Andrea provided more evidence for her mathematical understanding during this lesson. She moved from intense attention on the mathematics, to cleaning out her desk, always moving. John, on the other hand, was quiet and still, but appeared to be attending wholly on the mathematics. However, because he did not speak, we felt unsure of his level of engagement and/or understanding at times. In both cases, we felt that our tools for measuring mathematical understanding during the lesson (documenting discourse and collecting student work) were inadequate to the task of understanding their mathematical understandings. Should we devalue the quiet engagement of John? Or the intermittent attention of Andrea? Their differences in engagement remind us that there is not one way of learning mathematics, and no one way of being autistic, and that as we include students with autism in mathematics education, as we must, that we recognize and respect unique ways of engaging in mathematical practices.

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OPENING ACCESS TO ALL STUDENTS: STEMING SELF-EFFICACY

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In this mixed methods study, we examined how middle level students' self-efficacy towards STEM changed after participating in a week long summer day STEM Camp. Findings revealed differences in students' STEM self-efficacy exist among students who received a scholarship to attend STEM camp compared to those who did not. Continuous opportunities for authentic STEM opportunities to support more equitable education for students from low socioeconomic backgrounds are recommended.

Keywords: Equity and Diversity; Affect, Emotion, Beliefs, and Attitudes

Introduction

As the demand for science, technology, engineering, and mathematics (STEM) professionals continues to rise, calls from policy makers for improved diversity in mathematics and science education aim to broaden participation in STEM fields (NSF, 2010). Some studies cite the "STEM Pipeline", or limited access to rich academic coursework in K-12 education, as the cause of the shortage and lack of diversity (e.g., Fox & Hackerman, 2003). However, with PK-12 students in school less than 20% of their time awake (Afterschool Alliance, 2011), access to academic coursework is not enough to shift toward equitable representation in STEM fields. We must rely on partnerships beyond the formal school setting to motivate underrepresented groups and build their STEM self-efficacy.

In this study, we investigated the impact of an out-of-school environment, See Blue STEM Camp, on student's STEM self-efficacy. More specifically the research question underlying this study is *How does participation in the See Blue STEM Camp influence students from diverse socioeconomic backgrounds STEM self-efficacy of STEM?*

Conceptual Framework

Self-efficacy is defined as one's judgment about their ability to plan and execute a course of action to successfully complete a specific goal (Bandura, 1986). Because self -efficacy is a significant predictor of task motivation and performance beyond ability (Bandura & Locke, 2003), high self-efficacy is a key predictor of performance and persistence in STEM. Self-efficacy is developed through four primary sources: mastery experience, vicarious experience, social persuasion, and physiological reaction (Bandura, 1997; Pajares, 2005). In STEM, students must have access to 1) high-level STEM experiences in order to achieve mastery, 2) STEM role models similar to the students, 3) feedback and encouragement from influential people, and 4) a chance to build confidence through discussion in all environments (see Figure 1).

Traditionally underrepresented groups in STEM have less access to STEM learning experiences (Peters-Burton et al., 2014) and lower levels of self-efficacy (Hernandez et al., 2013), which perpetuates the cycle of blocked access to STEM fields (Carter, 2006). Out-of-school environments, such as the See Blue STEM Camp, have been shown to increase STEM interest and post-secondary matriculation (e.g. Mohr-Schroeder, Jackson, Schroeder, Miller, Walcott, Little, Speler & Schooler, 2014; MacPhee et al., 2013). However, there is limited knowledge regarding the development of

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STEM self-efficacy in students of low socioeconomic backgrounds in K-8 education (e.g., Lubienski, 2007). This study aims to contribute to the literature by investigating the self-efficacy of this population of students.

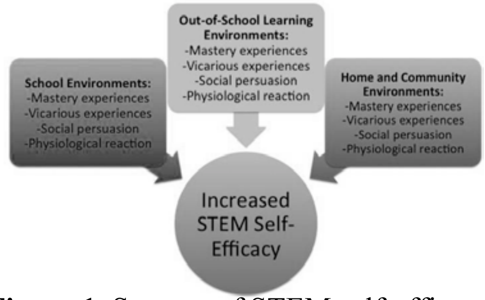

Figure 1. Sources of STEM self-efficacy.

Method

In this sequential, mixed methods design (Creswell & Plano-Clark, 2007), we examined middle level students' STEM self-efficacy before and after participating in See BlueSTEM Camp.

Participants. Participants (N=346) were rising fifth- through eighth-grade students from the southeastern region of the United States who attended a week-long STEM summer camp in either 2015 or 2016. Students participated in STEM content sessions, where they engaged in authentic, hands-on experiences in STEM that fostered the practices of engineers, mathematicians, and scientists. The camp targeted underrepresented groups in STEM (e.g., females, students of color, and students of low socioeconomic backgrounds). Scholarships were offered to students based on financial need. For this study, there were 72 (~21%) scholarship recipients.

Data. Students were administered the STEM-Career Interest Survey (Kier et al., 2014) at the beginning and end of STEM camp. The 44-item survey was based on a 4-point Likert scale. Semi-structured interviews, lasting around 3-5 minutes, were conducted with a sample of participants to discuss their concepts and connections to STEM before, during, and after camp. A total of 155 students (45%) were interviewed with 23 (6.6%) of the interviewed students receiving scholarships. Interviews were digitally recorded and transcribed.

Data Analysis. Exploratory factor analysis (EFA) was conducted on 346 matched pairs using weighted least squares with adjusted means and variance (WLSMV) estimator in *Mplus v.7.11* (Muthén & Muthén, 2013) to show unidimensionality was tenable, as supported by a ratio of the first eigenvalue to the second eigenvalue larger than three (Embretson & Reise, 2000). Then, omega reliability coefficient (McDonald, 1999) was calculated using bootstrapping method to obtain confidence intervals for the pre- and post-survey data using the WLSMV estimator to determine whether a raw score total could be used (i.e., the larger the omega, the greater the reliability in using raw score totals for analysis). Finally, the Wilcoxon Signed Rank Test was used to test whether change in students' responses from pre- to post-survey was statistically significant for the scholarship and non-scholarship students. To analyze the qualitative data, we used an inductive content approach to identify patterns and themes (Grbich, 2007). The themes were discussed until consensus amongst the categories was obtained.

Results and Discussion

In this study, we examined whether there was a difference in scholarship and non-scholarship students' STEM self-efficacy. The EFA results from the overall population on the survey revealed a single underlying construct related to interest in and awareness of STEM and STEM careers. Using the WLSMV estimator, the resulting omega values for the model results of the pre- and post-survey data were .961 and .969, respectively, with corresponding 95% Confidence Intervals [.961, .971] and [.972, .978] obtained from bootstrapping; thus indicating strong internal reliability and

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reasonableness to use raw score totals to pursue further analysis. The Wilcoxon Signed Ranks Test (see Table 1) suggest there is a statistically significant positive difference in students' interest in and awareness of STEM and STEM careers from pre to post for scholarship students, z = -2.62, *p* = .009, and non-scholarship students, z = -7.10, *p*< .001. When differences were tested for each group at the item level, scholarship students showed statistically significant change on some items related to self-efficacy (e.g., item 13 in Table 1). However, the non-scholarship students had more questions that were statistically significant across the survey, which could be attributed to needs scholarship students have that are not met by the camp or are not adequately measured by the instrument. In addition, the smaller sample size of the scholarship students could also affect the results.

Table 1: Descriptives and Wilcoxon Signed Rank Test Results for Self-Efficacy Items and Raw Score Total

	Scholarship (n = 72)			Non-scholarship (n = 274)		
Self-efficacy items	Presurvey Mean(SD)	Postsurvey Mean(SD)	z*	Presurvey Mean(SD)	Postsurvey Mean(SD)	z*
1. I am able to get a good grade in my science class.	3.28(0.63)	3.38(0.62)	-1.61	3.52(0.56)	3.59(0.55)	-2.16**
2. I am able to complete my science homework.	3.35(0.65)	3.35(0.75)	-0.43	3.66(0.55)	3.61(0.57)	-1.42
12. I am able to get a good grade in my mathematics class.	3.40(0.69)	3.51(0.73)	-1.57	3.58(0.60)	3.61(0.57)	-1.17
13. I am able to complete my mathematics homework.	3.48(0.67)	3.63(0.54)	-2.39**	3.64(0.62)	3.61(0.60)	-0.65
23. I am able to do well in activities that involve technology.	3.44(0.58)	3.46(0.65)	-0.6	3.46(0.66)	3.57(0.58)	-2.68**
24. I am able to learn new technologies.	3.50(0.55)	3.46(0.69)	-1.14	3.52(0.60)	3.57(0.57)	-1.23
34. I am able to do well in activities that involve engineering.	3.18(0.81)	3.43(0.69)	-2.39**	3.30(0.71)	3.53(0.57)	-5.76**
35. I am able to complete activities that involve engineering.	3.16(0.72)	3.36(0.72)	-2.45**	3.37(0.65)	3.54(0.58)	-4.56**
Raw Score Total	137.40(18.99)	140.22(21.58)	-7.10**	142.26(18.16)	147.51(19.07)	-2.62**

*test statistics for Wilcoxon Signed Ranks Test, **p < .05, two-tailed

To glean more insight into the scholarship students not specified by the quantitative data, we analyzed qualitative data from the student interviews. Three themes emerged from the data specific to the scholarship population: the thrill of achievement; positive interactions with peers, camp staff, and presenters; and increased confidence in STEM knowledge. Of the 23 scholarship students interviewed, 100% expressed joy over accomplishments they had achieved throughout camp. Building and programming robots and interacting with STEM professionals in their work environment were most frequently cited. Scholarship students communicated surprise at their own abilities to achieve such tasks while the non-scholarship population did not articulate the same sense of awe. Many scholarship students remarked they never had the opportunity to engage in the type of activities they experienced during STEM camp. Therefore, it was challenging for them to relate their successes at STEM camp back to their school environment (e.g., mathematics). Approximately 83% of the students shared how interacting with STEM professionals, camp counselors/teachers, and peers who look like them (sex, race, age, and location) made them feel like they belonged and were a part of the "team." Only three students commented on how their STEM knowledge connected to their families and friends outside of STEM camp. This may indicate that there is a need to provide more STEM experiences to students in order to increase their STEM self-efficacy beyond camp. Finally, 50% of the scholarship students reported a new sense of confidence for participating in classes related to STEM (e.g. mathematics), which strongly supports the statistically significant shifts in their self-efficacy.

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Results from this study indicate there were inconsistent increases in self-efficacy in STEM among students on scholarships compared to the ones who were not. The qualitative data indicate this could be due to missing connections between the STEM camp environment and the students' home and school environments. Access to See Blue STEM Camp benefited students from low socioeconomic backgrounds. Mastery experiences, vicarious experiences, social persuasions, and physiological reactions (Bandura, 1997; Pajares, 2005) were all shared, but only within the context of camp. This indicates access is a springboard toward equity in STEM, but it is not enough. Traditionally underrepresented groups need ongoing, year-round access to rich STEM experiences to increase their STEM self-efficacy and encourage them to enter into STEM fields. A singular episode, such as a camp, can make a difference, but more is needed.

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RELATIONSHIPS BETWEEN LOCUS OF CONTROL, LEARNED HELPLESS THROUGH PISA 2012: FOCUS ON KOREA AND FINLAND

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The purpose of this research is to gather empirical evidence for the attribution theory (Weiner, 1979) about students' learned helpless when doing mathematics. Korean and Finnish students' responses in PISA 2012 were analyzed with the ordinal regression analysis. For all observed cases, Korean students showed higher probabilities to feel learned helpless than Finnish students. Similar patterns between the two countries were found when students attributed their failure to either ability or task difficulty. For the other attributions, different relationships were found. The findings were generally corresponding to what the attribution theory claimed. However, the differences between Korea and Finland showed necessity of cultural factors in addition to the attribution theory to explain students' learned helpless.

Keywords: Learning Theory, Data Analysis and Statistics, Probability

Motivation has been a subject of research in mathematics education for several decades. As a difference between potential and performance, motivation could be one of reasons for underachievement. To resolve students' underachievement due to lack of motivation, researchers have scrutinized students' adaptive and maladaptive behaviors related to appropriate achievement. This research focuses on one of the maladaptive behaviors, *learned helpless* like challenge avoidance, give up, and lack of enjoyment when doing mathematics (McNabb, 2003).

Previous studies also have tried to describe conditions in which adaptive or maladaptive behaviors occurs. Particularly for learned helpless, social cognitive theories (e.g., Rotter, 1966) and the attribution theory (Weiner, 1979) discussed locus of control to answer who are likely to feel learned helpless. Rotter (1966) defined locus of control as "the tendency of people to perceive that outcomes in a particular arena were either within or outside of their control" (McNabb, 2003, p. 418). Diener and Dweck (1978) claimed that they are likely to feel learned helpless if students thought that the outcome is out of their control. Furthermore, Weiner (1979) argued the four attributions grounded locus of control and stability (see Figure 1). According to this theory, students believing their failure is because of their low abilities are likely to avoid challenges and low persistence.

		Locus of Control	
		Internal	External
Stability	Stable	Ability (X_1)	Task Difficulty (X_2)
	Unstable	Effort (X_3)	Luck (X_4)

Figure 1. Four attributions that explain academic outcomes (McNabb, 2003, p. 419).

The purpose of this research is to gather empirical evidence for relationships between the four attribution and learned helpless. The PISA 2012 data was analyzed to compare Korea and Finland. These two countries selected because both showed top-achieving countries, but they are different in motivational issues (e.g., Mullis, Martin, Foy, & Arora, 2012). The questions guiding this research are; (1) what is a probability to feel learned helpless by each attribution via PISA 2012? And (2) what are differences in tendency to feel learned helpless between Korea and Finland?

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Method

Data Description

Korean and Finnish data were collected from the PISA 2012 database. The total number of students in the PISA 2012 database is 8,829 for Finland and 5,033 for Korea. However, a large portion of students did not answer students' questionnaire. Data of students who had completed student questionnaire were gathered and analyzed instead of dealing with missing data. Thus, the final sample sizes were 2,789 for Finland and 1,681 for Korea.

Variable Selection

Specific five variables/questions with the four-level Likert scale were selected corresponding to the attribution theory. Table 1 shows details about those questions in the PISA questionnaire.

Table 1: Five Questions for Learned Helpless and Locus of Control

PISA Variable	Variable Code		Question	Coding
ST42Q08	Learned Helpless	(Y)	I feel helpless when doing mathematics problem.	1 Strongly agree 2 Agree 3 Disagree 4 Strongly disagree
ST43Q01	Effort	(X_3)	If I put in enough effort I can succeed in mathematics.	
ST44Q01	Ability	(X_1)	I'm not very good at solving mathematics problems.	
ST44Q04	Task Difficulty	(X_2)	Sometimes the course material is too hard.	
ST44Q06	Luck	(X_4)	Sometimes I am just unlucky.	

Data Analysis

The ordinal regression analysis was employed using SPSS. This regression was appropriate because all questions related to locus of control and learned helpless had four-level Likert scale, which produced ordinal variables. The specific five variables in Table 1 are all ordinal, but it should be noted that real distances between adjacent categories is unknown. I focused on describing the relationships between locus of control and learned helpless rather than conducting hypothesis tests for statistically significant differences between the regression models for Korea and Finland. This is because differences between the two countries may be evident, but not described.

Table 2: McFadden's R^2 for Each Ordinal Regression Model

	Ability	Effort	Task Difficulty	Luck
Korea	0.086	0.046	0.045	0.011
Finland	0.089	0.060	0.049	0.006

I reported pseudo R-squared to evaluate the ordinal regression models because model evaluation is essential in regression analysis. There were no clear recommendations about how to use pseudo R-squared and how to interpret them, this pseudo R-squared can help to evaluate the ordinal regression models in this research at some degrees. Particularly, McFadden's R^2, which has been preferred to other pseudo R^2s (Menard, 2000), was informed in Table 2. It need to be cautious to make a strong conclusion with a single index about goodness of fit although other indices were not available. All models were accepted because there is no evidence that McFadden's pseudo R^2 were zero. It is known that that index can be as low as zero and the value above 0.2 actually indicated excellent fit of models.

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Results

The analysis results showed that Korean students were likely to feel learned helpless in the following cases: students strongly agreed that their failure was due to their abilities; students disagree that they are able to success in mathematics with enough efforts; students strongly agree that course materials were difficulty; and students' failure in mathematics was because of misfortune. In those cases, the probability to agree or strongly agree with learned helpless was greater than 0.6. Korean students showed very high possibility of learned helpless when students only strongly agreed that their failure was due to their ability (0.837), task difficulty (0.771), or even misfortune (0.628). However, if students had any degrees of disagreement that they can success in mathematics with their sufficient efforts, they reported learned helpless (0.673 for "disagree" while 0.766 for "strongly disagree"). In addition, students' answers about their ability made the widest range of probabilities of learned helpless (from 0.114 to 0.837) while those about luck had the narrowest range (from 0.345 to 0.628).

Finnish students showed different patterns in the relationships between learned helpless and locus of control from Korean students. Finnish students were very likely to feel learned helpless in the two following cases: students strongly agree that they are not good at mathematics; and students moderately disagree that they can success in mathematics with enough efforts. Only these two cases of Finnish students showed higher probabilities than 0.6. The most interesting finding about Finnish students is the relationships between efforts and learned helpless. Students had the highest probability (0.611) in the relationships between learned helpless and efforts. However, if students strongly agree with that statement, the probability decreased to 0.359. Furthermore, Finnish students' answers about luck was independent from learned helpless.

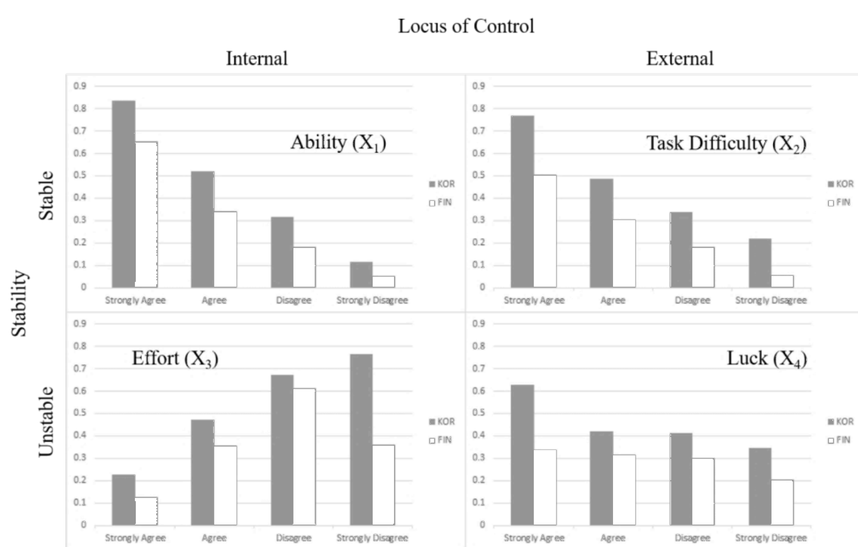

Figure 2. Probabilities to agree or strongly agree to feel learned helpless in Korea and Finland.

Figure 2 showed probabilities that Korean or Finnish students agree or strongly agree to feel learned helpless in learning mathematics. Whatever Korean students answer about their locus of control, they had greater chances for learned helpless than peers in Finland. Most cases for the Finnish students showed probabilities less than 0.5 to feel learned helpless while Korean students are very likely to do with strong beliefs about attributions to failure.

Similar patterns between two countries were found in the relationships of ability or task difficulty to learned helpless. As students' responses for learned helpless shifted from agreement to disagreement, the probabilities considerably decreased. Figure 2 shows that it is reasonable to assume

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a monotone relationship between students' learned helpless and strength of agreement that ability/task difficulty attributes to failure in both countries. The gaps of probabilities from strongly agreement to strongly disagreement were greater in Korea.

However, Korean and Finnish students showed different patterns in relationships of effort or luck to learned helpless. Particularly, Finnish students' beliefs about luck could be unconnected to their feeling of learned helpless. All responses about luck had similar probability for learned helpless from 0.203 to 0.340. However, if Korean students strongly agreed that they failed because they were unlucky, they were likely to feel learned helpless with the chance of 0.628. Moreover, Korean students showed monotone increasing probabilities of learned helpless from strong agreement to strong disagreement that they can success with enough efforts. As seen in Figure 2, Finnish students had a considerably different pattern, in which the highest probability was for moderate disagreement.

Discussion and Conclusion

The findings, particularly about the Finnish students, are matched to the claim of Diener and Dweck (1978). Student who attributed their failure to ability were most likely to feel learned helpless in both countries. For the other attributions, Korean students reported high probabilities for learned helpless, which were different from Finnish students. The attribution to luck is independent from feeling learned helpless for the Finnish students while the Korean students who agreed that they failed because of misfortune are likely to feel learned helpless.

However, Korean students had higher probabilities to feel learned helpless in all analyzed cases. The dissimilarities between Korea and Finland indicated that learned helpless cannot be explained only by the attribution theory. Particularly, the relationships between attribution to efforts and learned helpless differed most. This probably indicated that social and cultural factors can mediate the relationships between the attributions and learned helpless. The findings suggested further research to scrutinize social factors in addition to psychological ones. In addition, more comparisons among countries can contribute to better understanding about the attribution theory and learned helpless in learning mathematics.

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STUDENT AND SCHOOL LEVEL CORRELATES OF MATHEMATICS PERFORMANCE IN THE UNITED STATES REGARDING PISA 2015

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The purpose of this study was to investigate the effects of a set of variables that may explain the relationship between mathematics performance and equity. The Programme for International Student Assessment (PISA) 2015 data for the United States sample was analyzed using the hierarchical linear modeling (HLM) to determine student and school level correlates of mathematics performance. Based on the results, disparities in performance by students' background characteristics (e.g. socioeconomic status, gender) together with mediator factors (e.g. learning time, school location, class size etc.) and their implications for policy are discussed.

Keywords: Equity and Diversity, Data Analysis and Statistics, Policy Matters

Regarding mathematics education as a civil rights issue, Schoenfeld (2002) argued that mathematical literacy should be a goal for all students (p. 13). In accordance with Schoenfeld's argument, to examine countries' mathematics performances and their levels of educational equity outcomes, researchers have focused on analyzing large-scale international assessments such as PISA and the Trends in International Mathematics and Science Study (TIMSS). In this study, we investigated the relationship between the U.S. students' mathematics performance and equity-related factors in education by conducting the HLM analysis to the PISA 2015.

Theoretical Framework

According to the PISA equity framework, equity is defined as "providing all students, regardless of gender, family background or socioeconomic status, with high-quality opportunities to benefit from education" (The Organisation for Economic Co-operation and Development [OECD], 2016, p. 202). As extensively discussed in this framework, students' socioeconomic status and immigration background are two important factors regarding their school performances, and each country's education system should be more inclusive and fair to ensure high student performance. Hence, equity-related factors are to be associated with student achievement mainly because improving learning opportunities would increase student performance overall. In this study, we framed our investigation according to the PISA equity framework, and addressed the following research questions: (1) To what extent are background characteristics (i.e., socioeconomic status, immigrant background, gender) and mediating factors (i.e., access to educational resources, opportunity to learn, stratification policies) associated with the mathematics performance of students in the U.S. in PISA 2015? and (2) To what extent are school-level characteristics associated with the effects of background characteristics? Thus, this study aimed to provide evidence-based insights for policy-makers and researchers regarding the equity factors that are associated with students' mathematics performance in the U.S.

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Methods

The data for this study consisted of 15-year-old students' responses to the PISA 2015 Background Questionnaire and the Mathematics Test, and the PISA 2015 School Questionnaire.

After removing missing data, the final dataset in this study consisted of 4293 students from 159 U.S. schools. Because the PISA 2015 data were nested (i.e., students are nested within schools), HLM was used as a statistical technique so that the results could provide important empirical evidence on the decomposition of variance of performance by student- and school-level variables (Raudenbush et al., 2011). We followed two-level HLM by incorporating plausible values and sampling weights into the analysis so that each sampled student and school were represented accurately (Rutkowski et al., 2010). After building the unconditional model and the random-coefficients model in the HLM analysis, the full-contextual model that contains significant student- and school-level variables was developed.

Results

The results of the final model with statistically significant student- and school-level variables on student mathematics performance are shown in Table 1. In addition, it is important to note that this final model explains 65% of the variance between schools and 20% of the variance within schools.

Table 1: Final Estimation of Fixed Effects

Fixed Effect	Coefficient	S.E.	t-ratio
For INTRCPT1, β_0			
INTRCPT2, γ_{00}	495.38***	5.56	89.03
School Location (0=rural, 1=Small town, 2=Town, 3=City, 4=Large city), γ_{01}	-4.33*	2.18	-1.98
School Average ESCS, γ_{03}	38.15***	6.48	5.89
School Type (0=Public, 1=Private), γ_{04}	-37.84***	9.21	-4.11
For Gender (0=Male, 1= Female), γ_{10}	-23.43***	2.86	-8.02
Shortage of Educational Staff, γ_{11}	-4.98**	1.92	-2.60
For Repeat (0=Did not repeat, 1=Repeated a grade), γ_{20}	-36.76***	5.65	-6.51
For Learning Time (minutes per week), γ_{30}	0.05***	0.01	5.236
School Average ESCS, γ_{31}	-0.05*	0.02	-2.54
For Achievement Motivation, γ_{40}	9.94***	1.90	5.22
For COOPERAT, γ_{50}	10.03***	1.87	5.36
Percentage of immigrant students, γ_{51}	0.22***	0.06	3.78
For CPSVALUE, γ_{66}	-15.82	1.53	-10.31
For Perception of Teacher Unfairness, γ_{70}	-5.21***	0.78	-6.71
School Location, γ_{71}	1.24***	0.29	4.16
For ESCS, γ_{80}	22.48***	3.23	6.96
School Location, γ_{81}	-3.67**	1.42	-2.59
Average Class Size, γ_{82}	0.47*	0.27	1.722
School Average ESCS, γ_{83}	13.18**	2.72	4.85

*p<0.05 **p<0.01 ***p<0.001

Student-level variables. This section provides the extent to which background characteristics were associated with the mathematics performance of U.S. students according to PISA 2015. The gender gap within schools was found to be large (boys performed better), even after controlling for other variables ($\gamma_{10} = -23.43$, $p<.001$). Economic, social, and cultural status (ESCS), learning time, achievement motivation, and student's enjoyment of collaboration are positively associated with mathematics performance ($\gamma_{80} = 22.9$, $\gamma_{30} = .05$, $\gamma_{40} = 9.94$, and $\gamma_{50} = 10.03$, respectively, $p<.001$ for each). Moreover, mathematics performance was found to be lower on average for students whose

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teachers were reported as unfair by the students compared to the performance of those whose teachers were perceived as more fair (γ_{70}=-5.20, p<.001).

School-level variables. Considering the associations between school-level variables and mathematics performance, the school average ESCS, namely the school socioeconomic composition effect, was highly significant in predicting mathematics performance (γ_{03}=38.15, p<.001). Moreover, the model predicted that students in public schools scored higher than students in private schools after demographic and socioeconomic factors were controlled (γ_{04}=-37.84, p<.001). Students in urban schools performed less favorably than students in rural schools (γ_{01}=-5.21, p<.051).

Cross-level interactions. In terms of the significant cross-level interactions between background characteristics and school-level variables, the interaction of gender with a shortage of education staff (γ_{11}=-4.98, p=.010) showed that the negative effect of being a female student on mathematics performance was particularly higher for the students in schools whose capacity to provide instruction was hindered to a great extent by a shortage of education staff. The significant cross-level interaction between learning time and school socioeconomic composition effect indicated that the positive effect of the total learning time on performance was stronger for students who attend schools with a less advantaged social profile than those attending advantaged schools (γ_{31}=-.05, p=.014). Furthermore, the significant cross-level interaction of enjoyment of cooperation with percentage of immigrant students in schools demonstrated that the positive effect of enjoyment of cooperation on mathematics performance was stronger for students who attended schools with a higher percentage of immigrants (γ_{51}=.22, p<.001). In addition, there was a significant interaction between teacher unfairness and school location (γ_{71}=1.24, p<.001). The results suggested that the negative effects of high teacher unfairness on performance were stronger for students attending rural schools than those in urban schools.

Moreover, there was a significant interaction between ESCS and school location, class size, and the school socioeconomic composition effect. For school location, the positive effects of high ESCS were stronger for students in rural schools than for students in urban schools (γ_{81}=-3.67, p=.013). For the average class size in schools, the positive effects of high ESCS were stronger for students attending schools with a higher average class size (γ_{82}=.47, p=.091). For school socio-economic intake, the positive effects of high ESCS were stronger for students who attend schools with a more advantaged social profile than those attending less advantaged schools (γ_{83}=13.18, p<.001).

Conclusions and Implications

Results from this study indicated that the differences in mathematics performance observed across socioeconomic groups were significant. However, the U.S. has been successfully reducing the adverse effect of socioeconomic status on performance between 2006 and 2015 (OECD, 2016). The initiatives promoting equal opportunity (e.g., the No Child Left Behind Act and Common Core State Standards for Mathematics) might have played a major role on this reduction (Kitchen & Berk, 2016). On the other hand, the U.S. performance on mathematics was comparatively weak among the OECD countries. Thus, according to the PISA policy framework, policy-makers might consider more universal policies to raise standards for all students to improve performance along with equity (OECD, 2013). In addition, we found that school location and average class size influence the equitable distribution of achievement by socioeconomic status among students in a school. This result suggests that, beyond the universal policies, policy-makers and administrators should decrease the average class size in schools and focus on interventions specifically designed for rural schools.

OECD (2016) identifies *resilient students* as the ones coming from a disadvantaged socio-economic background, but scoring among the top quarter in a country/economy. Based on PISA 2006 and 2009 data, one factor for becoming a resilient student in science appears to be increased amount of time that the student spends in science class (OECD, 2011). Our study pointed out the

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same result for mathematics by providing evidence that the positive effect of learning time on mathematics performance is stronger for students in schools with lower ESCS than those attending schools with higher ESCS. The result implies a way to help students to overcome the adverse effects of their social background and, thus perform better in mathematics. A way of having disadvantaged students spend enough time in class is making mathematics courses compulsory. For example, in the U.S. in 2006, compulsory attendance made disadvantaged students score 40 points more in science, which is the equivalent of a full year of schooling (OECD, 2011). The result suggests that providing more opportunities for disadvantaged students to learn in class may be an integral way of fostering resilience.

Moreover, our analysis showed that girls tend to underachieve in mathematics in the U.S. Results from PISA 2015, however, showed that there is no significant difference between boys and girls in China, Singapore and Massachusetts in mathematics (OECD, 2015). This result shows that gender gap does not depend on student's innate differences in ability. In this study, the significant cross-level interaction of gender with schools' resources might reveal a potential way of narrowing the gender gap. That is, administrators should consider increasing the number of educational staff because the schools that were in need of teacher and assisting staff found to have a larger gender gap in favor of boys. In addition, another PISA data showed that girls tend to perform better in mathematics when they try to solve mathematics problems independently (OECD, 2015). Thus, teaching strategies that demand more of students might provide better learning opportunities for girls.

Future studies should continue focusing on the relationships between student and school correlates of mathematics performance by considering the random effects as well as the fixed effects because the magnitude of those relationship might vary from one school to another. Moreover, similar HLM analysis could be conducted using other international assessments such as TIMSS data to validate the findings.

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SUBJECT LEVEL ZOOM: A NEW LENS FOR STUDYING STUDENTS' PERCEPTIONS OF THE USEFULNESS OF MATHEMATICS

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This paper involves a methodological exploration into approaches used to assess students' perceptions of the usefulness of mathematics. I argue that it is crucial to attend to the level at which we ask students to report perceptions of usefulness. I present the construct of subject level zoom and a case study of one student to consider issues that might arise in measuring usefulness at the level of an academic subject versus a task or topic within that subject. Finally, I highlight important implications for both research and classroom practice moving forward.

Keywords: Affect, Emotions, Beliefs, and Attitudes; Research Methods

Introduction

In recent years, increasing emphasis has been placed on the usefulness of mathematics (National Governors Association Center for Best Practices, 2010). Existing research has illustrated that finding ways to enhance perceptions of usefulness (also referred to as perceived utility) is a worthwhile pursuit, as one's perceptions of usefulness are positively associated with academic achievement (Durik et al., 2005; Hulleman et al., 2008), course enrollment (Durik et al., 2005; Updegraff, Eccles, Barber, & O'brien, 1996), and interest in a subject (Hulleman et al., 2008). However, upon examining the ways in which perceptions of usefulness are measured in these studies, an unaddressed yet potentially important distinction arises. While perceived utility is conceptualized in the literature as a reason for doing a particular *task*, students are often asked about their perceptions of usefulness of an entire *topic* or *subject* area. While we might be interested in knowing students' perceptions at each of these levels of focus, I argue that it is crucial to explicitly attend to this distinction for two reasons: First, students' perceptions of usefulness might vary depending on whether they are considering the utility of a task, topic, or subject. Second, academic-related outcomes might differ depending on the level at which students perceive usefulness. In this paper, I consider this possibility, first briefly describing the existing literature and then presenting a new construct and a case study to consider potential implications of using measures with different levels of focus.

Sketching the Landscape

The expectancy-value model highlights the ways in which one's values and beliefs influence one's achievement-related choices and performance (Eccles & Wigfield, 2002). In this model, the degree to which one values a task directly influences achievement-related outcomes. Utility value, broadly defined as the degree to which a task is "useful and relevant for other aspects of [one's] life" (Harackiewicz, Rozek, Hulleman, & Hyde, 2012, p. 1), is one component of task value. However, many studies that measure utility value question students' perceptions of usefulness of entire academic subjects, rather than tasks (Anderman et al., 2001; Battle & Wigfield, 2003; Fennema & Sherman, 1976; George, 2006; Parsons, 1980; Xiang, Chen, & Bruene, 2005). For example, Anderman et al. (2001) asked students to respond to statements such as, "In general, how useful is what you learn in math?" In contrast, other studies zoom in on the usefulness of particular techniques or topics and ask participants to respond to statements such as, "This technique could be useful to me in daily life" (Canning & Harackiewicz, 2015).

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Subject Level Zoom

To consider potential issues resulting from this varying level of focus, I present the construct of *subject level zoom*. Subject level zoom refers to the grain size at which we examine a phenomenon related to an academic subject. In regards to the topic of usefulness, while researchers have previously questioned individuals about their perceptions of utility at multiple levels, the level of zoom has not been acknowledged as a factor that might influence individuals' reported utility value. I propose, however, that students' ideas about the usefulness of particular tasks or topics in mathematics might differ from their perceived usefulness of the subject of mathematics. Furthermore, perceiving utility at these different levels might differentially influence achievement-related outcomes. Below I present a case study that speaks to this issue.

A Case Study: Katie

Katie is a 13-year-old female who identifies as Hispanic. She is a member of a seventh-grade honors mathematics class at a school in a working-class suburb of a large Midwestern city. On a survey about usefulness, Katie reported that she thinks math is the most useful subject because "it mostly has everything in it and you learn a lot in math too." In her responses to a modified version of the Fennema-Sherman Usefulness of Mathematics scale (Doepken, Lawsky, & Padwa, 2004; Fennema & Sherman, 1976), Katie's mean rating was 3.83/5 across the 12 items, which corresponds to viewing mathematics as useful. Given these responses, one might conclude that Katie's perceived utility of mathematics is relatively high, which is likely to positively impact her mathematics achievement. However, applying the lens of subject level zoom highlights that Katie answered these questions about utility at the level of the subject. Will Katie's reported perceived utility differ if she considers the utility of particular topics within the subject of math?

During an interview, Katie was asked about six mathematics topics that her class explored during the year. Her responses regarding whether she expected to use the topics outside of class and if so, where, can be viewed in Table 1.

Table 1: Katie's Views on the Usefulness of Specific Mathematics Topics

Mathematics Topic	Use of Topic in Life (Katie's perspective)
Commission and Mark-up	"Commission and markup can be but I don't think it is—for like cashier work. 'Cause they're, like-they're, like, using a much, like, money they have to give or something. I don't know, yeah."
Adding and multiplying fractions	"Um, I don't think you use this for any jobs, I'm not sure."
Writing equations	"Um, no? I'm not sure."
Finding equivalent ratios	"Well, maybe those are like, for math teachers. But yeah, that's all."
Making graphs	"Uh... um, like, maybe making a graph for uh... for let me see, like-like for science? Like something-something that works for science. Like a job that is for science."
Calculating perimeter and area	"For construction, 'cause you need to see how much perimeter has and what area you're gonna use."

Zooming in to consider Katie's perceptions of usefulness of particular mathematics topics paints a more complex picture than seen earlier. For two topics – writing equations and adding and multiplying fractions – Katie could not think of any particular uses in life. While Katie did report that the other four topics "can be" or are "maybe" useful for particular jobs, she did not actually imagine herself pursuing any of those lines of work. For example, after Katie reported that she could imagine

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making graphs in “a job that is for science,” she was asked whether she was imagining that for herself or for others, and she clarified that she meant “just other people.” She was asked the same question after she reported that calculating perimeter and area would be useful “for construction,” and she again said that she imagined that for “other people.” Furthermore, at the end of Katie’s interview, she was asked whether she knew what she wanted to do for work when she got older, and she replied that she has “two choices” – either a lawyer or a veterinarian – and will “probably not” need mathematics for either. It is worth briefly mentioning that while Katie focused specifically on usefulness for jobs, the interviewer did not ask her to do so. Katie could have referenced ways in which she might use these topics for a variety of different purposes, yet for five of the six topics she spoke solely about uses – or lack thereof – for jobs. The potential significance of this will be considered in the next section.

Discussion and Conclusion

In this paper, I highlight a potential crossroads in research on usefulness regarding the way in which we measure students’ perceptions of utility. While a range of items has previously been employed to measure perceived utility of tasks, techniques, and subjects, I argue that we must begin to explicitly distinguish between these types of measures and their potential outcomes in our research. This work has important implications for mathematics education moving forward, as perceived utility is often used as a lever for enhancing student achievement and interest.

Drawing on the construct of subject level zoom and the case study of Katie, I propose two reasons for attending to the level at which we measure perceived utility. First, it might be the case that adolescents use different criteria to assess the usefulness of an academic subject than to assess the usefulness of specific tasks or topics within that subject. For example, Katie reported that math is useful for many things; however, when she was asked about how she might use particular mathematics topics, she only considered the topics in terms of their applicability for specific jobs and was unable to report any jobs in which she herself expected to use the topics. Katie’s responses raise the question of whether she always considers jobs when assessing usefulness, or if she only applies that criteria when considering the usefulness of particular topics. If asked to assess the usefulness of a particular mathematics task, would Katie consider its utility for future jobs, or might she consider other features, such as the task context or the form in which the task is presented? Reflecting back on Eccles and Wigfield's (2002) expectancy-value model, this potential difference in criteria considered raises another question: Will Katie’s perceived utility of the subject of mathematics positively influence her academic-related choices and outcomes, as the model suggests, if she is unable to view individual topics as useful?

As this question highlights, the second reason for attending to subject level zoom is that academic-related choices and outcomes might be differentially affected depending on the level at which students perceive usefulness. For example, it might be the case that many adolescents will report that mathematics is useful since they frequently hear such messages from valued adults, including parents and teachers. However, this belief might not improve students’ interest in mathematics or course enrollment because they do not see various aspects of classroom mathematics – including particular topics or tasks – as useful. In contrast, it might be the case that perceiving usefulness in particular tasks or topics is less likely to influence students to pursue a career in mathematics than perceiving usefulness in the subject of mathematics overall. Instead, perceiving usefulness in those tasks and topics might positively influence students’ engagement in the mathematics classroom and interest in the subject. In other words, different perceptions of usefulness might have very different outcomes for students’ performance, engagement, and achievement-related choices. Current research by the author is building on this work to examine the relationship between students’ perceptions of usefulness of different types of tasks and their engagement on those tasks.

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Moving forward, we must attend to the choices we make in measuring usefulness since they drive the interventions we create. This work will have direct implications for practice, as teachers regularly strive to help students see mathematics as useful. Conducting research that sheds light on the effects of different levels of perceived utility will allow teachers to more effectively influence students' perceptions of usefulness in order to improve mathematics achievement, interest, and classroom engagement.

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THE CROSSROADS BETWEEN HIGH SCHOOL AND COLLEGE LEVEL MATHEMATICS: PERSPECTIVES OF TEACHERS, INSTRUCTORS AND STUDENTS

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At the intersection between high school and college, is the mathematical knowledge students have learned enough to ensure their future mathematics success? This study looks at the expectations gap or the disconnect between what students know leaving high school and the actual knowledge and skills they need to be successful in college from the perspectives of high school teachers, college instructors, and remedial college mathematics students. Survey data from the high school and college instructors revealed agreement on seven of the top ten most important skills for student success. Remedial students' survey and interview data indicated they view their remedial placement as beneficial, yet identified the procedural emphasis of their high school teaching as contrary to how they best learn mathematics.

Keywords: Post-Secondary Education, High School Education, Teacher Beliefs

Objectives and Purpose of the Study

Throughout the United States, high school students who take a college preparatory curriculum are often surprised when they are placed into a remedial mathematics course in college, as they assume their high school courses have adequately prepared them to enroll in college level courses. The term "expectations gap" has been used to describe the disconnect between what students know leaving high school and the actual knowledge and skills they need to be successful in college (Achieve, 2010, p.7). This expectations gap is most noticeable the large percentage of students who require remediation upon entering college. In mathematics, 26 by % of the students who took the college preparatory mathematics courses of Algebra I, Geometry and Algebra II, needed to take remedial mathematics coursework in college (ACT, 2007). In fact, 17% of students who had taken one additional, advanced math course in high school required remediation (ACT, 2007). Of concern is that high schools are not giving students the adequate preparation needed for college even if they are taking college preparatory courses (Hoyt & Sorensen, 2001). Two specific research questions frame this work.

1. What are the expectations of high school teachers regarding mathematical college readiness and how do they correlate to the expectations of college readiness held by college instructors?
2. How do students placed in remedial mathematics courses view this placement and their prior mathematics teaching and learning?

In exploring these questions this study attempts to identify and clarify the expectations held by students, high schools, and colleges in regards to mathematical college readiness, thereby providing knowledge necessary to better preparing students for this transition.

Perspectives

The first research question asked in this study examines any discrepancies in expectations held by high school and college mathematics teachers concerning the skills necessary for college success in mathematics. Many college faculty members nationwide think that students are not adequately prepared for the intellectual demands and expectations of post-secondary education (Conley, 2008; Corbishley & Truxaw, 2010). The ACT (2009) found in their *National Curriculum Survey* that 91%

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of high school teachers feel their students are prepared for college-level work in their content area, but only 26% of the college instructors said their students arrived prepared. A survey of college professors found that 65% rated graduates' basic mathematics skills as "fair" or "poor" (Achieve, 2004). Any discrepancies in perception will be evidenced by what mathematics is ultimately taught and emphasized.

Although a strong conceptual foundation in mathematics is needed at the college level, one observation of entering college students is their lack of conceptual understanding in mathematics (Richland, Stigler, & Holyoak, 2012). "College-ready students possess more than a formulaic understanding of mathematics. They have the ability to apply conceptual understandings in order to extract a problem from a context, solve the problem, and interpret the solution back into the context" (Conley, 2008, p.8). It appears that many students have gained only a procedural understanding of mathematics at the high school level, and have been able to get by with this limited knowledge up to this point (Kajander & Lovric, 2005; Richland, Stigler, & Holyoak, 2012).

Methods

Research questions are answered using an explanatory sequential mixed methods design (Creswell, 2015). The research design includes both survey data (quantitative) and semi-structured interview data (qualitative). This mixed methods design integrates both data types and draws interpretations using the strengths of both sets to understand the research problem.

Question One

The first part of this study involved evaluating high school teachers' and college instructors' expectations for college readiness. Survey instruments asked participants to rate the importance of 57 factors for mathematical college readiness. A Likert scale ranging from 1 - not important, to 5 - extremely important, was used to rate each factor. An option to add any additional factors that were not mentioned in the survey was included, as well as the ability to list the five factors or skills they found to be the most important for college readiness.

The population for the first research question involved college instructors who teach remedial or beginning college-level mathematics courses at a large Midwestern university and high school mathematics teachers from the top 25 feeder high schools to this university who teach a college preparatory mathematics course. Approximately 415 high school teachers and 15 college instructors were contacted with 57 high school teachers (14% response rate) and 9 college instructors (60% response rate) responding to the survey.

Question Two

The second part of this study gathered and evaluated the views of students who have been placed into and taken a remedial mathematics class. An on-line survey was emailed to 2,638 remedial students, with 109 responding and 84, or 3%, completing the survey. Survey questions focused on how student's viewed their high school and college mathematics experiences as well as how students perceive they learn mathematics. Questions based on conceptual and procedural learning/teaching styles were used to determine how students felt these aspects affected their mathematical understanding.

Semi-structured interviews with a subset of the surveyed students added additional insight and clarity to the survey findings. Data gathered from the interviews was analyzed for common themes using a general inductive approach. Interview data from 13 students was used to validate, explain, and enrich the quantitative survey results in more depth.

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Results

Instructor Expectations

Findings from the high school teacher/college instructor surveys revealed that both groups rated "develop thinking skills" as the most important factor for students to learn in high school to be considered ready for college level mathematics. In fact, seven of the top ten highest rated factors were the same for both groups. Table 1 outlines these seven factors identified by each group and the average Likert scale score it received. Both groups also agreed on three of the five lowest rated factors: (a) "student journals", (b) "breadth over depth" and (c) "only teach/learn the procedures and process". Further analysis of the survey data revealed that the largest difference in mean importance occurred with "advanced math content knowledge" (1.339) followed by "reading ability" (1.000), "specialized curricula (AP, IB, etc)" (0.965), "cooperative learning" (0.959), and "memorize basic concepts" (0.947). In all cases the high school teachers ranked these topics as more important than college instructors and placed them on average approximately one entire importance level higher.

Table 1: Most Important Factors for Mathematics Success in College

Factors/Skills	Mean of High School Teachers	Mean of College Instructors
Develop thinking skills	4.74	4.67
Strong foundation of basic math skills	4.70	4.44
Understanding the concepts	4.56	4.11
Improve study skills and habits	4.47	4.44
Alternate representations (graphing, algebraic)	4.44	4.22
Time management skills	4.44	4.22
Providing regular feedback	4.42	4.33

High school teachers and college instructors were also asked to list in order of importance the five factors they feel are most important for college readiness in mathematics. "Understanding concepts", "perseverance", "problem solving", "critical thinking" and "number sense" were common themes in the high school teachers' list. Unfortunately with such a small college instructor sample, identifying any themes in the college instructors' list was not possible, although some themes identified by the high school teachers were also found in the college instructors' lists.

Student Perceptions

Student survey findings indicate that only 35% of the students felt their high school math background adequately prepared them for college level mathematics. Seventy-two percent of students say they were taught mathematics in high school using mostly computational problems. This is in opposition to 77% of the same students who said they need to understand the ideas and concepts in order to do well in mathematics. Over 78% of the students felt that their college math instructor explained the concepts first, with 81% stating that connections were made between different mathematical ideas and concepts in their college mathematics course.

In the interviews statements such as "I remember a lot of high school was just memorize" and "concepts were the biggest thing that I lacked until I got up here" were often mentioned. A common theme throughout the interviews was one of sense-making and understanding. One example came from a student who said when talking about high school, "I didn't understand why I'm doing it. I just knew how to get the answer. I didn't know why I was getting answers." A second student mentioned that in high school he felt like he "kind of learned bits and fragments" of math, but that his first semester in his remedial college math course "I was able to kind of connect the ideas and the big picture math concepts." Another interviewee, Alvin, also mentioned everything coming together for him in his remedial math course.

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I learned a lot in that course. And it was like I had seen all the parts that were shown to me but I was shown how they actually worked. So it's like I had this machine I'd been working with my entire life that I had no clue how it worked and then someone finally showed me where all the parts go. Which was, pretty nice!

Conclusions

Overall both high school teachers and college instructors held very similar views on what constituted mathematical college readiness. This was nice to see, as more disjoint views between these groups has been noted in the past. Of interest is that "advanced math content knowledge" had the greatest variability of any college readiness factor in this study.

Remedial students view their high school mathematics courses as being more procedural or computational, while they interpret their college mathematics courses as more conceptual in nature. Although they were not pleased they had to take a remedial mathematics course, 78% of the remedial students think they were accurately placed into their remedial course, and almost 81% thought taking their remedial course was beneficial. Additional analysis revealed time away from school/mathematics and memorization of steps instead of understanding of concepts were barriers to success, as well as contributing factors for students' remedial placement.

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DEVELOPING STEM LITERACY VIA AN INFORMAL LEARNING ENVIRONMENT

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STEM is "standards-based, meta-discipline… where discipline specific content is not divided, but addressed and treated as one dynamic, fluid study" (Merrill, 2009, p. 49). STEM literacy is the synergy of applying the knowledge and skills of STEM to "increase students' understanding of how things work and improve their use of technologies" to solve the greatest challenges of our era (Bybee, 2010). The purpose of this study is to examine the development of STEM literacy in rising middle grades students in an informal learning environment.

The See Blue STEM Camp is a week-long summer day camp for students (N=216) in incoming grades 5-8. The camp focused on authentic hands-on sessions where students were given opportunities to engage in a variety of STEM fields through the use of the eight Standards of Mathematical Practice (CCSSO, 2010) and the eight science and engineering practices (NRC, 2011). The language of these practices reveal an extensive overlap to support students solving complex problems and participating in authentic learning experiences, thereby increasing their STEM literacy. Data included pre- and post-surveys as well as interviews conducted throughout STEM camp. Data were qualitatively analyzed using the constant comparative approach (Glaser, 1965). Coding was used to generalize patterns (Strauss & Corbin, 1990).

Prior to participating in camp, most students gave a definition of STEM that focused on just one of the subjects that comprise STEM (e.g. an equation or science equipment), or by writing the words that make up the acronym. In interviews, students often defined STEM by listing the four individual subjects and struggled to give a broader conception of how STEM applied to their lives. After participating in See Blue STEM Camp, a majority of students defined STEM by copying the images from the camp logo or writing the words "Science, Technology, Engineering, and Math" with connections between the four siloed subjects. Over 50% of students included smiley faces, "fun," and other positive words or phrases. Nearly 15% students described STEM as a concept beyond its individual components and emphasized problem solving. This indicates students' conceptions of STEM started to evolve through attending camp. Their STEM literacy grew and progressed as they experienced the potential of the STEM field. These findings give insight into how we can move forward in developing STEM literacy for students. In order for students to develop their STEM literacy, they need to explore and experience STEM *overtime*. We must provide continued, extended opportunities and adequate time to build the STEM literacy of the next generation.

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FIXING A CROOKED HEART: EXPRESSING AND EXPLORING MATHEMATICAL IDEAS IN AN INFORMAL LEARNING ENVIRONMENT

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Playing and *doing mathematics* are often conceived of as mutually exclusive activities. One is dominated by positive affect and free choice, while the other is often perceived as highly abstract and predetermined. We wonder how children's play can provide an entrée into mathematical thinking? In play, children ask "what if…?" as they imagine new possibilities, and they often persist at solving problems. These activities resemble mathematical practices like *make sense of problems and persevere in solving them* or *look for and make use of structure*, potentially bridging children's spontaneous activity with desired activities in ambitious math teaching. We specifically ask: (1) How are mathematical ideas and practices expressed informally in children's play? (2) How is the body involved in children's sense-making during play?

Conceptual Framework

The math children use in a playful math learning context draws differently on meaning-making resources than either school math or mathematicians' math, requiring differing amounts of the following for meaning-making: situational meaning, personal meaning, disciplinary meaning.

Data Collection and Analysis

This study was set at a mathematical playground at the Minnesota State Fair called Math On-A-Stick. Our primary data collection strategy involved head-mounted Go-Pros on 345 children to collect video data on table-top activity. We engaged in data reduction using grounded theory (Corbin & Strauss, 1990) to identify *episodes of persistent problem-solving*. For this study, we present an analysis on one eight year-old female ("Elly"), selected for her unusually persistent problem-solving in play. Using interaction analysis (Jordan & Henderson, 1995) we examined Elly's talk, gesture, object use, with particular attention to how she used her body.

Findings

Elly defined an aesthetic goal of "making a heart" with colored plastic eggs in a 6x5 crate.

Mathematical Ideas and Practices in Play: (1) Everyday notions of "crookedness" and "middle" indicated *attention to and an increasingly refined understanding of structure* of both the symmetry of her heart and the parity of the grid of egg crate. (2)The playful nature of her activity allowed her to *persist in her problem-solving* as well as *maintain ownership* of the problem-solving process even with her mother's attempted interventions.

Role of Body in Sense-Making during Play: The use of *bilateral coordinated movements* aided her in realizing the problem of "middle," that she needed to rotate the crate so she could have a "middle," and continued be used to fix her "crooked" heart and achieve symmetry.

Implications

Elly's play involved approximations of mathematical practices like *make sense of problems and persevere in solving them* and *look for and make use of structure*. Children's personally relevant goals in mathematical environments can support the authentic use of math practices.

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ANALYZING THE RELATIONSHIP BETWEEN CLASSROOM ENVIRONMENT AND STUDENT BELIEFS IN 8TH GRADE STUDENTS

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Middle School Education

Student perception of mathematical abilities plays an important factor in in a student's decision to continue in their selection of rigorous course work and to study in the STEM fields (Tyson, Lee, Borman, & Hanson, 2007). Females are less likely than males to pursue a career in the STEM fields (National Science Foundation, 2014). There may be a relationship between the perceived mathematics classroom learning environment of high school students and their self-concept (Tosto, Asbury, Mazzocco, Petrill, & Kovas, 2016). Students need to feel competence, relatedness, and autonomy in the classroom to persist in challenging situations and to have higher achievement and interest levels (Gottfried, Marcoulides, Gottfried, & Oliver, 2013). This study examined the relationship between students' perceived competence and feeling of relatedness and autonomy as mediated by gender and autonomy of problem solving.

Methodology

TIMSS 2015 8th grade mathematics background questionnaire files were analyzed to answer the research questions. Scaled variables for relatedness and perceived competence were derived from questions 18 and 19 of the student questionnaire and had Cronbach alpha scores of 0.87 and 0.89. A linear regression analysis was performed with the dependent variable of competence and independent variables of relatedness and gender. An ANOVA analysis was performed with the dependent variable of competence and independent variables of autonomy to problem solve and gender.

Findings

Students' perception of relatedness in the classroom accounts for nearly 18% of the variability in their perceived competence. Being a female negatively affects perceived competence but not relatedness. Increased frequency of autonomy in problem solving increased competence perception, more so for males.

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GIRL TALK: USING GAME DESIGN AND ROBOTICS TO THINK, REASON AND COMMUNICATE MATHEMATICALLY

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Female student sets the robot to run the program. The robot partially navigates the course but doesn't knock the block down. At the "turn" in the course, the robot plays a short sound. F: And the crowd is cheering.

Keywords: Informal Education, Classroom Discourse, Gender, Equity and Diversity

In spite of well substantiated claims that culturally relevant pedagogies (CRP) help students develop voice and identity in diverse learning contexts, CRP remains a less-explored means for engaging a diversity of students in game design and robotics programs (Ladson-Billings, 1995; Leonard et al., 2016; Shah et al., 2013). Part of a larger study of middle school extra-curricular programs, the present analysis investigates girls' "math talk", cultural relevance, spatial visualization, computational thinking and problem solving during game design instruction. Building on the work of Sullivan & Heffernan (2016), our research supports the notion of learning progressions in technology education. Adding to their model, our data show that students' proportional reasoning is a critical step between sequencing and causal inference on the way to learning how to think about robotic systems, codes and their interplay. When students engaged in conversations or were asked to explain their programming choices, they often related numeric values within the code to specific measureable behaviors and outcomes, leading to increased understanding of causal relationships. While all students (individuals and within groups) developed similar communication habits around coding, girls tended to talk more frequently and openly about the connections between their rather haphazard codes, their observed results and connections to other phenomena, real or imagined. This observation is validated by our finding that girls had higher posttest scores (M=4.56; Std. Dev. 0.47) than boys (M=4.19 Std. Dev. 0.65) on the 21st century skills survey when pretest scores were used as a covariate. While the scores were not significantly different, the data show a trend that girls are more likely to verbalize their thinking than boys. Facilitated by a video journaling dimension of the program required by the teacher, the research team was able to document pathways to learning for students, chronicle changes in their reasoning and organize the development of computational thinking by analyzing qualitative data (journal videos, observation/field notes and focus group interviews). A rubric was developed that describes indicators of computational thinking as emergent (level 1), moderate (level 2) or substantive (level 3). An additional evaluation tool was developed to conceptualize how students engage the Next Generation Science Standards (NGSS) science and engineering practices most directly related to authentic communication of scientific ideas.

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EVALUATING COLLEGE STUDENTS' CONFIDENCE JUDGMENT OF FRACTION

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Assessment and Evaluation; Rational Numbers

A confidence judgment expresses how sure students are about the accuracy of their own answer to a test item. It plays a critical role in students' ability to successfully self-regulate their own learning (Dinsmore & Parkson, 2013). It also provides instructors with information to guide their instructional decisions regarding remediation (Stankov et al., 2012).

Knowledge of fraction concepts (e.g., decimals, percentages, ratios, rates, and proportions) and fraction arithmetic are essential skills for students' future success (Siegler & Lortie-Forgues, 2015) in probability (Garfield & Ben-Zvi, 2007) and other topics. However, half of middle and high school students and even many college students (Siegler & Lortie-Forgues, 2015) lack fraction sense and struggle with fraction applications to mathematics or real world settings. Erroneous fraction ideas have been found to exist in both conceptual understanding and procedural fluency (Panaoura et al., 2009). The present study investigates college statistics students' confidence judgments on fractions prior to their introduction to the study of probability. We look to identify fraction notions associated with high and low confidence.

Participants in this study were 57 college students from the statistics (GER) classes in a public 4-year college. Using a Rasch rating scale model, we analyzed their confidence ratings on a 5-point scale for all items on a 30-item instrument. The person and item reliability of the confidence construct were 0.93 and 0.90 (Ding & Moore-Russo, forthcoming). The study uses the item difficulty estimates from Rasch analysis to identify which fraction concepts are "easier" and "more difficult" for the participants to agree they are "completely confident." We found participants have increased confidence in their responses to the items about part-whole fraction, its applications, and fraction arithmetic computations. These items had a relatively lower item difficulty level. The study finds that 38% of items are at the above-average difficulty level, for which participants display low confidence. The fraction concepts underlying these items include fractions involving variables (e.g., TIMSS 2011 item M032662) and fraction division operation sense, etc. The findings reflect participants' procedural knowledge is more solid than their conceptual understanding of fractions and fraction arithmetic. The findings have implications for teaching and learning in developmental mathematics, college mathematics, and statistics.

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NARRATIVES OF MATH, SCHOOLING, AND IDENTITY IN THE MATHOGRAPHIES OF BROOKLYN YOUTH

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Keywords: Equity and Diversity, High School Education

Mathematics education is at a crossroads in terms of how we address issues of equity, moving from demographic-based measures of access and achievement to the pursuit of new understandings of how mathematics learning itself can be a racializing or minoritizing event (Gutierrez, 2013). We know that the terrain of mathematics learning can be particularly difficult for students from historically marginalized groups, at least in part because of the problematic discourses that circulate about these students with regard to success, failure, and questions of belonging in mathematics. At the same time, we have a deeply under-theorized understanding of how young people make sense of themselves in this space (Martin, 2009).

One way that people make sense of their lives and articulate a vision of the self is through stories. This study examines student narratives through the stories told in *mathographies* – written autobiographical narratives about a person's experiences learning mathematics. The data for this study consisted of 54 mathographies written by students in two differently tracked high school math classes in a large, diverse, public high school in Brooklyn, NY. The mathographies were examined using discourse analysis with a focus on positioning (Davies & Harre, 1990) in order trace the ways in which students positioned themselves and others by drawing on, or contesting, existing discourses of mathematics achievement, participation, and belonging in school as well as the broader social world. The research questions guiding the study were: (1) What is the typology of mathematics identities that emerge in a diverse urban high school? And (2) How do students draw on a range of available discourses - including of mathematics, schooling, race, immigration, and family - to position themselves as certain types of mathematics students?

Findings show that students draw extensively on discourses of schooling as they position themselves as students worthy of their teacher's investment. Students take up student positions that posit worthiness along axes of ability, effort, and resilience. They contest deficit narratives about themselves and their families; and they simultaneously reproduce dominant discourses of schooling by drawing heavily on institutional evidence of achievement, such as exam scores and homework compliance, and through stories of parental investment. These findings suggest that the mathematics education community must look closely at how it is implicated in the widespread reproduction of an American schooling context in which exam-based achievement scores, and the ability to out-perform others in order to get ahead economically, overshadow opportunities for participation and engagement on other grounds.

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GROUPWORK AND HIGH SCHOOL IMMIGRANTS

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Keywords: High School Education, Equity and Diversity

This poster presentation expands on prior research about bilingual students and English Learners (ELs), which has focused on how they best learn mathematics (Domínguez, 2011; Khisty & Chval, 2002; Moschkovich, 1999; Secada, 1991; Setati, 2005; Setati & Adler, 2000; Turner, Domínguez, Maldonado, & Epsom, 2010; Zahner, 2012). The purpose of this study was to explore how high school immigrants', who are also ELs, prior educational experiences influence their preferences for learning mathematics, specifically Algebra 1, in a new educational setting. Each of the three participants involved in this case study were interviewed and asked questions about their educational experiences in their home countries, how they best learn math, and how they prefer to be taught math. In analyzing this data, participants brought up ideas on working with others during math classes and whether they preferred to work independently or in a group. This poster shows this portion of the data analysis of the greater study.

Each of the participants' home countries is different, as is their experiences with formal schooling. While none of the participants has had interrupted schooling, they have been educated in the U.S. for varying amounts of time and have been in different types of schools in their home countries. The participants have shared different experiences with working with peers in both their home countries as well as in their U.S. high school. Each of the participants prefers to work individually. With varying prior experiences with working with peers in mathematics, participants have differing views about why they do not want to work with or the potential for working with peers to solve problems in the mathematics classroom. The analysis of these response shows some relationship between prior educational experiences and preferences for working with others in the classroom, however the sample size is quite small so generalizations cannot be made.

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PROVIDING ACCESS TO ALGEBRA FOR STUDENTS WITH AUTISM BY ELIMINATING BARRIERS CAUSED BY MATHEMATICS ANXIETY

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Proposal

Students with autism spectrum disorder typically achieve below their peers without disabilities due to the common characteristics of autism (Fleury et al., 2014). In addition, these students often experience high levels of anxiety (Bellini, 2004; Kim, Szatmari, Bryson, Streiner & Wilson, 2000), which can make access to and success with mathematics especially challenging. Specifically, mathematics that requires higher level thinking and mathematical reasoning is difficult for students with autism due to deficits in executive functioning (Barnhill, Myles, Hagiwara, & Simpson, 2000; Mayes & Calhoun, 2003). To provide access to abstract, secondary mathematics for students with autism, teachers need to utilize support strategies that eliminate barriers by reducing anxiety and addressing difficulties with executive functioning.

The researchers conducted an exploratory case study (Creswell, 2013) to describe teaching strategies to support executive functioning to reduce anxiety and increase the algebraic reasoning skills of a secondary school student with high functioning autism. We transcribed five teaching sessions, coded the data, organized the data into emerging themes, and utilized an independent rater to monitor interpretive validity during data analysis. The teacher utilized calming routines during strategically timed breaks within longer sessions of engaging in challenging mathematical reasoning. For example, when the student became anxious, she preferred to calm down by solving procedurally repetitive mathematics problems (at a skill level that she had already developed) and practicing deep breathing exercises. Then, when relaxed, the student would vigorously engage in challenging mathematics (e.g., Pythagorean Theorem and graphing inequalities) including problem solving, justification, and participation in discussions connecting her work across mathematical and other contexts. Findings from this study suggest that teachers of students with high functioning autism may be able to set high expectations for these students to access challenging mathematical content as long as part of their instructional approach includes strategies for reducing anxiety and addressing difficulties with executive functioning.

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HOW STUDENTS FROM THE BIOLOGICAL AND LIFE SCIENCES SOLVE CALCULUS TASKS INVOLVING ACCUMULATION

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Keywords: Advanced Mathematical Thinking

The crossroads of mathematics and biology are more important than ever as advanced quantitative techniques have become standard in biology (NRC, 2003). Additionally, 30% of Calculus I students intend on careers in the biological and life sciences (Bressoud, 2015). Furthermore, researchers have called for studies that investigate how students from the many disciplines that require calculus think about and use calculus concepts (Rasmussen, Marrongelle, & Borba, 2014). Existing research concerning the definite integral and accumulation is based primarily in physics and engineering contexts (e.g. Jones 2015) and little is known about how students from the biological and life sciences reason about such tasks. This topic is particularly important given the prevalence of modeling via differential equations in the biological and life science. Therefore, I sought out to answer the research question: How can we characterize the ways individuals in the biological and life sciences solve calculus tasks involving accumulation?

Methods

This qualitative study included task-based interviews with twelve undergraduate students majoring in the biological and life sciences. During the hour-long interviews, students were asked to solve five calculus tasks involving accumulation. Interview data was transcribed and open-coded in accordance with principles from constructivist grounded theory (Charmaz, 2000).

Results and Discussion

Results indicate that students developed local theories of how to solve the interview tasks. Students made conjectures concerning a solution strategy and then assessed the reasonableness of said strategies via their understanding of the problem context. In one task, students used the problem context (climate change) to judge whether their assumptions and mathematical approach were accurate. Familiarity with the problem context, as well as the representation of the rate of change function, affected how the students reasoned about accumulation. One significant implication of these results is the importance of using a wide variety of contextual settings in university calculus courses to better serve the undergraduate calculus community.

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THE IMPACT OF A DRAWING INTERVENTION ON THE SPATIAL VISUALIZATION SKILLS OF SIXTH-GRADE STUDENTS

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Background and Research Question

This study investigated the usefulness of a particular drawing intervention, *Quick Draw,* to better understand how sixth-grade students of varying abilities approach and interact with spatial visualization tasks, using a quasi-experimental, mixed-methods design. The concept of spatial intelligence has changed over time from being thought of as an innate ability to skills that can be developed (NRC, 2006; Sarama & Clements, 2009; Sorby, 1999). Studies of spatial ability and skills have grown from simply identifying mechanical ability to being a predictor of success in such academic fields as science, technology, engineering, and mathematics (Hegarty & Waller, 2005; Sorby, 1999). The power of spatial intelligence and its many subcomponents are an underappreciated and underutilized cognitive ability within many classrooms (NCTM, 2000; NRC, 2006; Sorby, 1999).

Method and Results

The research subjects were in a sixth-grade teacher's mathematics classes, and through pre-testing, four case study participants (two high and two low spatial ability students) were identified. A multiple holistic case study approach provided information concerning the differences amongst subgroups (high or low spatial ability) regarding their approach and interaction with spatial visualization tasks. The testing instrument for both pre- and post- testing consisted of a combination of five modified spatial visualization tests. Students participated in six weeks of *Quick Draw* interventions as five-minute warm-up activities to track their progress and to determine how their spatial abilities improved. Students briefly viewed a *Quick Draw* figure and were asked to draw what they saw, and follow-up discussions ensued. In a review of all of the data sources (quantitative and qualitative) three distinct differences were identified between the groups. One was how groups viewed the intervention activities' impact on other academic areas; two, the use of correct geometric terminology. However, the most distinctive difference was how groups appeared to view the figures. Students with high spatial visualization skills appeared to view images holistically, whereas students with low spatial visualization skills appeared to view images as components.

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INFORMAL STEM LEARNING: IMPACTING BLACK FEMALES SELF-EFFICACY AND INTEREST IN STEM CAREERS

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Keywords: Equity and Diversity; Informal Education; Affect, Emotion, Beliefs, and Attitudes

In most K-12 schools, Black girls face limited access to STEM course offerings and rigorous STEM learning experiences (Perry, et al. 2012). Black girls are often placed in mathematics classrooms dominated by non-engaging and non-rigorous curriculum devoid of meaning and any real connections to their lived experiences (Hill, 2010). With the growing interest in STEM at both the national and international level, as well as the persistence in racial disparities in educational achievement, it is crucial that educators provide learning experiences that support the positive development of Black girls as STEM learners. Previous research suggests that there is a relationship between informal STEM learning, self-efficacy and interest in STEM careers. For example, according to the Afterschool Alliance (2011), informal STEM learning experiences contribute to improved attitudes toward STEM fields and careers, increased STEM knowledge and skills, and higher likelihood of graduation and pursuing a STEM career. Kerr and Robinson Kurpius (2004) found that girls of color who participated in informal STEM activities increased their exploration of STEM careers, achievement and self-efficacy.

Building on this work, this study explores the impact of a four-week informal STEM program that utilizes a socially transformative curriculum model which values and draws connections between Black girls lived experiences and STEM content. To better understand the relationship between informal STEM learning and Black girls' self-efficacy and interests in STEM careers, we are conducting quantitative analyses of pre-post survey data from 55 Black girls (ages 9-17) who participated in the summer STEM program over two years.

Initial findings suggest that having access to socially transformative STEM curriculum in an informal setting increased participants' self-efficacy, interest in STEM careers and that learning in an informal setting with socially transformative STEM curriculum can help counter some of the detrimental effects Black female students may experience in more traditional settings.

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MOTIVATION AND SELF-REGULATION IN CALCULUS

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Metacognition

Recent studies have attempted to identify student characteristics that predict success in calculus (Hieb, Lyle, Ralston, & Chariker, 2015; Worthley, 2013). Prominent has been the use of Motivated Strategies for Learning Questionnaire (MLSQ) to measure motivation and learning strategies. Hieb et al. found ACT scores, algebra skills test scores, intrinsic goal orientation, time and study environment management, and test anxiety to be good predictors of student success in a first-year engineering calculus course. Worthley's model of calculus success incorporated variables such as math placement test scores, test anxiety and self-efficacy. While these studies indicate motivational and self-regulatory factors may impact success, they do not examine the interactions between success/failure and motivational/ self-regulatory factors without the added element of mathematics ability of incoming students. Our research aimed to examine the relationship among these constructs.

In autumn 2016, 545 Calculus I students at a large midwestern university were given the Calculus Concept Readiness (CCR) assessment (Carlson, Madison, & West, 2015) and the MLSQ. In addition, final grades as a percentage were collected.

In analyzing MSLQ results, each student's subscale average score was calculated. First, correlations with bootstrap 95% confidence intervals were ran between MSLQ subscales and final course grades. Then, controlling for CCR scores, partial correlations with bootstrap 95% confidence interval were calculated between each MSLQ subscore and final numerical grade. Using Bonferroni Correction, a significant *p*-value of .002, eight of the MSLQ subscales significantly correlated with final course grade: intrinsic goal orientation, task value, control beliefs, self-efficacy, test anxiety, elaboration, metacognitive self-regulation, and effort regulation. After controlling for CCR scores, only five of the subscales significantly correlated to final grade: instrinsic goal orientation, task value, self-efficacy, test anxiety, and effort regulation.

Results indicate that without controlling for incoming ability, additional motivational and self-regulatory characteristics correlated with final grade, implying these student characteristics may need to be addressed to assure success.

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PRETENDING WOGS ARE LOGS: EXPLORING CONTEXTUAL EFFECTS OF EQUAL SHARING WORD PROBLEMS IN FOURTH-GRADERS

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Keywords: Problem Solving, Cognition

Grounding mathematics word problems in realistic contexts (e.g., situations that involve everyday items, such as pizzas) can elicit informal, yet meaningful, strategies that enhance problem solving. These effects are observed when compared to problems that contain so-called idealized representations – i.e., arbitrary elements, such as alphanumeric notation, that do not trigger real-world knowledge as readily (Belenky & Schalk, 2014). What is less understood is the effect of different *types* of real-life problem contexts. Cutting a pizza into equal slices involves concepts and actions learned informally outside of school. In contrast, partitioning 12 m of rope entails measurement concepts (e.g., standard units) that are learned formally in school. When students' everyday knowledge is not activated, their strategies tend to suffer (Weyns et al., 2016).

Method and Results

In the present study, we examined students' performance on equal sharing problems as a function of context type. Fourth-grade students ($N = 36$) in 4 public schools in Canada solved 8 equal sharing problems with fractional remainders. Four problems were couched in *everyday* contexts that readily cue routine actions (e.g., slicing pizza), and two in *measurement* contexts (e.g., partitioning 12 m of rope) that cue formal school knowledge. Two problems contained items with no real-world referents ("wogs") and thereby constituted *idealized* problem contexts. We assessed the students' performance by scoring the appropriateness of the strategies they used.

Contrary to predictions, we found no difference between performance on the word problems with everyday contexts and those with idealized contexts. Video recordings and the students' drawings revealed that they ascribed everyday meaning to the idealized items ("I am going to pretend that a wog is a log."). Students' performance on the problems with the measurement context was significantly lower than performance on both the everyday problems (p =.008) and the idealized problems (p = .017). Descriptive analyses revealed that the measurement context appeared to impede the generation of meaningful strategies. Struggling to conceive of length as a quantity that could be partitioned, students relied on computation learned in school and guessing.

Conclusions

Our findings show that not all realistic contexts are equal. Some may contain elements that block the activation of informal knowledge that would support problem solving (Weyns et al., 2016). Children can be extremely resourceful in constructing meaning where it otherwise does not appear, but more research is needed on the conditions that elicit productive meaning making.

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UNDERSTANDING SCHOOL LEADERS' DISCOURSE IN REGARD TO MATHEMATICS ACHIEVEMENT

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The notion of discourse in school mathematics has been of considerable interest to researchers. Scholars largely associate mathematical discourse to classroom practices (Moschkovich, 2007) and tend to overlook the political dimensions of discourse that are largely the prerogatives of school leaders. There is a significant body of literature pointing to the fact that school leaders are critical in supporting effective schools (Fullan, 2011). Further, school leadership is widely considered to impact student achievement and success (Leithwood, Patten, & Jantzi, 2010). Despite that evidence, not much attention seems to be given to school leaders' discourse on math achievement even though it is well known that school leadership discourse impacts school culture (Webster, 2012).

This poster presents findings from a pilot project involving 10 school administrators and systems leaders representing 2 different jurisdictions: 5 from the Northern Haiti and 5 from the French-language schools in Ontario. The participants from Haiti were from private schools (religious and secular) and the ones from Ontario were from both the catholic and public systems. The research examines school leaders' discourse in the context of math achievement for all students. What are some of the commonalities of school leaders' discourse when it comes to math achievement? What do these leaders perceive to be success factors, roadblocks and challenges to students' math achievement? To what extent their discourse is a reflection of implicit inequities in school math? Do they allude to deficit assumptions in their understanding of math achievement for all students?

Using primarily semi-structured interviews, this research used a qualitative framework to explore some of these questions. Early analysis of the transcripts yielded to several themes including the importance of teachers' impact, the students' and teachers' attitudes effect, the fixed mindsets regarding math achievement and the challenge to make math meaningful to students. These four themes are examined in light of literature arguing that "effective school leadership is needed to support the transformation of teaching practice and school culture" (Vale, Davies, Weaven, Hooley, Davidson, and Loton, 2010, p. 47).

This research offers insights on school leaders' discourse related students' math achievement and addresses a gap in the literature (Herbel-Eisenmann, Choppin, Wagner, and Pimm, 2011, p. 5). This project initiates a framework for further studies on how school leaders' discourse interplay with math achievement. As scholars examine equity issues in school math and as policymakers discuss ways to increase math achievement for all students, it is critical to also consider how school leaders can be supported in developing discourses compatible to more equity in school math.

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WRITING TO LEARN MATHEMATICS: A STRATEGY FOR PROMOTING REFLECTIVE ABSTRACTION FOR STUDENTS WITH LEARNING DISABILITIES

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An estimated 6%-14% of students in the United States have persistent difficulties in mathematics and may be identified as having a learning disability (LD). These students often have difficulty connecting approximate number systems with symbolic number systems (Mazzocco, Feigenson, & Halberda, 2011), as well as challenges holding and storing information in short and long term memory. These specific difficulties, evidenced by challenges with foundational skills (i.e., counting on and subitizing), may relate to problem solving abilities.

We use Piaget's (2001) construct of reflective abstraction to explain how students cognitively reorganize information through reflection on their activity and its effects as they learn mathematics. First, students notice that the expected results of actions do not align with actuality, and try to resolve this perturbation by reorganizing their actions. Second, students compare mental activity records and their effects across similar situations (Simon, Tzur, Heinz, & Kinzel, 2004). Through repeated exposure, students internalize the concepts and processes used to select and execute appropriate strategies to solve problems. However, it is common for students with learning disabilities to struggle with identifying whether a solution to a given problem is reasonable and with keeping track of and reflecting on specific problem solving processes (Cuenca-Carlino, Freeman-Green, Stephenson, & Hauth, 2016).

Writing to Learn (the use of writing activities to aid learning in other content areas) can help students develop metacognitive strategies (Bangert-Drowns, Hurley, & Wilkinson, 2004). Reflective abstraction can be aided through metacognitive strategies (Simon et al., 2004). Therefore, WTLM may help students with LD engage in reflective abstraction. Psychological theories support the use of writing to develop connections between representations and the ability to communicate thought processes; both tenants are central to Piaget's theories on metacognition (Fox & Risconcente, 2008). In this poster session, we propose WTLM as an instructional strategy to promote reflective abstraction for students with LD, review the evidence for the use of WTLM to promote reflective abstraction, and provide suggestions for classroom applications as well as future research.

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Chapter 11

Teaching and Classroom Practice

Research Reports

Brief Research Reports

Posters

EXAMINING THE INTERACTIVE POSITIONS AND STORYLINES OF AN EMERGENT BILINGUAL LEARNER

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The U.S. storyline of emergent bilinguals has historically failed to highlight the mathematical and linguistic assets of this group; instead, it has primarily focused on providing support. To disrupt this narrative, a case study of one elementary teacher, Ms. Bristow, is presented. Ms. Bristow's discursive practices and pedagogy illustrate how she fostered the storyline of mathematical competence for an emergent bilingual Latina by positioning her in ways that called attention to her mathematical thinking. Ms. Bristow's creation of mathematical goals and participatory expectations provided the foundation for classroom interactions that enabled the student's storyline to come to fruition and be appropriated by peers.

Keywords: Equity and Diversity, Classroom Discourse

Although emergent bilinguals (EBs) are a diverse group of students with a wide array of mathematical and linguistic competencies, they are not positioned in the U.S. narrative as such (de Araujo, Smith, & Sakow, 2016). Frequently, EBs are positioned in ways that do not focus on their mathematical competencies, but on their linguistic deficiencies. Such narratives have repercussions in the classroom and can determine ways teachers interact with EBs (Wood, 2013; Yamakawa, Forman, & Ansell, 2009; Yoon, 2008).

Discourse is critical to mathematical learning (National Council of Teachers of Mathematics, 2014) and classroom discourse can facilitate or restrict this learning (e.g., Esmonde & Langer-Osuna, 2013; Turner, Dominguez, Maldonado, & Empson, 2013). Discourse that is used to control or silence EBs ultimately diminishes opportunities to learn mathematics and acquire English while maintaining the status quo (Battey & Leyva, 2016; Yoon, 2008). Therefore, teachers must be attentive to the ways their discursive practices and mathematical and participatory expectations influence EBs mathematical learning in the classroom.

Positioning theory offers one way to examine how teachers' discursive practices can facilitate mathematical learning for EBs and offer counter-narratives of who is mathematically competent. Through discursive practices, teachers position students in ways that contribute to their storylines as mathematics students. When teachers position students in ways that value their mathematical competencies and diverse cultural assets and experiences, a storyline of mathematical competence can be fostered. All too often, however, Latin@ EBs are shut out of such storylines (Brenner, 1998; Gutiérrez, 2008). To understand how teachers can establish storylines of mathematical competence for Latin@ EBs, a single case study was conducted of a teacher who had learned about positioning.

Positioning and Mathematical Learning

In mathematics education, positioning theory has been used to analyze social interactions at the individual (e.g., Yamakawa et al., 2009), class (e.g., Esmonde & Langer-Osuna, 2013; Turner et al., 2013), and national (Herbel-Eisenmann et al., 2016) levels. Although this research has demonstrated the importance of positioning to mathematical learning, mathematical identity, access to mathematics, and the field of mathematics education, it has not yet identified how classroom teachers establish storylines of mathematical competence for EBs.

In this study positioning theory (van Langenhove & Harré, 1999) was employed as a conceptual and methodological framework to examine discursive practices between a teacher and a Latina EB.

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Positioning theory is composed of three central components: communication acts (Herbel-Eisenmann, Wagner, Johnson, Suh, & Figueras, 2015), storylines, and positions. *Communication acts* are the ways people verbally or non-verbally communicate (e.g., gestures) (Herbel-Eisenmann et al., 2015). *Storylines* are the "broad, culturally shared narrative that acts as the backdrop" (Herbel-Eisenmann et al., 2016, p. 104) to social interactions. Oftentimes these refer to the categories that people give others in specific situations, such as teacher/student or man/woman, that define the expectations and conventions of interactions in the situation (Herbel-Eisenmann et al., 2015). Within each social interaction there are multiple storylines at play all drawn from and on participants cultural, historical, and political backgrounds and experiences that occur on different scales (e.g., utterance, episode, state, national, etc.) and define the expectations and conventions for interactions in the situation (Herbel-Eisenmann et al., 2015). For example, in the U.S. the storyline of mathematical success is often characterized by speed and accuracy, as opposed to deep conceptual thinking. Manifested in the classroom, this storyline is established by the teacher and fostered through communication acts in socially recognized ways (e.g., rewarding quick, correct answers).

The socially recognized ways people employ storylines are referred to as *positions*. Within storylines, people are metaphorically positioned or have a position, which refers to one's "moral and personal attributes as a speaker" (Harré & van Langenhove, 1991, p. 395). This position is relational, directly tied to the power one has compared to others, and is dynamic—each communication act results in a re-positioning of oneself and others. Moreover, one can position him/herself (reflexive position) or can be positioned by others (interactive position) (van Langenhove & Harré, 1999).

A teacher's position in the classroom situates them as the catalyst and leader for the establishment and maintenance of norms that determine EBs positions and storylines in mathematics (Yackel & Cobb, 1996). Furthermore, since peers reinforce positions and storylines designated by the teacher, he/she must position students in ways that call attention to and highlight EBs unique cultural backgrounds and knowledge bases in order to have opportunities to participate and learn (Turner et al., 2013; Wood, 2013; Yoon, 2008). Thus, within any classroom, teachers' positioning plays a key role by determining *who* has the right and duty to participate and learn. As a result, this study sought to answer the question: *In what ways does an elementary teacher use communication acts to interactively position and foster storylines of EBs in the mathematics classroom?*

Methodology

To answer the research question of this study, data are drawn from a large, longitudinal professional development intervention study that spanned three years and included four female, monolingual third grade elementary teachers. The intervention focused on EBs development of mathematics and language, enhancement of mathematics curriculum materials, and productive classroom interactions (see Chval, Pinnow, & Thomas, 2014 for more information). Over the course of the year, the researcher met with each teacher 9-12 times to discuss the themes in the context of lesson planning or debriefing. In addition, the researcher pushed each teacher to create mathematical goals for each EB in their classroom.

The present study used a single case study design (Stake, 1995) to examine one teacher, Ms. Bristow. Ms. Bristow taught in a Midwestern city with an approximate population of 115,000 in a school that was predominately white (>70%) with less than 10% of the student population Latin@ and over half of students receiving free and reduced lunch. At the start of the intervention Ms. Bristow had two years of elementary teaching experience with no prior education in pedagogy for EBs or experience teaching EBs. Thus, the first year of the study coincided with her first opportunity to teach EBs. Thereafter, in each year of the study, Ms. Bristow had 1-4 EB Latin@s.

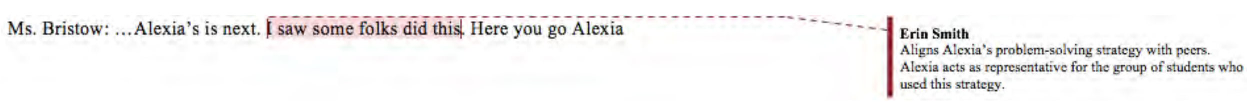
Data Selection and Analysis

To understand the ways Ms. Bristow constructed storylines for her EBs through interactive positions a subset of the data was analyzed—the first full month of the third year of the study. This subset was selected for two reasons. First, the establishment of storylines to facilitate EBs mathematical learning are heavily influenced by the teacher. Consequently, the teacher lays the foundation for future classroom interactions in the first month of the school year. Second, Ms. Bristow began learning about positioning and its importance in the classroom at the start of the intervention. Hence, after two years she had developed a greater knowledge base and mastery of positioning practices.

In the third year of the study, Ms. Bristow had one EB Latina, Alexia, who had relocated from a southwestern state two and a half weeks before the start of the school year. Ms. Bristow described Alexia as a quiet, shy, and reserved student who was often seen on task. Mathematically, Ms. Bristow explained Alexia had background knowledge that differed from her peers and, as a result, would bring up unfamiliar topics to peers in class.

Data analysis was conducted on a subset of data collected in September—the first month of the study (in the third year). Videos and transcripts of whole-class mathematics instructional interactions between Ms. Bristow and Alexia were analyzed. Classroom observations occurred five times, on September 6, 7, 15, 22, and 27. All classroom videos were reviewed and each instructional interaction between Ms. Bristow and Alexia was transcribed. Each transcription included verbal and non-verbal communication acts. In addition, transcripts of audio recorded one-on-one professional development intervention sessions were analyzed. These 40-50 minute sessions occurred on September 8 and 27.

The transcripts of Ms. Bristow's communication acts were open coded (Strauss & Corbin, 1990) at the utterance and turn taking levels to identify the interactive positions of Alexia (see Figure 1). These positions were analyzed sequentially to identify and construct the storylines Ms. Bristow fostered for Alexia as a mathematics student in the month of September.


Ms. Bristow: …Alexia's is next. I saw some folks did this. Here you go Alexia

Erin Smith
Aligns Alexia's problem-solving strategy with peers. Alexia acts as representative for the group of students who used this strategy.

Figure 1. Example of a coded communication act in a classroom transcript.

Findings

Ms. Bristow established and fostered Alexia's storyline as a mathematics student in multiple ways. First, she constructed pedagogical goals and participatory expectations for Alexia focused on her mathematical thinking and learning. Second, she employed communication acts to interactively position Alexia in whole class interactions where her mathematical thinking could be demonstrated. These interactive positions resulted in peers' appropriation of Alexia's storyline of mathematical competency.

Mathematical Goals and Participatory Expectations

In the intervention, the researcher consistently asked Ms. Bristow to create short and long term goals for her EBs at the start of the school year, but did not identify or specify the types of goals that these should be (e.g., what mathematical content, language competencies, and/or social competencies). This act of explicit goal setting for specific children facilitated Ms. Bristow's use of communication acts to position EBs in the mathematics classroom.

In September, Ms. Bristow described the goals she created for Alexia in her first meeting with the researcher on September 8.

Researcher: So what would be your goals for Alexia for this year?
Ms. Bristow: I think I want her to be a kid that is able to have strong mathematical thinking without just having to rely on just a set of rules. You know? I want her to be able to approach

> problems and be able to think of lots of different ways to solve a problem. And I don't...sometimes I worry with her that she will get trapped in an algorithm for everything because of the drilling that I think she is getting.

Ms. Bristow could have identified many different types of goals, however, as evidenced in her response the storylines and positions she wanted to facilitate for Alexia in her classroom were focused on her mathematical thinking. Specifically, a desire to increase Alexia's flexibility in mathematical thinking ("think of lots of different ways to solve a problem"), increase her mathematical reasoning abilities, and reduce her reliance on algorithms. These goals strictly attended to Alexia's mathematical thinking and learning and did not include or refer to her status as a language learner or newcomer in the school and community. In this way, Ms. Bristow began to construct a storyline of mathematical competence through her pedagogical goals, which were reified through classroom interactions.

Ms. Bristow was aware of the importance of participation for EBs' mathematical learning based on her involvement in the intervention. Consequently, Ms. Bristow intentionally positioned and solicited participation from Alexia in ways that would facilitate her mathematical learning. This was also described in the September 8 meeting with the researcher.

> As far as her [Alexia] participating, I try to get her to participate in some capacity in every lesson. I try to give her the opportunity to choose a partner that she is comfortable with. Because she is a little bit more reserved. And, I use her name in the problems. I used one of the problems that she had written in that word problem that the kids did with each other... I am trying to make her feel like a part of the class community and utilize her work and her name in as many different things as I can. I am trying to make her feel included.

Ms. Bristow clearly described the actions she took at the start of the school year to enhance Alexia's participation and mathematical learning. This included the incorporation of her name in mathematics problems, the option to select a partner, and the use of mathematics problems Alexia had written. What is most notable about Ms. Bristow's participatory expectations was her goal to seek out Alexia's participation in *every* mathematics class. Such actions worked to create a classroom community where Alexia—a newcomer and the only EB—could be successful in.

In addition to the above quotes, the analysis of video data demonstrated Alexia's participation took multiple forms. Ms. Bristow invited Alexia to the board to share her mathematical ideas, called on her in whole class discussions to explain her thinking, and displayed her work on the board to discuss in front of the class. To illuminate these practices, classroom examples are presented.

Sharing Ideas at the Board

Ms. Bristow commonly invited students to the board during discussions to share their mathematical thinking. In contrast to other teachers (e.g., Brenner, 1998; Yoon, 2008), Ms. Bristow frequently extended this invitation to Alexia and did not allow her to be a bystander. These actions were seen in the first day of classroom observations for the study, September 6. On this day, Alexia was invited to the board at the start of the lesson to share her mathematical ideas about a problem the class discussed. This communication act (i.e., Ms. Bristow's invitation to the board) interactively positioned Alexia in three ways: as a student who possessed mathematical ideas; as a student who would be an active participant in the global classroom conversation of mathematics; and as a student who was in the role of a teacher—a physical and metaphorical position of power. Together, these positions worked to establish the storyline of mathematical competence for Alexia publicly and counter deficit views of Latin@ students in mathematics (Gutiérrez, 2008).

What Are You Thinking?

At the start of the school year, Ms. Bristow established the expectation that *all* students—Alexia included—would share their ideas frequently and publicly. On the same day that Alexia was invited to the board to share her mathematical ideas (September 6), Ms. Bristow asked Alexia to share her mathematical thinking at the close of a lesson. In this lesson, students had played a game with a peer where they drew four cards each, made the greatest number they could, and then compared their numbers to determine a winner. The whole class discussion was framed around an example from a pair of students. To start the discussion, Ms. Bristow stated,

> Okay so…I have got two cards that I wanted to talk about today. Okay. Let's see…okay so Lamar and Adam. I thought that this was an interesting one. Adam got 9,760 and Lamar got 9,761 Why is that an interesting one? Why might that be an interesting one? Alexia what do you think?

At the start of this discussion, Ms. Bristow immediately asked Alexia what she thought about the selected numbers. This is an interesting question to pose Alexia as it asked her to not only consider how to compare the two values, but also the pedagogical decisions of Ms. Bristow (i.e., Why this example? What is important about this pair of numbers? What can we learn from them?). In this way, Ms. Bristow's communication act interactively positioned Alexia in two ways: as a student who can compare numbers in the thousands and as a student who can consider the pedagogical importance of this specific example. These two positions actively contributed to and reified Alexia's storyline of mathematical competence.

The next day, Ms. Bristow continued to invite Alexia into the classroom conversation of mathematics by soliciting her mathematical thinking. On September 7, Ms. Bristow began the lesson by modeling a game students would play with a peer where each player draws six numeral cards and makes two two-digit numbers "that, when added, give you a total that is close to 100." In the demonstration, Ms. Bristow first drew the cards, 1, 1, 5, 8, 3, 4 and wrote "_ _+_ _=__" on the board. Then, she invited the class to identify the best numbers to use, stating "Is there a number you could make or a combination you could make with just using those four of the six cards? Alexia, what are you thinking?" Like the day before, Alexia was the first student to be called on after Ms. Bristow initiated this whole class discussion. Ms. Bristow's question is interesting because it does not ask Alexia what are two numbers she could use, but what she was *thinking*. This communication act positioned Alexia as a student with mathematical thoughts who could strategically compose values to win the game. Moreover, it was the second time in *this* class that Ms. Bristow had asked Alexia what she was thinking. Furthermore, the questions posed to Alexia to describe what she was thinking were not low level or required a simple response, but challenged her to think critically and exercise a second language. Such instructional decisions may be directly tied to Ms. Bristow's pedagogical goal of developing strong mathematical thinking and reasoning skills for Alexia.

Sharing A Problem-Solving Strategy

Ms. Bristow often used student work (displayed on the board) at the end of a lesson to provide an opportunity to reflect on or share peers mathematical thinking. On September 15, Alexia was one of three students asked to discuss their problem-solving strategy (see Figure 2) of the peer written problem, "Sarah looks in her desk. She found 23 crayons, 14 pencils, and forty-nine crayons. How many crayons did she find in all?"

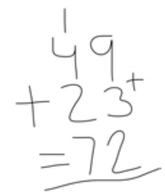


Figure 2. Alexia's problem-solving strategy.

Ms. Bristow: …Alexia's is next. I saw some folks did this. Here you go Alexia.
Alexia: *((comes to board))* I crossed out this. I crossed out this *((refers to "14 pencils"))* because if you say how many crayons are, did she find in all it doesn't say the pencils, so I crossed that out and I wrote this for this problem. *((points to algorithm))* First I wrote down the problem, then I put a plus sign here *((points to plus sign on right))* because I wanted to see 9 plus 3 is what. And it was, it was 12, so I put a 1 up there *((points to carried 10))* and I put a 2 there *((points to sum of ones column))*. Then I added these up 1 plus 4 plus 2 is 7 and I got 72.
Ms. Bristow: Questions and or compliments for Alexia. Keri
Alexia: Keri
Keri: I like how you thought about the, the answer.
Alexia: Mary
Mary: I like how you re-grouped.
Ms. Bristow: So that 1 where you say, that you put that 1 on top, what does that 1 representing? Is that one thing? Or what is that 1?
Alexia: It represents the 10 and so this is the tens place *((points to tens column))* I put it in the tens place.
Ms. Bristow: Yeah, so she actually took that 12 and put 2 of those loose ones on the side and then that one group of ten and added it to the 40 and 20 to make 70 and then we have the 2 loose ones to add to it. Alexia, I really appreciate you sharing. Thank you so much. *((class claps))*

This interaction represents another way Ms. Bristow solicited Alexia's participation. In this instance, Alexia was provided an opportunity for extensive mathematical talk. As a result, she was provided a chance to explain how she considered the information presented in the problem, determined what was erroneous, and then calculated her answer through an algorithm.

This opportunity to share her thinking was unique in the way it leveraged and elevated Alexia's status in the classroom and contributed to the storyline of mathematical competence. First, Alexia was one of only three students asked to share their thinking in this class. In this way, Ms. Bristow signaled Alexia was a competent mathematical student and her thinking was worthy of attention and discussion by her peers. Second, Ms. Bristow prefaced Alexia's discussion with the statement, "I saw some folks did this." As such, Ms. Bristow aligned Alexia's mathematical thinking with her peers and simultaneously enabled her to represent the group of students who used this problem-solving approach.

Regarding Alexia's mathematical thinking, Ms. Bristow also used this interaction to probe and clarify her ideas further in front of the class with her inquiry of the carried ten. This allowed Alexia to elaborate on her problem-solving strategy through mathematical discourse and illuminate how she understood the mathematical representation of the algorithm. An instructional decision that may be directly tied to Ms. Bristow's mathematical goal of increasing Alexia's flexibility in mathematical thinking and a reduction in the reliance of algorithms. Following Alexia's response, Ms. Bristow re-voiced her contribution, which acted to amplify her thinking to ensure all students had heard it and, reinforced her position as a student with valuable mathematical ideas.

This interaction also illuminates how Alexia's peers perceived her mathematical thinking and began to appropriate the storyline of mathematical competence. This is most evident in the targeted compliments Keri and Mary provided on Alexia's mathematical thinking. Unlike the two peers who also presented their mathematical thinking, Alexia was the only student who received compliments. This act of public recognition—a practice Ms. Bristow cultivated—is an example of how her interactive positioning of Alexia was taken up by peers, which fostered her mathematical success.

Discussion and Conclusion

Ms. Bristow offers a case of a monolingual elementary teacher who used of her position of power to construct a storyline of a mathematical competence for Alexia, a Latina EB, through communication acts and interactive positions. An examination of the data revealed Ms. Bristow initially established mathematical goals and participatory expectations focused on Alexia's mathematical thinking—not on her status as a newcomer Latina EB—that grounded future classroom interactions. Moreover, Ms. Bristow did not isolate Alexia in the classroom, allow her to be a spectator, or ask closed or simplified questions, but regularly invited her to share her mathematical thinking publicly, which provided opportunities for Alexia to use mathematical discourse—a critical aspect of mathematical learning. These communication acts worked to establish and foster Alexia's position in a storyline of mathematical competence and counter the narrative—and teachers' beliefs—that Latin@ EBs need mathematical support (Chval & Pinnow, 2010; de Araujo et al., 2016; Polat & Mahalingappa, 2013).

Given the growth of EBs nationwide, it is imperative teachers understand the connection between their communication acts, positions, and mathematics success and how their role in the classroom can be leveraged to create and manage storylines of mathematical competence for students that counter deficit views. Case studies—such as Ms. Bristow—can be used to illuminate how specific language practices and pedagogy can be employed by teachers to establish storylines of mathematical competence early in the year that position EBs for mathematical success.

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INTRODUCING MATHEMATICS TO INFORMATION PROBLEM-SOLVING TASKS: SURFACE OR SUBSTANCE?

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This study employs a cross-case analysis in order to explore the demands and opportunities that arise when information problem-solving tasks are introduced into college mathematics classes. Professors at three universities collaborated with me to develop statistics-related activities that required students to engage in research outside the classroom. This paper will focus on one aspect of the study: a comparison of how the teachers balanced mathematical content with information-problem solving in the tasks that they created. These tasks incorporated mathematics in a variety of ways, ranging from tasks in which the mathematical component was crucial to others where mathematics served solely as a marker of credibility. This research has the potential to provide tools for understanding how to productively incorporate information-literacy instruction into the mathematics classroom without losing sight of mathematical goals.

Keywords: Instructional Activities and Practices, Post-Secondary Education, Technology

Introduction

Mathematics students increasingly occupy an environment in which they enjoy immediate access to varied sources of information outside of the classroom. They can look up vocabulary terms on Wikipedia, find context for a statistics problem by accessing governmental statistics directly, and track down discussions of test problems on various Q&A web-sites. While mathematics instruction has historically been focused on sequestered problem-solving (Bransford & Schwartz, 1999), or the ability to solve problems without any outside aid, an increasing number of mathematics teachers have been actively encouraging students to engage with information-based problems (Walraven, Brand-Gruwel, & Boshuizen, 2008) or those problems that require "students to identify information needs, locate corresponding information sources, extract and organize relevant information from each source, and synthesize information from a variety of sources" (Walraven et al., 2008, p.623). I have previously argued (Erickson, 2015) that the use of information-based problems is not just preferable, but that it is also necessary if mathematics instruction is to help prepare students for the quantitative arguments that they may expect to encounter in their everyday lives (Paulos, 1988; NCED, 2001).

This paper presents a cross-case analysis (Stake, 2013) of three different undergraduate mathematics teacher who work with their students on statistics-focused tasks that require the students to seek out, evaluate, and make use of information that they find online. The use of information-based problems have been studied in the context of science education (Hoffman, Wu, Krajcik, & Soloway, 2003; Wiley, Goldman, Graesser, Sanchez, Ash, & Hemmerich, 2009) and history education (Britt & Aglinskas, 2002) before. These studies reported on experiments that took place apart from classroom instruction and so those who crafted the problems did not need to answer as to whether the problems provided evidence of learning, were congruent with the goals set out in the course syllabus, or incorporated the disciplinary topics to be covered in the class. Accordingly, the present study is unique for two reason: a) it takes place in the context of mathematics instruction, and (b) it was incorporated into ongoing mathematics courses. The analysis of the design and implementation of these information problem-solving math tasks provides a window into the balance teachers must strike between mathematical content and information literacy practices. In particular, the analysis addresses the following research questions:

1. How does mathematical content function as a component of an information-based problem introduced into a mathematics class which is meant to prepare students for the mathematics that they will encounter in their everyday lives?
2. What opportunities are students given to employ their mathematical knowledge through their work on these information-based problems?

Theoretical Framework

Three different conceptions of literacy have emerged which each highlight the importance of introducing information-based problems to mathematics instruction. (1) A move to attend to how reading and writing is conducted in the disciplines and to use this as a way of thinking about how to teach those disciplines means that mathematics teachers must think more carefully about how those who work in STEM fields locate and evaluate mathematical resources (Schleppegrell, 2007; Moje, 2007; Shanahan & Shanahan, 2012). (2) Quantitative literacy, sometimes described as a mathematical proficiency that can serve any individual in their everyday life, has been embraced as an educational goal, particularly at the undergraduate level (Cullinane & Treisman, 2010; Watson, 2013;MORE CITATION) and seen realization in courses offered by many colleges for non-STEM majors who need to fulfill a mathematics requirement as part of their liberal arts education. (3) Researchers in the information sciences have been arguing for the importance of instruction in order to facilitate college and high school students' ability to effectively research topics online (Rader, 2002).

While the importance of information-based problems for disciplinary literacy is easy to justify as long as one accepts that information-seeking is an important part of practice in the disciplines, it requires a little more unpacking to explain why this type of instruction might have a place in mathematics instruction. One way to begin such an explanation is to imagine an applied mathematics problem -- say students are given an editorial in which the author argues that federal guidelines on fuel efficiency will end up costing the country more money than it will save (Diefenderfer, 2009). Students are asked to read the editorial and then provided with several guiding questions that encourage the students to analyze the numerical argument contained in the article while noting some of the additional information that might be required prior to coming to a final verdict on the validity of the editorial's argument. While this activity is a legitimate applied mathematics problem, the real-world context (see Figure 1) suggests other directions that such a problem could be taken.

If a reader were to actually want to determine whether the editorial's claim was true or not, they would want to locate the relevant *epistemic community* (Haas, 1992), or that community that possesses the expertise to tentatively rule on the truth of the claim. In other words, they would need to engage in the practice of rational dependence by finding experts on whom the students have good reason to rely. The problem as originally stated does not afford the students an opportunity to engage in this practice. They are not asked to seek out and evaluate those sources of information that might either corroborate or challenge the argument found in the editorial. An *information-based problem* (Walraven et al., 2008)*,* on the other hand, requires that the student seek out and evaluate sources outside the classroom. In order to come to a better understanding of an information-based problem, the student must "identify information needs, locate corresponding information sources, extract and organize relevant information from each source, and synthesize information" (Walraven et al., 2008, p.2) in a process called *information- problem solving*. My inquiry can be framed, then, as a question about how mathematics teachers and their students cope with the introduction of information-based problems, and whether and how these problems afford opportunities for the practice of mathematical problem-solving skills.

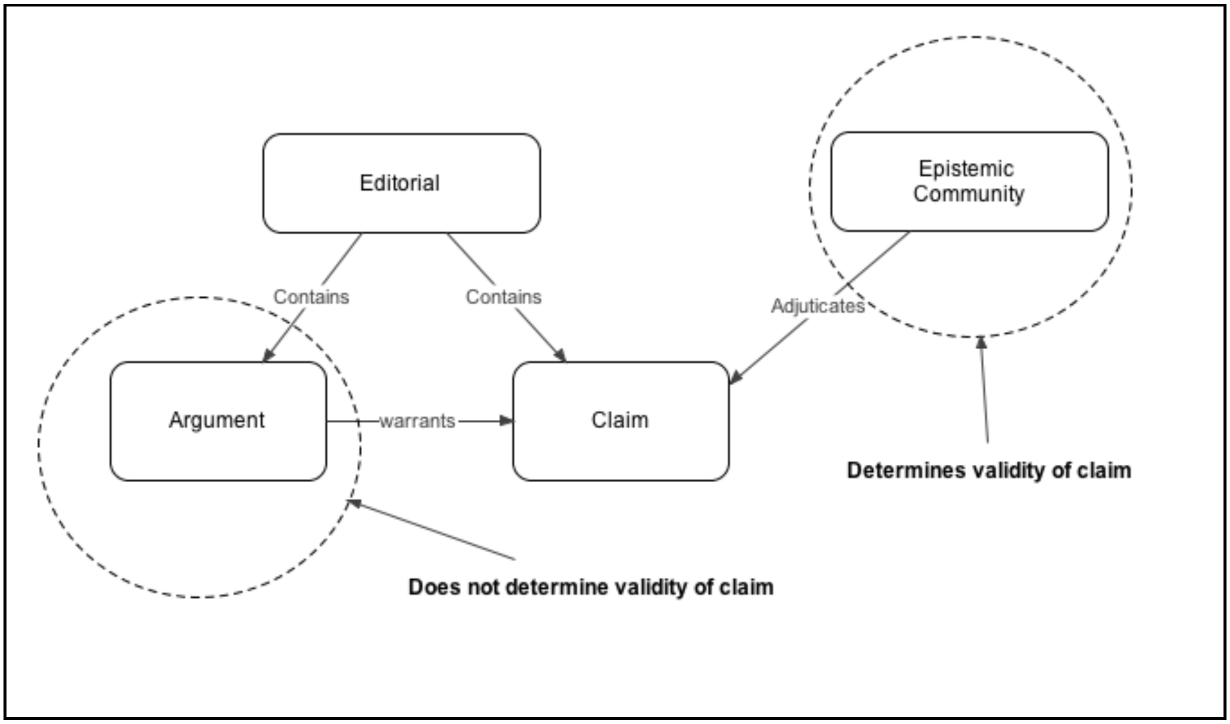

Figure 1. Relationship between editorial, claim, and epistemic community.

Methodology

I began this work by contacting instructors of terminal mathematics courses for non- STEM majors because my reading of the literature and interviews with mathematics educators suggested that this would be the most accessible site for this type of work. There were thirteen educators at ten different institutions who responded to this call and, while they all expressed interest in the idea when I explained it to them, I ended up with 4 collaborating instructors as the others were not able to work with me due to either institutional or timing constraints. I met with each of these four collaborating teachers and explained the rationale behind information-based problems and we worked together to design a couple of activities in which this type of problem would be introduced to their students. The analysis is focused on the three sites (see Table 1) where the instructors had the greatest role in designing and implementing the problems.

At Phi University students were assigned to argue one side in a classroom debate. To prepare, they were required to research their topic and provide some statistical evidence supporting their side of the issue. At Rho University we developed a two-part activity where students were asked to look for articles in which a conjecture about causation was being studied (e.g., vaccines and autism). They were asked to locate the quantitative evidenced used to claim that the two variables were or were not correlated, and then engaged in a small-group discussion with their peers about the topic. Their groups tried to come to a consensus on the issue at stake and then shared their verdict with the rest of the class. The students at Delta University also worked in small groups, but here they were asked to create a presentation in which they would analyze the way that statistics were used in a research article for the rest of the class. The focus of this analysis would be on the sampling methodology, but they were free to talk about other facets of the article if they so chose.

Table 1: Research Sites

University Name*	Course Name*	Students	Topics	Structure
Phi University (Research)	Topics in Mathematics	22 entering Freshman, Liberal Arts Majors	Gun Control, Marijuana Legalization, Single-Sex Education, Death Penalty	Debate Format
Rho University (Regional)	Quantitative Reasoning	14 Juniors and Seniors, many are prospective Nursing students	Autism and Vaccination, The Mozart Effect, Gun Control, Health Care Reform	Small-group Discussions
Delta University (Doctoral)	Mathematics in Today's World	24 Juniors and Seniors, many are prospective Nursing and Education students	Autism and Vaccination, Gun Control, Murder Rate, Vehicular Accidents, Employee Prospects	Small-group Presentation

*These are pseudonyms

This study takes the form of a multi-case analysis (Stake, 2013) derived from the activities described above. The *quintain*, or the phenomenon of interest for this cross-case analysis, is the introduction of information-based problems to an undergraduate mathematics course. The data for this study includes pre- and post-interviews with the instructors at each of the three sites, supplementary interviews with teaching assistants and students, field notes taken while observing instruction prior to the introduction of the information-based problems, video and audio-recordings of the in-class component of the activities, and copies of the work that the students submitted. These data sources informed the writing of individual case reports which were, in turn, used to develop the cross-case analysis. Following Stake (2013), I developed themes based on my research questions that I then used as an analytical lens for the development of case reports for each of the three sites. After writing up the case reports, I cross-referenced case-specific observations (see Table 2 for relevant examples) with the themes of the larger study. This allowed me to warrant theme-based assertions and used those to inform the final cross-case assertions about the introduction of information-based problems to undergraduate mathematics classrooms. The cross-case assertions related to the students opportunities to interact with mathematical content in the information-based problems will be presented below.

The teachers with whom I collaborated were taking on a unique challenge when they decided to introduce information-based problems to their mathematics classroom. While they agreed that their students would benefit from an opportunity to engage in information problem solving, they had to continue to teach their students mathematics through these problems. I intentionally provided very little guidance on this front and the teachers at each of the three locations approached the challenge in distinct ways. In order to describe the role that mathematical work played in these problems I describe the implemented problems as *academic tasks*, a term that encompasses both the perspective of the students as they try to meet the requirements of their assignment and the teachers as they manage the work of their students. I refer to academic tasks using the technical sense employed by Doyle and Carter (1984) where tasks are broadly understood as the "situational structures that organize and direct thought and action" (Doyle & Carter, 1984, p.130) in the classroom and more specifically as components of the curriculum that direct students to use *operations* on *resources* in order to achieve a *product* which is validated by a system of *accountability.* The different role of mathematics in each of the cases is clarified by articulating where it sits within the overarching academic task with respect to the component operations, resources, and product.

Table 2: Examples of Case-Specific Observations from Rho University

Case-Specific Observation A: In the first session, students were searching for articles and then looking for a correlation coefficient within the articles, but they rarely found a correlation coefficient.
Case-Specific Observation B: Students preferred to work with information resources that contained mathematics that they could understand.
Case Specific Observation C: In the first session, the teacher prioritized the identification of a particular piece of mathematical content, the correlation coefficient, to the exclusion of any discussion of the credibility of sources.

Results

The mathematical aspect of the debates at Phi University was fairly circumscribed; it almost existed as a freestanding task of its own within the larger task of the debate. Tim (a pseudonym for their professor) had decided that his students would provide a statistical chart or diagram as part of their argument and that this would tie the problem into the unit on statistics that the class was in the process of covering. In particular, the students would have to create their own graphic rather than downloading it from elsewhere,

> I made the decision at some point that I didn't want them to just download graphics off the internet because I was worried that if they did that, they would download a bunch of fancy graphics that they didn't really quite comprehend and then it would be too much information for anybody in the audience to comprehend too. And so I figured the way around that is that I would have them do it, they read the section on how to present data, that's section two of that chapter, how to present data, ideas on how to present data, and so let's make them do it. I think that was a good decision. (Tim, 8/14/13, Lines 73 - 79)

I refer to the creation of graphics as an *almost* freestanding task because it still was predicated on some information-seeking in order to locate the sources containing the statistics that would inform the graphics. One way to put this is that the operations that were part of the debate task (for example, using a search engine, scanning search results) provided the resources that would be used for the mathematical task. I refer to the mathematical work as an academic task of its own because it theoretically could exist independently of its role in the debate as long as equivalent resources were made available. Conversely, a debate could have taken place without students having to make use of statistical diagrams. Nonetheless, by introducing this mathematical task, Tim was able to simultaneously avoid the introduction of graphics that could be too mathematically demanding for his students and to create a relatively well-defined mathematical task within the larger information problem-solving task which he would subsequently be in a better position to appraise as mathematical work.

At Rho University, Anne (a pseudonym for their professor) also created a mathematical task that was part of a larger information problem-solving task, but in this case the mathematical work was necessary, at least as Anne conceived it, for the credibility assessment that constituted the larger task.

As with the students at Phi University, the students at Rho were to submit a product that included the mathematical work (i.e., the identification and interpretation of a correlation coefficient or confidence interval) alongside more ambiguous information problem-solving work (i.e., an assessment of the credibility of the different sources). It was also the case that the search for those mathematical markers could only occur after the students searched the internet in order to collect sources. Thus, we again have a situation where the information-seeking operations on the internet was used to provide resources for the students' mathematical work. There was one small but significant difference in the case of the second problem; those students were required to only collect sources that contained confidence intervals and so the mathematical operation of identifying a confidence interval became part of the information-seeking work. The subsequent classroom discussion constituted another part of the task, or perhaps a second task, which was much more ambiguous in terms of both its product and its operations. By breaking the information-based problems into two distinct activities, Anne was able to collect and evaluate concrete evidence of mathematical work with the written assignment and make that same mathematical work available as a resource for the students' discussions of credibility.

The first problem at Delta University as developed by Ivan (a pseudonym for their professor) was structurally similar to the mathematical sub-task at Phi University in that the students had to first find a source which they would then use as a resource for their mathematical work. The product at Delta University was the students' analysis of the sampling strategy used by poll or research article that they found. The second problem at Delta University was similar as well, but the use of sources was complicated by the fact that the students could not just pick a source and then analyze its contents. Instead, the students were held accountable for finding the data needed for their calculations which meant that their mathematical knowledge needed to mediate their information-seeking work in a way not seen in any of the other cases. By structuring the problems in this way, Ivan made the students' mathematical work the core of the product that they shared with the class. This was reflected in the type of feedback that Ivan provided for his students, he focused much more on the students' mathematical work than either of the other cases.

Looking across the cases (see Table 3 for cross-case observations) it was notable these students only questioned the source of the statistics when explicitly instructed to do so as part of the activity. When the statistics served as evidence for a point-of-view, as in the case of Phi University, students were only concerned with effectively communicating the information that they found to their peers in order to support their argument. Further, when the statistics were being used as evidence, students used the presence of statistical information as a token of credibility for the information source as a whole, and not as an element of the information source deserving critique in and of itself. Ironically, the activities at Phi and Rho University, in which actual mathematical work was very limited, were felt to be very successful by the Tim and Anne, while Ivan expressed some frustration with his activity. This appeared to be at least partly due to the opportunity the presentations afforded for Ivan to observe gaps in his students' knowledge of basic statistical concepts, an opportunity that did not exist at either of the other two locations. In a subsequent interview with Ivan, he expressed some satisfaction that he was able to realize that students did not fully understand some of the more basic statistical concepts that he had been trying to teach them and said that he would incorporate a similar activity into future iterations of the course in order to discover whether students could apply what they were learning to real-world scenarios.

Table 3: Cross-Case Observations About the Role of Mathematics in the Information-Based Problems

Cross-case Observation 1: Students only adopted a critical stance towards the statistics that they found when explicitly demanded by the problem.
Cross-case Observation 2: Students successfully drew on elements of a statistical knowledge base throughout the three cases, but those information-based problems in which mathematical work was prioritized presented a greater opportunity for revealing gaps in student knowledge.
Cross-case Observation 3: The students' mathematical knowledge sometimes conflicted with their ability to assess the credibility of a source. On the one hand, if the students are not asked to engage with the mathematics in question, then it may only serve as a superficial marker of credibility rather than providing insight into the mathematical argument. On the other hand, if a source contained mathematics that the students did not understand, then they might not use that source.

Discussion & Conclusion

These three cases afforded me the opportunity to see some of the ways that a practicing mathematics teacher can dispense their obligation to teach the discipline to their students while assigning non-traditional information-based problems. In all three of the cases, the students had to engage in some initial information-seeking in order to collect sources that would serve as a resource for their mathematical work. In the first case, this work was its own mathematical task, contributing to the larger information problem-solving task but not strictly necessary for its completion. In the second case, the mathematical work was a distinct task but it also served as a resource for the following discussion. In the third case, the result of the mathematical work was actually the product of the information problem-solving task and the students' information-seeking and evaluation played a supporting role.

This analysis of the dilemmas faced by mathematics instructors who wish to create opportunities for their students to build their information literacy skills is not intended to dissuade teachers from using information-based problems that bring students into contact with real-life quantitative claims nor to discourage the introduction of information problem-solving to mathematical tasks. Rather, it suggests that pedagogical choices must be made thoughtfully and deliberately if they are to meet their intended goals.

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INVESTIGATING THE RELATIONAL NATURE OF FEEDBACK PRACTICE

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At the crossroads of our prior research on prospective teachers' feedback to mathematics-learners and our mathematics teacher educator feedback practices, we study written feedback as part of relational practice. Using self-study methodology and an analysis of our narratives and conversations about written feedback, we identified factors that frame and motivate our written feedback. We argue that, assuming the central goal of teacher education is the development of relational practice, written feedback should support prospective mathematics teachers' skills and knowledge relevant to tasks involved in teaching mathematics and extend prospective teachers' views of mathematics teaching and learning by drawing on their experiences, insecurities, problems, and views of mathematics teaching and learning.

Keywords: Instructional Activities and Practices, Teacher Education-Preservice

Grossman et al., (2009) asserted that development of a relational practice is a central goal of teacher education and illustrated that mathematics teacher educators (MTEs) engage in activities in support of this goal. Buhagiar (2013) suggested that MTEs' practices serve as models for future mathematics teachers (teacher-learners). Thus, MTEs' activities should model relational practice, described by Fletcher (1998) as including "empathy, mutuality, reciprocity, and a sensitivity to emotional contexts" (p. 174). This paper focuses on MTEs' written feedback as one activity that models elements of relational practice. Feedback is a significant part of an assessment system and impacts learning and performance (Hattie & Timperley, 2007; Shute, 2008). Therefore, MTEs' written feedback and factors that frame and influence that feedback warrant study to improve understanding of written feedback as a model of relational practice.

Findings from a prior analysis of feedback that teacher-learners provided to mathematics-learners (i.e., K-12 mathematics students) through letter exchanges (e.g., Crespo, 2002) included a description of the ways mathematics teacher-learners used praise and attended to the learners' mathematics in their responses (Kastberg, Lischka & Hillman, 2016a). These findings raised questions about our MTE feedback practices. We wondered if our written feedback would stand up to the scrutiny leveled at teacher-learners. Did our feedback attend to ways teacher-learners saw learners' mathematics and build on their personal understandings of mathematics teaching and learning or did we direct teacher-learners to what we saw in mathematics-learners' work?

Considering this question brought us to a crossroads as we became conscious of a "living contradiction" (Whitehead, 1989) between our feedback practices and our expectations for teacher-learners' written feedback. In an effort to improve our feedback practice and identify ways in which such a practice modeled relational practice, we asked *What factors frame and motivate our written feedback as a model of relational practice?*

Literature Review and Theoretical Framework

Existing meta-analyses of quantitative studies of feedback (e.g., Hattie & Timperley, 2007; Shute, 2008) identified factors that mediate its effects, such as complexity of tasks and characteristics of praise on performance. Recent research (e.g., Evans, 2013) draws from a broader array of theoretical perspectives (e.g., socio-cultural, socio-critical, constructivist) not well represented in prior feedback discourse. From a synthesis of studies in higher education, Evans hypothesized a "feedback landscape" that "illustrates a two-way process in which feedback is moderated" (p. 97) by a collection of relational and information variables that include learners' "beliefs about learning and expectation of the learning environment" and teachers' "knowledge of the student" (p. 98). Evans' view of feedback applied to mathematics teacher education involves understanding written feedback as an element of relational practice.

There is much research on feedback in general with little attention to written feedback provided by MTEs to teacher-learners. Studies of feedback in teacher education have focused on feedback given during practicum (White, 2007), teacher-learners' perceptions of feedback (Dowden, Pittaway, Yost, & McCarthy, 2013), and self-studies focused on written feedback (Kitchen, 2008; Pittaway & Dowden, 2014). In *mathematics* teacher education, only Buhagiar (2013) explored written feedback. He reported that MTEs' feedback varied significantly and suggested beliefs about teaching and learning as the source of the differences.

Relationships with learners are important elements in effective feedback practices (e.g., Evans, 2013; Hattie & Timperley, 2007), allowing MTEs to leverage understandings of teacher-learners (Grossman et al., 2009) and contexts in which they work in support of teacher-learners' development of practices and understandings of mathematics teaching and learning. Kitchen's (2005a, 2005b) description of *relational teacher education* as teacher educators "knowing in relationship" (2005a, p. 18) is used to understand factors that frame and motivate MTEs' written feedback as a relational practice. Like Fletcher (1998), Kitchen drew from notions of empathy and vulnerability to describe relational practice and identified seven defining characteristics: understanding one's own personal practical knowledge, improving one's practice in teacher education, understanding the landscape of teacher education, respecting and empathizing with preservice teachers, conveying respect and empathy, helping preservice teachers face problems, and receptivity to growing in relationship. These categories are used as an analytical framework to explore factors that influence written feedback as a relational practice. Descriptions of each category are shared in the findings section.

Mode of Inquiry

To identify factors that frame and motivate our written feedback as a relational practice, we undertook a self-study. Identified by Borko, Liston, and Whitcomb (2007) as a form of practitioner research, self-study is aimed at improving one's practice (LaBoskey, 2007) and is characterized by openness, collaboration, and reframing (Samaras & Freese, 2009). Self-studies situate questions in existing research literature and suggest implications for "the larger audience of teacher-educators" (Borko, Liston, & Whitcomb, p. 9). Self-study involves the construction of narratives of experiences and conversations with critical friends sharing alternative perspectives on practice, described by LaBoskey as data in self-study methodology. This self-study was undertaken with the goal of improving our written feedback. We began by analyzing our written feedback using Hattie and Timperley's (2007) framework (see Kastberg, Lischka, & Hillman, 2016b for findings related to this analysis). This paper focuses on factors that framed and motivated our feedback, using transcripts of eight recorded online conversations about written feedback findings (May-December, 2015) and two self-constructed narratives (Clandinin & Connelly, 2000) as data. The first narrative described our feedback experiences as leaners and the second narrative described our experience creating opportunities for teacher-learners to provide written feedback to mathematics-learners (May and December, 2015). This data allowed for reframing experiences by taking the perspective of another on one's practice.

Narratives and discussions were coded using Kitchen's (2005a) characteristics of relational practice. Evidence of *knowing in relation to self and teacher education* was coded using Kitchen's (2005a) first three categories and evidence of *knowing in relation to teacher-learners* the remaining categories. Descriptions and exemplars are shared in the findings.

Blind review precludes specificity, so a sketch of the actors and context is included here.

Pseudonyms are used for the authors in the remainder of this paper. The authors work at three different institutions with Sandy and Jean both mid-career MTEs working with elementary teacher-learners and Pamela an early-career MTE working with secondary teacher-learners. All teacher-learners engaged in letter-writing activities, with MTEs providing written feedback.

Findings

Knowing in Relation to Self and Teacher Education

In restorying our experiences, we gained “self understanding” (Kitchen, 2005a, p. 19) of reasons for assignment structures and types of written feedback we provided.

Understanding one’s own personal practical knowledge. Kitchen (2005b), defined personal practical knowledge as “the ways in which past experiences inform present practice and intentions for the future" (p. 199). We unpacked our experiences as learners, mathematics teachers, and MTEs and considered how they informed our practices. Discussions focused on assignment structures and what teacher-learners’ approaches to feedback could teach us.

An example related to assignment structures involved exploring whether feedback on letters supported the teacher-learners to develop their views of mathematics teaching and learning or just to complete the task as we had conceptualized it. We wrestled with the question of whether the teacher-learners could use our feedback. Sandy and Pamela provided feedback on teacher-learners’ reflections on feedback provided in letters to mathematics-learners, while Jean had provided feedback on teacher-learners’ draft letters and requested revisions. Looking back at our experiences giving written feedback, Sandy and Pamela wondered whether the teacher-learners could make sense of the feedback.

> Sandy: I think that one of the fundamental assumptions that we operate under, [is that] if we give feedback, [teacher-learners] are actually going to operationalize it and use it. But the reality of the situation is that we know that really doesn't happen. In part, that is our own fault because … we don't provide opportunities to revise your work in light of feedback. When we do, Jean's work shows us that they attend to the letter of the law. “Oh, you told me I needed to add this … so I did those things.” (November 12, 2015)

We questioned whether we were supporting only those teacher-learners whose work aligned with our views. Retelling our experiences as MTEs, we developed empathy for teacher-learners trying to fulfill course demands while extending understandings of mathematics teaching and learning.

Our discussions of teacher-learners’ approaches to feedback focused on a contrast to our own pedagogical principles and strategies used in written feedback. We wondered if interpersonal relationships with teacher-learners would encourage them to use our feedback. Teacher-learners’ responses to mathematics-learners served as examples of relationship development. For example, some teacher-learners first attended to mathematics-learners as people and only highlighted elements of the learners’ mathematics after addressing unique characteristics of the mathematics-learner. In contrast, our written feedback focused on supporting the teacher-learners to complete the task at hand. Looking forward, we discussed if we should, or could build more personal connections into our written feedback.

> Pamela: Ok, then maybe I need to think differently about mine [feedback], because I looked at the few tasks that I have had a chance to look at and I’m thinking that I don't really attend to the PT [teacher-learner] as a person. … But I think I do a lot of that in class, it’s just not in my written comments. So I struggled with that one. (June 4, 2015)

Pamela’s comment illustrates that interpersonal relationships with teacher-learners may be developed face-to-face. We then wondered if our written feedback would be more effective if attending to interpersonal relationships before sharing feedback on the task and process.

Improving one's practice in teacher education. Kitchen's (2005a) category "improving one's practice in teacher education" includes exploring experiences and using insights gained to improve practice, such as teacher educators trying to communicate "understandings and structure meaningful lessons" (p. 23). Efforts to improve our feedback practices focused on the purpose of assignments, such as desiring to have teacher-learners explore mathematics-learners' reasoning.

> Pamela: My intent was just to have them thinking about [mathematics-learners'] thinking.
> Sandy: Yeah me too.
> Pamela: I wanted them to interact with a [mathematics-learner] because they don't have a field experience in the course. So I wanted student interaction. (July 28, 2015)

By comparing and contrasting our course activities, institutional contexts, and feedback practices, we proposed changes to our assignments and feedback practice. Pamela and Sandy drew insight from Jean's approach where feedback had resulted in teacher-learners' improving final versions of letters. Proposed improvements included providing feedback on drafts of teacher-learners' responses, initiating class discussions of MTE feedback, and adjusting letter exchange time-lines to allow teacher-learners to revise.

We evaluated proposed changes to our feedback practice based on whether a change was productive for teacher-learners and efficient for us. For example, we wanted to reduce time between the submission of work and teacher-learners' receipt of written feedback, but struggled with how to construct feedback quickly that attended to individual needs of teacher-learners.

Understanding the landscape of teacher education. Kitchen (2005a) identified the need to "frame individual challenge within a larger institutional and societal challenge" (p. 27) as "understanding the landscape of teacher education." We discussed motivations behind decisions about structuring assignments and crafting written feedback that included program assessments for accreditation, field structures, class size, the practices of supervisors in practicum, and the Common Core movement. Our feedback was part of teacher-learners' experiences in teacher education programs facing increased scrutiny and demands that graduates be expert teachers. In particular, Jean's motivation in developing the letter exchange between teacher-learners and mathematics-learners was for teacher-learners to develop ideas about providing written feedback, in response to data analysis from a program assessment related to accreditation that showed a need for improvement in the area of teacher-learners' feedback to K-12 students.

For example, Jean and Sandy shared stories about efforts to meet demands of accreditation organizations including preparing teacher-learners to collect "data" from their practices.

> Sandy: … because my [colleagues] are asking me for examples of my teacher-learners' work that I think are particularly good and that show that they can collect and have analyzed data and can make decisions about what to do next. (August 10, 2015)

Our conversations revealed the challenges in creating meaningful learning opportunities for our teacher-learners (e.g., constructing written feedback on mathematics tasks) while preparing them to document their work as required for accreditation.

Knowing in Relation to Teacher-Learners

Kitchen (2005b) described the last four elements of relational practice as modeling "respect for teachers as curriculum makers" (p. 200) with focus on the MTE/teacher-learner relationship.

Respecting and empathizing. Central to respecting and empathizing with teacher-learners is "a genuine belief that each prospective teacher must construct her or his own meaning as a curriculum maker" (Kitchen, 2005b, p. 201) by recognizing and supporting the needs of teacher-learners while encouraging them to probe issues of mathematics teaching and learning.

Discussions focused on gaining insight into teacher-learners' experiences and concepts of teaching. Knowing about teacher-learners' stories of experiences in schools could help us understand their views of mathematics teaching and learning. Yet, we typically had not asked teacher-learners about their experiences. When we did, we did not use that knowledge to inform our written feedback. For example, Sandy asked teacher-learners to reflect on their experiences as learners, but described still wondering about the sources of teacher-learners' insights.

> Pamela: So then how do you build on [teacher-learners'] own knowledge and experiences, when you haven't found out what those are?
>
> Sandy: I never really tried to understand where [teacher-learners] were coming from. So when I read Jean's [feedback] and then I read mine I was going: "Ok, well this feedback [responses to mathematics-learners] that they are giving could be interpreted so many ways." I wish I knew how [teacher-learners] were thinking about it. I had them do the reflections, so you would think I would know. But they would make statements about … the students' thinking, that I was just like: "I wonder where this is coming from." (August 10, 2015)

Not knowing the teacher-learners' motivations and experiences that may have influenced their interpretations of learners' mathematics made developing meaningful feedback difficult.

Although supporting teacher-learners' efforts to build conceptions of teaching and learning was a goal, we considered how to address their need to survive in practicum. We realized teacher-learners' responses to mathematics-learners, were a function of their conceptions *and* efforts to complete course assignments. To construct productive feedback, we conjectured about teacher-learners' conceptions of teaching and learning from evidence of their experiences. For example, Jean started asking "how are you doing?" (November 12, 2015) in individual meetings with teacher-learners before launching into feedback about lesson plan drafts. This simple question encouraged teacher-learners to relay stories from field experience that revealed not only their concerns with practicum, but informed Jean's understanding of their concepts of teaching and learning. Looking forward we developed other ways to gather evidence of such experiences.

Conveying respect and empathy. Describing his efforts to convey respect and empathy, Kitchen (2005b) suggested teacher educators can demonstrate their feelings by "acknowledging insecurities" (p. 204) and helping teacher-learners face challenges in programs of study. Further, teacher educators can express commitment through listening and responding mindfully.

We were uncertain about challenges teacher-learners faced since we had not invited teacher-learners to share experiences with us. To gain insights, we discussed our own experiences as teacher-learners and recalled challenges trying to "fit" into mentor teachers' classrooms. Using this experience, we considered our programs and course activities and the potential in these contexts for teacher-learner challenges. For example, in looking back to our letter-exchanges, we discussed the challenge of engaging with mathematics-learners whom teacher-learners did not know. This activity structure seemed misaligned with possible teacher-learners' views of teaching and learning situated in nurturance and care.

We discussed how to respond mindfully when teacher-learners shared challenges and insecurities. We wondered whether praise would count as part of a mindful response because without praise, teacher-learners might read our written feedback as lacking care or concern. Jean looked back at shifts in her written feedback from using "smiley face kind of stuff" in her written feedback to giving "specific comments" and whether this had impacted her relationships with the teacher-learners (August 10, 2015).

> Jean: But then, I wonder if that shifted relationship building because there just seems to be a difference between [teacher-learners] that I have had recently versus [teacher-learners] that I

remember from say years ago. (August 10, 2015)

Hypothesizing about challenges our teacher-learners faced was easy; knowing how to construct written feedback addressing the challenges was difficult.

Helping teacher-learners face problems. Kitchen (2005b) described helping teacher-learners face problems as identifying and supporting teacher-learners to confront tensions between their constructs of teaching and learning and the practical realities of classrooms.

Discussion focused on problems that could arise when a teacher-learner's goals did not align with her practices. Sandy described asking teacher-learners how to confront errors in mathematics-learners' work: "When I addressed dealing with incorrect responses with my [teacher-learners] … They were very sensitive to children being told that they are wrong. [Teacher-learners] really think there is no place for it" (July 28, 2015). Sandy strove to honor teacher-learners' perspectives, but felt the practice they described was inconsistent with their goals for mathematics teaching. The teacher-learners' position on error-handling was consistent with their determination to attend to mathematics-learners as people, yet seemed inconsistent with their goal of supporting the development of learners as mathematicians. Sandy recognized this tension with teacher-learners' perspectives on errors as a potential learning opportunity for them, but did not know how to use feedback to help the teacher-learners confront this tension.

Pamela's feedback was typically in the form of questions addressing what she identified as teacher-learners' problems of practice. For example, sharing a teacher-learner's response about being more clear and specific with mathematics-learners, Pamela illustrated how she used questions to help the teacher-learners face problems.

> My response was "I want you to consider whether it is the clarity and specificity that is important or the information on which you ask the students to build their thinking. How are you asking students to think about their own responses?" (Pamela, July 28, 2015)

Pamela hypothesized teacher-learners' conceptions of teaching and learning were surface-level and questions in her feedback would encourage teacher-learner reflection, even when directly disagreeing with teacher-learners' claims to help them unpack problems of practice. As Sandy wondered if her relationships with teacher-learners could withstand this approach, Pamela maintained that interactions with teacher-learners allowed giving critical feedback, asserting teacher-learners would attend to feedback due to collegial relationships with the teacher-learners.

Receptivity to growing in relationship. Kitchen (2005b) described receptivity as identifying one's own problems rather than "the 'expert'" (p. 206) defining the problem to be faced. MTEs discovery of new meaning and development of professional practice is then based on being receptive to needs of teacher-learners.

We discussed receptivity as components of our relationships with teacher-learners, yet having our own identities and values seemed to interfere with the development of our relationships at times. We discussed wanting intellectual relationships with teacher-learners.

> Sandy: I'm engaged with you because of the possibility of learning something new.
> Pamela: Because of the intellectual possibilities, not the interpersonal possibilities.
> Sandy: I want to know [teacher-learners] in an academic and an intellectual way, but I don't even think I do because I'm not taking up the ideas that they provide … except in the most superficial way.
> Jean: … Knowing them in an academic and intellectual way. So is there a way to think about empathy in terms of that … I mean what would that look like? (September 24, 2015)

We agreed our love of mathematics influenced our conversations and relationships with teacher-learners. Sandy felt teacher-learners might need more than our focus on mathematical thinking.

> What makes your classroom work are those relationships and those moments you have with the students where … you are connecting as human beings and the student is going, "Oh yeah, she gets me and I can talk to her." (Sandy, September 24, 2015)

Our discussions took up the need for human connection with teacher-learners, but the tension involved the power we had over their grades and the way that influenced the relationship.

> Pamela: But that is the challenge, I want them to get to the point where they are pushing back a little bit. (August 24, 2015)

Pamela viewed teacher-learners' questions about her motives and practices as an indicator of a mature relationship. We viewed human connection as important to demonstrating and supporting receptivity to growing in relationship through discussion of feedback.

Summary

Evidence of knowing in relation to self and teacher education showed assignment purposes and structures were factors motivating our written feedback. Improvements to our practice were viewed through a lens of efficiency, while considering mindfulness in our feedback. Program accreditation influenced the design of activities on which we provided feedback, responding to external demands of society and our respective institutions for teacher-learners to demonstrate proficiencies. We were not consciously attending to these factors as we wrote feedback, but they impacted our attention to what and how feedback was provided. Considering improvements, we turned to the teacher-learners' feedback as an example of attending to learners as people first.

Evidence of knowing in relation to teacher-learners revealed that we knew little about the experiences, challenges, and problems teacher-learners faced. To convey respect and empathy we attended to elements of teacher-learners' work on assignments, but without attending to teacher-learners' views of mathematics teaching and learning. To build relationships, we relied on in-person interactions to encourage teacher-learners to attend to our written feedback. Our love of mathematics and desire to have intellectual relationships with teacher-learners motivated attention to mathematics in our feedback, without attention to insecurities and problems of practice with which teacher-learners wrestled.

Discussion and Conclusion

Findings revealed factors that framed and motivated our written feedback as a model of relational practice. Our written feedback was influenced by knowing in relation to self and teacher education, and knowing in relation to teacher-learners. Discussions of our written feedback as a relational practice revealed attention to skills and knowledge relevant to tasks involved in teaching mathematics. This focus is essential for effective feedback (e.g., Evans, 2013), yet falls short when feedback is considered as a model of relational practice. Evidence that our written feedback was motivated by "empathy, mutuality, reciprocity, and a sensitivity to emotional contexts" (Fletcher, 1998, p. 174) was thin, suggesting a way forward in improving our written feedback. Considerations of teacher-learners' experiences and views of mathematics teaching and learning are needed to build written feedback as a relational practice.

Attention to the written feedback of our teacher-learners was a source of inspiration as we considered potential insights from the experiences of teacher-learners. Yet we recognized that it is necessary to move beyond the responses teacher-learners provided for course assignments focused on developing skills and knowledge. Aligned with the finding of Pittaway and Dowden (2014) that personal experience with feedback influences teacher educators' written feedback, our feedback experiences motivated activity design and ways in which we structured feedback. Further, our views of mathematics teaching and learning motivated our feedback, as suggested by Buhagiar (2013), and in some cases interfered with our relationships with teacher-learners.

As MTEs seek to contribute to teacher-learners' relational practice by modeling, the conceptions of the learner should be a central factor. Yet as part of an assessment system, feedback can focus on task performance without attention to the particularities of the learner. With relational practice as a goal, moving beyond the development of skills and knowledge needed to complete tasks in the work of teaching, toward gathering insights about and ways to use views of mathematics teaching and learning in our practice provides a way forward. Written feedback seen through the lens of relational practice should include empathy and build from experiences of learners in an effort to meet course, program, and learner goals.

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REALIZATION OF A LANGUAGE-AS-RESOURCE ORIENTATION IN LANGUAGE IMMERSION MATHEMATICS CLASSROOMS

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Researchers have argued for an orientation to language as a resource that values bilingualism in mathematics classrooms. However, little is known about what mathematics teachers can do to translate a language-as-resource orientation into productive classroom practice. In this study, I analyze video data from two language immersion classrooms to understand pedagogies that are possible in contexts where bilingualism is seen as a resource. I argue that teachers' purposefully devised discursive practices used students' languages in ways that enhanced mathematical learning opportunities. I provide examples from the classrooms and discuss implications for research in bilingual classrooms where a language-as-problem orientation dominates.

Keywords: Classroom Discourse, Equity and Diversity

Mathematics teachers in bilingual classrooms deal with competing language-related orientations. While some orientations regard bilingualism as a problem to avoid or overcome, others regard them as a resource. Mathematics education researchers have drawn on what Ruíz (1984) called language-as-problem and language-as-resource (Planas, 2014; Planas & Civil, 2013; Setati, Molefe, & Langa, 2008). An orientation toward language as a problem creates a hierarchy, namely, one language dominates communication while devaluing other non-dominant languages. This orientation emphasizes lack of proficiency in the community's dominant language as a handicap and, ultimately, marginalizes users of non-dominant languages. In contrast, an orientation toward language as a resource questions language hierarchies by valuing and encouraging bilingualism. Mathematics education researchers have analyzed implications of a language-as-resource orientation to mathematics classrooms (Planas & Civil, 2013; Planas & Setati-Phakeng, 2014). These studies have focused on student-student interactions, describing the benefits of allowing students to speak in their preferred languages. Less is known about what else, besides allowing students to use more than one language, teachers can do.

In language immersion classrooms, learning a language other than the community's dominant language is regarded as useful, and learning mathematics in such language is seen as possible. Therefore, language immersion classrooms provide opportunities to observe discursive practices that are possible when teachers look pass limiting language orientations. In this study, I explore different ways in which teachers translate the language-as-resource orientation in which their classrooms are embedded into specific discursive practices. I argue that teachers purposefully devised discursive practices that used both students' languages in ways that enhanced students' mathematical learning opportunities. I ask the following research question: What discursive practices do teachers in language immersion classrooms enact to enhance mathematical learning opportunities? Understanding this question may illuminate what teachers in bilingual classrooms embedded in contexts that hold a language-as-problem orientation can do to disrupt restrictive language practices.

Theoretical Framework

Instead of regarding discourse as a stretch of speech, Peirce (1989) viewed discourse as guiding what is considered possible:

> Discourses, in a poststructuralist theory of language, are the complexes of signs and practices that organize social existence and social reproduction. In this view, a discourse delimits the range of

possible practices under its authority and organizes how these practices are realized in time and space. (pp. 403-404)

Besides regarding language as part of discourse, this view also sees it as a mechanism to create, reproduce and enforce discourse. Language delimits and is delimited by the practices that a discourse organizes as possible. Those who participate in such practices are privileged while those who do not are marginalized, since “when participants cannot find subject positions for themselves within a particular discourse, they may be silenced” (Peirce, 1989, p. 405). Peirce (1989) proposed that teachers engage in a *pedagogy of possibility*, that is, a pedagogy that encourages teachers to reconsider which language discourse practices are possible. For the purpose of this study, the notion of a pedagogy of possibility draws attention to creative teacher practices that are consistent with a language-as-resource orientation. Attending to instances of intentional productive use of languages may support teachers in bilingual mathematics classrooms to develop pedagogies that are consistent with a language-as-resource orientation.

Teachers’ regulatory role may silence students’ use of their preferred languages, potentially muting all contributions from particular students (Planas & Civil, 2013). Alternatively, teachers may enhance learning opportunities through the strategic and intentional use of students’ languages. Following previous research on language-as-resource in mathematics classrooms, I focused on learning opportunities: “By referring to learning opportunities, we avoid fundamental claims about whether or not mathematics learning has actually taken place, but instead put the focus on the opportunities for communication and participation created by learners as well as by teachers” (Planas & Civil, 2013). I define *bilingualism-as-enhancer* as the strategic use of more than one language to enhance learning opportunities.

Methodology

Two Spanish immersion classrooms participated in this research. One was a third-grade classroom, with 14 students from middle class families. The teacher, señora Abad, is a US-born Latina, who considers both English and Spanish as her native languages. The second was a second-grade classroom, with 23 students from economically struggling families. The teacher, Ms. Griffin, is a US-born Caucasian teacher, whose native language is English. Both teachers conducted mathematics class in Spanish. All students’ native language is English.

Data Sources

I draw on audio-recorded interviews with each teacher and mathematics class video recordings. While video-recording class lessons using a handheld camera, I focused on the teacher during whole class discussions, and then alternated focus on different groups during small group tasks. I video recorded eight lessons of a geometry unit in señora Abad’s class, and three lessons of a number sense unit in Ms. Griffin’s class. Unstructured interviews took place after video recordings. The teachers and I discussed the interplay between language and mathematics in video recorded segments. All interviews were fully transcribed.

Data Analysis

Following Powell, Francisco and Maher’s (2003) video analysis model, I annotated videos to identify focal episodes. I engaged in repeated attentive viewing, which included watching the videos three time to refine interpretations and redefined episodes I interpreted as *bilingualism-as-enhancer*. Then, for each teacher I selected one lesson that seemed to provide more examples than other lessons of the role of using more than one language in enhancing the teaching and learning of mathematics. This process resulted in 10 focal episodes: 6 from señora Abad’s class, and 4 from Ms. Griffin’s.

Second, each teacher and I analyzed the video of their lesson collaboratively. In-depth focus on one video is consistent with the study's purpose of illustrating possible discursive practices, without claiming that what I describe is an exhaustive list. The teachers and I discussed interpretations of the role bilingualism played in supporting mathematics learning opportunities. I transcribed these discussions and the focal video episodes. I added quotes from the discussions with teachers to specific parts of the video transcripts. Finally, following a constant comparative method (Glaser & Strauss, 1967), I coded discursive practices in the transcripts and identified emerging themes. I followed Young's (2008) view of discursive practice in language immersion contexts. This view attends to the interplay between orientations at the societal level and interactions at the classroom level: "The aim of discursive practice is to describe both the global context of action and the communicative resources that participants employ in local action" (p. 3). This definition resonates with this study's purpose of attending to societal language orientations—*language-as-resource* in particular—in relation to what teachers do to enhance mathematics learning opportunities.

Findings

There were two main teacher discursive practices that relate to bilingualism-as-enhancer: (1) choosing the language that more transparently represents a mathematical idea, and (2) supporting students' inference of mathematical terminology. In this section, I present an example from the classrooms and an interpretation of what motivated the practice that illustrates bilingualism-as-enhancer. The transcript conventions are: **Emphasis,** <Speaker slows down>, *Translation*, and --- silence.

Choosing the Language that Represents a Mathematical Idea more Transparently

As Ms. Griffin's class (second grade, Spanish immersion classroom) worked on a mystery numbers task, each student wrote a number that no one else could see. Each student wrote clues for the rest of the class to figure out the number. For one of the cards, Ms. Griffin read out loud one clue at a time. The teacher solicited guesses and explanations from students. After each clue, the class discussed whether they had enough clues to come up with one unique number. In the following example (see Table 1), Ms. Griffin had read the clue, "Tengo tres dígitos" (*I have three digits*). After discussing this clue, she read and wrote on the board the second clue, "Soy impar" (*I am odd*). She asked for examples of odd numbers. Ms. Griffin asked Dereck and Karen—two African American students who had started in the Spanish immersion program that academic year—what they thought.

Ms. Griffin and I interpreted this episode as an example of intentionally choosing the language that represents a mathematical concept in a more transparent way. The teacher tended to use English to support the participation of students new to the Spanish immersion program, like Karen and Dereck. In this episode, however, she chose Spanish to support students' understanding. She used the Spanish words par (*even*) and impar (*odd*) because she could relate one word with the other and with the mathematical concept with which the class was engaging. It seemed like rather than not knowing whether a number was even or odd, Karen and Dereck had a hard time remembering to what set of numbers the word *odd* refers and to what set of numbers the word *even* refers (lines 3-4 and 20-22). In English, the words even and odd have different etymologies, and they seemed arbitrary and unconnected. In Spanish, the words par and impar seemed related and more transparent as the prefix im indicates negation: not even.

We interpreted this example as bilingualism-as-enhancer because the use of the two languages helped students clarify and express a mathematical idea. Applied linguists have highlighted the role semantic transparency plays in communication (Bell & Schäfer, 2016). Semantic transparency refers to whether the form of a word makes its meaning explicit. For example, the word *shoemaker* can be considered transparent because parts of the word (shoe and maker) describe its meaning, and those parts are easily discernible. In Ms. Griffin's example, the words even and odd seemed opaque for students, whereas par and impar seemed transparent. Although there might be other ways to support students' sense making of the words and concepts of even and odd, in this case the possibility of alternating between the two languages enhanced the learning opportunity.

Table 1: Discussing Odd and Even Numbers

Line	Speaker	Spoken utterances	Translation	Actions
1	Ms. Griffin	Dereck, ¿**im**par dice even or	*Does* impar *say* even or	Underlying im
2		odd?	odd?	
3	Dereck	Are three, five and seven even		
4		or odd?		
5	Ms. Griffin	¿Qué dice par? ¿Alguien	What does even say? Can	
6		puede explicar esto?	someone explain that?	
7	Javier	Par tienes algo que puedes	*Even you have something*	Gestures separating her
8		dividir.	*you can divide*.	hands
9	Ms. Griffin	Sí. En números iguales	*Yes. In equal numbers*	Holds two fingers
10		La clave dice soy **im**par. Soy	*The clue says 'I'm odd.'*	
11		impar dice soy **no** par.	*I'm odd says I'm not even,*	
12		¿Qué número debe de	*In what number does it*	Pointing at last of the
13		**terminar** en el lugar de uno?	*have to end in the ones*	dashes that represent
14			*place?*	each of the three digits
15	Dereck	**Im**par	*Odd*	
16		**No** par	*Not even*	Shakes head
17		Three		
18	Ms. Griffin	So, si este termina en tres, ¿es	*So if this one ends in three,*	Writes 3 on the board
19		par o impar?	*is it even or odd?*	
20	Karen	One, three and seven are		
21		something, and two, four, and		
22		six are something.		
23	Ms. Griffin	Remember, par means	*Remember, even* means *you*	
24		<puedes dividir en números	*can divide into equal*	
25		iguales.>	*numbers.*	
26		<And **im**par means **no** par>	And *odd* means *not even*	
27		Is three par or **no** par, **im**par?	Is three *even* or not *even*,	
28			odd?	
29	Karen	So two is par, and one, three,		Miss Griffin writes
30		five, and nine are **im**par		numbers under Impar
31	Dereck	And seven		
32	Ms. Griffin	And seven		Writes 7 on the board
33	Dereck	And two, four and six are par		

Supporting the Inference of Mathematical Terminology

Señora Abad's class (third grade, Spanish immersion classroom) was finishing a geometry unit focused on classifying two-dimensional (2D) shapes according to the shapes' attributes. During the final lesson of the unit, the class focused on three-dimensional (3D) figures. Señora Abad wanted the class to draw on what they knew about 2D figures to name 3D figures. Her goal was to draw attention on 3D figures' attributes, how the names of the figures represented some of those attributes, and the relationship between the 2D figures and the 3D figures. After a class discussion comparing and contrasting 2D and 3D figures, señora Abad projected on the board an image of a pentagonal prism.

Table 2. Naming 3D Shapes

Speaker	Spoken utterances	Translation	Actions
Señora Abad	¿Cómo tú piensas que se	*How do you think this is*	Points at drawing of
	llama esto? Voy a darte un	*called? I'm going to give you*	pentagonal prism
	minuto para pensar.	*a minute to think. Maybe*	
	Tal vez algunos saben la	*some of you know the English*	Holds a pentagonal
	palabra en inglés. Tal vez	*word. Maybe you remember*	prism
	recuerdas algo que aprendiste	*something you learned in*	
	en figuras **bi**dimensionales.	*two-dimensional figures*	
	¿Frank?		
	Largo	*Long*	Moves hand
Frank			horizontally
	Es largo ¿verdad? ¿Qué más	*It's long, isn't it? What else*	Moves hand
Señora Abad	ves en esta figura?	*do you see?*	horizontally
	Cinco esquinas	*Five corners*	
Frank	Cinco lados	*Five sides*	
Gloria	Ah. Ves <cinco lados>	*Oh. You see <five sides>*	Points at 5 sides of
Señora Abad			the pentagonal face
	Cinco pirámide	*Five pyramid*	
Mike	Tiene cinco lados. OK. ¿En	*It's got five sides. OK. How is*	
Señora Abad	qué se parece a una pirámide	*it similar and how is it not*	
	y en que **no** se parece a una	*similar to a pyramid?*	
	pirámide?		
	No tiene triángulos	*It doesn't have any triangles*	
Mike	Tiene **pentágono**	*It has* ***pentagon***	
Gloria	Ah. Ah-ha.	*Ah. Aha.*	
Señora Abad	Pentágono pirámide	*Pentagon pyramid*	
Frank	Pentágono cilindro	*Pentagon cylinder*	
Ismael	Qué interesante. Tiene un	*How interesting. It has a*	Points at pentagonal
Señora Abad	pentágono	*pentagon*	face. Moves hand
	y es larga como un cilindro	*and it's long like a cylinder*	along one edge
	¿Cómo se dice prism?	*How do you say* prism*?*	
Gloria	Pentágono prismo	*Prism pentagon*	
Frank	**Pentagonal** prismo	*Prism* ***pentagonal***	
Gloria	¿Por qué?	*Why?*	
Señora Abad	Tiene un pentágono y es	*It has a pentagon and it's like*	
Erika	como un prismo	*a prism*	
	¿Es como un prisma?	*It's like a prism?*	Moves hands
Señora Abad	Como una prisma	*Like a prism*	horizontally
Erika			

Señora Abad and I interpreted this episode as an example of supporting the inference of mathematical terms and phrases. Instead of providing definitions or translations, the teacher encouraged students to experiment with language and come up with terms and phrases that made sense to them. Mathematical ideas and words students already knew informed their guesses. By making and explaining informed guesses, the class explored mathematical concepts such as the attributes and names of 3D shapes. Simultaneously, the teacher expected students to author and assess mathematical ideas, therefore enhancing students' mathematical agency. The teacher conjectured that if she had conducted the task in English, perhaps Gloria would have said that the shape was a pentagonal prism from the beginning. In that case, *pentagonal prism* might have been a memorized expression that some students in the class would not relate to the figures' attributes. Bilingualism seems to have motivated the teacher and the students' exploration of the figures' attributes and how the name of the figure represents those attributes.

We interpreted this example as bilingualism-as-enhancer because students are used to using language creatively to figure out how to express their ideas. In this classroom, the coinage of words emerged as students recurrently asked señora Abad how to say certain words in Spanish in different subjects. To raise students' linguistic awareness and autonomous use of language, señora Abad started to ask students to make informed guesses. Applied linguists refer to lexical inventions as expressions that look and sound like a word in the language, but that are not formally defined or used (Dewaele, 1998). For example, Frank's 'pentágono prismo' (line 33) sounds like Spanish, although the expression formally used in Spanish is 'prisma pentagonal'. In the example, the teacher intentionally promotes lexical invention as a strategy to reinvent mathematical ideas. The bilingualism in this classroom motivated the inference of mathematical terms and phrases, and the discussion about the connection between those expressions and specific concepts.

Discussion

Researchers have argued that an orientation of language-as-resource at the societal level plays out in classroom language use. Few studies, however, have explored teachers' role in translating a language-as-resource orientation into classroom practice. In this study, I have explored two language immersion mathematics teachers' discursive practices that seem consistent with a language as resource orientation. I have drawn on a pedagogy of possibility as a theoretical framework to focus attention on teachers' intentional efforts to devise creative ways to use languages in their mathematics classes.

I have proposed the notion of bilingualism-as-enhancer to foreground two related issues. First, bilingualism-as-enhancer focuses on the bilingual dimension of debates about language orientations. Second, it draws attention to the possibility of enhancing mathematics learning opportunities when teachers purposefully integrate mathematics and communication in more than one language. This purposeful integration requires a pedagogy of possibility in which teachers expand and explore possible discursive practices. I have focused on teachers, extending previous studies that have focused on bilingual mathematics students' language use.

I described two overlapping discursive practices that exemplify bilingualism-as-enhancer: (1) choosing the language that represents a mathematical idea more transparently, and (2) supporting students' inference of mathematical terminology. These discursive practices illustrate how teachers can enhance mathematics learning opportunities by drawing on students' languages. I have presented teachers' perspectives on the use of these practices to describe their reflective process of exploring their particular pedagogy of possibility.

This study contributes insights on how language orientations at the societal level may play out at the classroom level as discursive practices. Language orientations play a role in whether teachers enhance or silence particular languages. At the same time, classroom discursive practices also inform language orientations at the societal level. Examples of bilingualism-as-enhancer may inform teachers in contexts where bilingualism is regarded as a problem to explore the benefits of specific bilingual discursive practices, developing their own pedagogies of possibility.

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REVEALING LAYERED MATHEMATICAL LEARNING GOALS THROUGH AN EXAMINATION OF MINDSET

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This exploratory case study investigated the role of mindset in the establishment of an elementary teacher's mathematical learning goals at different layers of her classroom and curriculum. Data from the critical case of a teacher displaying characteristics of the growth mindset and engaging in the processes of teaching change provided evidence for a unique goal structure that could prove useful to a variety of mathematics teachers and mathematics teacher educators. This layered goal structure featured goals at global, trajectory, and content levels that provided opportunities for further operationalization of the teacher's mindset.

Keywords: Instructional Activities and Practices, Research Methods, Teacher Beliefs

Introduction

In bridging the perceived divide that exists among the theories, research, and practices of mathematics education, we must begin by addressing ideas that are shared and fundamental to each of these facets of the discipline in a way that leverages the unique strengths of each. One such idea involves the mathematical learning goals established by teachers as they interact with learners across the days, weeks, and months that constitute the academic year. As others have observed, this is perhaps the most fundamental idea shared between the unique faces of our discipline, as "until learning goals are expressed clearly, further analyses are impossible" (Hiebert, Morris, Berk, & Jansen, 2007, pp. 50-51).

A growing body of literature documents that a teacher's establishment and sharing of appropriate learning goals is an essential component of learning mathematics in the classroom (National Council of Teachers of Mathematics [NCTM], 2014). The specific challenges that are experienced by both teacher and learner are defined by these learning goals (Hiebert & Grouws, 2007), and the degree to which these challenges are overcome is greater in classrooms in which these goals are clearly and explicitly defined for all involved (Haystead & Marzano, 2009; NCTM, 2014). Despite the array of knowledge regarding the value of mathematical learning goals, their genesis and interaction with an individual's beliefs about the teaching and learning of mathematics are difficult to measure. Additionally, theories that describe these goals in terms of teaching are in notoriously "short supply" (Hiebert & Grouws, 2007, p. 373). The research reported here attempts to bridge this gap by examining the influence of a teacher's mindset, operationalized through the tenets of self-regulation theory, on the mathematical learning goals she pursues in the classroom. In doing so, it establishes a crossroads between two important theoretical perspectives and the daily operation of the elementary mathematics classroom and promotes the synergy among these factors that is the theme of this year's conference.

Purpose

A complete model of mathematics teacher development must describe the teachers' motivations and dispositions for their teaching as well as the influence of these factors on areas such as the teacher's implementation of learning activities, interactions with students and the classroom environment, and interpretations of professional development experiences (Opfer & Pedder, 2011; Wagner & French, 2010). The principal purpose of this study was to explore one of these motivational factors, the teacher's mindset, within the contexts of the teacher's professional development experiences and classroom practices. However, as the study evolved, it became apparent that the teacher's mindset linked inexorably to the mathematical learning goals she established in her classroom and the manner in which she operated on and monitored progress towards these goals. These premises led to the primary research question of the study reported here: How do characteristics of the growth mindset influence the mathematical learning goals established by a mathematics teacher as she engages in professional development and mathematics teaching? To address this question, the influence of mindset on the teacher's learning goals as she observed, interpreted, discussed, adapted, planned for, implemented, and reflected on a demonstration lesson was examined.

Theoretical Framework

The unique strength of the theoretical framework of a research study is in its ability to provide an underlying structure for the research that allows the researcher to anticipate, understand, and attempt to explain the phenomena under consideration. Additionally, this framework shapes the research design process, and in a single-case study supports the analytic generalizations that are the primary product of the research (Yin, 2014). With this perspective in mind, two important theoretical constructs guided this study: the model of implicit theories, commonly referred to as mindset, provided the primary theoretical lens for the study, while tenets of self-regulation theory operationalized these mindset constructs. The remainder of this section contains descriptions of these theoretical elements.

The Model of Implicit Theories

Situated in their prior research on goal orientation and behavior, Dweck and Leggett (1988) described the social-cognitive model of motivation and personality that developed into the implicit theories framework. In this model, the authors posited that an individual's implicit assumptions about the nature of an ability lead directly to the type of goals he pursues regarding that ability and the behaviors he exhibits when faced with challenges to that ability (Dweck & Leggett, 1988; Dweck, Chiu, & Hong, 1995). These mindsets and their associated goal pursuits thus created "a framework for interpreting and responding to events" (Dweck & Leggett, 1988, p. 260) that promoted observable behavioral patterns when the ability under consideration is challenged. Two implicit theories, the *entity theory* and *incremental theory*, exist in this model.

Incremental theories. The model described individuals espousing an incremental theory as those who view attributes as malleable, with the potential for the related ability to grow over time. Subscribers to this *growth mindset* often establish learning goals that focused on improvement of the ability in question (Dweck, 1986; Dweck & Leggett, 1988; Elliott & Dweck, 1988). When faced with challenging situations related to this ability, individuals with growth mindset characteristics display adaptive, mastery-oriented responses characterized by engagement with the challenges and resilience to failure (Elliott & Dweck, 1988).

Entity theories. Individuals assuming an entity theory tended to view attributes as fixed, uncontrollable entities, for which ability depended on factors over which the individual had no control. Those with these *fixed mindset* characteristics adopted performance-oriented goals to gain positive judgments for skills they had already mastered or to avoid negative judgments regarding talents they had yet to acquire (Dweck, 1986; Dweck & Leggett, 1988; Elliott & Dweck, 1988). When faced with challenges, these individuals displayed helpless responses characterized by lowered performance and avoidance of challenges (Elliott & Dweck, 1988).

Generalization of the model. Although the tenets of implicit theory advanced through research regarding characterization of an individual's own intelligence (Dweck & Leggett, 1988), the authors soon generalized the model to other attributes and domains. Additionally, they predicted that for any attribute of personal significance, "viewing it as a fixed trait will lead to a desire to document the adequacy of that trait, whereas viewing it as a malleable quality will foster a desire to develop that quality" (Dweck & Leggett, 1988, p. 266). Applications of this prediction culminated in the validation of a simple instrument used to assess an individual's implicit theories for a variety of attributes (Dweck et al., 1995). Additional evidence supported the notion that the model holds for generalization to other traits, such as the character and attributes of other people (Erdley, & Dweck, 1993), or mathematical ability (Lischka, Barlow, Willingham, Hartland, & Stephens, 2015; Willingham, 2016; Rattan, Good, & Dweck, 2012).

Self-Regulation Theory and Mediated Pathways of the Growth Mindset

Burnette, O'Boyle, VanEpps, Pollack, & Finkel (2013) conducted a large-scale meta-analysis examining the relationship between implicit theories and self-regulation theory. Their analysis revealed a strong alignment between three key processes of self-regulation theory and the constructs of mindset. Goal-setting processes encompassed the performance versus learning goal orientations; goal-operation processes associated helpless and mastery responses; and goal-monitoring processes aligned the dichotomy of expectations of success and negative emotional responses. More specifically, the results described the relative strength of association for these mediators of incremental theory on an individual's goal achievement across a wide range of abilities, disciplines, and context (Burnette et al., 2013).

These findings aligned with prior research showing that an incremental theory regarding an ability associated with an affinity for learning goals, mastery strategies, and expectations of success regarding that ability, and negatively associated with the pursuit of performance goals, helpless responses, and negative emotions regarding that ability (e.g., Blackwell, Trzesniewski, & Dweck, 2007; Dweck & Leggett, 1988; Dweck et al., 1995). However, this analysis also revealed significant findings regarding the positive strength of association between mastery-oriented responses and expectations of success with goal achievement and the negative relationship between negative emotions regarding an ability and goal achievement. The most significant findings resulted from considering the mediated paths between an incremental theory and goal achievement. For instance, examination of these pathways revealed that the incremental theory's avoidance of negative emotions is more strongly associated with goal achievement than its expectations of success (Burnette et al., 2013).

Methodology

The unique strength of a well-designed qualitative research study is in its ability to answer specific research questions while at the same time testing the boundaries of those questions to discover significant relationships as the research evolves (Yin, 2014). When guided by a well-grounded theoretical framework, this approach allows important and perhaps unanticipated findings to emerge. The remainder of this section describes the design and contexts of the study, which allowed the meaningful goal structures found in the participant's classroom to surface.

The Study's Design

Overview. An exploratory, holistic single-case design (Yin, 2014) supported consideration of how characteristics of the growth mindset influenced an elementary teacher's mathematical learning goals. The study focused on Gale Martin, a second-grade teacher, deemed significant as evidence from her engagement in an ongoing professional development project, Project Influence, indicated that she represented the critical case of a teacher displaying strong growth mindset characteristics and engaging in the processes of teaching change. Over the course of four months of the fall of 2015, the researcher interview and observed Ms. Martin as she engaged in activities including the planning, delivery, and assessment of classroom instruction, classroom teaching, assessment, a demonstration lesson, and other professional development activities.

Data collection. The data collected throughout the study focused on how Ms. Martin's mindset characteristics influenced her other experiences, beliefs, and practices. More specifically, the researcher collected data regarding Ms. Martin's mindset and beliefs regarding the teaching and learning of mathematics, her described and observed mathematics teaching practices, and her adaptation of a demonstration lesson for use in her classroom. Data sources for the study included historical records of her mindset and beliefs, semi-structured interviews, classroom observations, artifacts of the observed and enacted demonstration lessons, reflective journal entries, and artifacts from Ms. Martin's lessons. The researcher collected this data across four stages, including a participant selection process, baseline classroom observations, Ms. Martin's engagement in the demonstration lesson through her professional development project, and her adaptation and enactment of the demonstration lesson in her own classroom.

Data analysis. The researcher then organized this body of data in chronological fashion, corresponding approximately with the data collection stages described above, and completed a simple time series analysis (Yin, 2014). A holistic analysis of themes, "not for generalizing beyond the case, but for understanding the complexity of the case" (Creswell, 2012, p. 101), was performed for the first stage of data through open coding and reduction of these codes into themes consistent with the theoretical framework. The themes emerging from the stage one analysis guided interpretation and coding of the stage two data, after which the stage one codes were revisited for completion. The researcher repeated this process through all four stages of data in order to produce a comprehensive set of themes to guide a written case description.

Research Contexts

The participant. The researcher selected Ms. Gale Martin, a Caucasian female in her mid-thirties, as the critical case for the study. Rationales for this selection included historical survey data indicating persistent growth mindset characteristics, a positive record of changes in beliefs regarding the teaching and learning of mathematics, and observational records indicating a change in classroom teaching practices consistent with the mindset and belief data. Ms. Martin was an elementary mathematics teacher in her second year of teaching second grade and her fifteenth year of teaching elementary school who taught in a rural elementary school of approximately 330 students in a southeastern state. Prior to teaching second grade, Ms. Martin had taught one year of kindergarten, two years of third grade, and 10 years of fourth grade, providing her some perspective in the mathematical content requirements of several elementary grades. During the course of this study, Ms. Martin was also engaged in her third year of ongoing professional development for mathematics teaching.

The demonstration lesson. As part of the study, Ms. Martin observed a second-grade demonstration lesson conducted by the faculty of Project Influence with a lesson goal of "engaging students in thinking about subtraction with regrouping, while potentially representing the process symbolically" (Demonstration Lesson, October 28, 2015). The lesson involved students interacting with the following task in a problem-solving format.

> On Thursday, Tara was at home representing numbers with base-ten blocks. The value of her blocks was 304. When she wasn't looking, her little brother grabbed two longs and a flat. What is the value of Tara's remaining blocks? Use pictures, words, and/or symbols to describe how you solved the problem.

During the lesson, students worked in pairs to solve the problem and participated in extensive mathematical discussions across pairs, small groups, and the whole group under guidance of Project Influence's expert teacher. Approximately 30 kindergarten to second grade teachers observed the demonstration lesson, and demographically, the county, school, and class were extremely similar to those of Ms. Martin.

Results

The unique strength of a teacher's classroom practices exists in their ability to mediate the outcomes of students' learning (Hiebert & Grouws, 2007). During the course of this study, Ms. Martin's students consistently displayed a sense of ownership of their mathematical ideas, communicated their thinking about these ideas to one another and Ms. Martin, explored relationships across their curriculum, and achieved lesson goals with varying degrees of success. This section contains a portion of the results of this study related to different goal pursuits Ms. Martin utilized to facilitate these outcomes.

Overarching Goals

Ms. Martin professed to believe that a focus on student thinking and ownership of their mathematical ideas was one of the most essential elements of her classroom environment, and one that allowed students to learn mathematics effectively. She described her thoughts on the matter, referring to how a student from the previous year's class was ultimately able to be successful in learning mathematics in her class.

> [The student] was able to share her ideas, because she heard somebody else share their ideas. So just building that community of "I am my own person in here, and that's okay. I can show you how I know something. It's just not Ms. Martin's way. I can have my way. So and so can have their way. Yeah, Ms. Martin shows us things, but if I don't see it the way Ms. Martin shows us, then I can still use my way." (Selection Interview, September 9, 2015)

Ms. Martin credited this student's willingness to value and share her own ideas to the fact that she first heard another student share their thinking and the community norms of her classroom.

This relationship between students' willingness to communicate their ideas and their increasing level of understanding of mathematics appeared to be one of the motivating beliefs behind much of Ms. Martin's thoughts regarding the teaching and learning mathematics. She spoke directly to the notion that understanding mathematics and communicating about mathematical ideas were inescapably linked.

> That's why I always tell them, "If you can explain, you can go home and teach mom or teach brother or come to me and show me and explain it in your own words, I think that's how you understand it." That's why I like for them to do a lot of talking, obviously, because I want them to share their ideas with each other and understand it, especially in kid terms. Because there are times when I have said something and a kid will say it differently, and I feel like we've said it the same way. If a kid says it, [other] kids are like, "Oh yeah, that makes total sense." (Selection Interview, September 9, 2015)

It appeared that Ms. Martin believed that this focus on student thinking and communication was essential to student learning and could help to reconcile differences between adult's and children's ways of thinking about and describing mathematics.

Lesson Goals Within a Learning Trajectory

Ms. Martin also spoke explicitly about her learning goals for her implementation of the demonstration lesson and how she believed these goals related to her students' past and future study of mathematics. In describing these learning goals, she referred directly to her students' current understanding of strategies for operating with two-digit numbers, a topic which directly preceded her implementation of the demonstration lesson.

> I would think that my students need to be able to represent the two-digit number in various ways to subtract. . . . I feel like the goals [for the demonstration lesson] are kind of the same, we're just kind of changing, we're moving from two-digit numbers to three-digit numbers. I feel like our goal is the same, can they look at these numbers and understand that I need to maybe represent the number in a different way in order for me to subtract. I guess that really works for this whole unit. (Planning Interview, November 10, 2015)

In this quotation, Ms. Martin established the specific mathematical goals for her enactment of the demonstration lesson as extending students' ability to represent numbers in a variety of ways into using these representations with purpose in the form of regrouping for subtraction. This was a substantial change from the goal of the lesson she originally observed.

Additionally, she described how this goal connected to her students' recent areas of study and her future goals for the semester.

> To me, it's definitely understanding that not only can we represent numbers in different ways, but we can do that when we are subtracting as well. I see where we were several weeks ago where we were really focused on the place value and representing those numbers in different ways. I really hope that they can see the tie-in with that to the subtraction. I really think if they make that big connection, then once we get to the algorithm of regrouping this will be no issue. (Planning Interview, November 10, 2015)

With these words Ms. Martin confirmed the goals described in this section and explained the connections among these goals, her emphasis early in the semester on representations of number and place value, and the future objective of having her students symbolically represent operations and utilize algorithms.

Reinforcing Goals Within a Lesson

The previous section presented explicit examples of Ms. Martin's lesson goals and her willingness to change the goals of a lesson she had observed to suit her lesson and trajectory needs. In addition to these planned changes, Ms. Martin also displayed the ability to adjust her classroom instruction to reinforce the goals she had established for her students. In closing her implementation of the demonstration lesson, Ms. Martin placed a final question on the board, which she read aloud with her class.

> All right, here is your last question. Let's read it together. Guys, listen, we've got five minutes, and this is our last thing. Ready? [Read aloud and recorded on the whiteboard] How could you represent 407 so that 3 longs could be taken away? Write a sentence explaining how you know. . . . You have to write a sentence, but I don't mind if you use a drawing to show it as well. (Classroom Observation, November 13, 2015).

As the demonstration lesson Ms. Martin observed ended with students reflecting on what they had learned from solving the original task, this extension question provided a substantially different close to the lesson that was more aligned with Ms. Martin's lesson goals.

In an interview after this lesson, Ms. Martin's overall assessment of the lesson was that it had been successful and that the majority of students had developed further understanding of the need to represent numbers in different ways in order to support operations such as subtraction.

> I'd say three-fourths of the classroom. . . were [able to address the exit ticket]. I could pick out maybe five or six that were not. I think, for the most part, they were able to. . . . They understood that you had to represent a number a different way in order for them to subtract, which leads to

the whole idea of the algorithm and regrouping. I feel like the trajectory is on target and moving towards the overall goal. (Reflection Interview, December 21, 2015)

This description further illustrates Ms. Martin's commitment to her lesson goal and her ability to adapt her instruction to continue to reinforce her student's progress towards this goal based on ongoing classroom assessment.

Discussion and Conclusion

Utilizing the self-regulation constructs of goal setting, goal operating, and goal monitoring to observe Ms. Martin's operationalization of her mindset in the classroom proved extremely effective. Perhaps one of the most important results of this framework was the revelation of three distinct layers of goals under which Ms. Martin operated throughout the semester. Although hierarchical language describes these goal structures in the following paragraph, this language relates only to the relationships among the goals themselves and not the value Ms. Martin ascribed to these goals as she spoken of them with equal importance (see Figure 1).

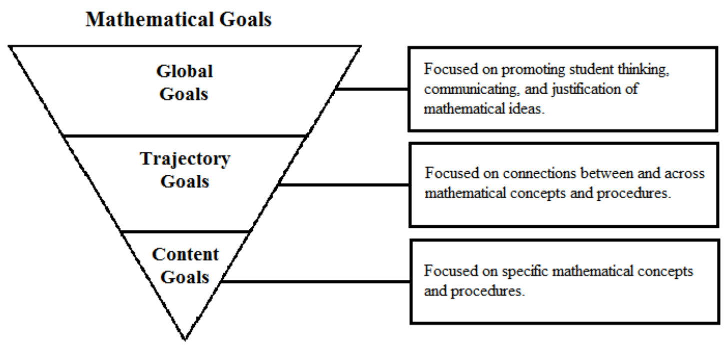

Figure 1. Ms. Martin's Layered Mathematical Learning Goals

At the highest layer, Ms. Martin established *global goals* that spanned the length of the school year. These goals tended to focus on widely applicable student mathematical practices such as thinking independently about mathematical ideas, communicating these ideas to others, justifying this thinking, and critiquing the reasoning of others. In an intermediate tier, Ms. Martin described *trajectory goals* that involved assessing and moving students along an evolving mathematical trajectory by helping them connect various mathematical concepts and representations throughout the semester. She spoke of these goals at the level of sequences of lessons and classroom activities, units of instruction, and conceptually related mathematical topics. These goals appeared to be more fluid than the global goals and evolved as the semester progressed based on her students' current understanding. At the lowest level were Ms. Martin's *content goals*, which aligned roughly with her learning goals within a lesson or brief sequence of lessons and could be adapted as needed within an individual lesson or student encounter.

These layered goals also offered a variety of opportunities to observe Ms. Martin engaging in goal operating and goal monitoring practices on a daily basis. In general, her goal operating practices aligned with utilizing specific instructional strategies to advance individual students, small groups, or the whole class towards her goals for them at different layers. Additionally, she focused heavily on her students' use of mastery strategies throughout her interactions with them. Ms. Martin's general goal-monitoring strategies focused on making students' mathematical thinking visible to her and the students' peers. She then used this thinking as evidence for assessment as she compared students' progress to her learning goals, for facilitation of group discussions, or for discussion and critique from the students' peers.

The most directly useful aspect of Ms. Martin's case for the classroom mathematics teacher is likely the fashion in which she operationalized her mindset through her goal-related practices. Although all classroom teachers may not operate under the tenets of a growth mindset on a daily basis, many of these goal-related practices can be easily adapted to any classroom. Setting goals that support student interactions about mathematics and that focus on mathematical concepts and strategies that transfer are broadly useful. Operating toward these goals by interacting with students via advancing, redirecting, and facilitating strategies appears to require little adaptation to the questioning approaches many teachers already use. Goal-monitoring practices such as focusing on student thinking and evaluating student progress against a mathematical learning trajectory align well with globally accepted assessment practices. These findings, which are useful to a variety of elementary mathematics teachers and mathematics teacher educators, represent the intersection the study's theoretical framework, research design, and observations of classroom practice, and would not be possible without the convergence of these facets.

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TEACHERS' RESPONSES TO A COMMON SET OF HIGH POTENTIAL INSTANCES OF STUDENT MATHEMATICAL THINKING

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This study investigates teacher responses to a common set of high potential instances of student mathematical thinking to better understand the role of the teacher in shaping meaningful mathematical discourse in their classrooms. Teacher responses were coded using a scheme that disentangles the teacher move from other aspects of the teacher response, including who the response is directed to and the degree to which the student thinking is honored. Teachers tended to direct their response to the student who had shared their thinking and to explicitly incorporate ideas core to the student thinking in their response. We consider the nature of these responses in relation to principles of productive use of student mathematical thinking.

Keywords: Classroom Discourse, Instructional Activities and Practices

Recommendations for effective mathematics teaching stress the importance of engaging students in meaningful mathematical discourse (e.g., National Council of Teachers of Mathematics [NCTM], 2014). Research has begun to help us understand how to effectively orchestrate discourse around written records of student work (e.g., Stein, Engle, Smith, & Hughes, 2008), but much less is known about how to effectively use the in-the-moment mathematical thinking that emerges during classroom mathematics discourse. One issue related to responding to student thinking is that not all student thinking warrants the same consideration. Rather, student thinking varies in the degree to which it provides leverage for accomplishing mathematical goals. Leatham, Peterson, Stockero, and Van Zoest (2015) described a framework to identify those instances of student thinking—MOSTs—that provide such leverage, but little is known, as of yet, about effective responses to MOSTs. The study reported here investigated teacher responses to a common set of MOSTs. Better understanding such responses will contribute to better understanding the role of the teacher in shaping meaningful mathematical discourse in their classrooms.

Literature Review

Research on classroom discourse has identified patterns in teachers' responses to student thinking. Mehan (1979) coined *IRE*—Initiation, Response, Evaluation—to describe a common pattern of classroom interaction where the teacher's main follow-up to an elicited student response is to evaluate it. An IRE interaction is an example of what Wood (1998) referred to as *funneling,* where the teacher's response is intended to corral students' thinking within predetermined and often narrowly-defined parameters. By contrast, Wood characterized certain other teacher responses as *focusing*; in these responses a teacher "keep[s] attention focused on the discriminating aspects of the solution" (p. 175).

Van Zee and Minstrell (1997) explored what they called a *reflective toss*—a pattern that consists of a student statement, teacher question, and additional student statements. Van Zee and Minstrell argued that changing the evaluation component of IRE to a question could positively impact the nature of classroom discourse by changing students' expectations for participation. These results are not unique; in general, research has found that teacher responses matter. Fennema et al. (1996) found that increases in teachers' focus on student thinking in their classrooms were directly related to improvements in their students' achievement. Kazemi and Stipek's (2001) investigation revealed that teachers in high-press classrooms—classrooms in which the teacher responded to their students' contributions to classroom discourse by pressing the students to further engage in thinking about important mathematics in their contributions—provided their students with increased learning opportunities.

Other researchers have looked at collections of teacher moves that accomplish a particular purpose related to student thinking. Lineback (2015), for example, investigated the construct of *redirection*—"instances when a teacher invites students to shift or *redirect* their attention to a new locus" (p. 419). This work generated a taxonomy of redirections to deconstruct teacher responses and analyze the contribution of different redirection responses to instruction. Bishop, Hardison, and Przybyla-Kuchek (2016) described the mathematical contributions of students, the moves teacher made in response, and the relationship between these contributions and moves, through the lens of *responsiveness*, which they defined as the extent to which teacher responses "mutually acknowledge, take up, and reflect an awareness of student thinking" (p. 1173). Connor, Singletary, Smith, Wagner, and Francisco (2014) developed a framework that includes teacher responses to student thinking that support collective argumentation in the classroom. Their work provides important information for focusing on a particular type of student thinking—that which involves mathematical argumentation.

In the work reported here, we narrow down the type of student thinking to MOSTs and consider the extent to which the teacher responses to those MOSTs accomplish the purpose of *building* on them.

Theoretical Framework

MOSTs (Leatham et al., 2015) are instances of student thinking worth *building* on—that is, "student thinking worth making the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36). To take full advantage of these opportune instances of student thinking, one would want to seek to build on MOSTs *in the moment*. Such use encapsulates the core ideas of current thinking about effective teaching and learning of mathematics (e.g., NCTM, 2014), including that student mathematics is at the forefront and that students are positioned as legitimate mathematical thinkers, engaged in sense making, and working collaboratively. These ideas serve as the principles underlying our conceptualization of productive use of MOSTs (see Figure 1).

1. The mathematics of the MOST is at the forefront.
2. Students are positioned as legitimate mathematical thinkers.
3. Students are engaged in sense making.
4. Students are working collaboratively.

Figure 1. Principles underlying productive use of MOSTs (Van Zoest et al., 2016).

We theorize that building on MOSTs is a particularly productive way for teachers to engage students in meaningful mathematical learning. Van Zoest, Peterson, Leatham, & Stockero (2016) put forth a conceptualization of the teaching practice of building on MOSTs (see Figure 2). Together the principles (see Figure 1) and building subpractices (see Figure 2) provide a way to assess the extent to which teacher responses to MOSTs instantiate the practice of building.

1. Make the object of consideration clear (make precise)
2. Turn the object of consideration over to the students with parameters that put them in a sense-making situation (grapple toss)
3. Orchestrate a whole-class discussion in which students collaboratively make sense of the object of consideration (orchestrate)
4. Facilitate the extraction and articulation of the mathematical point of the object of consideration (make explicit)

Figure 2. Sequence of subpractices of the teaching practice of building on MOSTs.

Methodology

The Scenario Interview (Stockero et al., 2015) is a tool to investigate how teachers think about responding to student thinking during instruction. During the interview teachers are presented with instances of mathematical thinking from eight individual students—four each from an algebra and a geometry context. The interviewee is situated as the teacher and asked to describe what they might do next were the instance to occur in their mathematics classroom and to explain why they would respond in that way. The Scenario Interview allowed us to compare teacher responses to a common set of student thinking. The analysis reported here focuses on responses to the four instances, two from each context, in which the student thinking was a MOST. The four MOSTs and their contexts are provided in Figure 3.

Scenario	Context	MOST
G1	Students were sharing their solutions to the following task (a corresponding picture was on the board). *Given two concentric circles, radii 5cm and 3cm, what is the area of the band between the circles?*	Chris shared his solution: "The radius of the big circle is 5 and the radius of the little circle is 3, so the gap is 2, so the area of the band is 4π cm^2."
G3		Pat explained how he got the same answer as Chris (4π cm^2) a different way: "π times r^2 for the big circle is π times 5^2, which is 10π and π times 3^2 is 6π for the little circle. I minused (sic) them and got 4π as my answer."
A2	Students had been discussing the following task and had come up with the equation $y = 10x + 25$. *Jenny received $25 for her birthday that she deposited into a savings account. She has a babysitting job that pays $10 per week, which she deposits into her account each week. Write an equation that she can use to predict how much she will have saved after any number of weeks.*	Casey said, "You could also change the story so the number in front of the x is negative."
A3	The teacher asked, "How do we find the equation given any table?" and put this generic table of values [to the right] on the board for the students to use in their explanation. X | Y 0 | 15 2 | 19 3 | 21 5 | 25	Jamie said, "I found the number in front of the x by subtracting the y-values in the table, 21 - 19, so that number is 2."

Figure 3. MOSTs that formed the basis of the teacher responses and their contexts.

Data Analysis

The data for this study consisted of video recorded interviews with 25 secondary school mathematics teachers from several sites across the United States. These teachers were representative of a set of 44 teachers who participated in our larger project. We used Studiocode (SportsTec, 1997-

2015) video analysis software to segment each interview into the instances of student thinking and the teacher responses to each individual instance—everything a teacher said about how they would respond to that instance. Transcriptions of the videos were used to facilitate the analysis. For the 4 instances and 25 teachers of this study, there were a total of 100 teacher responses. In one of those responses the teacher did not provide a description of how they would respond to the instance because they were not able to envision it happening in their classroom, thus 99 teacher responses were analyzed for this study.

The resulting teacher responses were then coded using the *Teacher Response Coding Scheme (TRC)* (Peterson et al., in press), a scheme that disentangles the teacher move from other aspects of the teacher response, including the Actor and the degree to which the student thinking is honored (Recognition-Action and Recognition-Idea). Figure 4 provides the TRC coding categories and codes that were included in this analysis.

Category	Coding Category Description	Codes
Actor	Who is publicly asked to consider the student thinking	teacher, same student(s), other student(s), whole class
Recognition-Action	The degree to which the teacher response uses the student action, either verbal (words) or non-verbal (gestures or work)	explicit, implicit, or not
Recognition-Idea	The extent to which the student is likely to recognize their idea in the teacher response	core, peripheral, other, cannot infer, not applicable
Move	What the actor is doing or being asked to do with respect to the instance of student thinking	adjourn, allow, check-in, clarify, collect, connect, correct, develop, dismiss, evaluate, justify, literal, repeat, validate

Figure 4. Subset of the *Teacher Response Coding Scheme* (*TRC*) used in this paper.

Results and Discussion

We discuss findings related to specific aspects of teachers' responses to MOSTs as well as interactions among those aspects. We first focus on the Actor and Move and their interactions, followed by the individual Recognition categories and their interactions. In doing so, we highlight how a response might adhere to the principles underlying productive use of MOSTs or contribute to enacting subpractices of building.

Actor and Move

With respect to the actor, the majority of teacher responses (66%) had the *same student* as the actor, meaning that the teacher proposed a move that was directed back to the student who had contributed the original thinking (see Table 1). In about 24% of the instances, the teacher move was directed to the *whole class*.

With respect to the moves, two occurred much more frequently than the others; together, *develop* (37%) and *justify* (18%) moves accounted for over half of the data. In a develop move, the teacher provides or asks for an expansion of the student thinking that goes beyond a simple clarification. In a justify move, the teacher asks for or provides a justification of the instance. Since our data showed that these moves had a *same student* or *whole class* actor, in both cases the teacher was asking for, rather than providing, the expansion or justification.

Table 1: Actor and Move

	Same Student	Whole Class	Teacher	Other Student(s)	TOTAL
Adjourn	0	0	3	0	3
Allow	0	3	2	1	6
Clarify	5	0	0	0	5
Collect	2	4	0	1	7
Connect	1	4	0	1	6
Correct	1	0	0	0	1
Develop	32	5	0	0	37
Dismiss	0	0	1	0	1
Evaluate	0	4	0	0	4
Justify	16	2	0	0	18
Literal	4	2	0	0	6
Repeat	4	0	0	1	5
TOTAL	65	24	6	4	99

Taken together, the Actor and Move findings suggest that teachers might instinctively respond to MOSTs by asking the student who provided the thinking to either expand upon or justify their idea. Because MOSTs are instances that a teacher can build upon to “engage the class in making sense of [student] thinking to better understand an important mathematical idea” (Van Zoest et al., 2017, p. 36), however, asking the student to develop or justify their idea may not always be necessary and may actually limit students’ opportunities to make sense of mathematical ideas. For example, consider scenario A2 (see Figure 3). It turned out that nearly half of the instances of *develop* moves with *same student* actor (15 of 32) occurred in response to this scenario. The most common teacher move in this instance was to ask Casey, the student who made the suggestion, to explain how they would change the story (e.g., “Well what do you mean? What sort of an equation, or what sort of a real life situation can you think of where that would be a negative?” (Teacher 6 [T6]). Contrast this response with a similar one directed instead to the whole class: “Interesting comment… who can come up with a story, a situation that would match what Casey is saying?” (T7). In this case, we would argue that directing the response to the whole class might be more productive, as it would engage all of the students in trying to come up with a situation where the coefficient is negative, likely advancing the entire class’s understanding of the mathematics of linear equations.

Similarly, consider scenario A3. More than two-thirds of the *justify* moves with *same student* actor (11 of 16) occurred in response to this scenario. The most common response to this instance was to ask Jamie why they used the numbers that they did (e.g., “Why did you do the 21 minus the 19? Why didn’t you do the 19 minus the 15?” (T14)). This response would allow Jamie to justify their idea, but does not engage the whole class in thinking about the importance of taking into account the differences between x-values as well as the y-values when calculating the rate of change. Consider an alternate response directed to the whole class, such as: “So [Jamie] got 2 from subtracting those two numbers, so what if I pick 19 and 15? If I subtract those, I get 4. Why did we get two different answers?" (T21). Such a response would allow all of the students to consider the mathematics of rate of change. We argue that teachers who respond to MOSTs by asking the student who shared the original thinking for justification may be focused on the details of the situation, whereas those who ask the whole class for justification may be more focused on the big mathematical picture.

In general, responses that turn the mathematics of a MOST over to the whole class instead of engaging a single student better adhere to principles underlying productive use of MOSTs. Such responses provide all students the opportunity to collaboratively engage in making sense of the mathematics of the MOST. In doing so, they put the students' mathematics at the forefront and position all students as legitimate mathematical thinkers. These responses may also demonstrate an ability to discriminate between those instances that need to be made precise before the teacher can turn them over to students (grapple toss) and those that do not.

Although the goal of building on MOSTs is to have the whole class consider the student mathematics of the instance, there are some cases where directing the initial teacher response back to the same student might be desirable. For example, in scenario G1, it is quite possible that other students in the class would not initially understand Chris' explanation, so the most common teacher response in our data, "ask him to explain by using…pictures and words, like how he came up with the 4π" (T18) may be the teacher helping to make Chris' idea precise before other students are asked to consider it. A move such as this could be an instantiation of the first subpractice of building (make precise)—an important first step in setting the teacher up to engage in the next building subpractice (grapple toss), in which they turn the now-precise student thinking over to the class for consideration.

Recognition of Student Actions and Ideas

The Recognition codes operationalize the extent to which the student who provided the instance would recognize their thinking in the teacher's response. As seen in Table 2, the majority of teacher responses either *explicitly* (54%) or *implicitly* (32%) incorporated the student's words (verbal) or gestures or work (non-verbal). Only 13% of responses would likely *not* be recognizable to the student as incorporating their own actions. Moreover, the vast majority of the responses (75%) remained *core* to the idea in the instance of student thinking. Together the results indicate that a large percentage of the teacher responses were both *explicit* and *core* (43%), meaning that the teachers in this study honored the student thinking by explicitly incorporating the student's verbal or non-verbal actions and staying focused on the student's core ideas in their described response. For example, the response to scenario G1, "I would want to know what he means by gap. Um, and maybe have him illustrate that visually, just to kind of picture that as a class," (T4) is *explicit* and *core* as it incorporates both the student's words (gap) and his ideas (having him illustrate his idea visually). A response such as this aligns with the principles underlying productive use of MOSTs, as it positions the student as a legitimate mathematical thinker by keeping the students' mathematics at the forefront. In general, many teacher responses that are *core* to the student ideas and *implicitly* incorporate student actions also adhere to the same principles, but may be problematic in that it may not be clear to the student(s) what mathematics is under consideration. For example, the response to scenario A3, "So I would want to ask her, 'Why did you do this? What are you thinking? Tell us a little bit more,'" (T24) fails to specify what mathematics the teacher wants to know more about. Among other things, the teacher could be wondering why the student subtracted or why they chose to select the numbers that they did.

Table 2: Recognition of Student Actions and Ideas

		Student Ideas			
		Core	Peripheral	CNI, Other, N/A	TOTAL
Student Actions	Explicit	43	10	1	54
	Implicit	26	4	2	32
	Not	5	1	7	13
	TOTAL	74	15	10	99

Conclusion

Our findings revealed that the teachers in this study most often responded to MOSTs by making a *develop* or *justify* move that was directed to the *same student* who had shared the initial thinking. Additionally, they did so in ways that stayed *core* to the ideas in the student thinking and often *explicitly* incorporated the students' actions.

Responses that either explicitly or implicitly incorporate core ideas of a student's contribution signal that these teachers value the students' contributions. We also see such responses positioning the students as legitimate mathematical thinkers who can make valid contributions to the development of the mathematics in the classroom. Hence the words and idea(s) teachers use in their responses to students' ideas could matter in terms of how students are positioned in the classroom. When the student action that is being considered is explicit, it is easier for the whole class to recognize that student thinking is being honored.

MOSTs are prime opportunities for teachers to enact the building practice, but teachers' tendencies to direct their responses to the student who had shared their idea could prevent them from doing so. As we have illustrated, directing a response to the same student could be productive in cases where the student's idea needs to be made precise before others can consider the idea, but many MOSTs do not require clarification. In these instances, rather than going back to the student, it would be more productive to *toss* the already precise student thinking to the whole class to provide all students an opportunity to collaboratively make sense of the mathematics.

The findings of this study advance research on teachers' in-the-moment responses to student mathematical thinking by moving beyond looking at what moves teachers make, to considering to whom those moves are directed and to what extent those moves would allow students' ideas to be recognizable to them or other students. In doing so, the study builds on the approaches taken in past research on teacher responses to explore more refined approaches that allow the field to look at teacher responses in new ways. Decomposing teacher responses in the way we have in this study has the potential to help teacher educators and researchers focus their development efforts. For example, if the majority of a teacher's responses honor student thinking, but engage only the student who contributed the instance, professional development work with the teacher could focus specifically on understanding the potential in directing a response to the whole class, and when it would and would not be appropriate to do so. Focused efforts such as this would allow professional developers to leverage teachers' strengths and thus develop teachers' practice more effectively.

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TOWARD MULTIMODAL POETIC ANALYSIS: A CASE OF PROPERTY NOTICING

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This paper uses multimodal discourse analysis to show that discursive form, in addition to words, gestures and sound dimensions of speech, is an important linguistic resource for expressing mathematical meaning. During a collaborative task, a student spoke an insight about an algebraic property five times over a few minutes, in slightly different ways. He consistently used repetition of grammar and words, a speech form known as poetic structure. These poetic structures were marked elaborately through discursive modes such as pause, intonation and gestures, suggesting that they form a meaning-making mode that is real to the speaker.

Keywords: Classroom Discourse, Problem-Solving, Algebra and Algebraic Thinking

Introduction

This paper contributes a methodological example to scholarship on language as a resource for learning mathematics. Use of home languages and the interplay of academic and informal discourses are known to be central resources (Barwell, 2015; Planas & Setati-Phakeng, 2014). This paper suggests that the discursive form of students' mathematical statements is also an important resource for learning. Grammatical repetitions within students' sentences are just as important as the words themselves in focusing attention on key issues within a task.

This study reports on a task in which two students, Sheila and Joseph, were asked to find a formula for the perimeter of a string of *n* hexagons. The key issue is that the interior sides of a hexagon string do not contribute to the perimeter. One of the students, Joseph, noticed this property (Pirie & Kieren, 1994), and over the course of about four minutes, stated the property five times before his partner agreed to it. Over these moments, we can consider which discursive strategies Joseph conserved and which he changed as he tried to make his point. Poetic structures, elaborately marked with pauses, changes in intonation and with gestures, were a discursive strategy that Joseph used in each property noticing attempt. This paper proposes that poetic structures can be considered as a discursive mode in multimodal discourse analysis.

What Is a Poetic Structure?

A poetic structure occurs when a speaker repeats a phrase or sentence that was spoken previously, while retaining some of the syntax of the prior statement, and at least one word (Staats, 2016a, 2016b). Repeating grammar helps students talk about math because the syntax establishes relationships among small mathematical ideas or images. Repeating the phrase lets students conserve or modify these relationships. Looking for repetitions that retain at least one word from the previous statement helps the researcher focus on continuity of topic.

Sometimes a poetic structure occurs within one student's turn at talk, as an "internal" poetic structure (Staats, 2016a). Here, Sheila is describing each diagram in Figure 1, from the n = 1 case to the n = 4 case, in which the number of interior sides are 0, 2, 4, and 6, respectively. In the layout of the transcript, I have used indentation to call attention to the grammatical repetitions so that they are arranged roughly in columns, one column for the phrase *would be*, one for *the total number of sides* and one for the number of sides that she sees:

So that would be	like a formula, right?
So this would be	6L,

and then this one would be, uh,
    the total number of sides   minus 2 .
And then this one would be
    the total number of sides   minus 4 .
This would be
    the total number of sides   minus 2, 4, 6

Poetic structures can also occur between students, or "across" students, when one student repeats and perhaps modifies a phrase that another student said in the past (Staats, 2016a; 2016b). For example, a short time after Sheila's statement above, over several of her turns at talk, she spoke fragments of a formula while trying to figure out how to subtract interior sides. At turn 76, she says, *...So, like in statistics the number of cases would be n and that would be your number minus 2...* and at turn 77, she says, *...Number of hexagons would be 1, 2, 3, 4. 4, uh, times 2....* At turn 78, Joseph uses an "across" poetic structure to recombine two of Sheila's phrases in order to request clarification, *Times 2 or minus 2?*

In this way, poetic structure analysis can focus on the form of a single student's mathematical commentary, or it can focus on the relationships of commentary that students share, sometimes over long stretches of a conversation. This study is concerned with the way Joseph used discursive form to explain a property of the hexagon diagrams, and so here we focus on "internal" poetic structures. As we see in the examples above, poetic structures can be shown on the page through indentation and through underlining. In order to conserve space in this research report, in the following examples, I use several forms of underlining instead of indenting to highlight repetitions within Joseph's property-noticing statements.

Theoretical Foundation

In recent research reports, I have outlined a methodological foundation for identifying and interpreting poetic structures in collaborative mathematical conversations. I've shown that poetic structures can facilitate activities such as organizing data, generalization, and shifting from a spoken mathematical formula to a written one (Staats, 2016a, 2016b). In this research report, I discuss how poetic structures contribute to a student's property noticing statements.

In Pirie and Kieren's theory of the dynamic growth of mathematical understanding, three of the early stages are image making, image having and property noticing (Pirie & Kieren, 1994). A student in the property noticing stage can coordinate multiple abstract images or ideas in order to identify a new property of a mathematical object or activity. Martin and Towers have extended this theory to describe collective property noticing, in which the insights are distributed across different students, with no single student expressing the full idea (Martin & Towers, 2015). Throughout the hexagon task, Sheila and Joseph had collaborated closely, and Sheila had voiced many key insights. But in the selections below, Joseph was in an individual stage of property noticing. Joseph seemed to be making a bid to have his insight validated by Sheila and brought into their collective work. The theoretical foundations advanced by Pirie and Kieren on one hand and Martin and Towers on the other allow us to locate Joseph's commentary just at the boundary of individual and collective property noticing. Joseph used multimodal discourse in varying ways over five moments of property noticing to gain collective engagement, with only some success.

Participants and Task

Sheila and Joseph were two undergraduate students who had recently completed a university class in precalculus. They participated in a paid, one hour video and audio recorded problem-solving session. The hexagon task was based on Wilmot et al (2011, p. 287). The task includes diagrams for strings of 1 to 4 hexagons, shown in Figure 1. It asks the students to fill a table of values for n = 1 to n = 5 hexagons with the corresponding perimeters. A correct answer is $p = 4n + 2$. Sheila and Joseph developed an answer of the form $\#H(6) – 2(N – 1) =$ in which H and N both stand for the number of hexagons (Staats, 2016a; 2016b).

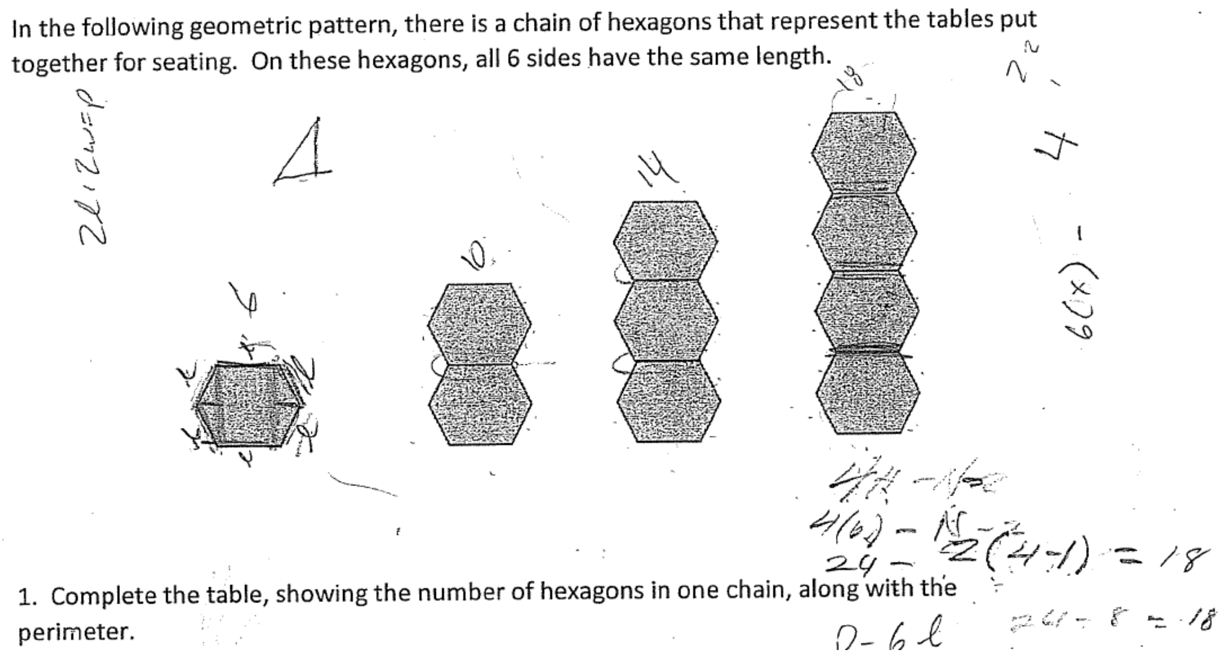

Figure 1. Task diagrams for n = 1 to n = 4 hexagons.

Methods

This study uses multimodal discourse analysis to understand a student's bid to propose a property as a collaborative activity. Because this paper is concerned with patterns in syntax and sound, as well as gesture, I use an approach to discursive multimodality that is drawn from conversation analysis (Bergmann, Brenning, Pfeffer, & Reber, 2012). This approach takes prosody—patterns of sound in language such as pitch, intonation, pause or loudness—combined with gesture and syntactic structures, as interactional modes that allow people to create meaning. Syntactic units are sometimes marked multimodally through prosody and gesture (Szczepek Reed, 2012). Multimodal analysis in mathematics education has shown that prosody, gesture, sounds created through gesture, and rhythm can be coordinated to express mathematical understanding (Bautista & Roth, 2012; Radford, Bardini, & Sabena, 2007). Alibali, Nathan, et al (2014) show that the coordination of speech and gesture is particularly important in linking mathematical ideas, particularly for new material. In this paper, I introduce poetic structure as an additional discursive modality for creating meaning.

When Joseph explains the property and Sheila fails to agree with it, he must explain it again but change something, in other words, he must deploy a new combination of discursive resources. Speakers could shift among many different discursive strategies to try to highlight important information for their idea. I consider four elements of speech that a speaker could combine in various ways to create a new strategy for explanation: introducing academic vocabulary; elements of prosody; gesture; and poetic structures. I represent these elements according to the transcription key in Table 1. This mode of transcription doesn't use punctuation such as commas, periods or question marks unless they are defined in the key as features of the sound. In the selections below, I transcribe Joseph's property noticing statements closely, and I present other statements in ordinary prose with ordinary punctuation. To identify poetic structures within one turn at talk, I looked for phrases that were repeated and that share some grammar and at least one word. If I could identify alternative ways of parsing the statement into poetic structures, I chose the approach that accounted for the longest stretch of words.

Table 1: Transcription Symbols

Symbol	Definition
↑ or ↓	High pitch or low pitch, respectively, compared to nearby words. Could occur at beginning, middle or end of word.
Bold text	Emphasis on a word through loudness, vowel lengthening, or stress
(x.y)	Pause, estimated, in seconds.
Underlining styles	Single, double or bold underlining represent the different poetic structures in one turn at talk
[*italics*]	Descriptions of gestures

Five Attempts to Explain a Property

In this part of the conversation, the students had already developed a numerical method for solving the hexagon problem that corresponds to the formula that they had written, *#H(6) – 2(N – 1)* = . The task now asks them to explain their method in terms of the diagram. After a brief discussion of whether this refers to the table of values or the images of hexagons on the handout, Joseph states a property of the hexagon diagram for the first time in the conversation.

First Attempt, Turn 171

171 J: Then this must be the diagram (1.0) so ↑how can we use ↓this (0.2) to prove our point (1.4) [*Pencil taps vertically on the diagram at each "this."*]

so the **hex**agon has ↑six sides (1.4) but when you put a hexagon in a **chain** they share **two** sides (0.6) so you're taking away **two** sides (0.8) from the chain↓ (1.6) so each time you **add** another hexagon in a chain↑ [S: But it's not asking, this one] you're losing two sides↑ [*During part of this segment, Joseph's hand obscures the paper, so we don't know precisely whether he used small gestures.*]

172 S: This one is not asking about the diagram. It says, how do we know this is true? Why does it work? Because the formula matches the diagram and table.

There are two repeated phrases in Joseph's first statement of the property, which can be summarized as: *you/put, add/hexagon in a chain* and *you're/taking away, losing/two sides*. The beginning and ending of each of these four underlined phrases is fairly strongly marked through prosody, either through pause, intonation or emphasis. For example, *so you're taking away **two** sides* is bounded on either side by a distinct pause. The poetic structure *when you put a hexagon in a chain* is marked as a bounded unit as well, but perhaps less distinctly, with an opening pause and a small level of sound emphasis on the word *chain*. It is important to notice that the beginnings and endings of many of the following examples of poetic structures are marked as a distinct unit, primarily through intonation or pause, and sometimes through less prominent forms of emphasis that are noted with bold text.

The poetic structure in Joseph's first explanation introduces a strategy that he uses in some of the following explanations: a cause/effect or if/then form of argumentation. Fairly often, the first repeated phrase is a cause or action in some sense, and the second one is an effect. This argumentative strategy is conveyed primarily through Joseph's poetic structures.

Though gestures are obscured, it appears from Joseph's hand position that there were probably some small gestures. Still, they did not result in Sheila's acknowledgement of his statement. In the next few turns, the students discuss what it means to explain a method in terms of a diagram. Sheila wonders if they need to draw a new diagram, which leads to Joseph's second statement of the property.

Second and Third Attempts, Turns 177 and 181

177.1 J: ↑I don't think so I ↑think you just have to explain so (1.8) a ↑hexagon has six sides↓ (0.4) [*just before "a hexagon", a heavy tap on the base of the n = 1 diagram*]

177.2 and as you **add** (1.2) an additional hexagon (1.2) [*at "add," light touch on the base; at "additional," light touch on top*]

177.3 you add six↑ (0.8) *[touch base before "six" and touch top after]*
177.4 because they ↑share two sides↑ (0.6) [*circling the top of the n = 1 diagram*]

177.5 you subtract↓ two sides↓ [*pencil is above the page. At "subtract," there's a beat in the air near the top of the hexagon and at "sides," there's a beat near the base*]

177.6 from their total↓ (0.8) number of sides↓

178 S: Um. I think this is, I think this is –
179 J: Which is essentially this.
180 S: The, the number of tables has six sides. And with that multiplied you minus, as they join together they lose one side. So for every-

181 J: One side for (0.2) **one** (0.2) hexagon. [*Overlapping with Sheila's 180 to 181. No gestures; Joseph's hand is away from the paper*].

182 S: -For every, for every two tables, one side is lost. So for the, for the, well, uh, the formula just states it, right? How do you state the formula?

At turn 177, Joseph realized that his property noticing was not taken up by Sheila, and so he made several prominent discursive changes. He used two mathematical poetic structures, each one advancing an "action/result" kind of argument: 177.2-177.3, and 177.4-177.5. Each of these four lines was strongly bounded by prosody, particularly pause, but also with some intonation and emphasis, except the end of 177.5 which was only bounded by intonation and gesture. In 177.2, 177.4 and 177.5, each was accompanied by a two-fold pointing gesture that emphasized the words *add*, *six*, and *subtract…sides*, respectively. Line 177.3 was marked by a different gesture, circling near one side of a hexagon. Generally, the poetic structures in attempt 2 included elaborate coordination with gesture and prosody. Joseph also shifted to the more precise mathematical terminology of *add* and *subtract* with this statement. These layers of multimodal cues suggest that these poetic structures were real to the student, rather than a researcher's analytical imposition.

Once a speaker establishes a unit of speech through a poetic structure, it opens up additional expressive possibilities. If we think of poetic structure repetitions as a kind of rhythm in speech, then prosody and gesture sometimes form a sub-rhythm that highlights words within the poetic structure in ways that enhance the mathematical idea. This is well known for gesture, but multimodal poetic structure analysis shows more clearly that gestures and prosody can express comparisons and contrasts in mathematical entities that are defined through the poetic structure.

For example, a subtle sub-rhythm emerges in 177.2, with a slight emphasis on the word *add*. With this word, Joseph introduces a new strategy in attempt 2, to use more precise computational terminology. The computational word *add* is a new idea, and this is enhanced with prosody (emphasis and pause) and the first part of a two part pointing gesture. In the repetition in 177.3, however, Joseph shifts the emphasis from *add* to *six*, with intonation, pause and a two part pointing gesture. This sub-rhythm controls and shifts focus within the poetic structure words. It shifts the hypothetical action on the diagram of *add/additional hexagon* and towards the computational technique of *add/six*. The poetic structure of 177.4 and 177.5 also contains a sub-rhythm that is coordinated with the repeated words of the poetic structures. Rising intonation on *share/sides* is contrasted with falling intonation on *subtract/sides*, which, like 177.2 and 177.3, emphasizes a shift from action to computation but also realizes that cause/effect argumentation structure. The sense of cause/effect was heightened as the circling gesture at 177.4 shifted to a two part beat gesture on *subtract/sides*. Within elaborately marked and bounded poetic structures, sound and gestures help Joseph to highlight mathematical actions and relationships.

Sheila's response to turn 177 was ambiguous: *as they join together they lose one side*, and so Joseph clarified again at turn 181. With each attempt to explain the property, Joseph uses a different combination of discursive tools. Here, his statement, *One side for one hexagon,* relied on a prominent poetic structure with no gestures. His strategy in attempt 3 was minimalism, to explain using as few words as possible. A poetic structure allowed him to do this. A subtle sub-rhythm used pauses and a small bit of emphasis to highlight the clarification of ***one*** *hexagon*.

Fourth Attempt, Turn 185

183 J: Well they want you to use the diagram. So using this we have to explain how that formula works. So there are six sides, which is, which corresponds to –

184 S: Let's see. Let's do 4. For every, for all these tables, four tables, there's only three intersections [*Joseph takes a piece of paper and writes "4."*]

185.1 J: But they **all** lose two sides at an intersection (1.6) so (0.4) a **hex**agon has six sides, right↑ (0.6) so 4 times 6 is 24↑ [*Joseph has now written 4·6 = 24*] (1.2)

185.2 um but each (1.0) table when each table touches the other one (1.2) [*Joseph has taken a second piece of paper, and at "when," he holds his hand up with palm down*]

185.3 they lose two sides↓ [*At "lose," he closes his palm into a fist*]

186 S: Oh, that doesn't work out mathematically.

Joseph's fourth attempt to assert property noticing used poetic structures less prominently, possibly because he was in the middle of introducing the new modality of writing mathematical formulas. The small level of repetition that exists still expresses the cause/effect argumentation structure, and this is supported through gesture. The open-handed gesture at 185.2 seems to be a gesture of asserting, and the closed fist at 185.3 seems to grasp a conclusion.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Fifth Attempt, Turn 187

187.1 J: Because ↑here's one and ↑here's one (0.8) **here's** one **and here's** one (0.4) **here's** one **here's** one (0.4) so you lose **six** sides (0.8) [*Joseph's hand has obscured the view again, but we can hear him scratching over the interior pairs of sides, and we can see him moving from the bottom pair to the middle pair to the top pair, making the marks that are visible on the interior sides of the n = 4 case in Figure 1. His hand comes away after the last "here's one."*]

187.2 so **24** (0.8) [*He writes a negative sign in 4·6 = 24 -]*

187.3 would be the number of ↑sides if they weren't touching (0.4) but because they're ↑touching (0.4) you lose six↓ [*He writes 6 after the negative sign*]

187.4 and that comes down to 18. [*Joseph has now written 4·6 = 24 – 6 = 18.*]

188 S: Yeah. Okay, go ahead and write it down. You're better at that than I am.
189 J: I'm just more visual.

The first part of turn 187 has a strong poetic structure based on *here's one and here's one*, in which the beginning and ending is marked and bounded off with prosody and a combination of gesture and drawing. Alibabi, Nathan, et al (2014) consider this action as a "writing gesture," though this is distinguished from writing linguistic or mathematical text. The various gestures that Joseph made near the base and top of a hexagon in several previous statement were here made lasting and tangible through a writing gesture.

At 187.2, when Joseph writes the minus sign, *4·6 = 24 -*, he creates a sense of incompletion. We could consider this as a sub-rhythm that is not completed until he writes the *6* as he says a part of the poetic structure at 187.3, *you lose six* and then completes the calculation of *18*. There is a slight dramatic feel to this moment, because he performs the subtraction while he completes the poetic structure. Another sub-rhythm is associated with the poetic structure at this moment, the rising and falling intonations on *touching* and *lose six*. The poetic structure, writing mathematical text, pause and intonation are all coordinated to reproduce the cause/effect argument that Joseph used throughout his property noticing statements. At 188, Sheila responded with a long, thoughtful sounding *Yeah*. It sounded as if she decided that she agreed with Joseph.

Conclusion

Across his five property-noticing statements, Joseph combined expressive modes into distinct discursive strategies that finally achieved acceptance if not extended collective engagement. Poetic structures were present in all attempts, prominently so in four of them. The poetic structures were very commonly marked near the beginning and end through prosody, especially pause and changes in intonation. This analysis shows that as syntactic units are repeated, they may continue to be marked multimodally, amplifying their expressive quality.

Poetic structures put units in parallel position into relationships of similarity— *as you add an additional hexagon/ you add six*; and of difference— *because they ↑share two sides*↑ *you subtract*↓ *two sides*↓. Bautista and Roth have similarly observed that sounds produced through gesture mark important mathematical similarities or differences (2012). Joseph capitalized on poetic structures to add layers of sub-rhythms through gesture, pause and intonation that did quite a lot of the mathematical work in Joseph's five property noticing statements. The co-occurrence of poetic structure and several modalities is strong evidence that poetic structures are not simply abstract grammatical mappings applied by a researcher, but rather, highly expressive tools of meaning-making. Poetic structure analysis is a powerful method to trace the moment-by-moment construction and expression of precise mathematical ideas through informal language.

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USING DISCOURSE ANALYSIS TO UNDERSTAND VARIATION IN STUDENTS' REASONING FROM ACCEPTED WAYS OF REASONING

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In this study, I use a systemic functional linguistics approach to examine mathematics classroom discourse with the aim of providing a plausible explanation of how students could actively participate in productive classroom discussions without adopting ways of reasoning that were accepted in the classroom community. In this way, I work in the crossroads of a research tradition examining classroom interaction and a research tradition that examines student learning. I found that even though particular ways of reasoning about exponentials and logarithms were advanced and accepted in the classroom discourse, the way these ways of reasoning were talked about in the class did not preclude students from maintaining less sophisticated ways of reasoning Specifically, I argue that the two exponential ways of reasoning were not explicitly contrasted, which may have contributed to students seeing them as essentially the same strategy.

Keywords: Classroom Discourse, Teacher Education-Preservice

Introduction

It is not uncommon for a novice teacher to be surprised by his or her students' poor performance on an exam or assignment after class discussions had seemed to have been going well (see, for example, Price & Valli, 2005). This surprise is not unreasonable. At times, students can be integral participants in productive class discussions that seem to advance the mathematical agenda, yet later still be confused or grappling with a concept that was thoroughly discussed in class. To understand this phenomenon in greater detail, I took the lens of the emergent perspective (Cobb & Yackel, 1996) to examine classroom interactions.

The emergent perspective coordinates social and individual aspects of the classroom community to explain students' learning. Specifically, it coordinates classroom social norms, socio-mathematical norms, and mathematical practices with their individual correlates. In this study I focused on the relationship between classroom mathematical practices, which are specific ways of reasoning that become adopted in a class community, and its correlate of individual students' personal ways of reasoning. According to the perspective, this relationship is indirect and reflexive. Accepted practices arise as individual students posit ways of reasoning. These are then discussed within the class community and are either collectively accepted or rejected. In this way, individual students' conceptions and ways of reasoning give rise to accepted mathematical practices. Then, students' conceptions are influenced as they continue to participate in established math practices. In this way the relationship is reflexive. It is also indirect, meaning that there is not a one-to-one mapping between accepted math practices and students' ways of reasoning. This is acknowledged by the perspective in at least two ways. First, a math practice is not defined as the conceptions held by the majority of students, but as the social status of a way of reasoning in the classroom community. Second, Cobb and Yackel (1996) were careful to point out that participation in a practice influences, but does not determine students' ways of reasoning.

Despite the fact that this acknowledgement of variation in thinking existed from the inception of the theory, the research community still does not have many images of the nature and extent of this variation, much less a well-developed theory of why these variations occur. Of those scholars that have explicitly investigated the reflexive relationship between individual cognition and the emergence of mathematical practices (e. g. Rasmussen, Wawro, & Zandieh, 2015; Stephan, Cobb, &

Gravemeijer, 2003; Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014), Stephan et al. (2003) gives the best image of the nature and extent of individual variation as they tracked two students' participation in a class community as various math practices emerged. While they documented significant variations in individual ways of reasoning from established practices, these differences seemed to resolve themselves through continued participation in the class. One explanation for this is that the individual ways of reasoning that varied from accepted practice did not yield correct answers, and thus, made it problematic to continue participation in class discussions while using their personal way of reasoning. This encouraged the students to reevaluate their way of reasoning and, eventually, to adopt ways of reasoning more consistent with the math practice. In contrast, my previous work (Gruver, 2016) found that mathematically significant variations in thinking can persist, even after instruction has ended. This finding shows that significant variations do not always work themselves out naturally through the course of instruction. This underscores the importance in further understanding these variations and their causes; this is the focus of the current study.

Nature of Variations From the Established Practice

In a previous study, I documented the ways that individual ways of reasoning varied from accepted classroom mathematical practices. I first determined which practices were established in a classroom of 29 prospective teachers and then compared those practices to the individuals' reasoning during a post instruction interview. I determined which practices were accepted in the class by analyzing students' arguments using the documenting collective activity method (Cole et al., 2012; Rasmussen & Stephan, 2008). Of the seven students interviewed, I found that four of them reasoned in ways that were different in mathematically significant ways from the established practice. In this section, I describe the established practice and the ways students reasoned in the interview.

The math practice emerged as students were developing an exponential number line, which would later be used to investigate exponential and logarithmic relationships. Early on, the students developed a number line in which powers of 10 were equally spaced. However, the spaces between the powers of ten were subdivided linearly. In this way, their number line had an exponential structure at the macro level, but a linear structure between the powers of 10. Eventually, this initial way of subdividing was eventually overturned in favor of a method that produced a fully exponential number line. The math practice, *Subdividing the Number Line,* consisted of two ways of reasoning that were accepted in the classroom community. These two methods for subdivision were cognitively distinct, but produced the same answer. The first method, *Subdividing Segments by Reasoning Linearly About Exponents* is characterized by students writing the number they wished to place on the number line in the form 10^b, ignoring the 10, and then determining the location of the number as if they were simply placing the exponent on linear number line. In other words, $10^{1.5}$ would go halfway between 10^1 and 10^2, because 1.5 would be halfway between 1 and 2 on a linear number line. The second method to subdivide the number line that became normative in the class was *Preserving the Multiplicative Relationship within the Segments.* This way of reasoning emerged as students noticed a constant multiplicative pattern at the macro level. Specifically, they noticed that the equally spaced powers of ten increased by a factor of ten. This differs from a linear number line where equally spaced points would increase by a constant sum rather than a constant factor. They then extended this pattern to apply to subdivisions. Thus, to determine the value of the half way point between 10^1 and 10^2, they would notice that between these two points there is an increase of a factor of 10; then, since the half way point divided the segment into two subsections, they would need to find a number that when multiplied by itself yielded 10. That number is the square root of ten. Thus, the midway point is 10 times the square root of ten. These two ways of subdividing segments on a number line emerged around the same time in class, but the first was talked about as a way to efficiently determine the value of subdivisions while the second was used as a way to explain why reasoning linearly with the exponents makes sense.

Three of the students coordinated these two ways of reasoning in the post interview. This means that while they may have determined the values of various points on an exponential number line using the numeric pattern in the exponents, they could also use multiplicative reasoning to justify their placements. However, the other four were not able to use the second way of reasoning, the multiplicative pattern, even when probed.

Since more than half the students interviewed did not include multiplicative reasoning in their interview responses, observers were left with the question: How could the students intellectually engage in class discussions, but not personally adopt ways of reasoning consistent with the classroom math practice? A partial response to this question will be developed in this report. In particular, I will focus on the nature of the classroom discourse as multiplicative reasoning was developed in this class to address the research question, *How might the nature of the discursive interactions in both whole class and small group settings give a plausible explanation for students' variations from the emergent math practice?* In answering this question, I examine the intersection of classroom interactions and individual student learning. This work is at the crossroads of two research traditions and contributes to a new path forward for using discourse analysis to give insights into the nature of individual knowledge construction.

Method

Data collection occurred in a math class for prospective secondary teachers (PSTs). The purpose of the course was to deepen the PSTs' mathematical knowledge of secondary topics. The current study focuses on a single unit where the PSTs explored exponential and logarithmic relationships. This unit was three weeks long. The class met twice a week for an hour and a half each time. Thus, the unit included nine hours of instruction spread over six days. Data included video and audio taped class discussions and approximately 1 hour problem solving interviews with seven students. The purpose of these interviews was to determine students' individual ways of reasoning about the content explored in class. These students were distributed among two small groups of four students each. The small group interactions of these seven focus students were also video and audio recorded.

To analyze the discourse, I used a modified version of Herbel-Eisenmann and Otten's (2011) method for thematic analysis (Lemke, 1990; Herbal-Eisenmann, 2011), a systemic functional linguistics (SFL) approach (Halliday, 1978; Halliday & Hasan, 1985). Central to this method is the assumption that words derive their meaning from their relationships to other words used in the discourse. To determine this meaning researchers examine the semantic relationships between words expressed in classroom discourse. For example, if a student said, "500 is at the midpoint," They are expressing a relationship between "500" and "midpoint." In particular, they express a located/location relationship. This helps determine the meaning of both 500 and midpoint, namely that 500 is something to be located and midpoint is a location.

I used this method to examine moments in the classroom where subdivision of an exponential number line was discussed to develop networks of semantic relationships between lexical items, words or phrases that came up repeatedly in the discourse. I developed a network for each method of subdivision based on arguments given in class as well as networks based on canonical arguments, those that are representative of how an expert might argue. Comparing the various networks revealed subtleties in the discourse and the meanings of various words.

I then analyzed discourse where students reflected on and talked about the methods of subdivision themselves. In these instances, students would explicitly refer to a particular method of subdivision as a method of subdivision. This contrasts with the other episodes of discourse where students were simply using a particular method. This means that in these instances of discourse, students were referring to a whole network of semantic relationships as a single lexical item. In SFL, this is called condensation (Lemke, 1990). In the episodes where students reflected on methods of subdivision, they tended to express semantic relationships in a different way than when they used the methods. In these episodes they tended to use equivalence and contrast strategies (see Lemke, 1990, p. 226) to show whether they thought two strategies were the same or different. This occurred during four episodes, though only two will be examined in this report. In the first episode, they contrasted linear and exponential methods using the discursive device of *parallel environments*. This means the speakers placed lexical items so that they have the same function in the grammar of two phrases. For example, a student might say, "in a *linear method* you do abc, but in an *exponential method* you do xyz". Here, the two methods are serving the same function in the grammar, in that they are both methods whose steps are being described. Furthermore, the methods are being contrasted and positioned as distinct, as one does something different in each instance. In the second episode, students explicitly said the two methods of subdivision were the same.

Results

Analysis of the four episodes in which students explicitly talked about the methods of subdivision themselves provided evidence for the following result: *When speaking, students distinguished between linear and exponential ways of reasoning, but did not distinguish between reasoning linearly with the exponents and multiplicative ways of reasoning.* In fact, students seemed to think of both of these methods as the same. In these instances, students may have referenced other lexical items, but these will be ignored as this analysis focused on references to the methods themselves.

Background to Episode 1

Students developed an exponential number line over the first three days of a six-day unit. On Day 1, they were asked to create a timeline that represented the history of the earth. Several approaches emerged on Day 1, but by Day 2 the teacher had encouraged them to focus on and develop a particular approach in which the timeline had a macro exponential structure, meaning powers of ten were equally spaced. However, to place events they subdivided the space between powers of ten linearly. Eventually, over Days 2 and 3, this method of subdivision was rejected and the two methods that were ultimately accepted were developed. The four episodes in which students in class reflected on the methods of subdivision occurred over these two days.

The first episode occurred near the end of Day 2. Students had already placed two dates on the timeline, the Renaissance and the Ordovician Periods, using linear reasoning to subdivide the space between two powers of ten. Presumably to problematize this way of reasoning, the teacher asked the students to place the Renaissance again, but using 1 and 1,000 and endpoints instead of 10^2 and 10^3. As part of her question, she specifically asked if 500 would end up in the same place. This led to the realization that using the different endpoints resulted in a different placement for the Renaissance. The students then considered the idea that the method they were using to subdivide was problematic. As they reflected on their method they contrasted how they subdivided the segments with the macro exponential structure, using the discursive device *parallel environments*. This contrast helps support the main claim in this paper, that over several discussions in class linear and exponential strategies were contrasted in the discourse, while the two exponential strategies for subdivision were not.

Discussing the Problem of the Renaissance Moving

The first instance of using *parallel environments* to contrast the linear relationships with exponential relationships came up as Nathan described why their linear method might be problematic:

> Yeah, so ultimately, the issue is, it seems like we're trying to apply a method that's completely linear in nature, when our graph [timeline] is not, it's exponential. That is, that's the problem, so that means that right there, the solution will not work, because why would it?

In this quote, Nathan referenced the idea that they were using a "method that's completely linear in nature" and a "graph… [that's] exponential." While Nathan argued that there was a mismatch between the nature of the method and the nature of the graph, Danna provided even more detail as to what the problem might be. She argued that a linear placement would not work, highlighting that it inaccurately predicts the placement of the known point 10^3.

> I started with first doing basically what we did up here [the linear way of reasoning]. So, we looked at the difference between 10^4 and 10^2, which was 9,900 years. And half of that [points to halfway between 10^4 and 10^2] should have been the 48 or should be…4,950, but we know that it's actually 10^3, so that right there told me linear doesn't work and it pushes the halfway mark closer to the 10^4 side. So applying the 500 to this one, I knew it was going to be closer to 10^3, just `cause, 500 it's halfway if it's linear, but when it's exponential, you know it's not, based on this [points to the number line that she used to argue "linear doesn't work"]. Then, I realized you can do it this way.

Again, Danna used the discursive device *parallel environments* to contrast linear and exponential ways of reasoning. She began using the linear method to predict the placement 10^3, a year whose placement was already determined from the macro pattern. She then extrapolated, "500 years, it's halfway if it's linear, but when it's exponential, you know it's not." Here the linear and exponential relationships have a similar grammatical function, in that she basically said, if it's linear, then 500 is halfway, but if it's exponential, then 500 is not halfway. She did this again when she said, "We were trying to look at it linearly in between each chunk, but the entire timeline is exponential." Here, she contrasted linear and exponential ways of reasoning by saying that in each chunk the structure was linear, but the structure of the whole timeline was exponential.

Background for Episode 2

Talk about methods of subdivision continued on Day 3. The day began with a student, Lacey, introducing the first way of subdividing that eventually became normative. She determined that the halfway point between 10^2 and 10^3 should be $10^{2.5}$, using the subdivision method of *Reasoning Linearly with the Exponents*. While discussing this task, Nathan justified her placement by introducing the second way of subdividing that eventually became normative, *Preserving the Multiplicative Relationship within the Segments*:

> Well, the way I did this one was I was looking at it where, in the more general sense, each tick was, …each thing apart on the bigger one is the same distance… multiplicatively apart, so we're going to do the same thing here. We have two so, we have two sections that when multiplied all together are 10. So, each side we'd had better have the square root of ten, because that's the only thing that's gonna give us 10 when we multiply it again, so I…just took the, I just figured it was, the distance away was 10^2 and then times the $\sqrt{10}$, which is 3.162. So I got 3.162 times 10^2.

Here, Nathan argued that by extending the multiplicative pattern that existed at the macro level, one can find the halfway point to be 100 times the square root of 10, or $10^{2.5}$, as Lacey had said. Presumably because the teacher noticed this was a distinct way of reasoning, she asked the students to talk about it in small group, specifically asking them to explain where they see the square root of ten coming up. However, during their small group discussions, the students said that Nathan's method and Lacey's method were the same.

Reactions to Nathan's Ideas.

In both small groups that included focus students, they failed to see the difference between Nathan's and Lacey's ways of reasoning about the subdivisions. In the first group, instead of engaging with the ideas of factors and multiplication, one student, Tanya, started the discussion of square roots by talking about the exponents. She said, "Well, the exponent one half is the square root right? ... So if it's 10^2, multiplied by $10^{1/2}$, right? So it's 100 times the $\sqrt{10}$." Here we see Tanya following the teacher's prompt to attend to the square root, however Tanya is arriving at the square root in a much different way than Nathan did. Instead of continuing the multiplicative patterns that existed at the macro level, she is arriving at the square root via a previously known rule that $10^{.5}$ is $\sqrt{10}$. This allowed her to still preserve her linear ways of reasoning about the exponents, while at the same time explaining where the square root is coming from as the teacher asked. This made it so she did not have to distinguish between the two ways of reasoning. This analysis is consistent with her groupmates' comments. Kathy said, "[Nathan's way of reasoning is] the same thing, because if you're doing 10^2 times 10 to the square root that's the same thing as .5." Rachel concurred saying, "He just thought of it as square root instead of…one half." Kathy summarized by saying, "Yeah, it's the same thing, he just wrote it differently."

In the second group, the students also continued to focus on exponents. However, instead of engaging with Nathan's idea, they explicitly said they did not understand it and ignored it. Farah said, "Well, I don't understand what [Nathan] said, but this is how…I thought of it." She then continued with her own idea.

In these small group interactions, the students explicitly said Nathan's and Lacey's way of reasoning were equivalent. Even though the teacher prompted them to attend to the square root, an idea that was central to Nathan's idea and absent from Lacey's, the students treated this as simply a notational difference. Tanya began by asking "the exponent one half is the square root right?" Rachel echoed this connection when she said, "He just thought of it as square root instead of…one half." This interpretation may have allowed them see Nathan's idea as simply another expression of Lacey's ideas rather than a new idea worthy of examination.

Summary

Analysis of the discourse in the four episodes where students reflected on methods for subdivision provide evidence for the claim: *When speaking, students distinguished between linear and exponential ways of reasoning, but did not distinguish between reasoning linearly with the exponents and multiplicative ways of reasoning*. In the first episode, when students discussed the problem of the point representing the Renaissance moving, they distinguished between linear and exponential ways of reasoning. They talked about subdividing the segments as a linear process while the macro structure was exponential in nature. This contrast between linear and exponential came up again in the third episode when students contrasted halving the values on the line with halving the values of the exponents. While this contrast is important, it does not help to disambiguate between the two exponential ways of reasoning. Furthermore, when the students talked about Nathan's multiplicative way of reasoning in small groups, they referred to it as the same as Lacey's method, which relied on linear patterns in the exponents. Thus, it is possible that students participating in the class discussion could see the two exponential methods as the same, which leaves little intellectual encouragement for students who can reason successful by focusing on the exponents to adopt multiplicative ways of reasoning.

Discussion

This study focused on how the nature of the classroom discourse can help explain how students could participate in a classroom where multiplicative ways of reasoning were developed and accepted by the class community, but not adopt those ways of reasoning as individuals. Through discourse analysis I discovered that exponential and linear ways of reasoning were contrasted, but reasoning multiplicatively and reasoning linearly with the exponents were not. This may mean that students thought there were primarily two ways of reasoning, a wrong way and a right way—linear reasoning and exponential reasoning. Thus when students heard multiplicative explanations, they may have thought that what they were hearing was no different from reasoning linearly with the exponents, since both were exponential.

It is reasonable to think that strongly contrasting linear and exponential methods in the classroom discourse is a desirable outcome, since previous research on exponential reasoning suggests that making the transition from linear to exponential reasoning is difficult (Berezovski, 2004; Liang & Wood, 2005). However, tackling this issue in this classroom seemed to background the subtler difference that exists between the two methods for subdividing exponentially. Being able to distinguish between the two methods is key to developing conceptual understanding of the relationships among numbers on the exponential number line. To develop this understanding, students need to coordinate the two methods of reasoning linearly with the exponents and continuing the macro multiplicative pattern. If students think of these as the same, then they can simply reason linearly with the exponents to get the right answer, without thinking exponentially at all. This means that, ironically, since the only contrast between methods of subdivision that existed in the discourse were between exponential and linear methods, students were able to participate in classroom discourse about exponential subdivision while reasoning only linearly.

This work raises the question of how to encourage students to see the differences between two ways of reasoning, especially when both ways of reasoning yield the same answer, so that they can then explore their relationships. In this paper, I have argued that the two exponential ways of reasoning were not explicitly contrasted, which may have contributed to students seeing them as essentially the same strategy. As such, an implication for teachers of this specific unit is to consider asking students to explicitly name and contrast the two exponential strategies. This would make it problematic for students who thought of the two methods as the same to continue to participate in the discourse. This may encourage them to disambiguate between the two methods, positioning them well to explore the relationships between them.

More generally, teachers teaching any unit should think about various ways of reasoning that may arise in the class and if they should be named and contrasted. However, it should be noted that determining which methods should be contrasted can be difficult to predict. While the research presented here underscores the point that the two exponential methods of subdividing a number line should be contrasted, this was not obvious before instruction. While the teacher recognized the complexity of transitioning from linear to exponential ways of reasoning and appropriately orchestrated the discussion to contrast those two ways of reasoning, she seemed to underestimate the difficulty students would have with disambiguating and coordinating the two exponential strategies. This highlights that to some extent, familiarity with and competence in executing general discourse moves, such as those involved in orchestrating discussion in such a way that strategies are contrasted, only goes so far in teaching and even highly effective teachers need support garnered through research that illuminates the conceptual difficulties of particular topics and gives insights into how to teach those topics. This suggests that teacher education focused on discourse should be paired with professional development focused on understanding the cognitive difficulties students face as they learn specific topics.

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A QUALITATIVE METASYNTHESIS ON CULTURALLY RESPONSIVE TEACHING & CULTURALLY RELEVANT PEDAGOGY: UNPACKING MATHEMATICS TEACHING PRACTICES

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This qualitative metasynthesis synthesizes published research papers using qualitative methodologies focusing on Culturally Relevant Pedagogy (CRP) and Culturally Responsive Teaching (CRT) published between 1994 through February of 2016. Twelve articles were synthesized to understand how researchers interpret mathematics teaching practices that support CRP and CRT in pre-kindergarten through 12th grade. There were six findings: a) caring; b) knowledge of contexts and teaching practices using contexts; c) knowledge of cultural competency and teaching practices using cultural competency; d) critical consciousness; e) high expectations; and f) mathematics instruction/teacher efficacy and beliefs.

Keywords: Research Methods, Teacher Knowledge, and Teacher Beliefs

Since 1994, there has been a large body of research focused on Culturally Relevant Pedagogy (CRP) and Culturally Responsive Teaching (CRT) across all content areas. Mathematics education has benefited from teaching and research using the tenets of CRP and CRT. Much of the research using the tenets of CRP and CRT in mathematics education employs qualitative methodological approaches. However, this work has yet to be synthesized and interpreted using methodologies for qualitative metasynthesis. Aronson and Laughter (2016) conducted research synthesizing CRP and CRT across all subject areas. This research uses a qualitative metasynthesis as a methodological approach for mathematics education research discussed by Thunder and Berry (2016). This research focuses on understanding how researchers interpret mathematics teaching practices that support CRP and CRT in pre-K through 12th grade.

Theoretical and Methodological Frameworks

Gloria Ladson-Billings', *Dreamkeepers* (1994) outlined CRP as a framework and Geneva Gay's *Culturally Responsive Teaching: Theory, Research and Practice* (2000) outlined a framework for CRT. Both CRP and CRT are frameworks for unpacking teacher practices embedded in classrooms as sites for social change and social justice. Both connect cultural framing to academic skills and concepts, build cultural competence through teaching, and use teaching as a way to critique power discourses and representations. Published peer-reviewed research papers between 1994 and 2016 using qualitative methodologies focused on CRP and CRT were sought for this qualitative metasynthesis. Prior to conducting database searches, inclusion and exclusion criteria were developed based on four parameters: topical, population, methodological, and temporal. All papers use CRP and/or CRT as frameworks (topical) and the research focused on mathematics teaching and learning in preK-12 contexts in the United States (population). Qualitative research was the methodological framework for all papers; however, mixed methods research studies were included if the qualitative findings were distinguishable. Subject term searches were conducted using EBSCO to simultaneously search five databases for peer-reviewed journal articles. Figure 1 shows the flowchart of inclusion and appraisal to determine articles for the qualitative metasynthesis.

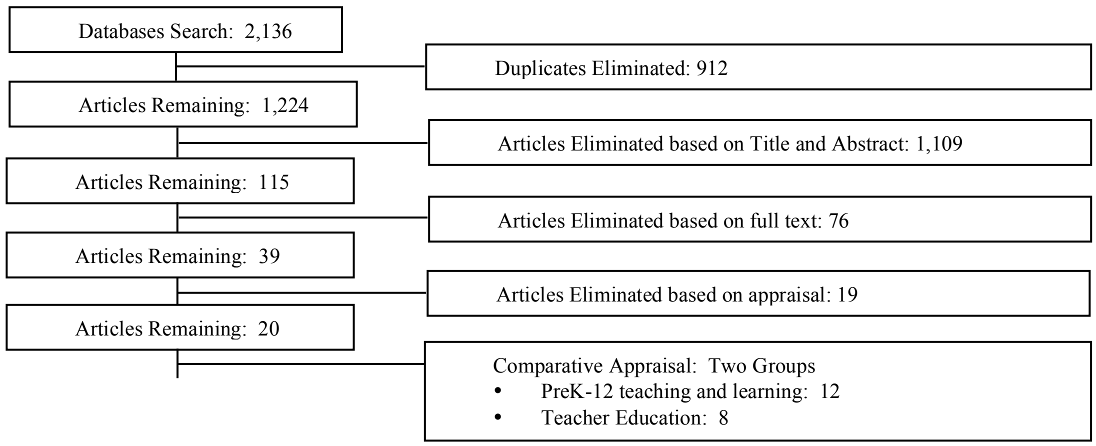

Figure 1. Flowchart of inclusion and appraisal.

The initial EBSCO search produced 1,224 articles. Following our initial search, we worked through a validation process by looking at the titles, abstracts, subject terms, and full text for published peer-reviewed journal articles. This process left further 39 articles fitting the inclusion criteria. Book reviews, reports, chapters, and dissertations are examples of items that were excluded. We then performed individual appraisals for each article, appraising the quality of the research methodologies using the rubric published by Thunder and Berry (2016, p. 329-330). Following their appraisal process, 20 articles were identified. Further, we did a comparative appraisal, dividing the articles into two groups: 1) preK-12 teaching and learning; and 2) teacher education. This qualitative metasynthesis treats the findings from 12 articles focused preK-12 teaching and learning as informants (the 12 articles are marked with an * in the reference). Dedoose, a data analysis software, was used to support data analysis and initial codes were developed and defined. Six initial codes with eight child codes were used to code the data; we re-read, re-coded, and unpacked the data to synthesize and interpret for reporting.

Findings and Excerpts

The six findings focus on teacher practices, classroom interactions, and student experiences. Each finding is highlighted with representative excerpts in Table 1.

1. **Caring** is a continuous cycle of working to establish a rapport, using knowledge gained from that rapport to inform teaching practices, and then, reflecting upon teaching and learning to understand students' mathematical knowledge.
   - Teachers created positive learning environments where students saw themselves as participatory.
   - Teachers took an active role in seeking out knowledge about students and communities.
   - Teachers supported students emotionally and academically by making mathematics content accessible and empowering students mathematically.
2. Knowledge of **context** is related to space and place in the ways teachers gained knowledge of home, community and neighborhoods. Teachers integrated mathematics instruction and knowledge of **context** by making meaning of the mathematics curricula and tasks.
   - Teachers actively engaged in communities to work with students' parents and families.
   - Teaching practices included mathematizing contexts, creating and adapting mathematical problems, utilizing questioning strategies to elicit students' local knowledge, requiring explanation and justification as it relates to context knowledge, and creating project-

based opportunities incorporating funds of knowledge.

Table 1: Findings and Excerpts

Findings	Excerpts
Caring	*When establishing relationships, teachers cannot merely go through the motions because students know when teachers are genuine and really care about them…The teachers realize they must have a relationship before they can make mathematics lessons relevant to the students. They take the opportunity to know their students and discover their motivations and interests. They tailor their instruction with this knowledge.* (Jackson, 2013, p. 7)
Context	*Ms. Finley often ''walk[ed] the neighborhood'', taking time out in the evenings to visit with students and their families. She knew that this type of connection with the community was important, and she was able to weave the knowledge that she gained through these interactions into the mathematical content that was the basis for her lessons.* (Bonner & Adams, 2012, p. 30)
Cultural Competence	*Inga decided to interview some of her students to better understand their experiences with money when shopping for their families. From this, Inga learned about her students in ways she did not expect, finding that those students who shopped with their families were able to quickly solve problems regarding currency…The strategies children use with money are often non-routine, and this might have offered an opportunity to gain a deeper knowledge of students' understanding.* (Wager, 2012, p. 16)
Critical Consciousness	*…the teacher is explaining an adjustment he made to the map's display to make the trends that they are examining more visible…The teacher superimposed the dots over the average income of each census tract with lighter colors corresponding to lower incomes…This display and discussion is part of the group's overall argument that de facto segregation still exists in Los Angeles.* (Enyedy & Mukhopadhyay, 2007, p. 161-162)
High Expectations	*…Ms. Bradley's classroom was highly structured and disciplined, focusing on high expectations and success through "tough love." When a student did not have his or her homework, for example, Ms. Bradley would take the student in the hallway to call his or her parent or guardian. Furthermore, if a student was not participating in the group chants or problem-solving activities, Ms. Bradley would "call him [or her] out and take him [or her] to church," meaning she would stop the lesson and "preach" about the decisions students were making and the importance of academic success…she indicated that this type of culturally connected communication and maintenance of high expectations allowed students to develop racially and culturally "so that they don't have to give up what they are used to for the sake of passing class...* (Bonner, 2014, p. 395)
Mathematics Instruction/Teacher Efficacy and Beliefs	*…Chela loved math. Chela turned this passion for math into a professional strength—she took advantage of all math professional development opportunities and she made mathematics a central part of her practice...Weaving math into daily activities was what Chela did best. As she designed different games or visual supports she looked for the math hook…* (Graue, Whyte, & Delaney, 2014, p. 308)

3. **Cultural competency** was found in the ways teachers developed knowledge and competencies associated with various forms of communication and funds of knowledge.
    - Teaching practices and strategies primarily focused on classroom discourse including storytelling, utilizing call and response, and dynamic forms of interactions.
    - Teachers promoted engagement by incorporating nonverbal communication through proximity and by integrating music and movement into teaching practices.
    - Teachers made mathematics accessible by unpacking and connecting cultural artifacts.
4. While **critical consciousness** and critical reflections are significant parts of the CRP and CRT, only one study (two articles) examined critical consciousness explicitly. Teachers in four studies made reference to critical consciousness and reflection.
    - Teachers and students critiqued and mathematized societal issues.
    - Societal inequities were acknowledged by framing challenges within communities. It was not clear how students were using mathematics as an agent for social change.
5. Teachers must have **high expectations** both for their students and for themselves.
    - Teachers made necessary teaching revisions based on their students' needs, interests, and understandings as they relate to mathematics.
    - Teachers were warm-demanders by establishing learning environments in which students were held accountable and empowered by taking an active role in their own learning.
6. **Mathematics instruction** highly correlates with teaching practices and strategies for both context and cultural competency. This finding is specific to mathematics teaching practices.
    - Teachers utilized technology, incorporating tools and manipulatives in their instruction.
    - Teachers modeled their thinking for students.

- Teachers with high confidence in teaching mathematics and high self-efficacy believed that mathematics instruction should be student-centered, open-ended, inquiry-based, highly interactive, and impromptu, based on students' needs and interests.
- Teachers with low confidence in teaching mathematics, or low self-efficacy struggled with CRP and CRT. This led to limiting tasks focusing on high cognitive demand.

Implications

There are significant gaps in the literature focused on mathematics teaching related to CRP and CRT. The gaps primarily exist around unpacking mathematics teacher actions focusing on CRP and CRT. More work is needed in the field for unpacking teaching practices that promote access, equity, and empowerment. The findings of this research suggest that CRP and CRT teachers know their students and the communities of their students; more work is needed to unpack the continuous cycle teachers use to develop rapport. The findings from this work suggest that mathematical knowledge for teaching positively impacted teachers' lens for CRP and CRT; more work is needed to understand and unpack the interactions of teachers' knowledge of context and culture with knowledge of mathematics and teaching mathematics.

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CONSTRUCTION AND JUSTIFICATION OF CENTRAL ANGLE THEOREM IN DYNAMIC GEOMETRY ENVIRONMENT

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Understanding how students construct abstract mathematical knowledge in classroom setting and the use of various instructional tools in facilitating such abstract constructions have been central themes in mathematics education research. This study used Abstraction in Context (AiC) as a theoretical framework to study studying high school students' processes of constructing and justifying of central angle theorem in dynamic geometry environment. Data analysis revealed that the although the central angle theorem was proposed and verified through dragging and measuring tools enabled by a GeoGebra driven simulation, the need to produce a formal proof establishing its validity was not realized by the group, suggesting that the dragging and measuring features of the dynamics geometry environment may pose challenges for teacher and students to construct abstract formal proofs.

Keywords: Geometry and Geometrical and Spatial Thinking, Reasoning and Proof, Technology

Understanding how students construct abstract mathematical knowledge in classroom settings and ways to facilitate the development of abstract generalizations and justifications among learners has been a persistent theme in research in mathematics education for decades (Mitchelmore and White, 2000). Indeed, as it pertains to abstracting and generalizing regarding geometric properties and relationships, scholars have argued, with empirical data, that the presence of dynamic geometry software (DGS) (e.g. GSP, Cabri, GeoGebra) can provide productive means for allowing students to explore, notice relationships, conjecture about potential relationships and construct generalization regarding geometric properties and relationship, with the understanding that such environments also presents challenges when moving students from inductive verification to deductive formal proof (González and Herbst, 2009). However, a majority of work in this area has considered knowledge construction of individuals or small groups of students in presence of DGS. Not many studies have been reported with a focus on exploring the dynamics of generalizing and justifying within a whole group setting, taking into account the impact of peer-teacher interactions on what might be extracted from the DGS and what is assumed collaboratively. The current study aimed to address this gap and is guided by one question: how do students and teacher in a high school geometry classroom construct and justify central angle theorem while working in DGS?

Theoretical Framework

Hershkowitz, Schwarz, Dreyfus (2001) proposed Abstraction in Context (AiC) as a theoretical-methodological framework to investigate the genesis of abstraction. The core of the model consists of three epistemic actions: *recognizing*, *building-with*, and *constructing*. *Recognizing* a familiar structure occurs when a student realizes that a structure is inherent in a given mathematical situation. Recognizing is often, though not always, at the level of empirical thought. *Building-with* consists of combining existing artifacts in order to meet a goal. When operating at the *building-with* level, the student is not enriched with new, more complex structural knowledge. She or he uses available knowledge to produce a viable solution to the problem at hand. *Constructing* consists of assembling knowledge artifacts to produce a new structure, which could be new methods, strategies, or concepts. The three epistemic actions are dynamically nested in such a way that building-with includes recognizing, and constructing includes both recognizing and building-with. This model has been used to investigate processes of abstraction in a wide range of situations (Dreyfus, 2012). This study will use AiC to examine the process of constructing and verifying central angle theorem in DGS.

Methodology

Data for the study comes from an ongoing program of research which aims to understand how students and teacher in a regular classroom settings interactively work towards constructing and justifying generalizations pertaining to geometric properties and relationships. The participating site is a high school geometry class consisting of 21 students and 1 teacher. All mathematics lessons have been video-recorded daily to capture interactions taking place during instructional activities. All student artifacts have been collected to examine the types of inferences and connections students make individually/collectively based on group interactions.

The data used in this report comes from a particular lesson whose objective was for students to learn central angle theorem. Prior to this lesson students had explored and seemingly learned about (1) all the radii of a circle are congruent, (2) two radii and a chord make an isosceles triangle and the base angles of the isosceles triangle are congruent, (3) inscribed angles subtended by a diameter are right angles. A GeoGebra simulation was created as in Figure 1a for students to explore the relationship between the measure of the central angle and the measure of the circumference angle. In the exploration, the center, point C and the diameter of the circle were locked and point D and E were not fixed. Students were first assigned to work in small groups to explore in the pre-made GeoGebra file and answer the following: (1) As you move D around on the top half of the circumference, what happens to the measure of angle E? (2) As you move E around on the bottom half of the circumference, what happens to the measure of angle O? (3) What do you think is the relationship between angle E and O? And then the teacher initiated a whole-class discussion around their observations.

Findings

When exploring in GeoGebra simulation all the 21 students have *recognized* when moving D around both angle E and angle O become larger or smaller depending on the moving direction and that when moving E around the measures of angle E and angle O stay the same. And when asked the relationship between angle E and O, 18 students had *constructed* the idea that angle E is half of angle O. However, it is unclear students understood why angle E is half of angle O. In the whole group discussion, the teacher asked students to share their observations and tried to help students to see why angle E is half of angle O. The following transcript demonstrates how central angle theorem had been collectively verified in whole group discussion.

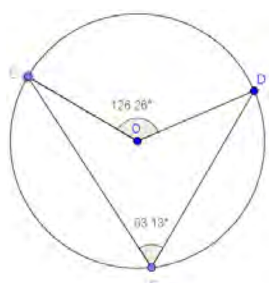

Figure 1a

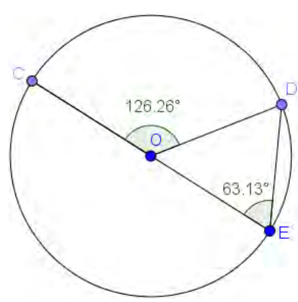

Figure 1b

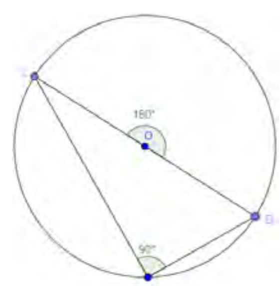

Figure 1c

Teacher: If we move angle D, what happens in general to the two angles that are given to you? (move D around the circle)

SS: They change.

Teacher: Yes, they are changing. So they are not remaining the same. What about when I move around angle E (move E around the circle)?

SS: They stay the same.

Teacher: They stay the same. Ok, let's put this at…here is a good one, because as just said we can move around E and nothing will change even if I make it overlap like this. That means I can make this as a diameter if I want to. Now it's a diameter [move E to make CE as a diameter as in Figure 1b]. What can I say about what's going on here? Think about last week and raise your hand. [no hand raising in 15 seconds] What congruence marks can I put here? [a few hands raising] Chase, what congruence mark can I put here?

Chase: The middle point and from the middle point to the other side [the teacher then put congruent mark on OE and OD].

Teacher: Anything else, Oliva?

Oliva: OD [the teacher put congruent mark on OD].

Teacher: Ok, these are all congruent. So what's going on here now? Does it help us connect? What do you see Jodi?

Jodi: Isn't like since both them are congruent you can put both angles would be equal?

Teacher: So which angels would be the same?

Jodi: The 63

Teacher: This one here 63.13. Now could I find this angle here [point to angle DOE] suppose I don't know this is 126.26 [point to angle DOC]? How would I do that? Andrew, what do I do?

Andrew: add up 63.13.

Teacher: 63.13 plus 63.13 which is 126.26. Is 126.26 already on there? Hmm, so the sum of these two angles give me this [point to angle DOC]. Are these two angles [point to angle OED and ODE] always going to be congruent no matter where I move that?

SS: Yes.

Teacher: Yes, these two should be congruent. So what do you think the relationship between this angle here E and this [point to angle DOC] we call this a central angle because it comes from the center. What do you think the relationship might be between angle E and the central angle? What do you think, Ethan?

Ethan: It's half of the central angle.

Teacher: So the central angle is twice as much as the circumference angle.

Ethan: Yes

Teacher: So even if I have it looking funky like this [move E up] is that relationship still the same? Is the relationship down here still the same [move E down]?

SS: Yes

Teacher: Yes, that will hold. Here is another way we can prove it. Last week when we looked the diameter as the leg of a triangle [move D to make CD a diameter of the circle as in Figure 1c]. What did we say about this angle here [point to angle E]? What would it always be if this is a diameter? Carol, what do you think?

Carol: 90 degrees.

Teacher: So this is 90 [point to angle E] and this is 180 [point to angle COD]. Does that relationship hold? Is the central angle twice the outside angle?

SS: hmm.

To help students understand why angle E is half of angle O, the teacher created the first special case of central angle theorem and then asked students what congruence marks could be put on the diagram (turn #5). The question triggered students to *recognize* that OC, OE and OD are radii of the circle (turn # 6 and #8) and therefore triangle DOE is isosceles. *Building-with* these ideas, students concluded angle ODE and angle OED are congruent (turn #10). The teacher then drew students' attention to connect the sum of the two base angles and the central angle (turn # 13 and #15). And students finally articulated that the circumference angle is half of its central angle (turn #18). The teacher then introduced a second special case to connect central angle theorem with inscribed right triangle (turn #23 and #25). Students *recognized* that E is 90 degrees, which is half of 180 degrees. This special case again verified that a central angle is twice as much as its circumference angle.

Discussion

Studies have shown that dragging and measuring features of DGS provides students opportunities to make and empirically verify conjectures and therefore construct new mathematical ideas (González and Herbst, 2009). However, the dragging and the measuring features of this environment also pose challenges for students to produce an abstract formal proof. Even though the teacher introduced two special cases to demonstrate why central angle theorem holds, a more general proof was missing in the whole-class discussion. The first special case has the very potential to be developed into a general proof had the teacher made it clear that why the sum of the two congruent bases angles of the isosceles triangle equals to the central angle. It's unclear in whether the teacher and students used the angles measures or a theorem (i.e. an exterior angle of a triangle is congruent to the sum of the two angles of the triangle that are not supplemental to the given angle) to conclude the sum of the two congruent bases angles of the isosceles triangle equals to the central angle (turn #13, #14, #15). If the conclusion was drawn based on the theorem, a general formal proof can be more easily developed from this special case. However, if the conclusion was drawn based on angle measures of the three angles, a deductive proof might be more difficulty to be obtained.

The data revealed even though central angle theorem was empirically constructed and verified, an abstract mathematical proof of the theorem was not collectively constructed by the teacher and students. In the process of verifying the theorem the teacher and students were mostly *recognizing* previously learned knowledge and partially *building-with* existing mathematical ideas. It seemed that the measuring and dragging functions prevented the teacher and students to construct an abstract formal proof. Further research is needed to explore how to move students to construct abstract proof in dynamic geometry environment.

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CONTEXTUALIZING DIFFERENT PATHWAYS AT THE CROSSROADS OF CRITICAL MATHEMATICS IN URBAN CLASSROOMS

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A wide range of findings related to Critical Mathematics (CM) have been reported alongside descriptions of diverse sets of CM activities and challenges and successes that teachers have experienced. Little research, however, has considered how teaching contexts may influence the teaching outcomes of CM. The intent of this paper is to highlight the importance of teaching contexts through a comparison of three cases of CM. In the cross-case comparison, I examine how the three CM teachers take different pathways at the crossroads that I frame as the tension of teaching mathematics vs. teaching students (Gutiérrez, 2009). The comparison offers possible explanations of how the teaching context may have influenced such contrasting outcomes of CM, and this adds social and cultural contexts (e.g., teacher's status and power, students' academic goals, and school culture) as another dimension to CM theory.

Keywords: Equity and Diversity; Instructional Activities and Practices

In North America, rooted in Frankenstein's (1983) pioneering work, Critical Mathematics (CM) in K-12 settings has gained traction as a way to support marginalized students' identities and power. Although different terms are used for CM (e.g., Teaching Mathematics for Social Justice, Radical Mathematics), approaches to CM are commonly characterized by efforts to interweave two types of goals: mathematics and social justice, and by aims to synergize two areas of learning: mathematical and sociopolitical.

As an increasing number of CM educators have been examining cases of CM, they have reported a wide range of outcomes in secondary school settings. Often, the discussions focus on how the social justice tasks are created and implemented and what challenges the teachers faced within the classroom; yet the discussion on how the variety of teaching contexts may have influenced CM has been lacking. Without considering the contextual influence, we may unfairly attribute the success/failure of CM to the teachers and their teaching.

My intent of this paper is to highlight the importance of teaching contexts for successful CM teaching, and I will do so by comparing three cases of CM. Some scholars have used cases of CM (e.g., Gutstein, 2003) and comparisons of cases (e.g., Esmonde, 2014) to expand theoretical understanding of CM. To expand theories of CM to account for connections between CM teaching and contexts, I analyzed three existing cases of CM from the research literature using a framework for pivotal moments of teaching, and in particular, moments when tensions arise between teaching mathematics and teaching students (Gutiérrez, 2009).

A Theoretical Framework for Understanding Inherent Tension in Critical Mathematics

CM scholars maintain that teaching CM is not a straightforward task; it requires facing many sorts of tensions and negotiating them (Enyedy & Mukhopadhyay, 2007; Gregson, 2013; Gutstein, 2003). Because CM aims to synergize two areas of learning (mathematical and sociopolitical), negotiating the tensions is at the crux of its success. To frame the tensions that emerge while teaching CM, I borrow Gutiérrez's (2009) three types of tensions inherent in teaching mathematics from an equity stance. In her essay, Gutiérrez (2009) describes the different types of tensions her pre-service teachers experienced when they were interacting with marginalized students in urban school settings. She categorized those tensions into three types: "1) knowing your students and not knowing your students, 2) being in charge of the classroom and not being in charge of the classroom, 3) teaching mathematics and not teaching mathematics" (p. 12). Later in the essay, she rephrased "not teaching mathematics" into "teaching students" to highlight the focus of one side of tension: students. In this manner, I will call her third type of tension as "teaching mathematics versus teaching students."

In this paper, I will focus my discussion on the third: the tension between teaching mathematics and teaching students, because it reflects the complexity of teaching CM in a classroom context with a rigid curriculum while confronting the gatekeeping role of school mathematics. CM teachers need to teach the required curriculum (i.e., teaching mathematics) to help their students advance their education and careers, but they also need to engage with their students around issues beyond the required curriculum (i.e., teaching students) to support their identities and power.

Comparing Three Cases of Critical Mathematics Teaching

To make meaningful cross-case comparisons, I selected three cases of CM teachers within seemingly similar contexts (i.e., minimized variation of context) with varying degrees of reported tensions (i.e., maximized variation of tensions). The selected teachers were: (1) Mr. Rico (his classroom name) in his own study (Gutstein, 2016); (2) Mr. Brantlinger in his own study (Brantlinger, 2013); and (3) Ms. Myles in Gregson's (2013) study. They all taught in urban secondary school settings where they predominantly served students of color from low-income communities.

In each case, I looked for the descriptions of social and cultural characteristics of the teachers, the students, and the schools from the articles. In addition, I looked for incidents of the teaching mathematics versus teaching students tension, and how the teachers negotiated this tension. The summary of the findings is presented in Table 1.

Discussion and Implications for Expanding CM Theories

Due to the seemingly similar teaching contexts of the three cases, I was surprised by the contrasting outcomes. Mr. Rico performed a skillful "dance" (p. 469) between mathematics content and critical and community content. Gutstein (2016) presented powerful evidence that illustrated how his students were able to use mathematics to understand the unjust society and to advocate for changes. On the other end of the spectrum, Brantlinger (2013) presented his unsuccessful experience of teaching CM to the students from similar demographic groups as Mr. Rico's students. He initially blamed his lack of experience and skills for the failure of his CM activities, but toward the end of the paper, he questioned if CM is a good fit for his urban students and his geometry course. Considering these two ends of the spectrum of CM teaching, I ask "What could have made their outcomes so different?" For one possible answer to this question, I now turn to the contexts of their teaching and consider the role that subtle differences in context might have played in shaping the tensions teachers experienced.

Differences in Teacher's Status and Power

Table 1 shows the various statuses of the teachers. Mr. Rico is a university professor, which affords him a certain degree of power within society. In general, professors are seen as people with more knowledge and professionalism than secondary teachers. Moreover, Mr. Rico was involved in the founding process of the school, and the school had consulted him for its teacher development programs. Being such an important figure in the school may have come with certain privileges: Mr. Rico was allowed to design his own curriculum for his course and to select mathematical topics that would fit well with his CM projects and his students.

Table 1: Summary of Three Cases

	Mr. Rico	*Mr. Brantlinger*	*Ms. Myles*
Teacher	University professor who was involved in founding process of the school and teacher development activities	Graduate student with 9 years of teaching experience	Full-time teacher with 12 years of experience with teaching and community organizing
Students	12th-grade students who chose Mr. Rico's class over 2 other options; Mr. Rico met with them twice prior to the school year.	Generally, older (18-19 years olds) students who are motivated to pass the course but not necessarily to learn	8th-grade students
School	Social justice focused high school	Night school with a vocational focus. It mainly serves students who failed a course in the day school or who have disciplinary issues.	Social justice focused school (grade 6-12) Ms. Myles plays a critical role to meet the federal requirement for the standardized exam.
Standardized Exam	Not required	Unknown	High-stake
Tensions	Not apparent. Mr. Rico skillfully interweaves mathematics and social justice goals.	Geometry content is not relevant to social justice topics; Students bypassed mathematics during the social justice discussions.	Time demand for social justice activities conflicts with the time needed for preparing for the standardized exam.
Negotiation	Moved his emphasis between mathematics and social justice so that these two types of learning support each other.	Incorporated statistic concepts that are out of the scope of the required curriculum; Separated mathematics and social justice portions of the lesson.	Downsized social justice project into a small problem-based lesson; Focused more on the test as the test gets near.

Mr. Brantlinger (a graduate student) and Ms. Myles (a full-time teacher), on the other hand, did not appear to have the luxuries that Mr. Rico had. In contrast to Mr. Rico, Mr. Brantlinger had to design his CM activities around the required geometry concepts although he did not see them as a good fit for his social justice activities. A similar tension was present in Ms. Myles's case: If the social justice activity required mathematics that was not emphasized on the gatekeeping assessment, the time needed for a social justice activity could consume the time needed to prepare for the assessment. In these two cases, a meaningful negotiation between teaching mathematics and teaching students required teacher autonomy to select mathematics content often not included in the required curriculum and on high-stake exams.

Differences in Students' Academic Goals and School Culture

Although all teachers predominantly had students of color from low-income communities, I noticed differences in students' academic goals in the respective mathematics classes and different foci of students' schools. Mr. Brantlinger's students being "more concerned with getting through his course for accreditation than for learning mathematics" (p. 1057) suggested that his students did not willingly choose the geometry class but, instead, were required to take the course for high school graduation. Moreover, the focus of his school was not social justice but vocational training. Considering the school's focus and the students' desire to obtain their geometry credits to move on with their post-secondary lives, Mr. Brantlinger's social justice activities likely came as a surprise for his students. In this sense, it appears reasonable for some of his students to perceive his social activities task as a distraction from their academic goals.

In contrast, the schools of Mr. Rico and Ms. Myles have an explicit vision and mission for social justice. Hence their students had more likely experienced social justice related topics in their previous classes, were less likely to resist social justice activities. Moreover, Mr. Rico's students chose his class out of three options for their fourth-year mathematics courses. He also met with the students twice prior to the school year to discuss the topics; thus his students were well informed about political contexts of Mr. Rico's CM course, and his students' academic goals were more likely aligned with the CM activities compared to the students of Mr. Brantlinger. This illustrates how the particular school culture and students' academic goals may support CM teachers to negotiate the tension in a meaningful manner.

The above comparisons of the three cases of CM offer the possible explanations to how the teachers took different pathways at the crossroads of CM in relation to their teaching contexts. This paper adds one more dimension to the theory of CM: teaching context, in particular, differences in teacher status and power, students' academic goals for the course, and school culture, may heavily influence the success/failure of CM. Similar to Brantlinger (2013) blaming himself, we often attribute the instructional success/failure to the teachers' knowledge, experience, or actions; however, this comparison offers another way to look at how the teachers negotiate the tensions of CM. The context, which is sometimes out of the teacher's control, can funnel the teacher into decisions that are far from ideal. This suggests we need to work not only from the inside (teachers and teaching) but also from the outside (broader teaching context), so that innovative teaching methods, like CM, can evolve and flourish.

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DEWEY ON EARLY CHILDHOOD TEACHERS' EXPERIENCES LEARNING AND TEACHING MATHEMATICS

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Using initial and debriefing interviews around a photo visual narrative inquiry (Bach, 2007), this pilot study aimed to create space for 10 early childhood teachers to explore and make sense of their experiences learning and teaching mathematics, as well as to hear their voices sharing detailed and nuanced views of their strengths, needs, and areas for growth related to mathematics teaching and learning. Components of Dewey's (1938/1998) experience construct, including continuity, interaction, social control, and subject matter were used to understand nuances of the teachers' experiences learning and teaching mathematics. Initial findings indicate that these teachers had negative mathematics learning experiences, but are now enthusiastic early childhood mathematics teachers, indicating that their experiences should be valued and explored in training, and by those who determine their training.

Keywords: Early Childhood Education; Teacher Education – Inservice/Professional Development; Instructional Activities and Practices; and Policy Matters

Purpose

Early childhood teachers ask why they are required to take college algebra courses in which they struggle and whose content is not related to the work they do with young children. Their question implies that expectations for their training and professional development are not well-aligned with their experiences of mathematics preparation or teaching work. I argue the teachers' perspectives and voices are not included in the dialogue about their mathematics training and teaching work. I aim to create space for early childhood teachers to explore and make sense of their experiences with mathematics. Also, I aim to give early childhood teachers a voice and platform from which to share a more detailed, nuanced view of their mathematical learning and teaching strengths, needs, and areas for growth. I report initial findings from a pilot study in which I joined with a community of early childhood teachers to explore their experiences learning and teaching mathematics with young children.

Perspectives and Theoretical Framework

Standards for professional preparation and early mathematics are well-aligned with the mathematics teaching work that teachers do (NAEYC, 2009, 2016; NAEYC & NCTM, 2002; NAECTE, n.d.). However, goals put forth in the standards do not always reach preservice and in-service teachers (Sarama & DiBiase, 2004). Accounting for that gap is a set of policy, funding, and social viewpoint quagmires (Grunewald & Rolnick, 2010; Heckman, 2008, 2011). Early childhood education policy and funding hail from all levels of government and not-for-profit agencies (US Dept. of HHS, 2014; United Way Worldwide, 2016). Social viewpoints range from understanding early childhood education as a salve for educational and social strife to babysitting (Phillips & Shonkoff, 2000). Researchers make suggestions for early childhood teachers' mathematics practices based on their own research observations (e.g., Parks & Blom, 2013). Missing is the perspective of early childhood teachers' professional voices. This multi-faceted research puzzle considers gender and mathematics (Brooks, 2007; Fennema & Sherman, 1977; Hancock, 2001), early childhood education as "women's work" (Kim & Reifel, 2010) and the social situatedness of teaching and learning (Edwards, 2003), which creates a need for a multi-pronged investigation approach.

A framework for delving deeper into teachers' lived experiences with mathematics, in order to understand nuances of their experiences (Clandinin & Connelly, 2000) and empower participants as early childhood mathematics is used. Dewey's (1938/1998) *experience* construct, including components *continuity*, *interaction, social control,* and *subject matter,* is used to understand nuances of early childhood teachers' experiences learning and teaching mathematics.

Research Questions

What role, if any, do mathematical learning experiences play in early childhood teachers' mathematics teaching practice? In what ways do their voices contribute to the professional dialogue?

Methods

This study used visual narrative inquiry methods (Bach, 2007) to situate the participants' experiences learning and teaching mathematics within the three-dimensional inquiry space, situating it within the teachers' mathematical history, classroom context, and social milieu.

Participants

Ten teachers in an early childhood lab school at a Midwestern state university participated. Participants included seven lead teachers, two assistant teachers, and one student teacher. Teachers had been at the center from one to 22 years. One assistant teacher has a four-course early childhood certificate, seven teachers have bachelor degrees related to teaching young children, and two have master degrees. The children in the classrooms range in age from two to 5 years, with 14 to 20 children in each classroom depending on age.

Data Sources

This study included three data collection activities: (1) initial semi-structured interviews with classroom teaching teams, regarding their experiences learning and teaching mathematics; (2) teaching teams and researcher photographing mathematical activity and materials; and (3) debriefing interviews during which each classroom teaching team described the mathematics captured in photographs taken by themselves and the researcher.

Analysis

Several rounds of analysis were conducted to gain a sense of the underlying relations and tensions (Clandinin, Murphy, Huber, & Orr, 2009) that teachers experience as they learn and teach mathematics. Initial interview questions were answered from the transcripts for each teacher. Passages from all initial and debriefing interview transcripts were marked for the main principles of Dewey's (1938/1998) *experience* construct, *continuity* and *interaction*. Additional components of Dewey's experience construct, *social control*, and *subject matter* were marked in each transcript. Anecdotal snapshots of each classroom were developed, and short narratives of each teacher were created that included a timeline and dominant themes found in their individual transcripts. From these pieces, narratives were developed of the teachers as learners and teachers of mathematics with young children, along with their philosophies. Photos related to transcript passages accompany the narratives.

Preliminary Findings

Here, data are related to components of Dewey's (1938/1998) experience construct.

Initial Interview: Union of Continuity and Interactions

E1 and D2 did not like geometry as students and had trouble learning it (internal condition), but as early childhood teachers, they have learned about geometry with and from the children and since developed engaging geometry activities (objective condition) (see Figure 1).

Figure 1. Geometry Activities.

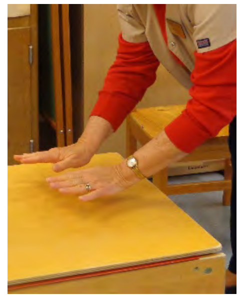

Figure 2. Table Rhythm.

Figure 3. Letters & Math

Photographing Mathematical Activity: Social Control

The researcher photographed S1 pounding rhythmically on the lid of the sensory table (see Figure 2). When asked about this photo, she explained that this is a routine used for multiple purposes: a signal to the children to clean up (which they do eagerly when they also get to pound), the children practice imitating and following, and they learn patterns and counting. S1 used her knowledge of the children's interests and subject matter in this routine that encourages social organization.

Initial and Debriefing Interviews: Subject Matter

D2 shared an activity that illustrates several facets of subject matter. She incorporates patterning into the lining up routine to both organize the children and connect an everyday occurrence to mathematics content. Children "keep track" (Dewey, 1938/1998, p. 110) of the pattern, reconstructing the experience daily, and self-correct as they line up or notify the teacher when they no longer have enough children to keep the pattern going as defined (i.e., more girls than boys mean a girl-boy pattern will end with a group of girls).

While examining photos, S1 and D2 were surprised as they noticed previously unrecognized instances of mathematics. Referring to the Table Rhythm photo (see Figure 2), S1 said, "Gosh, you know what? You don't think about these things. You do it so many times, you don't think about that. I guess I've never seen a picture of that before." D2 said (see Figure 3), "Look how many letters I have, and I was like, oh, my gosh. We've got math! So, I quickly found the camera and took that, and I guess this research project to me, more than anything, has made me more aware of math is in lots of places other than one plus one is two."

Discussion

Teacher interviews and photos analyzed using Dewey's (1938/1998) *experience* construct and components indicate that teachers' mathematical learning experiences influence their current practices, in their dispositions to learn mathematics from and with the children, and to make math "fun" so the children associate mathematics positively (Lenz Taguchi, 2005). The study afforded teachers an opportunity to uncover unforeseen mathematics practices, nurturing curiosity in their practices and understanding of children's mathematics. Teachers eagerly taught mathematics with young children, despite many having negative feelings from their own "mis-educative" (Dewey, 1938/1998, p. 13) experiences learning mathematics.

Early childhood teacher educators should respect the knowledge early childhood teachers bring to training, and help these teachers unpack meanings from their experiences. Also, this might indicate that grounding math methods or content in experiences *with* young children could more productively engage early childhood teachers who feel less competent in mathematics training settings (Philipp, 2008). A next step is for early childhood teachers to communicate their experiences to the funders and stakeholders who determine their training.

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DO YOU SEE WHAT I SEE? CONNECTING MATHEMATICS TO THE REAL WORLD

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This study explored how three high school mathematics teachers thought about using real world connections to teach mathematics. Each of these teachers expressed beliefs about mathematical objects as entities that could be seen in the real world. This belief was connected to ideas about knowing mathematics as seeing these mathematical objects. This paper elaborates this epistemology of visibility and describes some of the implications.

Keywords: Learning Theory; High School Education; Affect, Emotion, Beliefs, and Attitudes

Most teachers, educational researchers, and policy makers agree that students learn best when mathematics problems are embedded in real world contexts. These real world connections support students by allowing them to use their prior knowledge and experiences to make sense of new mathematical ideas (e.g., NCTM, 2000). These connections can also prepare students for their careers and adult responsibilities.

In spite of widespread agreement about the benefits of real-world connections, there is limited research about teacher understandings of and uses of these connections in their classrooms. Gainsburg (2008) reported on *ways* in which teachers used real-world connections including the contexts they referenced, the format of problems, and the constraints they noted. Her detailed study provides an important overview of real-world contexts in secondary mathematics classrooms. Two other studies sought to make sense of *how* teachers understood and presented the real-world contexts in mathematics word problems. Chapman (2006) and Depaepe, De Corte, and Verschaffel (2010) examined the epistemological framings, or ways of knowing, teachers brought to the teaching of word problems. Both studies considered whether teachers focused on paradigmatic knowing, in which the key to knowing was to strip away the context and focus on minimal necessary information to reason to an answer, or on narrative knowing, in which the focus was on the social context of the word problem. The authors argued that students needed opportunities to consider word problems from both paradigmatic and narrative perspectives and that it was challenging for teachers to juggle these two different perspectives within a lesson.

These studies begin to paint a picture of teacher understandings of real-world contexts. However, they are necessarily only a small glimpse into what is an important component of mathematics education. In our study, we sought to add to this understanding of teachers' work with real world contexts. In particular, we wanted to better understand how teachers thought about mathematics in the real world and about what it would mean for students to come to know that mathematics.

Theoretical Framework

Earlier studies of epistemology and mathematics teachers' real-world connections focused specifically on paradigmatic and narrative framing of ways of knowing. While this provided interesting insights into teachers' perspectives, it was also a narrow lens on the nature of knowledge. In our study, we sought to add to this picture by considering how teachers' epistemological beliefs might be connected to their ontological beliefs.

Epistemology is the study of beliefs about knowledge and ways of knowing (Olafson & Schraw, 2010). For example, some people believe that knowledge is created or constructed by the mind and has no existence in the real world. Others believe that knowledge comes only from what one can

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sense. These beliefs are influenced by one's ontological beliefs, or beliefs about the nature of objects and of their existence (Kang, 2008; Packer & Goicoechea, 2010). For example, teachers may hold the ontological belief that mathematical ideas are entities that can be treated as though they were real-world objects. Ontological beliefs are connected to understandings of the nature of knowledge and of knowing so that some teachers might think that students need to interact with these mathematical objects in order to gain knowledge about them. These beliefs also influence teachers' instructional practices so that if teachers assume that mathematical ideas can be treated as real-world objects, they then design lessons in which students have opportunities to find real world examples of mathematical ideas.

In our study, we investigated the ontologies and epistemologies of mathematics teachers as they considered the role of real-world connections in teaching students.

Methods

This research was conducted in the context of a three-year master's degree program in mathematics and science education at a large research university in the Southwest. Three high school mathematics teachers from the master's program accepted our invitation to participate in the study. All teachers were within the first seven years of their teaching careers. They taught in suburban public, urban charter, or urban public schools. The student populations at the schools ranged from 28% to 59% receiving free or reduced-rate meals and were from 33% to 83% non-white. All teacher names are pseudonyms.

Data Collection

We conducted initial interviews with each teacher, asking about their goals and purposes for teaching mathematics, their ideas about what makes mathematics instruction relevant or authentic, and their perceptions of how real-world mathematics might relate to school instruction. We also observed and recorded field notes on a mathematics lesson in each teacher's classroom. We conducted pre- and post-observation interviews in order to learn more about each lesson. All interviews were recorded and transcribed.

Other data sources included selected written assignments from a graduate course taken the first summer of the program. Teachers wrote two short essays on prompts that asked them to reflect on (1) what curriculum is and what teachers should do to make connections among students, the disciplines of mathematics, and the real world; (2) a lesson that they had taught in the past that made connections between students, the school curriculum, and the "real world."

Analysis

For each teacher, we parsed their talk and writing into examples of real-world connections. We defined an example of a connection as a description of an activity, lesson, event, or student interaction in which the teacher attempted to involve students in learning mathematics or science by referring in some way to the real world. This process produced 18 examples.

We then open-coded (Esterberg, 2002) the data for each teacher, seeking themes in how the teachers talked about knowing mathematics in the real world. As we found patterns in these themes, we developed a set of categories and then recoded the data using those categories. The findings in this paper elaborate just one theme that emerged from the data.

Findings

Knowing is Seeing

When the teachers talked about students and mathematical ideas, they frequently used discourse that either literally or metaphorically referenced seeing as a way to know mathematical objects. This talk aligns with other expressions used to describe knowing or understanding. For example, we might exclaim, “I see” when we suddenly grasp an idea. This phrase suggests that we understand knowing, at least metaphorically, as a kind of seeing.

For example, one teacher, Brian, described the difference between teaching as giving and teaching as showing:

> I’ve gotta teach them the basic principles, I need them to *see* why that they can do what they’re doing, ya know. For instance, the quadratic formula, I could give that to them. I could say here is the quadratic formula, let’s divide the square root. But then if I tell them, let’s use it in this problem, they don’t know what it means. They don’t know where it comes from, and they just don’t *see* the use of it. Whereas if I go though it with them and actually *show*, here, here’s how you do a *y*, the actual quadratic formula, they *see* the purpose behind it. They *see* the understanding and so … they know how it works. [Italics added.]

Brian started this response by describing the problems with a transmission model of learning in which he would merely *give* the quadratic formula to students. In contrast, he felt it was important for students to *see* where the formula comes from. If they saw the purpose of the formula, they would then understand it and could use it to solve problems.

Another teacher, Rohit, said,

> I show them [students] one video where the parabolas can be used. For example, most of the antennas and antennae dish are parabolas and so we can *see* the focal point. Or the headlights of the car and porch lights; all these are to *see* the parabolic shape inside and built there. And then they connect the parabolas, what they learn in here, to the real life applications and it’s kind of a little bit fun…It makes a little bit [more] sense to them why they’re learning parabolas and all that stuff. [Italics added]

As students learned to see parabolas in a variety of real world objects, they would understand why parabolas were important to learn about.

These are just two examples of seeing. Of the 18 total examples of real world connections, seven contained instances of what we are calling an *epistemology of visibility*. In other words, these teachers seemed to think about knowledge as seeing mathematical objects. While this epistemology of visibility was not the only framing of knowing used by these teachers, it was striking that each of the teachers made numerous references to seeing as knowing.

Mathematical Ideas Exist

This framing of knowing as seeing seems to arise from the teachers’ thinking about the nature of mathematical knowledge. When teachers discussed their efforts to connect mathematics to real world perspectives, they consistently described mathematics in terms of real objects that could be found in the world. For example, in a lesson on congruent triangles, one teacher (Jessica) pointed out examples of congruent triangles in structures surrounding the school, including bus stops and high-rise cranes. In his quotation above, Rohit talked about parabolas as though they were real entities that could be visibly perceived.

The notion of a visual aspect to knowing mathematics is not new. However, studies of visualization in mathematics are typically focused on visual representations and contrast those to nonvisual representations such as algebraic equations (Rivera, 2011). In contrast, these teachers talked about equations as though they were objects that existed and could be known through seeing them.

Conclusion

In this study, we noted a novel epistemology, one that we are calling an *epistemology of visibility*. In this epistemology, teachers hold ontological beliefs that mathematical ideas are objects that are visible (either literally or metaphorically) in the real world and that knowledge of these objects is gained by seeing the objects in the world. In other words, the work of teaching is showing mathematical objects and processes to students so that they might come to know them.

This epistemology is an important variation on the commonly discussed epistemology of knowledge as received. Received knowledge epistemologies frame knowledge as something that can be conveyed or transferred from the teacher to the student. Knowledge is information like facts that can be organized into discrete bundles. Learning is about taking in these facts. In contrast, an epistemology of visibility is less about knowledge transfer and more about coming to see or notice objects. The knowledge is not contained within the mind of the student, but is instead present in the world and noticed by anyone who looks.

This finding offers important insights into teachers' attempts and frustrations in using real world examples to teach mathematical ideas. Teachers are sometimes puzzled when students, whose mathematical vision is not the same as the teachers, fail to see, or become enthusiastic about, mathematical objects that are so obvious to teachers. Teachers may also fail to understand how students cannot grasp a mathematical explanation that is so clearly illustrated in real world phenomenon.

Real world connections can be an important learning resource for students. However, if teachers are going to make the most of these connections, they may need to reflect upon their epistemological beliefs and consider how their teaching relies upon a way of seeing that may not yet be possible for their students.

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EXPLORING THE RELATIONSHIP BETWEEN TEACHERS' NOTICING, MATHEMATICAL KNOWLEDGE FOR TEACHING, EFFICACY AND EMOTIONS

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Observing and responding meaningfully to students' thinking is a key component of effective teaching. In this regard professional noticing of students' mathematical thinking has become an important area of study in mathematics education. Researchers have examined the effect of teachers' experiences on their abilities to notice, however less work has been done examining the role of cognitive and psychological factors on what and how teachers notice. In this study, we sought to explore if and how these constructs, specifically teacher efficacy, emotions and mathematical knowledge for teaching, influenced how and what teachers were able to notice.

Keywords: Mathematical Knowledge for Teaching; Affect, Emotion, Beliefs, and Attitudes

Purpose of the Study

To teach effectively, teachers must attend to students' thinking during the act of teaching and make in-the-moment decisions about the best ways to respond to what they observe (Ball and Cohen, 1999). In efforts to better understand what teachers observe, how they respond to it and how it can be influenced, over the past few years a significant body of work (e.g. Sherin & van Es, 2009) has emerged documenting the role teacher noticing has played in teacher learning and subsequently improvement of instructional practices (Sherin, Jacobs & Philipp, 2010). Some of the work on teacher noticing has focused on the differences in what and how teachers' notice concluding that expert teachers tend to interpret and recall classroom events with greater detail and insight than novice teachers.

Although researchers have examined the effect of teachers' experiences and professional development on teachers' abilities to notice students' thinking, less work has been done examining the role of cognitive and psychological factors on what and how teachers notice. In particular, we were interested in the relationship between the teachers' mathematical knowledge for teaching (MKT) – specifically their common content knowledge (CCK) and their knowledge of content and students (KCS) -, emotions, efficacy, and their level of noticing. In this study, we explore if and how these constructs influenced how and what teachers were able to notice.

Theoretical Framework

We know from research that teachers' mental lives greatly influence their teaching experiences (Schutz, Hong, Cross & Osbon, 2006), commitment to the profession and overall wellbeing. However, absent from the literature are the ways in which specific cognitive and psychological constructs, such as MKT, TE and emotions, influence teachers' instructional activity, specifically noticing. A growing body of literature on emotion and cognition has shown that one way in which emotional valence influences an individuals judgment is by foregrounding and making accessible emotion-congruent information in memory (Forgas, 2001). So individuals experiencing positive emotions tend to identify and recall positive memories more easily than negative memories, and vice versa (Bower, 1981; Forgas, 2001;). Subsequently, individuals in a positive emotional state tend to evaluate people, places, and events more favorably than people in a negatively emotional state (Forgas, 2001). As such, the emotions teachers experience as they engage in noticing may influence what and how they notice. Teacher efficacy (TE) is defined as a teacher's belief about her capacity to affect how students learn and their overall performance (Tsachannen-Moran, Woolfolk-Hoy & Hoy, 1998) and has been shown to influence teachers' willingness to adopt and enact particular instructional practices. Knowledge efficacy related to mathematics describes a person's confidence in his/her understanding of mathematics content (Roberts & Henson, 2000). Personal efficacy describes a person's confidence in his/her ability support students' learning through their teaching. Teachers with a strong sense of efficacy tend to exhibit greater levels of planning and organization, are more open to new ideas and more willing to experiment.. Mathematical knowledge for teaching (MKT) encompasses deep knowledge of math concepts and the knowledge and skills to attend to students' thinking during the act of teaching and make in-the-moment decisions about the best ways to respond to what they observe (Ball and Cohen, 1999).

Methods

This study involved eight elementary teachers who were involved in a professional development program. The teachers were each involved in coaching with each coaching cycle consisting of a pre-meeting, post-meeting lesson planning, the coaching session and a post-coaching meeting. All meetings with the teachers were audiorecorded and the coaching session, in which the coach supported the teacher during the teaching of a lesson, was videorecorded. Data for this study was drawn from the videorecordings of the coaching sessions, and the audiorecordings from the post-coaching meetings. Teachers also stated the emotions they experienced prior to, during and after the coaching experience. They also completed the adapted SETAKIST (Self-Efficacy Teaching and Knowledge Instrument for Science Teachers) for Teachers of Mathematics. The 16-item, 5-point Likert scale survey was scored to determine a score for personal (8 items reversed scored with a score of 8 being high) and knowledge efficacy (8 items with a score of 40 being high). All teachers completed the MKT surveys. The raw scores were converted to IRT scaled scores then aligned with a percentile value. Scores above the 80^{th} percentile indicated high MKT, between 61^{st} and 79^{th} percentile medium MKT, and below 61^{st} indicated low MKT. To classify their noticing, we first watched the coaching video and selected clips that focused on students' thinking. We also asked the teachers to select three videoclips he/she wanted to discuss with the coach in the post-coaching meeting. Then we used Van Es (2011) noticing framework (Fig. 1) to analyze the clips the teachers selected to determine what and how teachers noticed and compared the emotions they stated, their efficacy related to mathematics teaching and their level of MKT.

Findings

For the purposes of this proposal, we present the results of two cases to show the relationship between teachers' emotions, TE, MKT and *what* and *how* they noticed.

Table 1: Results of Analyses of Teachers' Noticing, MKT, TE and Emotions

	Noticing		MKT		Emotions			Efficacy	
	What	How	KC	KCS	Before	During	After	KE	PE
Bill	Level 3	Level 3	High	High	Nervous/ Anxious	Nervous/ Uncomfortable	Nervous/ Uncomfortable	Mid	Low
Kathy	Level 3	Level 1	Low	Low	Indifferent/ Excited	Calm	Calm	High	High

We observed that the teachers were attending to students' mathematical thinking to some extent. In the cases of Bill and Kathy who both scored level 3, although they were attending to students' mathematical thinking, they were not consistent in making explicit connections between what they noticed and the teaching strategies they observed. In contrast to the score on *what* they noticed, there was variation in how they noticed. Bill identified significant events that showcased students' thinking, and explained why it was noteworthy drawing evidence from the videoclip. For example, one of the clips Bill selected to discuss was one where a student drew connections between the organization of base 10 blocks into recursive groupings of 10 to the way stickers, packets (10 stickers in a packet) and envelopes (10 packets in an envelope) were being used in a place value activity. Bill selected this clip because the student was able to recognize the similarities between the groupings of smaller units into larger units around the base of 10 showing his growing conception of place value. Although Kathy identified a clip that showed students' reasoning about number patterns as noteworthy, she did not discuss how her teaching strategies were linked to the students' ability to identify number patterns. She also made general comments about the difficulty grade one students experience in identifying number patterns, without any evidence, justification or suggestions of pedagogical approaches that would support their learning.

	Level 1	Level 2	Level 3	Level 4
What teachers notice	Attend to whole class environment, behavior and learning to the teacher pedagogy	Primarily attend to teacher pedagogy	Attend to particular students' mathematical thinking	Attend to the relationship between particular students' mathematical thinking and between teaching strategies and mathematical thinking
		Begin to attend to particular students' mathematical thinking and behaviors		
How teachers notice	Form general impressions of what occurred	Form general impressions and highlight noteworthy events	Highlight noteworthy events	Highlight noteworthy events
	Provide descriptive and evaluative comments	Provide primarily evaluative and some interpretive comments	Provide interpretive comments	Provide interpretive comments
	Provide little or no evidence to support analysis	Begin to refer to specific events and interaction as evidence	Refer to specific events and interactions as evidence	Refer to specific events and interactions as evidence
			Elaborate on events and interactions	Elaborate on events and interactions
				Make connections between events and principles of teaching and learning
				On the basis of interpretations, propose alternative pedagogical solutions

Figure 1. Noticing framework.

We observed that there was alignment between high MKT and strong noticing abilities. Bill, who had high MKT also scored on average at level 3 with respect to what and how he noticed. However, Kathy who scored low on MKT, was level 3 on what she noticed and level 1 how she noticed. These results would suggest that to be able to identify significant incidents of students' mathematical thinking and to attend to the relationship between such thinking and teaching strategies (key aspects of high level noticing) requires deep understanding of mathematical concepts and how students make sense of mathematical ideas (key aspects of MKT). On the other hand, low MKT may not be a significant in order to attend to particular students' thinking but essential to making connections to instructional practices.

The relationship between teachers' efficacy, their MKT and how they noticed was interesting. In particular, Bill had mid-low scores for both knowledge and personal efficacy, yet he had high levels of MKT and noticing. In contrast, Kathy had high scores for efficacy but demonstrated low levels of MKT and noticing. We also observed that unpleasant emotions, such as anxiety and nervousness was positively related to how teachers noticed, while more pleasant emotions aligned with lower scores on how they noticed. In this context, we believe teacher efficacy may have a significant influential factor, in that Bill had fairly low confidence in his knowledge of math and teaching which may have encouraged him to pay more attention to students' thinking in the class as well as how his teaching actions supported or hindered students' reasoning. It may also have contributed to his feelings of anxiety throughout the experience. On the other hand, Kathy was very confident about her knowledge of mathematics and teaching so she may have attributed any struggles students had to the individual student, or students in general, therefore not focusing attention on how her teaching actions may have influenced their thinking. Overall, low knowledge and high efficacy aligned with more pleasant emotions, while high knowledge and low efficacy aligned with unpleasant emotions (e.g., nervousness, anxiety).

Discussion

It's not surprising that a teacher with strong MKT is able to identify significant mathematical events, interpret them meaningfully and connect them with instructional practices. This finding aligns with existing research that suggests that high quality teachers are skilled at effectively utilizing students' thinking to drive instruction. Of note however is that a teacher with low MKT can also attend to students' thinking but may struggle to interpret the thinking or meaningfully connect it with pedagogical solutions. Our findings suggest there may be other factors at play or there are dimensions of MKT that may better support noticing. In this regard, the relationship between teacher efficacy and noticing provides some insight. Being overly confident about one's mathematical knowledge and teaching may negatively influence a teacher's response to observed students' thinking, in that the teacher may overlook the ways teacher moves influenced students' thinking (particularly in the case of errors or misconceptions) or may not have the knowledge needed to make connections to principles of teaching and learning or to propose alternative solutions. Further examination of the full corpus of data should provide greater insight into the nature and strength of the relationship among these constructs.

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HOW STUDENT QUESTIONS IN MATHEMATICS CLASSROOMS ARE RELATED TO AUTHORITY DISTRIBUTION

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This paper analyzes the questions students ask in two very different middle school mathematics classrooms. Both classrooms were taught by experienced and respected mathematics teachers, but the instruction was very different between the two classrooms. Mr. Cordoba characterized his instruction as student-centered, project-based, and focused on understanding. Mr. Ezzo used a more traditional approach. Classroom instruction was videotaped, student questions were highlighted and categorized, and instructional moves were analyzed. Results show that authority distribution in the classroom is connected to the types of questions students ask in mathematics lessons.

Keywords: Classroom Discourse, Middle School Education, Equity and Diversity

Objectives and Framework

Studies show that teachers who distribute authority within their classrooms promote more productive discourse than teachers who maintain all the authority with themselves and with the textbook (Herbel-Eisenmann, 2007). An important component of classroom discourse is student question asking. This study seeks to build on Herbel-Eisenmann's work by showing that the types of questions students ask in mathematics lessons are linked with the way authority is distributed in the classroom.

Methods

Two middle school mathematics teachers, Mr. Cordoba and Mr. Ezzo, were selected for this study because their students frequently ask questions in whole-class lessons and their instructional styles are very different from one another. Mr. Cordoba engages in complex instruction (Cohen & Lotan, 2014) and asks students to collaborate (more than half of class time is spent in small group work). He presents his students with tasks that require high cognitive demand (Stein & Smith, 1998) and states that students should be "active participants in mathematical activity through generating and discussing mathematical ideas in class." Mr. Ezzo's approach is what some might call traditional: every day class begins with a scoring of the previous night's homework followed by a lecture on new material. Each night for homework, students in Mr. Ezzo's class solve mostly context-free problems in the precise way they were taught by their teacher.

Data for this study consisted of approximately 15 hours from each teacher of transcribed videotape of whole-class lessons (i.e. small group work was not transcribed). Extensive field notes, semi-structured interviews with each teacher, and student surveys were also used. All student questions were highlighted in the transcripts. These questions were then counted and coded using a categorization schema (Kemmerle, 2016) designed to highlight the different kinds of student questions asked during mathematics lessons. See Table 1 for a selection of the categories that showed up most frequently in the data.

During whole class lessons, students in Mr. Cordoba's classroom asked, on average, 11 questions per hour and these questions most often fell in the Mathematical Curiosity/Extension, Visual Representation (Conceptual), and Seeking Conceptual Information categories. Students in Mr. Ezzo's classroom asked, on average, 56 questions per hour and these questions most often fell in the Correct Answer, Form of Answer, and Meeting Teacher's Expectations categories.

Next, the video transcripts and interview transcripts were open-coded (Glaser, 1978) for instructional moves or instructional philosophies that might affect the questions students ask, and it was found that distribution of authority (Cobb, Gresalfi, & Hodge, 2009; Boaler, 2012) appeared to be related to the types of questions students asked during their mathematics lessons.

Table 1: Abbreviated Categorization of Student Questions in Mathematics Classrooms

Question Category	Examples from Transcripts
Form of Answer	-Do I have to show it as a repeating decimal, or can I round up? -I didn't simplify, is that okay?
Correct Answer	-I got 74%. Is that right? -But what is the answer?
Assessment/Grading	-Could I get extra credit for this? -Can we do any of these ways on the test? -How many points are on the quiz?
Meeting Teacher's Expectations	-Wait, do we copy down the percents bar model too? -Do you want us to draw it or label all the points?
Seeking Procedural Information	-How did you get 16? -What is your shortcut for filling in your t-table?
Seeking Conceptual Information	-Wait, how is it 3 out of 5? -Why did you have to cross out the 40 and the 10?
Visual Representation (Conceptual or Informational)	-If your x were to get bigger and then the trend line, would your points, can it ever cross the x line? -Where would it be on the graph?
Mathematical Curiosity/ Extension	-Can *k* be an odd number, like 27, and could you still have *x* and *y* be the same number? -If k were to be bigger and the k was negative, where would it be?

Findings and Conclusion

Mr. Cordoba and Mr. Ezzo handled authority very differently in their respective classrooms. During whole class discussions, Mr. Cordoba almost always stood in the back or at the side of the classroom and asked students to present ideas to each other. During one lesson, a group of four students gave a presentation to their classmates about indirect variation ($y=kx$). After their presentation, a few students in the audience asked fairly simple questions such as "Could you please repeat your summary statement because I couldn't hear?" and "What was your *k* because I don't see the point on the graph?" At this point, Mr. Cordoba instructed the class on what kind of questions he expected from them. He said,

> You guys have done a good job asking clarifying questions. Like I especially appreciate whoever asked Sophia to clarify her [ideas] because the first time I didn't know what she was talking about, but the second time I understood it much better. Now I'm wondering if you have questions that would, not just clarifying questions, but questions that would push them further. Let me say why I'm saying that. Three of you have really clear summary statements. Remember I am looking for the summary statements to include statements about the graph, t-tables, and reasons and I don't think any of you covered all three of those. So [to the audience] can you think about what you heard from each of them and ask them a question that pushes a little bit, so they can get full credit just like everybody else?

There are multiple norms depicted in this excerpt. First, Mr. Cordoba thanks the audience for being helpful. He also makes it clear that they are a community of learners, responsible for each other's success. The audience is expected to listen carefully and take their peers' explanation of mathematics seriously. Mr. Cordoba thus shares his authority with the four group members who are presenting; he expects them to explain their thinking using "graphs, t-tables, and reasons." His expectation that the presenters use mathematical representations and reasons requires the presenters to first understand the content and second to explain it in a convincing way. Mr. Cordoba requires them to be the experts who have real mathematical knowledge to present to the class. In addition, he expects the audience to also have knowledge and understanding and to share in the responsibility of pushing the community forward in their mathematical understanding and their ability to explain it to the class.

Immediately after Mr. Cordoba's statement, audience members proceeded to ask the presenting group questions that fell mostly into the Mathematical Curiosity/Extension and Visual Representation—Conceptual categories. Here is a partial excerpt from the transcript:

Student from Audience: Did you notice any patterns on your t-tables?
Student Presenter: I noticed that when x was smaller, it had a bigger y and when x was bigger it had a smaller y.
Student from Audience: Is there a way that your point could cross the x and y axes?
Student Presenter: No, it could not, because in order for it to cross, a number times zero would have to equal 16 and a number times zero is zero, so it couldn't cross.
Student from Audience: Are any of you guys, do you know what would happen if your k was negative?
Student Presenter: What would happen to what?
Student from Audience: If your k was negative, would you notice any patterns?
Student Presenter: My graph, if my k value was negative, then I would have to multiply a negative number by a positive number to get my negative k so it would be down here (points to graph), depending on if the x number is negative or the y number is negative.

The shift in authority is clear here. Students are the source of mathematical ideas for each other. Student questions from the audience guide and direct the discussion. Mr. Cordoba is not the source of all mathematical knowledge; instead, he encourages students to ask each other and answer meaningful questions, thus shifting expert authority away from himself. On other occasions, Mr. Cordoba said the following statements to students who were asking questions to the presenting group, "Can you try your question again because what you are asking is really important and here's why…" and "I'm really glad you asked that, Luis" and "That's a good question, Carlos, did you understand her answer?" These statements encourage students to ask meaningful and productive mathematical questions.

In Mr. Ezzo's classroom, in contrast, the teacher stood in the front of the classroom almost 100% of the time and was the only person in the classroom who wrote on the Smartboard. When a student Emily asked, "I didn't simplify, so is that okay?" Mr. Ezzo replied, "No, that's wrong. Sorry." This simple statement implies that Mr. Ezzo is the ultimate authority on whether or not an answer is correct or incorrect. During a daily homework check, the following dialogue occurred:

Monique: When you do 21 over 25, is that the fraction?
Mr. Ezzo: Yes, that's the fraction and you have to convert it to a decimal and a percent.
Emily: And you already have the fraction, but no, [the textbook] says to write a fraction and a percent, so the answer would actually be…
Mary: Do we have to draw it like that?
Emily: It says to write a fraction and a percent, so would the answer be 21 over 25 *and* 84%?

Mr. Ezzo: I'm going to answer Mary's question first, Emily, and then I'll come to you.
Emily: Sorry.
Mr. Ezzo: That's okay. So how did you do it, Mary?
Mary: I just wrote the steps…
Mr. Ezzo: Perhaps you did it like this…(writes on board). My friend Mary punched 21 divided by 25 into her calculator and the calculator said 84 hundredths, and she converted the 84 hundredths into 84%.
Emily: But wait, what's the answer then?
Mr. Ezzo: 84%
Emily: Yeah, but it says fraction *and* a percent, so would it be 21/25 *and* 84%?
Mr. Ezzo: Uh yes, so I'm looking for this (points to the two forms of the answer on the Smartboard).

In this excerpt we see that not only is Mr. Ezzo considered the authority on mathematics in the classroom, but also the textbook is revered as authority. Throughout his lessons, Mr. Ezzo encouraged students to participate and he was kind, but his statements such as "That would be wrong, dear" and "Yes, that is correct" indicate that students should look to him as the authority and rely on him to determine the mathematical validity of their work. Mr. Ezzo's students asked many more questions per hour than Mr. Cordoba's students, but this is, in part, because most questions in Mr. Ezzo's class can be answered in a quick and concise (sometimes yes or no) way.

The data above show that authority distribution in a mathematics classroom impacts the types of questions students ask. Through focusing on authority issues as well as reflecting on the types of questions we want students to ask, we can help teachers begin to see themselves and their students as mathematicians at work together. This does not mean that teachers need to relinquish authority completely, but rather to share authority in a community of inquisitiveness, respect (for each other and for mathematics), and productivity.

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MASTERY-BASED GRADING: AN EXPLORATION OF ONE TEACHER'S IMPLEMENTATION OF REFORM GRADING PRACTICES

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The following paper is a brief report of a qualitative case study of a mathematics teacher as he planned for and implemented instruction in a mastery-based learning classroom. This study took place at a college preparatory high school in a large community in the Rocky Mountain region. Interviews, classroom observations, and lesson reflections were collected over the course of one week. The data were analyzed for common tensions that arose within the teacher's activity system. A rich description of the teacher's experiences includes a discussion of how timing and content coverage, lack of student preparation, and mastery-based grading impacted how the teacher planned for and implemented instruction.

Keywords: Teacher Beliefs, Instructional Activities and Practices, High School Education

Successful mathematics teaching involves navigating a complex series of decisions and consequences with respect to how teachers engage in classroom instruction (Schoenfeld, 2008). Schoenfeld (2008) argued that mathematics teachers mold classroom instruction by identifying goals and designing classroom activities to achieve those goals. Specifically, teachers are constantly making decisions, some obvious and other less so, which impact the flow of classroom activity. The purpose of this qualitative case study was to explore the planned and unplanned decisions made by a mathematics teacher, and how those decisions impacted instruction and student learning opportunities as classroom instruction continued from one day to the next. This paper will focus on the first of three research questions: *What do secondary mathematics teachers attend to during the lesson planning process?*

Methods

Mr. Jones is a secondary mathematics teacher who was working towards implementing reform teaching practices. During the time of the study, Mr. Jones was in his fourth year of teaching at a college preparatory high school in a large community in the Rocky Mountain region. This research study focused on Mr. Jones' planning and implementing instruction in an upper-level, college preparatory mathematics course, which covered topics studied in a traditional college algebra course. Students in the course included sophomores, juniors, and seniors of whom Mr. Jones described most as being of advanced ability level.

A semi-structured initial interview was conducted to understand Mr. Jones' initial plans for the observed lessons. In addition to instructional plans, the interview focused on better understanding the classroom norms, students, and typical lesson design for the class under consideration. That is, the initial interview was intended to gain an understanding of Mr. Jones' initial *activity system* (Engestrom, 2015). Data collection involved audio- and video-recording lessons during one week of the focus course. Additionally, Mr. Jones was asked to record a lesson plan summary prior to teaching each lesson his goals and plan for the upcoming lesson, his rationale for the flow of the mathematical concepts, and other lesson plan decisions. During the day following the last classroom observation, Mr. Jones participated in a stimulated-recall interview during which he was asked his perspective on the success of the observed lessons as well as to reflect on specific, video-recorded episodes of classroom instruction.

Cultural-historical activity theory (CHAT) is a perspective of human cognition which takes as its minimal unit an *activity system* consisting of six components: subject, object, instrument, community, division of labor, and rules (Engestrom, 2015). Activity systems transform when tensions or contradictions emerge between or within the components of an activity system. With respect to the mathematics classroom, these tensions or contradictions have a direct impact on what the teacher attends to when planning for and implementing instruction. In the present study, all interview and classroom observation data were transcribed and coded based on the CHAT coding scheme developed by Herbst and Chazan (2012). Specifically, coded data were analyzed for common themes in the form of tensions or contradictions within the transformation of Mr. Jones' classroom *activity system* over the course of the observed lessons.

Mastery-Based Learning

A significant characteristic of Mr. Jones' teaching philosophy and classroom policy is his use of something he called *competency-based grading*; this is commonly called *mastery-based grading* (Vatterott, 2015). Mastery-based learning differs from more traditional learning philosophies in that "instead of teach, test, and move on in one large group, learning is a series of masteries for individual students – teach, check for understanding, apply learning, get feedback, revise learning, and get more feedback until mastery is achieved" (Vatterott, 2015, p. 29). Students are given a list of learning targets on which they must demonstrate mastery by the end of a learning term (e.g., a semester). Over the course of that learning session, students will be given multiple opportunities to demonstrate mastery with only their best attempt being used as part of final grade calculation (Vatterott, 2015). Implementation of this learning model requires a fundamental shift in classroom activity from a teacher-centered classroom to a more individualized, student-centered classroom (Vatterott, 2015).

Summary of the Activity System

Analysis of data self-reported by Mr. Jones during his initial, semi-structured interview provided insights into his activity system (Engestrom, 2015). The following is a summary of the subject, object, instruments, community, division of labor, and rules.

Mr. Jones expressed enjoyment in learning mathematics. He believed that this stemmed from mathematical problem-solving and the tools mathematics provides for problem-solving in the real-world. He believed that it is important to make meaningful connections between the mathematical concepts being taught and the real-world applications of that content.

Mr. Jones expressed both short- and long-terms goals. These are the object of his activity system (Engestrom, 2015). For the short-term, he wanted to focus on two main goals: (1) review of previous content and (2) introduction of new content. Specifically, he wanted to give his students an opportunity to go back and review the mathematical concepts, or learning targets, that were missed on the most recent quiz as well as move forward with an exploration of functions with the ultimate goal of discussing inverse functions. As for long-term goals, Mr. Jones acknowledged that there was a set amount of curriculum that he was expected to cover during the semester. This was especially the case in the observed course as it was the first in a two-course sequence.

Mr. Jones noted several instruments that he utilized during lesson planning and instruction. These instruments, or tools, included a list of objectives for which students needed to demonstrate mastery, a school resource website, and the course curriculum. Based on prior courses, Mr. Jones had a complete map of the course over the entire academic year. This resource served as a pacing guide and as a list of objectives for achieving mastery. With the goal of being able to spend more time in class on exploration, Mr. Jones utilized an online course website where he posted class materials and links to videos for students to watch. In addition, this website included discussion boards that Mr. Jones used to identify student misconceptions and prior knowledge before class instruction. As a college-preparatory course, Mr. Jones was restricted by an official curriculum. However, he used this curriculum as a resource for identifying mathematical tasks and course activities. In addition, he used previous exams as a resource for developing new assessments as well as homework assignments.

Mr. Jones identified multiple members of the community which had either direct or secondary impact on his lesson planning process. In addition to himself, this community included his students, their parents, students in another section of the same course, the members of the mathematics department, and the administrators in the school district. His students consisted of multiple grade levels who, in Mr. Jones' estimation, had varying degrees of academic maturity, desire to learn, and mathematical understanding. Based on his philosophy of mastery-based grading, this was expected and encouraged. Mr. Jones believed that his students' parents were supportive of this new grading policy because many expressed a dislike for learning mathematics and, therefore, valued any attempts to teach it better than what was their experience. Mr. Jones perceived support for his implementation of mastery-based learning practices from both the other members of the mathematics department as well as administrators in the school district.

With respect to his classroom, Mr. Jones described the division of labor in terms of his role as the teacher and his students' role as a class or individual students. As the classroom teacher, it was his responsibility to present the content in such a way that makes mathematical connections clear as well as makes the mathematical concepts relevant to students. In contrast, Mr. Jones expected his students to be active participants during class instruction. Mr. Jones believed that it was important for students to ask questions. Mr. Jones encouraged his students to be reflective learners. Students were expected to work continually towards mastering the course content.

Mr. Jones focused primarily on how the philosophy of mastery-based learning influenced his classroom expectations. He argued that students shouldn't stop learning following a test and that students should be able to improve their grade. As a result, it was the expectation that students would continue to work towards mastering the learning targets covered by a specific test. Mr. Jones implemented a *soft deadline* when it came to collecting homework. That is, homework was never truly due and students had access to a complete answer key. In Mr. Jones' view, homework was a formative opportunity for students to measure their progress towards mastery.

Lesson Planning Tensions

Data analysis revealed three *lesson planning tensions* that were pervasive over the course of the week of instruction. These tensions involved timing vs. content coverage, lack of student preparation, and mastery-based grading.

Timing vs. Content Coverage

Mr. Jones expressed concerns about taking the time to explore concepts in depth while making sure that all important mathematical concepts were addressed. Adding to the complexity of this issue, the mastery-based learning model, adopted by Mr. Jones, emphasized that learning is individualized where students are given the opportunity to work at their own pace. This manifested itself as a tension among the object, instruments, and rules within Mr. Jones' activity system. Mr. Jones found himself having to make instructional decisions during lesson planning that took into consideration a balance of coverage, timing, and individual student needs. This resulted in a fewer application problems than he would have liked to have done.

Lack of Student Preparation

In order to help reduce the amount of content coverage needed during class, Mr. Jones implemented an online course website where he posted resources for students to learn basic skills prior to attending class with the hope that class time could be spent focusing on more advanced problems and topics. Students were expected to complete discussion board questions in order to help

Mr. Jones gauge student understanding. During the observed lessons, students were not completing these pre-class tasks. This tension among the object, division of labor, and rules within Mr. Jones' activity system forced Mr. Jones to modify his desired lesson plan. Instead of focusing on exploratory activities aimed at helping students grow their understanding as well as provide time for students to go back to prior concepts, he spent more class time engaging in direct instruction in order to cover all of the concepts necessary for class.

Mastery-Based Grading

A significant component of Mr. Jones' mastery-based learning model was the policy that students would only be graded when they were able to achieve mastery of a particular concept. When a student achieved the basic level of mastery, they were given an 80% in the gradebook, otherwise they were given a 0% in the gradebook. Students could improve their grade above an 80% by working beyond basic mastery. This policy caused a tension between the community and the rules within Mr. Jones' *activity system*. For many of his students, this was their first experience with this type of grading system. During the week of observations, the students were returned their first quiz. On this quiz, there were quite a few missing, or below mastery, scores. As a result, there was a significant amount of stress surrounding their grades. Mr. Jones spent time during every lesson reassuring students that their grades will improve as long as they put forth the effort to obtain content mastery. This caused Mr. Jones to reduce the amount of time spent on instruction than he had planned.

Discussion and Future Research

This case study focused on the experiences of a single mathematics teacher while planning for and implementing instruction in a mastery-based learning classroom. For Mr. Jones, timing and content coverage, lack of student preparation, and mastery-based grading impacted how he planned for and implemented instruction over the course of the week of observed instruction. His instructional issues were supported by general education literature focused on *mastery-based grading* (Vatterott, 2015). However, there is limited available literature focused on implementation of these models in mathematics classrooms. Future research is needed to explore how *mastery-based grading* models can be implemented in mathematics classrooms.

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MATHEMATICS LESSON PLANNING PRACTICES OF NOVICE ELEMENTARY TEACHERS

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Lesson planning is a focal activity in elementary teachers' teacher preparation, however existing research suggests that teachers' lesson planning practices shift as they enter the profession. In this paper, we report results from a large-scale project that interviewed 99 early career teachers (ECTs) about their planning for mathematics lessons twice during the 2015-2016 academic year. We analyzed transcripts to determine factors related to lesson planning, including when they conducted their planning, who they planned with, and what resources they used when planning. Our findings also shed light on district-level influences on teachers' use of resources when planning.

Keywords: Elementary School Education, Instructional Activities and Practices, Policy Matters

Introduction

Following the widespread adoption of the Common Core State Standards (NGA & CCSSO, 2010) and the accompanying federal reforms such as Race to the Top, early career teachers (ECTs) have faced unprecedented pressure to fulfill expectations of "highly effective" teachers, such as producing strong results on standardized assessments with their students, early in their careers. The supports ECTs receive from their schools and districts, including mentoring offered through interactions with school-based colleagues, can help them face such pressures. As lesson planning is a common practice shared among novice and experienced teachers alike, albeit manifesting in different forms (Borko & Livingston, 1989), the broad focus of our inquiry is to understand the nature of ECTs' planning for mathematics instruction in this "high-stakes" era, including how planning practices shift as ECTs gain more experience and the role of institutional factors, such as mentoring from school-based colleagues, on ECTs' planning.

Researchers have long faced the challenge of understanding what teacher planning is and how teachers conduct it. The focus of existing research on planning has largely considered the time units that characterized teachers' planning (e.g., daily, weekly, unit) (Clark & Peterson, 1986) and concluded that unit planning was often unproductive because of the unpredictability of classroom life (e.g., McCutcheon, 1980). While most teachers do not produce written plans on a regular basis (McCutcheon, 1980; Morine-Dershimer, 1979), teachers' routine planning often involves practices such as envisioning connections within the academic content and determining activities students will engage with to learn the content (Morine-Dershimer, 1979; Mutton, Hagger, & Burn, 2011). However, ECTs may struggle with these planning practices, as they lack the experience to visualize how their mental plans are likely to play out (Norman, 2011) and their teacher preparation does not fully prepare them to plan capably given the pressures and time constraints of full-time classroom instruction. Thus, novice teachers are likely to need targeted assistance for planning (Mutton, Hagger, & Burn, 2011).

Given this conceptual foundation, the research questions guiding our work are: *What is the nature of elementary ECTs' mathematics lesson planning practices? What district-level influences do ECTs mention as influential on their lesson planning practices?*

Methods

This study was a part of a larger study investigating the mathematics instructional practices of elementary ECTs in the Common Core era and the relationship between instructional practices and factors such as collaboration with school-based colleagues and the instructional practices of those colleagues. We conducted a lesson planning interview with each participating ECT (n = 99) in conjunction with observations of their mathematics instruction over the course of two consecutive days in both fall 2015 and spring 2016. All participants were within the first four years of their careers, teaching grades K-5.

The research team members received training on interview techniques prior to conducting interviews, and shadowed an experienced team member for the first few observations before conducting observations and interviews on their own. Each team member was instructed to follow the interview protocol closely and only use follow-up questions to ensure that questions were answered clearly. The interview data was audio-recorded, and each interview was fully de-identified and transcribed prior to analysis. The results in this paper focus on participants' responses to the following set of questions, asked in this order: *How did you start? How did you decide what content to focus on? What resources did you use when you planned this lesson and why? How did you decide to use these resources? When did you do the planning of today's lesson? Did you talk with any other teachers or staff to help you prepare for this lesson?*

Analyses of responses began by having individual research team members review a sample of 5 responses to a single question. From their review, they generated a set of codes they felt captured the essence of the majority of responses to the question. Then, the entire research team analyzed additional responses to assess the utility of the initial coding frameworks for each question. The team met bi-weekly to discuss individuals' reflections on the emerging codebook and to suggest improvements. Then, in the final stage, individual team members were assigned to focus on all responses for a single question and review the codes assigned to make sure codes had been applied consistently and revise coding if necessary. This method allowed for collective input and review of the codebook for *all* questions by *all* research team members, rather than solely obtaining inter-rater reliability between two of the raters for analyses of each question.

Results

With Whom Are ECTs Planning Their Mathematics Lessons?

Overall, most ECTs in our sample engaged in some aspects of mathematics lesson planning with other school-based colleagues (n=73 in fall 2015) rather than planning solely on their own (n=26). The nature of these discussions, however, varied among the participants and could be described as one of four types: (1) informal collegial conversations; (2) formal collegial conversations; (3) informal planning and (4) formal planning. These four categories describe the level of engagement (collegial conversations tended to be more general compared with planning activities which focused specifically on what to do and what resources to use) and routine (informal interactions were not arranged in advance compared to formal interactions that happened as the result of meetings being set by teachers/teams/schools/districts).

Although many ECTs described interactions with others when planning, some answered "no" when asked specifically: *Did you talk with any other teachers or staff to help you plan this lesson?* For example, one ECT described talking with a grade-level peer regarding "if they typically find this is a more difficult lesson" for the purpose of gauging what to expect from her students during the lesson, yet she did not consider this conversation to be planning with a colleague. Overall, we found our participants did not consider discussing resources, struggling students, or the curriculum in general with others to be planning with a colleague.

When Do ECTs Plan for Mathematics Lessons?

We received three types of responses to the question *When did you do the planning of today's lesson?*, either (1) day-by-day; (2) weekly overview planning, with some modifications day-by-day; or (3) weekly, with little to no modification. One trend we noted was shifts in ECTs' responses from the fall interview to the spring interview. Notably, the majority of ECTs (fall, n=48; spring, n=63) indicated that they planned for their mathematics lessons on a weekly basis, usually with their fellow grade-level teachers in their building (as noted in the findings above). Approximately 30 ECTs indicated that, in either semester of observation, they also planned weekly but with modifications based on assessment of student understanding on a daily basis. Yet, in the fall, 24 of the 99 participants indicated that they planned primarily day-to-day, whereas, in spring, only 11 indicated that they planned day-by-day. While not surprising, the findings suggest that ECTs may start the school year planning day-by-day, but are likely to switch to either planning weekly, or weekly with some daily modifications as the school year continues.

What Resources Do ECTs Use When Planning Mathematics Lessons?

Participants mentioned a variety of resources they utilized for mathematics instructional planning in response to the question: *What resources did you use when you planned this lesson and why?* as well as questions such as *How did you start?* Overall, a total of eight coding categories arose from ECTs' responses to the interview prompts: (1) prior classroom experience; (2) professional development; (3) pacing guides or curriculum maps; (4) students' performance on in-class assessments; (5) students' standardized test performance or specific items from standardized tests; (6) learning standards (e.g. Common Core State Standards or the specific content objectives of a unit or lesson); (7) teachers' knowledge of students' understanding, based on students' work on non-assessment-oriented classroom tasks; and (8) supplemental materials from sources online such as Teachers Pay Teachers. Results show that the ECTs primarily draw from supplemental materials when planning (fall, n=64; spring, n=60), along with pacing guides or curriculum maps (fall, n=57; spring, n=54). In general, the number of ECTs using each type of resource decreased from fall to spring, except for the number of teachers using learning standards as a resource, which increased (fall, n=34; spring, n=40). Ongoing analysis will unpack the different types of resources coded into the category of supplemental materials.

What External Factors are Salient for ECTs' Mathematics Lesson Planning?

Throughout the interview, many (though not all) ECTs discussed how various external factors, particularly pertaining to the institutional context in which they work, influenced how, when, and with what resources they conducted their planning of mathematics lessons. We characterized their descriptions as aspects of fidelity, using a notion of fidelity that resonates with the definition "faithfulness to a person, cause, or belief" (Fidelity, 2017). Three codes related to fidelity to the curriculum emerged in ECTs responses, namely: (1) Fidelity to scope of curriculum, with option to modify sequence; (2) Fidelity to scope *and* sequence; (3) Fidelity to "script" of curriculum.

ECTs' responses were most frequently coded as fidelity to scope of curriculum, with option to modify sequence. For example, ECTs made statements such as: "I can't tell you specifically if we follow directly the, the pacing guide. We get to choose the order" (231.062, spring). To a lesser extent ECTs described fidelity to scope *and* sequence of curriculum, with statements such as: "So we did a scope and sequence, one for the first half of the year and one for the second half of the year and then we just kind of follow that to see. It goes weekly what we are suppose[d] to be doing this week, what are we suppose[d] to be covering" (121.053, spring). Finally, some ECTs (n=28) stated following the curriculum in a scripted fashion ("everything is all scripted. It's all laid out for you," 120.011, fall).

District directives pertaining to curriculum use appeared to influence ECTs' fidelity to the curriculum to a larger extent than the nature of the curriculum provided. For example, while District 230 provided the Go Math curriculum for teachers, a curriculum that provides step-by-step guidance for how teachers are to implement the lessons, the majority of participants from that district indicated that they closely followed the scope but took liberties to adapt the sequencing of the content. In contrast, teachers from District 210 all described that they followed the scope and sequence of the curriculum (Everyday Math) closely. Ongoing work involves investigating the fidelity of implementation during the video-recorded lesson observations to triangulate evidence of teachers' stated fidelity to the curriculum.

Conclusion

Taken together, the results present a snapshot of mathematics lesson planning practices for ECTs in an era of accountability. Some trends are promising; ECTs increasingly seek out school-based colleagues as resources when planning. However, other findings suggest that teachers may not modify plans on a daily basis in response to student progress. As we continue to develop this line of inquiry, we plan to analyze how these variables interact with other data that we have gathered from surveys and observations. Most importantly, this descriptive work can inform those working in instructional improvement through policy and professional development to consider ways to change institutional supports and constraints so as to create more effective mechanisms for ECTs to prepare to engage in high quality instruction for all students.

Endnotes

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MATHEMATICS PEDAGOGY AS SOCIAL JUSTICE ACTIVISM: THE CASE OF MS. LARA

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This case study occurred in a culturally, ethnically, and linguistically diverse fifth grade classroom. Ms. Lara's (pseudonym) mathematics pedagogy exemplified equitable teaching practices. Data from video recordings of lessons and interviews were coded using equity and social justice pedagogy codes to identify nuances of social justice mathematics pedagogy. The findings present the teacher's activism for social justice, and have implications for teacher practice and teacher education.

Keywords: Equity and Diversity, Elementary School Education, Instructional Activities and Practices

Introduction

Teaching mathematics for social justice is understood in multiple ways - teaching about social justice issues using mathematics, teaching mathematics with a social justice lens, or teaching students to use mathematics to challenge social injustices (Gutstein, 2006). Rarely do we find a discussion about a teacher's pedagogy as work for social justice. Imagine a classroom where students are accepted for who they are and respected for what they bring, especially when, some of what they bring is pain and struggle. Not all students have such an opportunity and students thus marginalized continue to under-perform. Achieving equity involves creating opportunities for all students to access engaging mathematics including high quality teaching and pedagogy. Equity in mathematics education remains elusive for various reasons, such as teachers with fewer mathematics experiences and lower Mathematics Knowledge for Teaching (MKT) (Hill, Rowan, & Ball, 2005). This study identifies pedagogical practices, and argues that development of such pedagogy is the teacher's activism for social justice.

Theoretical Framework

Four key features of teaching for equity are commonly agreed upon by multicultural education theorists: 1) recognizing that racism exists at individual, institutional, and cultural levels, 2) racism is perpetuated in educational settings, both in curriculum and practice, 3) the purpose of education must go beyond content knowledge and passing standardized tests, and 4) building students' critical consciousness must be an expected outcome of education. Banks and Banks (1995) describe equity pedagogy as teaching that is dynamic, by being strongly student-centered and flexible in being able to cater to individual student needs. Ladson-Billings (1995) posits a *culturally relevant pedagogy*: "specifically committed to collective, not merely individual, empowerment" (p. 160); a pedagogy that places high expectations on all students, and helps them achieve academic excellence, cultural competence, and critical consciousness. Three dimensions of a teacher's pedagogy are considered critical to equitable mathematics instruction: focus on rigorous mathematics content, learner-responsive pedagogy, and application of real-world contexts (Erchick, Joseph, & Dornoo, 2014; Gutstein, 2006). Students also need the opportunity to understand the world critically and become agents of social change (Banks & Banks, 1995; Gutstein, 2006). Studies are needed to inform the "theoretical understanding of the issues as well as [our] practical efforts to reduce existing disparities" (Gutstein, 2006, p. 95). This study, which is part of a larger study (Joseph, 2013), fulfills such a need.

The Study

This interpretive ethnographic study took place in a fifth-grade classroom in a low performing urban school. Student-demographics showed 61.4% non-white, 89.9% on free/reduced lunch, and 20.5% limited English proficiency. The participant, Ms. Lara, provided a potential exemplar of teaching for equity and social justice. She has a certification in special education and Montessori training. She completed part of a doctoral program in Educational Philosophy, has 30+ years of teaching experience, and was invited to Liberty Elementary because of her commitment to students.

Data Sources and Analysis

Multiple sources of data informed the study (Joseph, 2013): audio-recordings of interviews, video-recordings of teaching sessions, student work samples, field notes, and participant reflective notes. Data was analyzed using a research-based codebook (shown in Table 1.)

Table 1: Equity and Social Justice Mathematics Pedagogy Codebook (Joseph, 2013)

Code	Concept		Research support (e.g.)
Category I Content objectives (Attention to mathematics)	ETL	**E**xplicit **T**alk about the meaning and use of mathematical **L**anguage	Ladson-Billings (1995)
	ETR	**E**xplicit **T**alk about ways of **R**easoning	Ladson-Billings (1995)
	ETMP	**E**xplicit **T**alk about **M**athematical **P**ractices	Ladson-Billings (1995)
Category II Pedagogical Orientation (Purposeful support of students as learners)	EST	**E**xplicit **S**tudent **T**asks and work	Ladson-Billings (1995)
	IT	Quality **I**nstructional **T**ime spent on mathematics	Ladson-Billings (1995)
	EDC	**E**ncouragement of a **D**iverse array of mathematical **C**ompetencies	Ladson-Billings (1995)
	AU	**A**utonomous student work opportunities	Boaler & Staples (2008)
Category III Contextual Relevance (Awareness of instructional context)	RWP	**R**eal-**W**orld **P**roblems or examples	Gutstein (2006)
	ESE	**E**mphasis of **S**tudent **E**ffort and message that effort will eventually pay off	Ladson-Billings (1995)
	EE	**E**xpressed **E**xpectation that everyone can do the work	Ladson-Billings (1995)
Category IV Social Justice Objectives (Intentionality for student self-empowerment)	OCK	**O**pportunity for **C**o-construction of **K**nowledge	Esmonde (2009)
	SVA	Fore-grounding **S**tudent Voice and **A**gency	Freire (1973, 1989)
	EMR	**E**xplicit attention to **M**utual **R**espect	Bartolome (1994)
	ECT	**E**ncouraging **C**ritical **T**hinking	Gutstein (2006)

The research-based codebook consists of instructional practices in teaching mathematics for equity and social justice. They are grouped into four categories: *content objectives* (teacher's attention to mathematics), *pedagogical orientation* (purposeful support of students as learners), *contextual relevance* (awareness of instructional context), and *social justice objectives* (intentionality for student self-empowerment). Transcripts were chunked collaboratively coded, and reached inter-rater reliability of 85%.

Key Elements of Ms. Lara's Work

1. *Developing students' critical thinking as a personal* process – Ms. Lara's activism was seen in providing students opportunities to develop critical thinking skills. She required students to reflect on their realities, make choices, voice their decisions, and make a conscious decision to learn.
2. *Having informal and informative assessment strategies* – Assessments tend to be framed in ways that disadvantage students who are from cultural and linguistic backgrounds different from the school (Morgan, 1999). Ms. Lara's informal continuous assessment of her students helped her to 'keep her finger on the pulse' of her classroom and learners.
3. *Intentionality in getting to know her students* – Ms. Lara made deliberate choices in her commitment to her students – work in a high-need environment, understand, and connect with students. She communicated with her students to know them – their context, their hopes and aspirations, and made instructional decisions based on that knowledge.
4. *Helping students build mathematical and social identities* –Ms. Lara's practice characterized student identity building through collaboration. Her classroom represented a 'learning community' in which students from all ethnic, cultural, language and ability groups engaged in learning and legitimate participation (Lave & Wenger, 1991) to produce knowledge and benefit in positive ways both personally and as a community.
5. *Students as the locus of authority for both mathematics and instruction* - Having students as the locus of authority meant focusing on student thinking. Ms. Lara's students were invigorated to think deeply and take responsibility for their participation. From being often considered 'deficit' in their mathematical knowledge, her students became endowed with credibility and had their ideas valued.
6. *Paying explicit attention to "connected knowledge"* – Making connections is required in the process of helping students learn mathematics. Ms. Lara not only made connections through multiple representations of content and across content areas, but she also made connections with students' lived experiences.

Implications for Teacher Practice and Teacher Education

Providing equitable learning environments for students in mathematics classrooms is a challenging endeavor. Besides needing in-depth MKT, teachers need to facilitate 'bringing the world into the classroom' (Gutstein, 2006). Teachers often reject the idea based on the perception that attending to social justice means moving away from rigorous content (Erchick & Tyson, 2011). Another reason for rejecting the idea of teaching mathematics for social justice is that teachers believe that it requires curriculum and resource changes over which they have little control. This study provides a vision of social justice mathematics pedagogy where subtle actions can produce classrooms with no barriers to learning: classrooms where every child is provided the unique opportunity to explore, learn, know their voice is heard, and follow their dreams.

Challenges to Social Justice Pedagogy

Teachers like Ms. Lara face many challenges to their work, such as policies that call for strict adherence to a curriculum detailing step-by-step instructions, standardized testing, and programs that are not necessarily aligned with student needs. Standardized tests continue to show students from linguistically diverse and poverty backgrounds as academically unsuccessful. This has grievous impact on student confidence and leads to students accepting this as a sign of their inability and giving up. Students' lack of effort and teachers' acceptance of their lot may lead to a vicious cycle of a society that continues to marginalize underserved communities. In such a context, a teacher's role is critical. Ms. Lara found ways to negotiate these challenges by using her own means of assessing students through verbal communication and representations, and finding ways to help students regain confidence in themselves, their abilities, and even take pride in what they know.

In understanding the kinds of knowledge teachers bring to their work for equity that include mathematical and pedagogical knowledge, Gutierrez (2012) added political knowledge: "negotiating the world of high stakes testing and standardization, connecting with and explaining mathematics to community members and district officials, and buffering oneself, reinventing, or subverting the system in order to be an advocate for one's students." (Gutierrez, 2012) This aspect of a teacher's work for social justice is perhaps the most crucial.

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MEASURING RECOGNITION OF THE PROFESSIONAL OBLIGATIONS OF MATHEMATICS TEACHING

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We show validation data of surveys that estimate high school teachers' recognition of four obligations of the mathematics teaching profession. Measures of internal consistency show three instruments reliably measure three of the four obligations, while the fourth has lower internal consistency. Factor analyses support a 3-factor model for the disciplinary obligation and 2-factor models for each of the individual, institutional, and interpersonal obligations. We inspected correlations between recognition of obligations and teachers' beliefs: Low correlations found suggest recognition of obligations and beliefs are different constructs.

Keywords: Instructional Activities and Practices, Measurement, Teacher Beliefs, Research Methods

The Study of Mathematics Teaching: Background and Theoretical Framework

Our research contributes to theoretical and methodological progress understanding the work of mathematics teaching. Teaching has often been described as the expression of teacher characteristics or as the enactment of behaviors (Shulman, 1986). In mathematics education these have led to studies of teachers' beliefs and teacher knowledge (e.g., Even, 2009; Leatham, 2004), on the one hand, and studies of classroom discourse, norms, and patterns of interaction (e.g., Cobb, 1998) on the other hand. These two approaches have complemented each other, often drawing data from classroom observations, but seeing it alternatively as projection of an individual teacher goals, beliefs, and orientations (Schoenfeld, 2010) or as adaptations of the teacher to the context of his or her interactions with the students and the content (Voigt, 1985).

Less prominent has been attention to how the environments of instruction (Cohen, et al., 2003) frame both what it means to be a mathematics teacher and what a teacher is required to do in mathematics teaching. Yet these environments warrant the encounters among teacher, students, and content. How do those environments create expectations that frame the position of mathematics teacher? Herbst and Chazan (2012) proposed the notion of *professional obligations* to identify those expectations. The position of the mathematics teacher obligates mathematics teachers to stakeholders that look at mathematics teaching from four different perspectives, which Chazan, Herbst, & Clark (2016) call Knowledge, Client, Society, and Organization. From the Knowledge perspective, mathematics teachers are obligated to the discipline of mathematics--to engage students with mathematically correct knowledge and practice. From the Client perspective, mathematics teachers are obligated to the individual students--to tend to their cognitive, emotional, physical, and other needs. From the Society perspective, mathematics teachers are obligated to the interpersonal collective of their class--to promote social values such as fairness and respect. From the Organization perspective, mathematics teachers are obligated to institutional policies and practices of the system, district, school, and department.

These obligations are hypotheses; confirmation requires looking at how much mathematics teachers themselves recognize being under those obligations in contrast with other people that might not be so obligated. We developed the PROB surveys to measure the extent to which teachers recognize each of the four obligations and describe its properties below.

The PROB Surveys

There are four PROB surveys, PROB-MATH, PROB-INDV, PROB-INTP, and PROB-INST, designed to measure recognition of the obligations to the discipline, the individual student, the interpersonal collective of the class, and the institutions of schooling respectively (see also Herbst et al., 2014). Each of the items in all four surveys asks participants to consider a statements that avowedly describes mathematics teaching (e.g., "Mathematics teachers take time to discuss school policies") and then asks participants to “Rate the degree to which mathematics teachers are expected, as professional educators, to act in the manner that this statement describes” using a 4-point Likert-type of scale that ranges from (1 = Teachers are never expected to act in this manner to 4 = Teachers are always expected to act in this manner). We developed the survey through several iterations that included brainstorming, item writing, cognitive pretesting, internal and external vetting, piloting with teachers, and examining the collected pilot data using classical test theory (Crocker & Algina, 1986). The rating prompt, resulted from a design process informed by cognitive pretesting, oriented to elicit the participant’s sense of whether mathematics teachers were expected by others to act in the way described.

Method

We administered the PROB surveys to a national representative sample of U.S. high school mathematics teachers (497 teachers, 47 states), along with other questionnaires using the Lesson*Sketch* online platform. Participants were majority Caucasian (83%) and female (59%), which is consistent with nationally representative data obtained from the NCES database. On average, participants had been teaching mathematics for 14.1 years (SD = 8.7), and had taken 14 college-level mathematics courses (SD = 7.25). The analysis looked at the internal consistency of the surveys and dimensionality of the constructs we attempted to measure.

Analysis

Reliability as Internal Consistency

To document the reliability of the PROB surveys, we evaluated internal consistency of retained items using both Cronbach’s Alpha and the mean inter-item correlation (MIIC). Cronbach's Alpha values over .7 are usually seen as acceptable and over .8 as good, while acceptable mean inter-item correlation values range between 0.15 and 0.25 (Kline, 1995; Clark & Watson, 1995). As can be noted in Table 1, the disciplinary, interpersonal, and individual surveys had good internal consistency, but the institutional survey had acceptable Cronbach alpha and low MIIC.

Table 1: Internal Consistency of PROB Surveys

Obligation	Number of items	Mean inter-item correlation	Cronbach’s Alpha
Disciplinary	18	0.273	0.8711
Institutional	20	0.1117	0.7154
Interpersonal	29	0.2226	0.8925
Individual	18	0.3082	0.8891

Dimensionality Validation

Cronbach’s Alpha is a good measure of internal consistency if it is possible to assume that items are unidimensional, that they are all equally good to measure the construct, and that their errors are uncorrelated. Because the possibility existed that one or more of those assumptions was not met, we examined the factorial structure of disciplinary, individual, and interpersonal scales using factor analysis. The WLSMV estimator, which is optimal for categorical variables with a small sample size was used to test the factor model. To find the best model that is not only meaningful but also satisfies fit criteria, we considered the Root Mean Square Error of Approximation (RMSEA) looking for a value of RMSEA <0.6, the Tucker-Lewis index (TLI) and the Comparative Fit Index (CFI), in both of these looking for values greater than 0.95 (Hu & Bentler, 1999). Factor means were set to 0 and factor variances were set to 1. The specific factor models tested are described in detail below.A three-factor model, where 7 items have the same estimated loadings (discrimination) and two pairs of items are correlated due to the same wording, fits our PROB-DISC data well with all standardized factor loadings greater than 0.5. Three suggested factors are interpretable in regards to the item statements (see Figure 2).

F1: Obligation to the discipline insofar as member of a community contributing to increase and extend appreciation of knowledge outside of the classroom (9 items)
F2: Obligation to the discipline insofar as responsible for its correct representation in classroom interaction (5 items)
F3: Obligation to the discipline insofar as responsible for its correct representation in study resources (3 items)

Figure 2. Factors of the disciplinary obligation.

Using similar procedures we determined that the items in the PROB-INDV survey could inform a two-factor model of recognition of the individual obligation. The two individual factors we found are defined in Figure 3a. The items in the PROB-INTP survey were also best accounted for by a two-factor model which are defined in Figure 3b. Items in the PROB-INST also loaded in a hypothesized two factor model (Figure 3c). The results above show a mostly positive outcome of the PROB surveys. It is of interest to investigate how these measures relate to other constructs being used in research on teaching, particularly other measures of teacher characteristics. Years of experience teaching showed significant positive correlation with the all three PROB-DISC factors though no significant correlations with either of the others.

F1: Obligation to the academic needs of individual students (12 items)
F2: Obligation to the socio-emotional needs of individual students (6 items)

Figure 3a. Factors of the individual obligation.

F1: Obligation to support social interaction among small groups of students (16 items)
F2: Obligation to support social interaction in the whole class (8 items)

Figure 3b. Factors of the interpersonal obligation.

F1: Obligation to support policies for school-wide events and activities (4 items)
F2: Obligation to support school policies that concern classroom activities (9 items)

Figure 3c. Factors of the institutional obligation.

Our participants had also taken the survey by Stipek, et al. (2001), which measures 7 different aspects of teachers' beliefs. We were interested in correlations between factor scores in the obligations and mean scores in belief factors. Significant correlations were found, yet the most important finding is that those correlations are uniformly low. This suggests that recognition of obligations does not measure the same thing as this measure of beliefs.

Endnotes

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SECONDARY MATHEMATICS AND SCIENCE TEACHERS' DATA USE WITHIN AN ASSESSMENT-AS-ACCOUNTABILITY CONTEXT

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The test-based accountability movement has had profound impact on classroom practice, particularly at the high school level where teachers' experiences may differ from elementary and middle schools where students are tested on a yearly basis. We use a multiple case study to examine how high school mathematics and science teachers use data and what challenges they face in the current accountability context. Our findings reveal unique aspects of data directly related to test-based accountability. Expectations for what mathematics looks like (subject area expertise) as well as testing schedules (state and federal policy) influence their data practices, along with district structures and norms. Individual teachers' beliefs about data and what it means to know a student also had a significant impact on their day-to-day work.

Keywords: Assessment and Evaluation, High School Education, Teacher Knowledge

Background

Data-driven decision-making (DDDM) has become a central focus for educational policy and practice at all levels as a strategy to support teachers in developing the skills and knowledge needed to engage in these key tasks of teaching (Gill, Borden, & Hallgren, 2014; Luo, 2008; Mandinach, 2012). DDDM is broadly defined as "the use of data analysis to inform choices involving policies and procedures" (Gill et al., 2014, p. 338). Educators must integrate assessment results with data from other sources, such as attendance and discipline data, to fully understand student development. Data-literate educators can be a driving force of student learning because of the emphasis on data-based evidence and decision-making (Orland, 2015).

Teachers' subject areas likely influence their data practices, especially for secondary teachers whose practices are based largely on their content-specific education (Daly, 2012). Within a building, high school teachers are most likely to meet with their subject area departments on regular bases, whereas elementary school teachers are more likely to meet with teachers who teach the same grade level. In addition, different content areas have criteria for student success, which translates to differences in assessing students, identifying relevant data, and measuring knowledge (Datnow, Park, & Kennedy-Lewis, 2012).

As mathematics has been at the forefront of the high-stakes accountability culture, data use practices of mathematics teachers are a viable subset to detail. They are expected to skillfully handle data such as scores from external, interim, and benchmark tests, district-level program assessments, classroom-level assessments, classroom behavioral, demographic, and attendance data (Schleppenbach, 2010). Additionally, science's heavy emphasis on scientific practices, hands-on inquiry activities, and scientific knowledge merits additional investigation into the specific data use practices of science teachers (Rangel, Monroy, & Bell, 2016). The existing literature, however has not focused on content-specific distinctions (Rangel et al., 2016). There is also little research on how high school teachers experience the current accountability context in ways that are similar to or different from elementary and middle schools, in which students are tested on a yearly basis. In this study, we address the research questions: *(1) How are high school mathematics and science teachers using data? (2) What challenges do high school mathematics and science teachers face in the current test-based accountability context*?

Framework

Systems theorists (Bronfenbrenner, 1994) argue that in order to understand learning and practice, we must attend to the influence of larger systems in which individuals are situated. Bronfenbrenner's model identifies a series of "nested structures" that situate individuals within multiple contexts. *Microsystems* refer to individuals and institutions that immediately impact the individual, which likely includes teachers' peers, building administrators, students and community, and their teacher preparation. The *mesosystem* highlights that all of the elements within the microsystem are in constant interaction with each other with the high school teacher at the center. The *exosystem* includes norms around district practice and policy related to data use, curriculum, and standards. The *macrosystem* involves dominant ideologies and beliefs around teaching and learning and state and federal policy. In considering teacher data practice, it is necessary to consider policies that mandate specific types of data use, different school and district practices and norms around data use, subject-specific practices and norms around data use, and individual teacher practices, beliefs, knowledge, and dispositions around data.

Methods

We address the research questions through a case study of data use amongst science and mathematics teachers at one high school conducted over two years. We conducted 30-50 minute interviews with the mathematics and science teachers and principals to gain an in-depth understanding of practices, beliefs, knowledge, and dispositions around assessment, accountability, and data use. We first coded the teacher interview transcripts using line-by-line coding (Charmaz, 2006) and then looked for pattern codes within each teacher's transcript. Examples of pattern codes include "online assessments," "role of intuition," and "school expectations." We then read the codes across transcripts to identify similarities and differences with teachers from the same district. Using these pattern codes, we next analyzed the transcripts of principals and developed case narratives (Stake, 2013). We also analyzed through the lens of systems-theory, using the systems identified as the nested structures in which teacher data use occurs. We inductively coded data within each system to develop a set of pattern codes, which we overlay to understand how the nested structures influenced data use.

Findings

The case study revealed that specific aspects of expectations for what mathematics looks like (subject area expertise) as well as the testing schedules (state and federal policy) in particular influence their data practices. District structures and norms also influence teacher data use, though individual teachers' beliefs about data and what it means to know a student also had a significant impact on their day-to-day work. We share a sample of findings here.

Showing My Work, Online: The Role of Computers in Test-Based Accountability Systems

To monitor students' progress and prepare them for on-line assessments, the district invested in various software programs for teachers and students. This presented a significant challenge for Patricia, an Algebra 1 teacher, who identified several reasons that online assessments were not a useful tool. Patricia shared that online assessments are "a little deceiving because you really don't know what's going on behind the scenes." Patricia uses the online assessment platform provided by the district "once a year," which she finds nice because the assessment platform scores the assessment. However, she finds this problematic because "I don't really know much about the score because I didn't physically grade it. I don't know what they really did when it says 72."

In addition to concerns about a numerical final score on an assessment, Patricia also found that using online platforms that were open-ended were tedious in the mathematics classroom:

> We just had a chapter on exponents. We did the whole chapter on paper by way of worksheets and the reason, I told the kids, "Here's the reason we're doing it. Because it's going to take you forever to type to the power of, to the power of." Like (4m^2)(y^3)(z). Well, that'll take forever, when they could write it so much faster.

For Patricia, it does not make sense to have students complete certain types of mathematics problems on the computer because of the inefficiency of having students type specific equations.

Alexa, however, also an Algebra 1 teacher, finds great value in an online program that creates reports showing "each student and what they're struggling with...That helps to see if everybody's struggling with this concept, maybe I need to re-teach it again." Alexa increased her use of online quizzes "to make sure the kids are all hitting these standards" that will most likely be tested. Alexa reported that she uses data from these assessments to determine who needs extra help and who to "keep an eye on." She also appreciated that online platforms give students "immediate feedback" if they get a problem right. However, she expressed concern, "because if [students] do keep getting [a problem] wrong and they don't write their work down, they're just trying to do it in their head, I can't see the steps."

Mary, a biology teacher, spoke about,

> A variety of ways you can structure tests and quizzes and assessments, where they are matched to other certain objectives or standards. Then you can get a report through [the online program]…so I utilize that a lot…I want the kids to know their own data, and I want them to know where they fall along with the rest of the class. Usually after a test, I'll ask them to assess whether they've mastered a certain standard…

Mary acknowledged that "for biology, everything I do is standard-based... because we have a state test tied to the biology, we push it a little more." This push includes more regular use of online assessments to monitor student mastery.

The test-based accountability context has led to an emphasis on online assessments that provide immediate, easy, and efficient data to teachers and students about their work; however, for teachers in the same school, these online assessments are used differently.

Intuition Versus Tests: How Teachers Determine Student Growth Objectives (SGO)

As part of the state's guidelines, all teachers were required to identify Student Growth Outcomes (SGOs) — rigorous learning goals aligned to standards-based, common assessments that are specific and measurable. To develop SGOs, teachers are to use previous standardized assessment data as a baseline. However, several teachers reported that their intuition led to the same conclusions about student mastery as students' prior performance on standardized exams. Kate, a biology teacher, said that the "[summative assessment data] does usually correlate when I'm looking at who are my low kids in a class." Patricia agreed, noting that she initially develops her SGOs based on the knowledge of students developed over the first two weeks of school, and then when she compares her informal assessments to the state standardized assessments scores later in the year, she shares, "It's the same list. I spend an hour fighting through the data when I already knew what the data was." Sean, a science teacher, had a similar attitude toward "data" and intuition, and wondered, “what could the data tell me that I don't already know?” He developed his SGOs in part based on students’ prior science and math grades, but he does not “place a lot of emphasis on the standardized tests.” He is more interested in how students perform in various settings, and would rather know about students’ learning styles, interests, and comfort levels with content than their prior performance. Overall, teachers’ beliefs about a teacher’s intuition and what it means to know a student lead them to question the usefulness of data in informing their practice, even within a context that requires certain data practices.

Discussion and Implications

Not surprisingly, mathematics and science teachers face unique challenges related to data use in the test-based accountability context. The heightened use of online platforms for regular assessments of student mastery, for example, is a challenge for teachers to know how students are thinking. The expectation to use specific assessments to evaluate student growth also is a place of challenge, particularly for science teachers who students have not taken a science assessment since 6th grade. Different mathematics and science subjects draw on different types of knowledge, and a student who did well in a prerequisite may not do well in the next course. It is unclear what valid prior standardized data can be used to help teachers to develop useful SGOs, and what prior data can be used as the comparison point at the end of the year.

This study highlights the need for greater research into how different subject areas are impacted by different mandates and policies related to test-based accountability. While all courses are aligned to state standards, teachers and the principal give that extra push to the tested subjects. The teachers in this study, with an average tenure of 22 years, shared how their assessment practices have changed over time and the specific impacts of the test-based accountability on their practice. It was clear that the macrosystems of federal and state policies were not totalizing forces that drove all of their decisions, but it was also clear that these policies influenced how they view data and its role in education. While some embraced the changes, they largely viewed “data” as something tedious and not as a meaningful way to improve practice.

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STORIES OF AGENCY: DO GRADUATE STUDENTS PERCEIVE THEMSELVES AS PART OF THE MATHEMATICAL COMMUNITY?

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Graduate student teaching assistants (GTAs) are responsible for the instruction of undergraduate students in critical introductory courses, but are not yet in the position of professors. Given their unique status, we ask if there are differences in how graduate students and professors express their agency when speaking about their responsibilities and how graduate students position themselves as members of the community of mathematicians. We use tools from systemic functional linguistics (Halliday, 1994) to analyze 16 interviews with graduate students and professors from research I universities. We found important differences in how graduate students and professors perceive their agency, and agency varies according to whether it concerns disciplinary or institutional responsibilities. Future research can investigate how to create more opportunities for developing the agency of GTAs in institutional decisions.

Keywords: Post-Secondary Education; Affect, Emotion, Beliefs, and Attitudes

Each year approximately 743,000 undergraduate students enroll in calculus courses and 834,000 enroll in introductory level courses (e.g., pre-calculus) taught in mathematics departments (Blair, Kirkman, & Maxwell, 2013). In some departments GTAs are only responsible for leading recitations and grading, but in other departments they teach over a third of all course offerings (Lewis & Tucker, 2009). Graduate students benefit from the experience because teaching is an aspect of being a mathematician, the community into which they are entering. As Lave and Wenger (1991) stressed, teaching and learning do not only occur in an individual's mind, but rather are mediated by social situations in a community of practice. Teaching positions, as part of the graduate education, assist in the socialization of graduate students into the faculty positions they may eventually take (Austin, 2002). However, GTAs have little to no experience and are given little training. Preparation programs range from a few hours' orientation to weeklong workshops (Ellis, Speer, & Bookman, 2016). How can we support the apprenticeship into teaching for graduate students while fostering quality instruction? This study seeks to understand differences between how professors and GTAs perceive and manage their roles as instructors. We compare their social positioning and *agency*.

Researchers have noted the significance of agency, or "who has control over the way mathematics is done and expressed" (Wagner, 2007, p. 36), for the doing of mathematics in the context of voice in textbooks (Herbel-Eisenmann, 2007; Herbel-Eisenmann & Wagner, 2007; Morgan, 1996) and in classroom discourse (Wagner, 2007). Previous research has addressed teacher and student agency in their interactions with each other and the discipline, but it has not addressed agency in two key areas: comparing agency of different groups of teachers and comparing agency in different aspects of teaching. We will compare graduate students and professors in two aspects provided by our theoretical framework.

Theoretical Framework

Herbst and Chazan (2012) proposed a framework of four *professional obligations* that mathematics teachers must respond to as professionals: towards representing the discipline of mathematics appropriately (disciplinary), treating individual students as persons with unique assets and needs (individual), creating a socially and culturally appropriate environment for students to share space and resources in a class (interpersonal), and respecting institutions such as the school, department, district, State, or unions in matters including curriculum, assessment, and policy (institutional). We use the institutional and disciplinary obligations as lenses for agency because they are most relevant to how a graduate student socializes into the mathematics community - the departmental community (institution) and the work of a researcher (the discipline). We explore the remaining obligations elsewhere.

Our study of agency uses tools from systemic functional linguistics (Halliday, 1994), which is a theory of language that enables us to explore how meaning is construed by the language people choose. In this study we draw from what Halliday refers to as *processes*, which are aspects of a clause that report about "the event or state that the participants are involved in" and are canonically realized by verbal groups (Thompson, 2013, p. 87). Processes are typically categorized as material, relational, mental, verbal, existential, or behavioral. We wanted to identify processes that revealed happenings with actors or doers, so we focused on *material* processes, which are processes of physical actions (e.g., I taught…, if you are writing..., students have to solve…), and *verbal* processes, processes of saying (e.g., I can talk about..., we have to tell our students that..., I asked them to...) (Thompson, 2013). We contend that analyzing the actors that instructors identify in material and verbal processes will reveal important insights to who feels agentive in different contexts, leading us to ask the following research questions:

1. Are there differences in how graduate students and professors express their agency when speaking on their responsibilities to represent the discipline of mathematics and to their institutions?
2. How do graduate students position themselves as members of the community of mathematicians and/or the departmental community?

Methods

The source of the text analyzed in this study was a set of sixteen hour-long interviews with eight doctoral graduate students and eight tenured or tenure-track faculty members from large midwestern research universities. We focused on responses to questions about the institutional and disciplinary obligations. Participants read and listened to a full definition of each obligation (taken from Chazan, Herbst, & Clark, 2016) and were asked to respond to the question, "Given this description, how does this obligation play a role in your own teaching practice?"

We analyzed the interviews by transcribing them and identifying material and verbal processes with their corresponding actors and sayers. In the following results, we have italicized actors and underlined material and verbal processes. We counted the actors of almost every material and verbal process and decided to count instances of *I*, *we*, *you*, *the institution*, and *students* because they were most frequent. Infrequent actors or processes with ambiguous actors were counted under "other" to create accurate percentages (see Table 1). Certain material and verbal processes were excluded or indicated a lack of agency. Details on the specific ways these were determined are available in a longer report.

Results

The largest distinction between graduate students and professors was in their use of 'we' in material (portraying physical action, e.g., "we break [problems] into simple pieces") and verbal processes (saying, e.g., "we talk about why it's wrong") when speaking about how the institutional obligation plays a role in their practice. Professors used the pronoun *we* in 24% of their clauses, compared with 3% from graduate students (see Table 1). The complement to this observation is that graduate students referred to the institution as an actor for 18% of their material processes, compared with 9% by the professors. For instance, student 2 explained, "There are certain due dates that are part of the, that are already designed and built in by the institution." These observations suggest that when GTAs enact the institutional obligation, they perceive the institution as an external actor while professors perceive themselves as part of the institution. Professor 7 elaborated,

> So in terms of institutional obligation…at least as far as the university is concerned, is to ourselves. *We* decide a policy, and *we* enforce them... And those things are our obligations to policy that *we* set ourselves and the policies are ones we deem reasonable.

Here, the professor has situated himself as a member of the departmental community.

A different story of community emerged in discussions around the obligation to the discipline (see Table 1). We did not find as large of a difference in how graduate students (7%) and professors (13%) used *we* to express agency. The following quote illustrates a graduate student speaking for mathematicians:

> Whatever *we* have done, say for centuries before, that subject is built on truth and truth only. At every step we had this choice, zero or one, and every time *we* choose one, and the whole *subject* is built upon it. So that I feel that *I* must impart to students.

He situated himself as one of the mathematicians who knew what was important to represent about the discipline.

Between the two obligations, the largest difference in agency is in the use of ‘I’. Both sets of instructors use ‘I’ to represent the agents of verbal and material processes more often in the disciplinary obligation (36% and 37%) than in the institutional obligation (19% and 15%). This signals that they have much more personal agency when acting on behalf of representing mathematics than when addressing institutional practices.

Discussion

To address our first research question, our findings suggest that there are important differences in how graduate students and professors perceive their agency, and agency varies according to which obligation is at stake. Both professors and graduate students had more individual agency speaking with ‘I’ in responses to the disciplinary obligation. Graduate students use ‘I’ as often as the professors, which may indicate that the students feel as agentic as the professors when representing the mathematical content at stake--this suggests that the issue is not reducible to developmental differences in the sense of agency, or, that agency varies depending on the obligation undergirding what instructors feel responsible to. One valued purpose of course policies is to create a uniform experience covering the same core material at least for the students that pass through a given institution (Rasmussen & Ellis, 2015). But we question whether this is achieved, and why graduate students are disproportionately assigned to these introductory courses that are critical for so many students.

In response to the second research question, we did find evidence that students position themselves as members of the mathematics community by their use of the pronoun *we*. Wagner (2007, p. 42) said that “student[s] who want to show that they are members in this collective of people who do things right have the *we* voice at their disposal.” Both students and professors were able to position themselves as members of the group that holds knowledge of mathematics. However, graduate students were much less able than professors to speak for the community of the department or institution. Professors may feel they can represent their institution due to the stability of their tenure or tenure-track positions and as a byproduct of other work they do to serve their institution. Although the graduate students did not seem oppressed by institutional constraints, they were not affiliating with the choices made by the institutions. Future research should investigate how to create more opportunities for developing the agency of GTAs in institutional decisions.

Table 1: The use of actors or sayers as indicators of agency

Obligation	Actors/Sayers	GTAs		Professors	
		Frequency	%	Frequency	%
Disciplinary	I	27	36	33	37
	Students	19	25	11	12
	We	5	7	12	13
	You	16	21	12	13
	Institution	0	0	0	0
Institutional	I	22	19	12	15
	Students	24	21	9	11
	We	3	3	20	24
	You	6	5	3	4
	Institution	19	17	5	6

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TOWARDS A SHARED LANGUAGE OF INSTRUCTION: EXPLORING TEACHERS' LEXICON FOR MATHEMATICS TEACHING AND LEARNING

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This paper reports on the development of a national lexicon for describing what happens in middle school mathematics classrooms in the United States. Using primarily surveys and focus groups with mathematics teachers, 100 terms were widely agreed upon as familiar to the teachers. The terms varied on a number of features including whether they referred to particular classroom activities or broader instruction-related concepts, whether they focused on teacher or student actions, and the degree to which teachers reported using the terms. Challenges experienced by teachers in the process of reflecting on their own language use are discussed, and implications for research and classroom practice are addressed.

Keywords: Classroom Discourse, Instructional Activities and Practices, Middle School Education, Teacher Knowledge

Introduction

Over 30 years ago, Lortie (1975) lamented the absence of a "common technical vocabulary" (p. 73) for describing teaching. More recently, Grossman and McDonald (2008) continued to advocate for a "framework for teaching with well-defined common terms for describing and analyzing teaching" (p. 187). Despite these important claims and some recent work investigating teachers' adoption of new terms (e.g., Davis & Boerst, 2014; McDonald, Kazemi & Kavanagh, 2013), teaching in the U.S. still lacks a precise common language for describing what takes place in classrooms. The work we present here is a first step towards identifying a lexicon used by middle school mathematics teachers in the U.S. to describe mathematics teaching and learning.

Theoretical Framework

Gaining insight into the common language of U.S. middle school mathematics teachers is important for several reasons. In writing about social communication, Carroll (1980) discusses the difference between the linguistic forms of naming and describing. Carroll explains that names come to stand in for descriptions once items become "nameworthy," that is, once they become familiar enough to warrant a name for more efficient reference. As he explains, "Shorter name expressions are substituted for more lengthy descriptions when a social and referential familiarity is established" (p. 321); therefore, names provide a means of communicating socially shared ideas. In the same way, mathematics teachers' names for what happens in their classrooms indicate what they deem as communally important. Furthermore, research suggests that names can also help us to recognize and make sense of phenomenon of interest, such as teaching and learning in mathematics classrooms. Sapir (1958) wrote of the "language habits" of a community, claiming that "we see and hear… largely as we do because the language habits of our community predispose certain choices of interpretation" (p. 69). Thus, naming an object can help to make the object more salient to those familiar with the name. In the domain of secondary mathematics teaching, Milewski and Strickland (2016) worked with teachers to create a framework that defined and characterized the different ways that teachers responded to students in the moments of instruction. The researchers found that once teachers had names for particular "moves," the teachers were better able to shift their practices in ways that increased their responsiveness to students. Thus, names have the potential to shape both communication and practice. Here we explore this idea through the development of a teacher-generated lexicon representing terms that U.S. teachers currently use to describe and discuss their classrooms. To be clear, our goal is not to focus on a set of ideal instructional practices. Rather, we aim to be fairly comprehensive in the terms we include, as long as they are familiar to teachers.

Methods and Data: Three Phases

This research was conducted as part of an international study in which teams from nine countries sought to construct local lexicons of mathematics teaching and learning at the middle school level. This paper reports on the findings from the U.S. team.

Initial Generation of Terms

To begin, two middle school teachers and two researchers watched classroom videos from eight countries. The videos served as a stimulus for the teachers and researchers to note things that happened for which they had a name. This process generated approximately 70 terms that the researchers and teachers agreed were familiar terms used to describe what happens in a mathematics classroom in the United States. Next, approximately 20 former and current mathematics teachers known to the researchers were asked to brainstorm 5-10 terms that they would use to describe middle school mathematics teaching and learning. In all, 157 terms were identified. The initial team of teachers and researchers then wrote definitions for the 157 terms.

Local Validation

A process of local validation was conducted with three focus groups comprised of middle school mathematics teachers with a range of teaching experience. One group included three teachers from different schools (one suburban public, two urban private), a second group included four mathematics teachers from a suburban public school, and the third group involved four mathematics teachers at an urban religious-affiliated private school.

Teachers first completed a Q-sort task (Block, 2008) in which they separated lexical terms into familiar and unfamiliar. Familiar terms were subsequently sorted into somewhat familiar and very familiar. Teachers then provided feedback on the definitions of those terms with which they were very familiar. Teachers were also invited to propose new terms that they would use to describe what happens in a mathematics classroom but that had not been included in the lexicon.

Teacher ratings of familiarity were compiled and terms were ranked by the number of teachers who were very familiar with each. Any terms with which three or fewer teachers were very familiar were removed. Following this analysis, 103 terms remained in the lexicon. Definitions of those terms were then edited by the researchers based on teachers' suggestions.

National Validation

Finally, a survey was developed to gather national data regarding the lexicon. The survey asked teachers to review a subset of the lexicon and rate their familiarity with the terms, as well as how frequently they use each term in conversations with colleagues. Teachers also identified whether the written definitions matched their own understanding of the terms, recommended changes if needed, and proposed new terms that were familiar to them but did not appear in the survey. In total, the survey was completed by 241 teachers (131 female, 49 male, 53 did not respond) from 28 states across the U.S. Fifty-nine teachers reported teaching in urban schools, 67 in suburban schools, and 40 in rural schools. Of the 183 teachers who recorded their years of teaching experience, 31 reported 1-3 years of experience, 84 reported 4-14 years of experience, and 68 teachers reported 15 or more years of teaching experience.

To determine the final lexicon, we sorted terms by the percentage of teachers who rated each term as very or extremely familiar (1 or 2 on a 5-point scale). Three of the 103 terms did not meet the

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

threshold of 75% of teachers rating the term as very or extremely familiar and thus were removed from the lexicon.

Findings

The resulting U.S. lexicon consists of 100 terms, a subset of which is listed in Table 1. Several key distinctions were noted. First, the terms represent different aspects of mathematics classrooms, including general classroom activities (*offer feedback, note taking*), more specific practices (*compare multiple strategies*, *pattern recognition*), participation structures (*partner work*, *student presentation*) and assessment-related terms (*remediation*, *differentiation*). Second, some terms refer to activities that occur during instruction (*homework check, warm up*), while others refer to broader educational concepts (*high expectations*, *student accountability*). Third, some terms refer to actions that are more typical of either the teacher (*assign seats*, *give directions*) or the students (*struggling*, *memorizing*). Fourth, terms represent actions and events that occur over different time scales (*aha moment*, *whole-class discussion*). This range of terms seems to highlight the complexity of teaching, which involves a wide array of activities, is multimodal, and requires attention to many things simultaneously (Sherin & Star, 2011).

Table 1: Sample of Terms in U.S. Lexicon

Asking Questions	Posing questions to either a teacher or student(s).
Justifying	Providing evidence to support one's explanation, idea, or solution in order to illustrate that the solution or explanation is reasonable.
Scaffolding	Teacher providing a series of supports, hints, or questions to move students forward on a problem, assignment, or concept.
Wait Time	Several second pause provided by the teacher to allow students time to respond to a question.
Warm Up	Brief activity used at the beginning of class, often for review.

Several challenges emerged for teachers as they worked to identify the terms they use to describe mathematics teaching and learning. First, questions arose about what it would mean to say that a term was familiar and should be included in a national lexicon, as many teachers had never before reflected on what terms they use to discuss their teaching. Teachers in the focus groups sometimes recognized terms as familiar but did not consider them to be part of their own practice, and thus questioned whether these terms belonged in the national lexicon. For example, Eliza acknowledged that *assign seats* was a familiar term, but she categorized it as outside of her lexicon and thus unfamiliar. This example illustrates a key challenge related to evaluating the familiarity of terms: How should teachers treat a term that they recognize and understand if it does not describe the teaching and learning that takes place in their own classrooms? Related, teachers were frequently familiar with multiple terms that describe the same action or idea. For example, when discussing what term reflects the idea of "a sudden discovery made by a student," the focus group teachers suggested both *aha moment* and *light bulb*. Some teachers then raised the issue that they might be more likely to use the term *light bulb* but that *aha moment* was, in fact, more familiar. While our lexicon includes only the most familiar term for each practice or idea, we acknowledge that many alternative terms exist that teachers use to describe those same practices.

Finally, differences emerged in teachers' familiarity with terms as compared to their usage of terms. While all of the 100 final terms were very or extremely familiar to at least 75% of the teachers, not all terms were used frequently. In fact, only 35 of the 100 terms were used daily or weekly in conversations with colleagues by at least 75% of teachers (e.g. *asking questions*, *clarifying, collaborating, differentiation, partner work, proving, reasoning, struggling, warm-up*).

While discussing differences in usage of the terms, some focus group teachers reported using different terms when speaking with different communities. For example, John mentioned that he uses the term *critical thinking* “a lot more with parents and… colleagues,” while he would talk to his students about “how you’re thinking.” Mike and Christina, in contrast, do use the term *critical thinking* with students. We suspect that some differences in usage might be related to teachers’ preparation and professional development experiences and thus plan to examine such connections in the future using existing data.

Discussion and Conclusion

We believe this lexicon has the potential to serve as both a window into teachers’ practices and a lever for change. On the one hand, we gain insight into how teachers describe mathematics teaching and learning today and how they define key components of their practice. Doing so at different points in time and with different populations of teachers might provide valuable information about the state of mathematics teaching in the U.S. In addition, interacting with this lexicon can prompt teachers to reflect on both their language use and teaching practices. Because naming a practice can promote the spreading of that practice, this lexicon can be an important resource for teacher education and professional development. Developing a shared language is a key component of that work, both because it enables teachers to communicate with each other about their teaching and because it can highlight for teachers gaps in their language – and perhaps even their practice. In future work, we plan to expand the national validation to a broader sample of U.S. teachers and compare the current teacher-generated lexicon with one developed by teacher educators. Once all international teams have completed their lexicon generation, we will also begin a phase of comparative work. These comparisons will help us to both identify gaps in the language that teachers use to describe teaching and learning in the U.S. and reflect on unique features of the U.S. lexicon.

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AT THE CROSSROADS OF MATHEMATICS AND LIVED EXPERIENCES: INCREASING YOUNG CHILDREN'S ACCESS TO RIGOROUS MATHEMATICS

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Equity remains an enduring challenge for mathematics education. The experience of many students is one of marginalization such that school mathematics is disconnected from their lived experiences (Chazan, 2000). This is consequential because such disconnect, together with systemic factors of oppression, contribute to disparities in achievement (Ladson-Billings, 1994). Research has demonstrated the benefit of drawing on students' community-based knowledge and experiences in mathematics instruction and the importance of attending to both children's mathematical thinking (CMT) and their community knowledge (FOK) to support mathematics learning (Aguirre, et al., 2013). This poster examines how connecting to authentic student experiences supports students' opportunities to learn rigorous mathematics by studying the work of teachers and students around the following task: How many bottles of water does our class need for one day, based on recommended daily amounts of water for children (40 oz/day)? Students' FOK included their experiences of their community's ongoing water crisis, which required students to use only bottled water for washing, cooking, and drinking at home and at school. CMT included eliciting student thinking, use of discourse moves, etc. Data sources were videos of two classroom sessions. In multiple passes, we looked for evidence of attending to CMT and attending to FOK. In subsequent passes, subcodes were identified. We used Studio-code to identify every instance of each subcode, to indicate the duration of each instance of a subcode across the two classroom, and to see the back and forth across the two broad codes. Our analysis revealed that teachers attended to student thinking in a variety of ways, connected to students' funds of knowledge throughout the lesson, and effectively cycled between the two to provide a rich context for student learning. The teachers' decisions to connect mathematics learning to students' lived experience provides mathematics teachers and mathematics teacher educators with evidence of practices to do this work effectively. The lessons were mathematically rigorous and the deep connection to the lived experience provided a context for teachers and students to refer back to. The research reported in this article was supported with funding from the NSF (Award #1417672). Any findings, conclusions or recommendations expressed are those of the authors and do not necessarily reflect views of the NSF.

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CHARACTERIZING TEACHERS' INFORMAL CONCEPTIONS OF LEARNING TRAJECTORIES IN MATHEMATICS

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Keywords: Teacher Knowledge, Learning Trajectories

While there has been a considerable increase in research related to learning trajectories (LT) in mathematics (e.g., Clements, Wilson & Sarama, 2004), more recently, researchers have begun to explore LT-based professional development (PD) programs (Wilson, 2014) and how teachers use LTs in their instruction. Little attention, however, has been paid to teachers' own conceptions of LTs, conceptions that are often (re-)formed from their instructional interactions with students over time. In this poster, we characterize teachers' informal conceptions of LTs in middle school math to better understand ways of supporting teachers to effectively navigate students' LTs during instruction.

Data from this study is taken from the BLINDED Project, a multi-year project focused on developing LT-based instructional resources and PD for middle school teachers. Ten participating teachers were interviewed at least 3 times across the first year of the project about their instructional and assessment practices. Our analysis focuses on how teachers think about LTs in the context of these interviews. We define LTs as an empirically-supported description of ordered experiences students progress through instruction, moving from informal to formal ideas with increasing sophistication over time (Confrey, 2008). We are particularly concerned with the landmarks and obstacles that define such a progression.

Results of our analysis raise several important issues. First, teachers' described *sequences* of student learning and *obstacles* students typically encounter as part of these sequences, which comports with definitions of LTs. However, teachers described these *sequences* and *obstacles* at different *grain sizes* of specificity. For example, three types of grain size emerged from the data: LTs at grade level, LTs at instructional unit level, LTs at math topic level. We found that teachers generally talk about LTs at a large grain size (e.g., students struggle with fractions), but seldom attend to finer grain size levels of LTs, especially levels of sophistication in students' thinking, which echoes those of Wilson (2014). Second, we identified the concept of *gap* in teachers' reflections about LTs, which we define as a lack of prior math knowledge that students should have become proficient in previously. While teachers help students tackle obstacles from newly-learned topics, they have to address the knowledge gaps relevant to the math concept understudy. We argue that it is important for LT research to not only on develop LTs, but also on the development of instructional resources that support teachers in managing obstacles and gaps.

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CROSSROADS TO STEM CAREERS: MATH AS A BRIDGE NOT A GATEKEEPER

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Keywords: Post-Secondary Education, Curriculum, Equity and Diversity

At a four-year public university in the western United States with over 5,000 students and 40 degree programs, more than half of the population of students are students of color, 51% are first generation college students, and a large proportion of students come from low-income backgrounds or are non-traditional students. With this population, many students enter the university underprepared for the rigors of college level mathematics and are placed in courses below Calculus. As a STEM focused campus, a significant population of students enter the institution with the intention of majoring in disciplines for which Calculus is a prerequisite. Thus, many of these students find themselves at a crossroad of whether to stay straight on their path to a STEM career or to turn because of a lack of mathematics background. By the time students actually declare a major, the number of students choosing STEM disciplines has declined by over half. These proportions are even higher for women and students of color. Although there are many reasons for which students change their intended major, mathematics has been consistently cited as a contributing factor. In an effort to make STEM majors more accessible to all students, faculty are reforming mathematics curriculum of courses below Calculus using ambitious strategies outlined in recent research (Bressoud, Mesa, & Rasmussen, 2015) to include more active learning, group work, multiple representations, as well as metacognitive strategies.

The Study

The purpose of this research is to study the effects of these changes to add to the knowledge of effective practices for educating all students in mathematics for the 21st century. The full study seeks to begin to answer the question: *What effect does using active learning, group work, multiple representations, and metacognitive strategies have on students' beliefs about mathematics, their ability to do mathematics and their attrition from STEM fields*? The subset of the study featured in this poster specifically examined the effect of group interactions on students' understanding of graphical, tabular, symbolic and verbal representations of mathematical concepts and the connections between them.

The study uses a mixed-methods design with observation instruments, video analysis of group interactions and discourse patterns, and student work. The data provides insight into how groupwork contributes to students' understanding about different mathematical representations. Preliminary data suggests that student discourse during group interaction plays a significant role in students' sense-making of different representational forms. Although research on the relationship between multiple representations and student understanding is abundant, this study seeks to connect theory and practice by considering the impact of pedagogical practices and student discourse on students' sense-making of representations. Pedagogical practices are significant because, as first year courses are revised, consideration should be given to planning opportunities for student communication when learning concepts that are heavily loaded with multiple representations.

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EXAMINING DISCOURSE STRUCTURE IN CHINESE AND U.S. ELEMENTARY FRACTIONS LESSONS

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Keywords: Classroom Discourse, Elementary School Education, Rational Numbers

Introduction and Background: Horizontal, Vertical, and "Teacher-Facilitated" Discourse

Public policy, theory, and evidence suggest that mathematical learning is supported in U.S. elementary classrooms when students voice their understanding directly to each other, following *horizontal* discourse patterns. In contrast, in Chinese classrooms, where students have been reported to excel in mathematics, both Confucian tradition and cultural expectations dictate that teachers remain in control in guiding their students, thus following *vertical* discourse patterns. The question examined in this investigation was: To what extent do we find these two patterns and an additional, hybrid pattern—*teacher-facilitated horizontal* discourse—in U.S. and Chinese classrooms and what roles might they play in students' construction of mathematical knowledge?

Method

We videotaped one lesson on equivalent or adding fractions from each of 31 4th- and 5th-grade classrooms (14 from a mid-size city in the U.S. and 17 from Beijing, China). These topics were chosen because they are central to the mathematics curricula in both countries. Next, we transcribed all classroom talk and then coded each student statement to identify to whom the student was directing that statement. We identified three forms of discourse: vertical (the student statement was directed only towards the teacher), horizontal (the student statement was directed towards another student), or teacher-facilitated horizontal (the student statement was directed by the teacher to respond to another student). We also coded the mathematical terms mentioned by the student in each response statement.

Results

Using a generalized linear mixed model with log transformation, we did not find significant differences in vertical discourse between the U.S. lessons and the Chinese lessons, $F(1, 56) = 1.36$, $p = .74$. We did, however, find significantly more horizontal discourse responses in the U.S. lessons than in the Chinese lessons, $F(1, 56) = 8.53$, $p < .001$. We found significantly more teacher-facilitated horizontal discourse responses in the Chinese lessons than in the U.S. lessons, $F(1, 56) = 17.09$, $p < .001$. Moreover, Chinese students used more mathematical terms than U.S. students in both their vertical and teacher-facilitated horizontal discourse.

Discussion

The hybrid "teacher-facilitated horizontal" discourse—in which the teacher in control, but the students to address each others' ideas—may potentially be significant in helping to understand how Chinese teachers appear to engage their students to learn mathematics successfully, in ways theoretically supported, but at the same time maintaining control of classroom discourse, in concert with cultural expectations. This hybrid form may act like horizontal discourse, in that students must reckon with each other's ideas and, in this way, provoke cognitive discord that provides fertile ground for cognitive change.

STUDENTS' ENGAGEMENT WITH THE SCIENCE AND ENGINEERING INTEGRATED CALCULUS TASKS

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Calculus acts as a filter to the STEM pipeline, which blocks students' access to STEM careers (Steen, 1987). Therefore, a strong foundation and understanding of calculus concepts is an important requirement for all STEM degrees (Young et al., 2011). Students who are engaged during learning activities, achieve better grades and educational activities are positively related to academic performance (Kuh et al., 2008).

Schools should provide opportunities to learn about mathematics by working on problems arising in contexts outside of mathematics (NCTM, 2000). Literature shows tendency towards integrated science and mathematics education; however, more empirical research grounded in these theoretical models is clearly needed (Berlin & Lee, 2005). Therefore, this study aims to investigate the following question:

- How students engage with the Science and Engineering Integrated Calculus Tasks?

The Science and Engineering Integrated Calculus Tasks refers to the calculus tasks that includes ideas from science and engineering fields and requires students to use calculus tools to perform.

The Science and Engineering Integrated Calculus Tasks refers to the calculus tasks that are science and engineering related in nature. Since this study was a small-scale study for those tasks, physics and computer science tasks were selected and piloted. Participants were students who were enrolled to calculus courses at a Southwestern university in the U.S. Data come from task-based interviews involving the participants working on the tasks.

Strong evidence showed that the tasks supported the participants in connecting physics and science to calculus. One participant states that:

> It's to me it's coming up, its creating and designing a solution to something that could be a real-world problem and so I think that I think that adds more to the experience. It certainly gives a lot. Like I feel like I am doing something I feel like I am not just doing a bunch of math you know?

Evidence shows that the tasks were interesting and enjoyable for the participants and that they felt motivated through this experience. This finding suggests that interdisciplinary approaches might increase students' engagement and thus contribute to positive learning experiences with calculus.

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A NARRATIVE INQUIRY ON THE EARLY TEACHING EXPERIENCES OF POSTSECONDARY MATHEMATICS TEACHERS: A PILOT STUDY

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Keywords: Post-Secondary Education, Research Methods

In the context of a doctoral project, this poster presents a pilot study on the process of becoming a postsecondary mathematics teacher. More precisely, the context is set in cegep institutions (general and vocational colleges), the first step in postsecondary education in the province of Quebec, Canada. The program in those institutions covers about the same mathematics as the last year of high school and the first year of university in the United States. Our focus is on new cegep mathematics teachers and how they negotiate the transition from being a mathematics students (graduate or not) to teaching at postsecondary level, without any formal training in teaching or education. Little is known about the process of becoming a mathematics teacher at postsecondary level (e.g. Speer & Hald, 2008) and our work is meant to contribute to this issue. We also hope to contribute to the discussion on the mandatory (or appropriate) education to teach in those types of institutions.

Framed by Dewey's philosophy (1916, 1938), our goal with this work is to shed some light on their experience as new teachers, in order to know more about the process of becoming a teacher.

With a methodology based on narrative inquiry (Clandinin, 2013; Clandinin & Connelly, 2000), weekly meetings were arranged with two cegep teachers during a whole semester. The first one was staring his third year as a teacher, the other was starting his second. They shared stories of events they lived and identified as significant for their teaching, and of reflections they made about these events. An account was written of these stories as the interviewer heard them, and of their reflections about them. An analysis was then performed according to the three-dimensional framework designed by Clandinin & Connelly (2000), with a focus on the teachers' relationship with mathematics and education. This poster discusses what this pilot taught us about our choice of method and about our general topic. It also addresses possible changes in the method and what are the next steps to be taken to get closer to our goals.

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SUPPORTING COLLABORATIVE TEACHER REFLECTION BY VISUALIZING PRACTICE WITH DATA

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Keywords: Classroom Discourse, Design Experiments, Teacher Education/Professional Development

Facilitating whole class discussions is highly valued in math classes because they can support deeper conceptual understanding by making the elements and connections within mathematical systems visible to students (e.g. Ball, 1993; Cobb, Wood, Yackel, & McNeal, 1992; Kazemi & Stipek 2001). In addition, discourse during whole class discussions can provide opportunities to develop new epistemic practices that more closely resemble the epistemologies in STEM professions (Ford & Forman, 2006, p. 3). However, the value of class discussions varies widely, and high-quality discussions require facilitation from a teacher skilled in "effectively guiding whole-class discussions of student-generated work toward important and worthwhile disciplinary ideas" (Stein et al, 2008, p. 319). Teachers need significant support to develop the practices necessary to do this challenging work. Because of this, their effectiveness can vary due to differences in teacher facilitation.

This poster describes an innovative approach to supporting teachers to continuously reflect on their facilitation of classroom discussion called Visualizing Practice with Data (VPD). The VPD approach systematically supports teachers to reflect on practice through exploring data visualizations that represent their whole class discussions. This new routine for teacher collaborative reflection is designed to orient teachers towards specific problems of practice, and to link these problems to conceptual framings that can support continual improvement (Horn & Little, 2010).

We will report on a pilot design study (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2004) conducted with two 7th grade teachers over a three-week period. The author co-planned a three-week unit on probability, modeling, and inference with two teachers. The author created data representations of the discussions, and the teachers used the visualizations to collaboratively reflect on their lessons. VPD sessions provoked replays of classroom practice and rehearsals of future lessons with strategies for improvement, which we will share and elaborate on (Horn, 2010).

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CONNECTING CONTEXTUAL AND MATHEMATICAL KNOWLEDGE FOR BUILDING THE COLLECTIVE MEMORY OF THE MATHEMATICS CLASS

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The main question of this study is: *what linguistic resources did a teacher use to bridge contextual elements of a problem and mathematical ideas during public discussions?* I analyze a teacher's choices for creating the collective memory using Systemic Functional Linguistics (Martin & Rose, 2007). I focus on choices for keeping track of people or things (i.e., participants) using options from the system of identification: *presenting*, *presuming*, *possessive, comparative,* and *text reference*. Madeline (a pseudonym) taught a problem-based lesson about the theorem stating that the set of points that are equidistant to two given points are on the perpendicular bisector of the segment connecting the two points. The problem was situated in the context of finding a fair location for an after-school center, considering two schools.

Madeline used more contextual participants during the launch than in the summary (Table 1). In contrast, the summary focused on mathematics. In addition, Madeline's references to the geometric diagram during the summary connected the problem's context and underlying mathematical ideas for solving the problem. The diagram illustrated the segment connecting the two schools, its midpoint, and the perpendicular bisector of the segment.

Table 1: Participants and Tracking

Activity	Contextual				
	Presenting	Presuming	Possessive	Comparative	Text Reference
Launch	7	35	0	0	1
Summary	0	2	0	0	0
	Mathematical				
Launch	1	0	0	0	0
Summary	8	37	0	2	1

The study shows a case of how a teacher created the collective memory of the mathematics class by shifting students' attention from the problem's contextual to mathematical ideas, using the context as a starting point to reach the mathematical goal of the lesson.

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TEACHER BELIEFS: UNAWARENESS OF AND CONFLICTS WITH EQUITY

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Keywords: Teacher Beliefs, Equity and Diversity, Elementary School Education

This study examines teacher beliefs about good mathematics teaching viewed through the lens of social justice teaching [SJT]. I define SJT as teaching that considers diversity for inclusive learning community and promotes equitable learning for all students, which is grounded in anti-oppressive teaching (Kumashiro, 2008). Teachers' beliefs serve as filters through which they view education; mediate knowledge; and work as a guide for teaching (Thompson, 1992). Given the powerful influence of teaching on learning, what a teacher believes results in equitable or inequitable learning; therefore, attending to teaching mediated by beliefs is important. Employing a case-study design, I investigate two Korean teachers' common sense of good mathematics teaching through espoused and enacted beliefs, using SJT perspective. Despite a Korean contextualized study, it could provide an opportunity for the dissemination of SJT to improve school mathematics because equity is a matter for everyone, regardless of countries.

In this study, the key notion is equity: *All* students should not receive identical instruction but access to meaningful mathematics through appropriate accommodations, along with uniformly high expectations (NCTM, 2000). For this, previous studies suggest that mathematics instruction considers all students' active participation in learning process; helps find out their mathematical strengths and hold positive identity as learners; and includes beyond classical knowledge and connects funds of knowledge. Also, mathematical authority should be shifted from teachers or textbook to learning community (e.g., Boaler, 2002; Moll et al., 1992). These all contribute to developing the conceptual framework to analyze espoused and enacted beliefs of teachers.

According to their interviews and teaching practices, the teachers could not well recognize the notion of equity and its importance and necessary in mathematics education, which is related to a general conception of mathematics (e.g., absolute, value-free) and a Korean educational context (e.g., excessive competition, emphasis on students' uniformity, rather than diversity). Also, the teachers felt tensions between efficient, effective, meaningful, and equitable instructions, and a priority presented in their teaching lacked consistency. Last, their teaching processes/goals were not often aligned with their students' learning processes/goals in terms of equity. Findings suggest that teachers need learning to teach for productive beliefs and knowledge against oppressiveness and inequity in teaching and learning mathematics.

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DEVELOPING A PROTOCOL FOR DESCRIBING PROBLEM-SOLVING INSTRUCTION

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Schroeder and Lester (1989) distinguish between three approaches to problem solving instruction: teaching *about*, *for*, and *through* problem solving. Each approach has distinct tasks and emphases in students' learning. Teaching about problem solving (TAPS) focuses on learning of heuristics and/or processes of problem solving; teaching for problem solving (TFPS) helps students to understand concepts and procedures in problem solving first thin use the knowledge gained to solve problems; in teaching through problem solving (TTPS), students learn a concept or procedure through an experience as they engage in problem solving. As states and districts adopt new standards such as the Common Core State Standards for Mathematics (CCSSM) that focus more on reasoning and sense-making, teachers should align their instructions to meet these standards. Since TTPS contexts are feasible with CCSSM and demonstrate ways to align these standards with a problem-solving focus (Bostic, 2011), I used the following research method to develop a protocol to support teachers in implementing a problem solving approach based on TTPS. This protocol answered the research question: "What are descriptors of teaching for, about, and through problem solving in grades 4-6 mathematics teachers' instruction?". In this protocol, I provided detailed descriptions of strategies for each approach in problem-solving instruction and presented many examples of each strategy to elucidate the practices that would support a change to TTPS.

Participants in this study were teachers from fourth-, fifth-, and sixth-grade classrooms. These teachers participated in a PD program to improve their understanding of the CCSSM and thereby their pedagogy overall. Data were collected using two sources: teachers' lesson plans and video tapes of the lessons being taught after attending the PD program. I used the constant comparative method of qualitative analysis to compare newly-collected teacher data with previous findings elaborated in literature about TAPS, TFPS, and TTPS in order to develop an observation protocol to describe each of the approaches to problem solving.

This protocol can be used to determine which type of teaching a teacher uses and which practices he/she might change to move toward TTPS approach. This protocol can be useful for mathematics teachers who are looking to develop their classroom practice in relation to problem solving. It can also serve as a self-directed orientation for teachers' professional development. Finally, it can be useful to administrations and teacher educators who work for practicing teachers in professional development settings.

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TEACHER NOTICING: A QUALITATIVE STUDY OF NOVICE AND EXPERIENCED SECONDARY MATHEMATICS TEACHERS

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Recent research has examined teacher attention, studying activities in the classroom that teachers notice (Jacobs et al., 2010; Russ & Luna, 2013), such as a pivotal teaching moment (PTM), defined as a student-generated disruption to the flow of discussion where a teacher can respond flexibly to extend or enhance student thinking (Ruiz-Primo & Furtak, 2007; Stockero & Van Zoest, 2013). Questioning is an important consideration within the context of noticing.

This multiple case study compared three novice and three experienced secondary mathematics teachers' noticing of PTMs and use of questions during PTMs. Six participants from five schools took part in two interviews and five lesson observations. The research question examined was: What similarities and differences exist in questions and responses to students during PTMs between novice and experienced teachers?

Analysis of lesson and interview transcripts and field notes revealed: 1) Both novice and experienced teachers emphasized proper procedures but reflected this emphasis differently; 2) One novice teacher and one experienced teacher placed emphasis on connections within and outside of mathematics; 3) One experienced teacher emphasized efficient problem solving.

The first finding showed participants emphasized mathematical procedures differently when using questions during PTMs. Tom's procedural focus was coupled with conceptual understanding, and Samantha emphasized correct procedures without emphasis on related concepts. Dana steered her students tightly toward correct procedures to avoid misconceptions.

The second finding showed one novice teacher and one experienced teacher emphasized connections within and outside of mathematics. Amy made connections to pop culture and other disciplines. Kathleen used questions to connect current topics with previously learned concepts.

A third finding relates to one experienced teacher's emphasis on efficient problem solving strategies. Amy used questions to prompt students in sharing their strategies with classmates and explain why a particular strategy was or was not useful for solving particular problems.

The topic of PTM noticing and response has many avenues for further research in mathematics education. The current literature in mathematics education could be enhanced with more study of teacher questioning in relation to the noticing and response to PTMs. Empirical studies on a larger scale can examine teacher noticing and PTMs with regards to how a teacher frames the questioning before, during, and after a PTM.

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EXPLORING DIFFERENTIATION WITH MIDDLE SCHOOL TEACHERS

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Despite the promise of differentiated instruction (DI) as a way to address cognitive diversity, it is least likely to be used in secondary mathematics classrooms (Gamoran & Weinstein, 1998), due in part to the many challenges of implementation. DI requires proactive planning to meet diverse needs of students while still maintaining a cohesive classroom community (Tomlinson, 2005). To follow through on the promise of DI it is vital to investigate the how secondary math teachers—starting with middle school—understand and implement it. This poster presents preliminary findings from a year-long Teacher Study Group (TSG) with 15 middle school math teachers from around the state of Indiana that, together with a university research team, investigated DI.

This research is drawn from the third year of a 5-year study of DI for middle school students and their teachers. The members of the TSG met for a 3-day workshop during July 2015, for 8 monthly meetings from August to April, and for 1 day of sharing and presentations during June 2016. Between monthly meetings, participants completed and shared assignments about DI, drawing on student thinking from their classrooms. The analysis for this poster was drawn from a comparison of questionnaires completed by teachers in July 2015 and June 2016.

Over the course of the TSG, we saw development in two domains: teachers' ideas about students' thinking and their ideas about DI implementation. At the beginning, their discussion about both reflected grand goals but somewhat superficial details. For example, their discussion about students was about innate ability or procedural accuracy rather than student reasoning about particular mathematical ideas. They discussed DI as a key to closing achievement gaps among students, and their ideas about implementation often involved giving students work that differed in appearance or quantity but not necessarily in depth or complexity of the mathematical thinking required. By the end of the year, teachers had made some changes in how they thought about the endeavor of differentiation. Teachers commented on the need to build on what students were thinking, which involved gathering careful evidence of that thinking, posing tasks that could allow students to make small steps, and focusing on mathematical meaning. With regard to their implementation of DI, they recognized more clearly what they were already doing in their classroom to differentiate, what was meaningful to them about differentiation, and focused on making small changes to better meet the needs of their students.

As researchers, our ideas about DI evolved alongside those of the teachers; we began to understand how important it is to focus on student thinking as a foundation. We began to understand how important it is to make small steps, tailored to the needs of both students and teachers.

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TEACHERS' NUMBER CHOICES FOR EQUAL SHARING PROBLEMS

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Equal Sharing story problems can be used to teach fractions when the context involves quantities that can be individually partitioned. For example, the Equal Sharing problem of 11 bars shared equally among 4 children results in a remainder of 3 bars that can be partitioned to exhaust the bars and give each sharer a fractional amount. The specific numbers in Equal Sharing problems can influence the accessibility of problems for a child, the knowledge a child might use to solve problems, and the mathematics that can be addressed. Teachers' choices for numbers used in Equal Sharing problems are therefore a potentially powerful component of planning for fraction instruction.

Studying teachers' number choices for whole-number story problems, Drake and Land (2014) identified the need to investigate how teachers select numbers for problems used during instruction. We answered their call by exploring teachers' reasoning for number choices in a fraction story problem by interviewing 43 teachers of grades 3-5 after a classroom observation where they posed an Equal Sharing fraction problem of their choice. These teachers had participated in 1, 2, or 3 years of professional development focused on children's fraction thinking. We asked: *How did you decide on this problem for these students? How did what you know about your students inform your number choice(s)?*

We found that teachers' number choices were informed by children's mathematical thinking in two main ways: (a) Sometimes teachers drew on general information about children's fraction thinking (e.g., children often draw to solve such problems); and (b) sometimes teachers drew on detailed information about specific children's fraction thinking (e.g., Rosie's repeated halving). In addition, teachers' number choices were informed by the mathematical content they wanted to address, also in two main ways: (a) Sometimes teachers described how their target content emerged in children's strategies (e.g., a strategy where each brownie was split into 8 parts and the parts combined for each share could be used to address addition of unit fractions); and (b) sometimes teachers described target content with minimal connections to children's thinking (e.g., "I chose 10 sharers because I wanted to work with decimals"). These findings highlight critical components in teachers' reasoning about number choices for fraction story problems and lay the groundwork for research on such choices in instructional planning.

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EXPLORING TEACHERS' SCAFFOLDING OF STUDENTS' MATHEMATICAL EXPLANATIONS IN SECONDARY SCHOOLS

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One way to communicate and clarify one's thinking is through giving mathematical explanations (Clark, Moore, & Carlson, 2008), which has been connected by researchers to students' understanding of mathematics (Goos, 1995). Yet, studies have shown that students struggle with giving mathematical explanations (e.g., Brunstrom & Fahlgren, 2015). To understand the genesis of this struggle, it is important to explore how teachers scaffold students' mathematical explanations. Through scaffolding, the anticipation is that the support offered will enable the learner to internalize the strategies required to complete a similar task independently (van de Pol, Volman, & Beishuizen, 2010). That said, if students are to demonstrate the competency in explaining their work, then studying how teachers foster such competency is worth considering.

In this study, we employed a case study design to explore what two high school mathematics teachers perceived as a sufficient mathematical explanation, what they thought are scaffolding methods that support students in giving mathematical explanations, and their methods of scaffolding students in giving mathematical explanations. The two participants from a suburban independent school and a suburban public school were selected following their practice of asking students to give mathematical explanations during their lessons. Data were collected using two semi-structured interviews and four lesson observations, each roughly one hour in length. A constant-comparative data analysis method was employed, where analysis involved three phases: (a) transcribing all recorded interviews and lessons observed; (b) through open-coding, identifying names, events, and actions that seemed to address the research questions; and (c) categorizing all codes and creating analytical memos. We identified three major themes: (1) teachers' perception of a sufficient mathematical explanation; (2) methods perceived to scaffold students' mathematical explanations; and (3) enacted methods of scaffolding mathematical explanations.

Our poster will present findings and their implications. In sum, teachers in this study perceived a sufficient mathematical explanation to reference the quantities of the problem in context (Clark et al., 2008). They also perceived diverse and common methods of scaffolding students' mathematical explanations, which some were observed during their lessons. Our poster will provide examples of these methods.

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TEACHING NORMS FOR THE CONCEPT OF DERIVATIVE IN HIGH SCHOOL AND COLLEGE LEVEL CALCULUS COURSES

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Keywords: Instructional Activities and Practices; High School Education; Post-Secondary Education

From its inception, Advanced Placement calculus was implemented to not only bridge the gap between high school and college level courses by providing a challenging curriculum, but to promote equity by providing opportunities for motivated students who excelled in mathematics in the public school system. Certain teaching norms may differ from one context to another (Herbst & Chazan, 2012), from an Advanced Placement calculus classroom environment to a college calculus course in a university setting with respect to practical rationality and instructional activity. The purpose of this study is to explore whether this has an impact on student learning with respect to the concept of derivatives, and whether certain teaching norms e.g. inquiry-based tasks, are either less prevalent in a university setting, in upper level mathematics courses, or possibly both.

This study is situated with the perspective that instructional strategies in the calculus classroom vary with regard to procedural and conceptual understanding of the content (Törner, Potari, & Zachariades, 2014). Educators' perceptions of effective practice with respect to teaching derivatives can impact the depth of student learning, which will be explored with the following research question through the lens of practical rationality to examine aspects of procedural and conceptual understanding: How do college and high school level calculus instructors describe teaching norms for the concept of derivative, and what justifications do they use for endorsing or departing from these norms?

Two high school AP calculus teachers and two college calculus instructors are participants in this study. Participants were interviewed with the objective of gaining insight into instructional methods, curricula, and perceptions of effective practice. This study could impact the way calculus is taught in both high school and college classrooms, and provide insight into the alignment between AP calculus and college calculus courses.

In each interview lasting approximately 45 minutes, four scenarios were developed using the platform LessonSketch.org to see how participants respond to two teacher focused scenarios and two student focused scenarios, designed to elicit certain responses regarding instructional practices. Preliminary analysis of interview responses revealed that college calculus instructors recognized more instructor-centric teaching as normative at the collegiate level, regardless of whether they endorsed these norms individually. AP calculus instructors were more likely to have higher expectations of their students, responding immediately with confidence that their students were able to describe the relationship between a function and its derivative.

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FIRST GRADE WRITTEN MATHEMATICAL EXPLANATIONS

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Keywords: Elementary Education, Classroom Discourse

Communicating about their thinking in mathematics is challenging for young children. The Common Core State Standards for Mathematics requires children, beginning in Kindergarten, to *construct viable arguments and critique the reasoning of others*. While the CCSSM (2010) does not specifically require children to construct their arguments in writing, children in grades three and up are asked to construct arguments on standardized assessments, requiring them to *write* about their mathematical reasoning. However, while writing in mathematics helps children consolidate their thinking (NCTM, 2000), undeveloped metacognitive awareness frequently limits their ability to put that thinking in writing. Since the early elementary grades are a foundation for future success, exploring ways to help young children develop writing about their mathematical thinking is warranted (Cohen et al., 2015).

The CCSSM (2010) specifically state that first grade students explain the reasoning used when adding and subtracting two-digit numbers and multiples of ten. To study the impact of writing on students' ability to explain their reasoning, two classes of first graders engaged in eight 45-minute lessons in which they solved simple word problems using a strategy of their choosing, such as direct modeling, counting or an invented algorithm (Carpenter et al., 2015). Students then shared their solution strategy with a partner. After sharing with a partner, students wrote about how they solved the problem. Finally, the whole class engaged in a conversation about the students' solution strategies and the mathematical goals of the lesson.

To determine change, the students participated in a pre/post assessment in which they explained in writing how they solved a Join Result Unknown problem. The written responses on the pre-assessment were vague and mainly limited to describing their drawings, such as *I drew ten blocks and one blocks.* The written responses on the post assessment included more mathematical vocabulary, including add, more, counted, tens and ones. Students demonstrated conceptual understanding of adding two-digit numbers and multiples of ten by writing statements such as " *I used 3 tens 6 ones and made 36. I used 4 tens 0 ones and made 40. I counted them and made 76.* Students also demonstrated mathematical reasoning by explaining that they knew to add because "*he went to the store and got more*." The student responses on the post assessment demonstrate that *very* young children are able to write about their mathematical thinking in deep, thoughtful ways.

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MY CALCULUS, YOUR CALCULUS: TEACHING MATH THROUGH SOCIAL JUSTICE IN COLLEGE CALCULUS

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Teaching mathematics through social justice supports students' perceptions of relevance in mathematics (e.g., Gutstein, 2006). However, while some research reports tension between mathematics and social justice (e.g., Enyedy & Mukhopadhyay, 2007), Gutstein (2017) describes it, instead, as an interweaving dance, and this study investigates the choreography of teaching mathematics through social justice. How do tasks support students' development of critical dispositions toward social topics, or their views of mathematics?

A growing body of scholarship has informed our understanding of teaching mathematics through social justice and has emphasized the usefulness of real-world contexts in creating opportunities for math and sociopolitical learning. The data were collected in a semester-long Calculus course. This study contributes to literature by addressing the following questions:

1. What capabilities and constraints are observed when teaching and learning mathematics through social justice?
2. How can teachers and students "dance" between social justice pedagogy and mathematical content?

University calculus students (n=106) conducted social justice projects. Classes were audio-recorded, transcribed, and analyzed. In questionnaires, students tended to identify constraints in correlation with mathematical content ($p<.05$) and capabilities in correlation with cultural knowledge ($p<.05$). While these correlations inform the tensions between content and culture, the qualitative analysis reveals more about the dance between mathematical and sociocultural knowledge. For example, a thematic analysis reveals a humanization of mathematics. After analyzing homeless rates among psychiatric patients, one student reported, "It puts a face on the numbers." Other themes included access and action, encouraging critical dispositions.

By analyzing the instructor's perspective, practical implications include affordances in developing engaged dispositions toward Calculus and social justice-themed activities. Theoretical implications include refinement and extension of Gutstein's framework.

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SIMULATING SPREADING IDEAS ACROSS MATHEMATICAL CLASSROOMS

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Keywords: Classroom Discourse, Teacher Education/Professional Development, Teacher Beliefs

This paper explores how students' mathematical ideas can spread across the mathematics classroom space during seatwork. Recent reform recommendations emphasize the importance of a well-designed challenging task forming the core of students' in-class mathematical experiences (e.g., NCTM, 2000). Maintaining each individual student's cognitive demand while solving this central task is a major part of teacher work during the implementation phase of a mathematics lesson. However, in a class where communication, collaboration, and joint cognition are expected, valued, and encouraged, students' own interactions can paradoxically and ironically undermine the teacher's attempts to maintain the cognitive load of a teacher-posed task for each individual student. As students listen to each other's thoughts, successful solutions to teacher-posed cognitive task can rapidly promulgate across the classroom space. When students adopt successful ideas of their peers, their cognitive demand (relative to the central task) is diminished.

To better understand this dilemma, this study explored ways in which students' mathematical ideas can spread across the mathematical classroom space. Using an idealized classroom space of 36 student desk locations (six desks across by six desks deep), 3,568 separate computer simulations were run on a computer program based on the initial parameters of (a) spread initiating from a single location and (b) propagating in eight possible directions (c) based on a pre-set percentage chance of idea spread from one student location to the next. 1,561 simulations allowed for (d) possible new student understanding to potentially and spontaneously crop up at other locations in the classroom (to simulate the possibility of unique student understanding developing that was unconnected to the original student's idea, as seatwork progressed). Results from these computer simulations include (1) empirical justification for the theory of placing more able students on the periphery of the classroom space slowed the spread of mathematical ideas (which allowed slower students the opportunity to experience longer the vital element of struggle so important to mathematics learning), (2) the distributions of the number of time iterations for full classroom understanding were not normally distributed, and (3)

This study contributes to the growing body of research on the development of authentic mathematical communities in mathematics classes, although this study is limited in scope. Better understanding how student ideas can spread across the mathematics classroom landscape will enable more robust theory generation for the dilemma of maximizing students' cognitive demand during seatwork of teacher-posed challenging tasks while simultaneously respecting the need for appropriate communication. More research utilizing more precise computer simulations that better model how student ideas spread across mathematics classroom spaces is needed to better understand how to maintain students' cognitive load during mathematics task activity.

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UNPACKING SECONDARY MATHEMATICS TEACHERS' FORMATIVE ASSESSMENT PROCESSES SUPPORTED BY TECHNOLOGY

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Keywords: Assessment and Evaluation, Instructional Activities and Practices, Technology

Existing literature on formative assessment focuses on the characteristics, cyclic processes, and ongoing conceptualization of formative assessment (Black & Wiliam, 2009; Cowie & Bell, 1999; Heritage, 2010; Wiliam, 2014). The purpose of this qualitative case study research was to identify the formative assessment processes of three secondary mathematics teachers who used technology to gather feedback/answers from students during instruction. Data collection included audio-recorded non-participant classroom observations, audio-recorded semi-structured pre- and post-observation interviews, and when applicable, screen capture of student responses through technology. The conceptual framework of formative assessment by Black and Wiliam (2009) was used to analyze transcribed classroom observations to determine how each teacher utilized, implemented and sequenced the five key strategies (KS) of the framework during instruction.

Results indicated that all three teachers incorporated *eliciting evidence of student understanding and learning* (KS2), *providing feedback to move learning forward* (KS3), and *activating students as instructional resources for one another* (KS4) into instruction. All three teachers verbally shared learning objectives with students, but only one *clarified and shared learning intentions and criteria for success with students* (KS1). Two teachers *activated students as owners of their own learning* (KS5). Only one teacher implemented all five KS into their formative assessment process. In addition to identifying the KS that emerged during the formative assessment processes, the implementation and sequencing of strategies was also revealed. Each teacher demonstrated their own unique process of formative assessment. Diagrams depicting the sequence of KS implemented by each teacher during their formative assessment process will be shared.

Future implications include continuing to explore teachers' implementation and sequencing of KS of formative assessment using technology, how each teacher's mathematical disposition impacts their process of formative assessment, and the impact of each formative assessment process on student achievement. In the three cases presented, some instruction occurred prior to using technology to elicit evidence of student thinking. How did this implementation and sequencing of KS impact student understanding and achievement? How would varying the sequence of KS impact student understanding and achievement? Additional work is needed to learn more about how teachers use the key strategies of formative assessment in practice and the impact these formative assessment processes have on student understanding and achievement in mathematics.

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Chapter 12

Technology

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Brief Research Reports

Posters

DEVELOPING PRESERVICE TEACHERS' UNDERSTANDING OF FUNCTION USING A VENDING MACHINE METAPHOR APPLET

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The purpose of this study is to examine the use of a Vending Machine applet as a cognitive root for the development of preservice teachers understanding of function. The applet was designed to purposefully problematize common misconceptions associated with the algebraic nature of typical function machines. Findings indicated affordances and limitations of the applet as a cognitive root, motivating revisions to the applet for further study.

Keywords: Algebra and Algebraic Thinking, Instructional Activities and Practices, Technology

Functions are a critical base for mathematical understanding of STEM disciplines and are often regarded as the unifying element of much of secondary mathematics. This is recognized in the CCSSM where the study of functions is given its own domain, separate from Algebra, in grades 9–12 (National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010). The conceptual obstacles with respect to putting function front and center have been well documented in the literature (e.g., Even, 1990, Tall et al., 2000) and have proven particularly hard to overcome. Rather than constructing their own definition of function based on tasks, students are often presented with a highly theoretical definition, resulting in a disconnect between their concept definition and their concept image (Even, 1990). The same is true for undergraduate students (Carlson, 1998), including preservice mathematics teachers (PSTs). Thus it is important that PSTs have an opportunity to develop a deep conceptual understanding of functions that remedies any existing misconceptions and understand how to engage students in tasks to develop and test their own definitions.

Theoretical Framework

There is evidence that PSTs often have a view of function that is limited to algebraic expressions and their associated graphs (e.g., Carlson 1998; Even 1990). Such understandings typically result in a "vertical line test" related definition of function (e.g., Carlson, 1998). Furthermore, those working with a vertical line test definition of function or an equation view of function often have the misconception that constant functions (horizontal lines) are not functions (Bakar & Tall, 1991). One suggested strategy for mitigating these common misunderstandings is the use of a function machine as a cognitive root.

The idea of a *cognitive root* was introduced by Tall as an "anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" as he was developing a cognitive approach to calculus (Tall et al., 2000, p.497). One common example is the use of a *function machine* (sometimes referred to as a function box) as a cognitive root for the development of a definition for function as well as for building understanding of independent/ dependent variables and domain/range. The machine metaphor suggested by Tall and colleagues was typically a "guess my rule" activity. In such activities the inputs and associated outputs are provided, and students are challenged to determine what happened in the function machine (i.e., determine the function rule). While students are presented with a machine to embody the function concept, the rules used by the machine are algebraic in nature. Using such machines proved quite promising, yet some students still struggled with connecting representations and determining what is and is not a function (McGowan et al., 2000).

Given the potential of the machine metaphor as a cognitive root for function, in our work designing learning experiences for PSTs that incorporate technology, we originally followed Tall and colleagues' recommendations and created a function machine using GeoGebra to engage PSTs in "guess my rule" activities, including designing their own function machines. Our PSTs demonstrated misconceptions that were similar to those in the literature, namely, that all functions needed to be represented by a formula, difficulty differentiating between the terms unique and exactly one output, and identifying constant functions as non-functions.

As a result, we set forth to design a set of machines that would not only be a familiar anchoring concept, but would hopefully also push and probe PSTs current understandings about functions in ways to intentionally problematize these misconceptions. To do this we considered Carlson's (1998) key aspects of function as they related to the function machine metaphor: interpreting and characterizing independent and dependent variables, ability to identify and describe the domain and range, using one representation of function to make sense of another, and distinguishing between functions and non-functions. The function machine as a cognitive root and these aspects of function as they related to common misconceptions framed both the design of our applet and our analysis of PSTs' work with the applet.

Design of the Applet

Unlike previous machine metaphors PSTs had experience with, the function machine applet we designed contains no numerical or algebraic expressions, rather the applet was built on the metaphor of a vending machine. The Vending Machine applet (https://ggbm.at/LdtLR0ex) is a GeoGebra file that contains five vending machines each with buttons for: Red Cola, Diet Blue, Silver Mist, and Green Dew. The directions state to explore the five machines and determine which are functions (Figure 1; McCulloch et al., 2015). When the user presses a button (input), one or more cans appear in the bottom of the machine (output). To remove the can(s) from the machine, the user clicks a reset button. The functionality of each machine was designed to address misconceptions from the literature on distinguishing functions and non-functions that our PSTs had previously demonstrated. Machine A is the identity function; each button produces a can of the corresponding color.

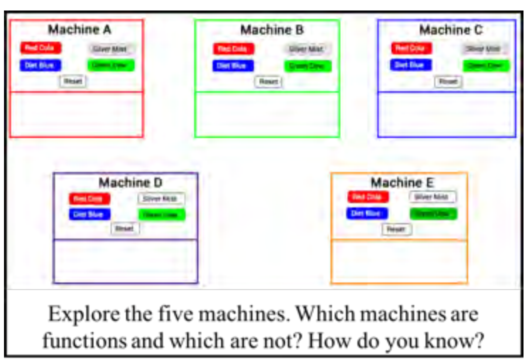


Figure 1. Screenshot of function machine applet.

Machine B is the same as A except when Silver Mist is selected, it produces two silver cans. This machine requires students to wrestle with the notion of what represents an element in the range. For Machine C, every button results in a single green can. The purpose of which is to present PSTs with a constant function to consider (i.e., the same number of cans of the same color for each button). For each button on Machine D a single can is produced, but the color is different from the color of the button pressed. This machine was designed to problematize their occasional use of the term "unique" when thinking about outputs. Finally, Machine E is similar to D, except the Silver Mist button randomly produces cans of different colors each time it is pressed. The purpose of Machine E is to provide a context in which testing the buttons on the machine once is not sufficient for determining whether or not the object is a function. Thus machines B, C, and E explicitly address the misconceptions we found when using function machines with associated algebraic expressions or graphs.

The purpose of this study was to examine the effectiveness of the Vending Machine applet as a cognitive root for function. In this paper we specifically address the following research question: What understandings of function do PSTs develop from engaging in a task using a vending machine metaphor applet?

Methods

To answer our research question, we engaged PSTs enrolled in a content-focused methods course in a task exploring the Vending Machine applet. Nine PSTs (referred to as S1-S9) engaged in the task; of the nine, seven were undergraduate secondary mathematics education majors (three of the seven were also dual mathematics majors), and two were enrolled in the Master of Arts in Teaching (initial licensure) program.

Data Collection and Analysis

The study began by asking PSTs to individually write a definition of a function, including examples and non-examples. After doing this independently, a whole class discussion was facilitated using their definitions through which the class agreed upon the following definition: *A function is a mathematical relationship such that each input has exactly one output.* Then, they were asked to engage in the Vending Machine applet task as a homework assignment; they were to explore the machines to determine which were functions and which were non-functions. Each PST captured a screencast of their work as they followed a "think aloud" protocol while working on the task. Simultaneously, they completed a worksheet to provide written documentation of their thinking. Following the task, PSTs completed a written reflection where they were asked to revise the agreed upon definition of function based on their experience with the task, to reflect on the different representations of functions presented in the task, and to discuss aspects of function highlighted by different machines with which they engaged (including possible uses with students). The PSTs uploaded their screencasts to a shared, secure online folder and submitted their written work during the following class session.

We began by examining the data to develop a code book based on four key aspects of function (Carlson, 1998) we previously identified that informed our theoretical framework. We chose two students' data (screencast, written worksheet, and reflection) and used open coding to identify themes related to each key aspect. We used a constant comparison method (Strauss & Corbin, 1998), which allowed for emerging categories within each key aspect and the refinement of these categories as they were contrasted with new data. Since we were particularly interested in how PSTs made sense of the applet, we also looked for evidence of the affordances and limitations of the applet, as well as misconceptions/errors, and an "other" category to capture any unanticipated themes. Then, to check for reliability, each author individually coded the two original students' data, plus an additional student's data, using the code book.

For the screencasts, we recorded the code and machine the PST referenced. For example, when working on Machine E, one PST commented "Machine E is a non-function because the silver button points to different colored cans" This was coded as "Distinguishing between functions and non-functions" and "Machine E" since the PST was justifying why that particular machine was a non-function. For the written artifacts, we coded their written explanations and drawings. For example, on the worksheet, several PSTs drew mappings that described each vending machine. These drawings were coded as “Interpret and characterize independent and dependent variables” and labeled with the letter of the corresponding machine that the drawing was in reference to. The researchers met to discuss the codebook and emerging themes and to clarify the codebook.

Once the codebook was finalized and reliability achieved, we coded all of the remaining data. We then looked within each code to understand the ways in which the PSTs engaged with the function machines to make sense of each key aspect. Finally, we examined the data by each function machine (for example, we looked at all codes associated with Machine A) to understand how each particular machine supported (or conflated) PSTs understanding of function, in particular the key aspects, as well as the affordances and limitations of the applet.

Results

Of the nine PSTs that completed the function machine task, five of them provided correct responses for all five machines. The other four PSTs’ errors were related to Machine B and/or Machine E (see Table 1). Their interpretations and characterizations of the independent and dependent variables (referred to as “inputs” and “outputs” by most PSTs) were central to their determination of whether or not each machine was a function. Here we focus on the ways in which the PSTs made sense of two specific machines, B and E. These machines are the only two for which PSTs provided incorrect answers and as such they provide insight to the aspects of function that the machines were designed to elicit. For the purposes of this paper we discuss two specific code categories, *distinguishing between functions and non-functions* and *interpreting and characterizing independent and dependent variables*, and the ways they provide insight to student sense making.

Table 1: PSTs’ Worksheet Responses: Function or Non-Function

Machine	S1	S2	S3	S4	S5	S6	S7	S8	S9
A	F	F	F	F	F	F	F	F	F
B	F	N	F	N	N	N	F	F	F
C	F	F	F	F	F	F	F	F	F
D	F	F	F	F	F	F	F	F	F
E	N	N	N	F	F	F	N	N	N

The Case of Machine B

As a reminder, pressing the Silver Mist button on Machine B results in the production of two cans of Silver Mist. The other three buttons act as the PSTs expected, producing one can of the soda noted on the button. Noting that Machine B is not a function was the most commonly made error in this task (n=4). The characterizations of the independent and dependent variables were quite different for those students that determined B was a function and those that determined it was not.

When examining the machines, the PSTs that described the independent and dependent variables (called inputs and outputs from here on out) only attending to how many elements there were in the output for each input determined that the machine was not a function. For example, S4 explained, “Red cola has one output, diet blue has one output, silver mist has two outputs, and green dew has one output...since silver mist has two outputs, I would say that is not a function.” Similarly, S5 noted “So you put in an input and you get two outputs, in a function you are not supposed to get two outputs.” Notice that both PSTs were focused on the number of outputs for each input.

In contrast, other PSTs were attending to not only the number of elements in the output, but also what they were and if the mapping from input to output was “predictable”. S3 stated, “Exactly one output for each input, 2 silver mist cans is the output for silver mist.” S9 went further and explained the importance of the output being “predictable” when he said, “Even though silver mist comes out with two cans versus one can but it always comes out with two silver cans, instead of three silver cans one time and two silver cans another time.” While the idea of two cans being the output was not problematic for some students, there were a few that grappled with it as they made sense of the situation. For example, S1 noted

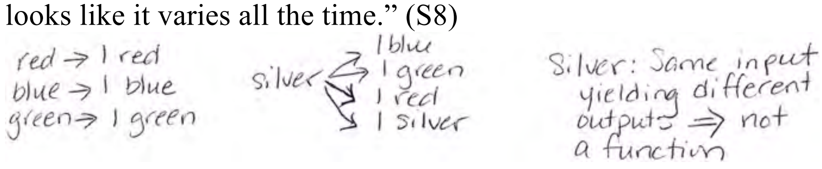
> So silver mist comes out with two sodas...hmmm...[keeps pressing]...that's interesting...cuz it depends on what you consider the output. The output could be how many different kinds of sodas there are...so it could be like you're putting in red you're getting out one, you're putting in blue you're getting out one, you're putting in silver you're getting out two, it doesn't necessarily mean it's not a function...it's just a different value.

The difference between these responses and those from the PSTs that focused on only the number of elements was that these students identified a mapping.

The Case of Machine E

Like we saw for Machine B, attending to only how many elements in each output was also problematic when trying to determine whether or not Machine E represented a function. As a reminder, Machine E included a button for Silver Mist for which the output was random, so it was not a function. S4 explained, "Red cola has an output, diet blue has an output, silver mist has an output, and green dew has one output...again [presses buttons]...they each have one output, silver mist has one output but it is a blue soda as well...each choice has one output...so it is a function." Even though she clearly identified that the Silver Mist button did not always produce the same output, she determined it was a function because it always produces one output. S5 only tried each button once, so he did not see that the Silver Mist button was unpredictable. He said, "For Red Cola you get a Red Cola, for Diet Blue you get you a Diet Blue, for Silver Mist you get a Green Dew, for Green Dew you get a Green Dew. I still think it's a function...you can get two of the same outputs because you are still getting one output for each input." While he did not have the opportunity to see the different outputs, the fact that he only tested the button once suggests that he does not understand what it means for an input to map to exactly one output.

The PSTs that interpreted outputs to have characteristics beyond just "how many" elements, correctly identified Machine E as a non-function. We have taken to describing this coordination between inputs and attending to multiple characteristics of the outputs as "mapping thinking". The following explanations are examples of what we are referring to as mapping thinking:

- "Silver mist can give you any soda. Yeah this one is not going to be a function because the silver mist button can give you all kinds of sodas. Machine E is not a function because it has different outputs for the same input." (S9)
- "Every input is supposed to have one output. It's not supposed to change all the time...The assignment of inputs to outputs can't change all the time. It has to be a unique prescription. It looks like it varies all the time." (S8)
- 

(S7)

These explanations not only include an interpretation of independent and dependent variables beyond simply how many elements in an output are associated with a given input, but also show evidence of a deeper understanding of the definition of function. This understanding goes beyond "one input gives one output" and is not attached to an equation or graph.

Revising Definitions Based on Machines B and E

In a reflection assignment the PSTs were asked if they would suggest any changes to the agreed upon definition: *A function is a mathematical relationship such that each input has exactly one output*. The PSTs that misidentified both B and E did not suggest any changes to the definition. S1, who misidentified only B, wrote, "I think that to be a function, it had to map colas to one other cola, not change the colas it was giving out or producing more. So, I think that our definition is still a good one." Notice that he is still focused on the number of elements in the output. In contrast, all of the PSTs that correctly identified all 5 machines suggested some changes to the definition to address the misconceptions that might be associated with working in the context of vending machines. For example, S3 suggests,

> I would like to change it to a "function is a mathematical relationship such that each input produces the same output at all times." [...] I noticed that in Machine B, each input produced one can except Silver Mist, which produced two cans, and this made me initially say it wasn't a function. However, I know that you can have a function that is not uniform, so producing two cans each time is an appropriate thing for Silver Mist to do without violating the function rules. Thus, the "exactly one" language is misleading… Then, Machine E threw me off because each input produces exactly one output, but Silver Mist produces a different output each time...Because of that discovery, I realized that the language "exactly one output" could be very confusing to students and does not exactly explain what is going on in a function, which is that no matter when you place an input into the function, you will always get the exact same output that you got the last time you put that particular input into a function. That is what we mean by "exactly one output," but the language restricts students from really understanding what the function is doing with that input.

It is clear from these suggested revisions that the context of a vending machine was not only a powerful cognitive root for the PSTs' own understanding, but it also resulted in an understanding of ways to mitigate their future students' possible misunderstandings.

Discussion

As this was our first use of the Vending Machine applet as a cognitive root for function with PSTs, it is important to note affordances and limitations of its design. Machines B and E, the two most commonly misidentified machines, are nice examples of how these PSTs make sense of the definition of function within the context of the Vending Machine applet being used as a cognitive root and provide insight to the ways that they interpret independent/dependent variables and apply them to their existing definition of function. Results showed that through these experiences, PSTs recognized that the use of the commonly used phrase "exactly one output" in the definition of function can be problematic. Furthermore, the fact that the applet used non-expression objects seems to have been a helpful context for this realization.

The need to press the reset button to remove the output was first seen as a limitation as PSTs often forgot to do so and ended up with multiple outputs showing at once. However, this feature was also an affordance in that it offered insight to their thinking and, for those that did it intentionally, provided a representation of the range. Another limitation was the number of machines that were included in the applet. There were not enough examples of non-functions or functions with more than one element as an output. In more than one instance PSTs got to the final machine and assumed it was a non-function simply because "at least one of these must be". Finally, it is possible that agreeing upon a class definition of function prior to engaging with the task might have changed the way that PSTs interacted with the applet. It was not clear if all PSTs were actually in agreement with the adopted definition. If one did not understand the agreed upon definition, it would be difficult to apply it to an analysis of the machines.

From our research on the PSTs' engagement with the applet and the affordances and limitations identified, a revision to the applet has been made. The new version has been created to address PSTs' interpretations of Machine B and Machine E and the limitations noted above. It consists of four pages (Figure 2). The first two pages contain two vending machines, on each page one machine is labeled as a function and the other is labeled as not a function. The two non-function machines each have at least one button that produces a random can when pressed. The new applet provides the opportunity for PSTs to make a conjecture, after page two, on why Machines 1 and 3 are functions and Machines 2 and 4 are non-functions. They then test their conjecture on the pages three and four, that each contain five machines. The five original machines are included, along with five new machines. These new machines provide additional opportunities to examine machines that are not one-to-one, produce random pairs of cans as an output, and have random outputs for all four inputs.

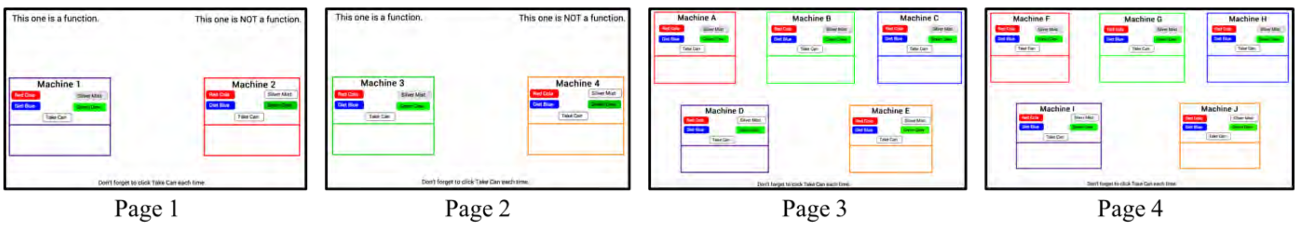


Figure 2. New version of the applet (https://ggbm.at/MAEdhkH6).

Conclusion

Our findings suggest that the Vending Machine applet might be a powerful tool (cognitive root) for building understanding of key aspects of function as identified by Carlson (1998). In thinking about the theme of this conference, this work suggests a potential change in route in the ways we might consider engaging PSTs with concept of function. However, to determine whether or not these findings are generalizable the use of the applet needs to be studied on a larger scale. We are currently conducting a larger study using the new version of the applet with approximately 40 PSTs from five universities across the country. Our hope is that through further study and revision, this work results in a solid cognitive root that remedies existing misconceptions and on which deep conceptual understanding of function can be built.

Acknowledgments

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DEVELOPING TEACHERS' COMPUTATIONAL THINKING BELIEFS AND ENGINEERING PRACTICES THROUGH GAME DESIGN AND ROBOTICS

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This research report presents the final year results of a three-year research project on computational thinking (CT). The project, funded by the National Science Foundation, involved training teachers in grades four through six to implement Scalable Game Design and LEGO® EV3 robotics during afterschool clubs. Thirty teachers and 531 students took part in the Year-3 study that blended game design and robotics. Eight of these teachers and 98 students participated in a large urban city in Pennsylvania, while the remaining 22 teachers and 433 students participated in rural Wyoming. This paper reports on the results as it pertained to teacher outcomes, specifically, teachers' development of CT beliefs and engineering practices.

Keywords: Technology

Computer science is a rapidly growing field (Bureau of Labor Statistics, U.S. Department of Labor, 2016) that when coupled with engineering offers youth a broad, multidisciplinary pathway to occupations that provide both intrinsic and extrinsic rewards and benefits. Computing is integral to science, technology, engineering, and mathematics (STEM) in general and roughly 67% of computing jobs will be in non-STEM industries (Israel et al., 2015). Underrepresented minority students have limited opportunities to engage in high-quality STEM education. As access to school-based technology increases, opportunity expands to out-of-school time (OST) programs. Given the data-rich context in which we live, this expansion fills a critical need to identify the tools, pedagogy, and practices deemed essential for promoting STEM learning.

This report describes our Year-3 study, which blended game design and robotics to provide elementary school teachers with a robust curriculum that would allow them to engage students in computational thinking (CT) at high levels. While several teachers were familiar with robotics, few had previously worked with game design. Because competency in computer science and engineering demands an understanding of mathematical and scientific processes and problem solving, we believe our pedagogical approach—to couple game design and robotics within the context of culture—is both necessary and innovative. Culture is defined as "a group's individual and collective w ays of thinking, believing, and knowing, which includes their shared experiences, consciousness, skills, values, forms of expression, social institutions, and behaviors" (Tillman, 2002, p. 4) and can be used to motivate students to learn difficult concepts. Game design and robotics were used to promote CT as students engaged in abstraction, logical thinking, and debugging to create game conditions using simple commands and to make the robot perform a task. Specifically, we were interested in how teachers' beliefs and practices changed as a result of their work in STEM with rural or urban elementary students during OST.

The research questions that guided the study reported here were as follows:

How did teachers' beliefs about computational thinking change as a result of the study?

How did teachers' engineering beliefs and practices change as a result of the study?

How did teachers' instruction change in comparison to baseline observations?

How did focal teachers describe their teaching practices during OST?

Theoretical Framework

The theoretical framework is grounded in theories of cognitive development that emphasize a developmental and constructivist view of the relationship between cognition and culture. The Saxe (1999) model is grounded in developmental and constructivist theories of learning and emphasizes the relationship between cognition and culture. Saxe argued that individuals create new knowledge while participating in culturally influenced goal-structured activities that occur in social settings. The model focuses on three areas: (a) goals for learning structured by common cultural practices, (b) particular cognitive forms and functions created to reach goals, and (c) identifiable characteristics involved in the interplay across learning in different cultural contexts.

First, the goal structure of cultural practice consists of the tasks or activities that must be carried out. "Games are inherently artifacts of culture through which cultural roles, values, and knowledge bases are transmitted" (Nasir, 2005, p. 6). In African American communities, for example, goal structures may be communal as success of the group is valued over individual success (Coleman et al., 2016). Intricate roles of help-seeking and help-offering strategies occur during game design, revealing the intertwining characteristics of individual and sociocultural systems of cognitive processing (Nasir, 2005). These roles may also be evident in game design and robotics, particularly if teams of students work together. In game design, goal structures are associated with creating unique agents and developing functional games to maximize points; in robotics, goals are associated with movement and carrying out specific tasks.

Second, according to the Saxe model, sign forms—such as counting systems and cultural artifacts — are needed to execute and influence goals that emerge in cultural practice. "Practice-linked goals are influenced by many dynamics of activity, including social interaction between those engaged in a practice, the organizational structure of a practice, individuals' prior goals and understandings, and artifacts, norms, and conventions of the practice" (Nasir, 2002, p. 216). There are shifting cognitive processes in game design to adapt the game to meet gamers' needs and goals (Nasir, 2005). These cognitive processes may be noticeable in game design and robotics as students developed CT strategies via learning progressions. The role of the teacher as a facilitator is critical to students' development of these skills.

Literature Review

While STEM encompasses a wide range of disciplines, the literature review focuses on digital game design, robotics, and computational thinking as teachers used CT strategies and engineering practices to broaden equitable participation in STEM.

Digital Game Design

K-12 students are part of a digital and gaming culture. Game design and simulation have been used to address elementary and middle school students' motivation and interest in computer science courses and careers (Webb et al., 2012). Software tools like Scratch (Mouza et al., 2016; Israel et al. 2015) and Scalable Game Design (SGD) (Repenning et al., 2015) have been used successfully to help children design digital games. SGD uses instructional units to support game design and simulations through the use of AgentSheets and AgentCubes, allowing students to engage in higher-level thinking skills as they use code to move the agent through obstacles in a game (Repenning et al., 2015). Scratch, AgentSheets, and AgentCubes were used in this study to provide teachers with robust curriculum.

Robotics

Sullivan and Heffernan (2016) conducted a review of research on robotics construction kits (RCKs) as computational manipulatives in P-12 settings. They define computational manipulatives as those that have internal computing capabilities, programming, or microcomputers embedded in the hardware. Grounded in the LOGO (i.e., logic oriented graphic oriented) programming language, computational manipulatives allow children to engage in analytical and embodied cognition (Papert, 1993). Sullivan and Heffernan assert that RCKs have a dual function as students engage in both cognitive and physical aspects of learning. For example, during LEGO® robotics, students may mimic the physical motions of the robot as it travels along a path while also engaging in reasoning, reflection, discussion, and problem solving to complete a robotics task. Most importantly, students may follow a learning progression, such as sequencing, causal inference, conditional reasoning, and systems thinking, as they interact with RCKs (Sullivan & Heffernan, 2016).

Computational Thinking

CT is an evolving field that also emerged from the work of Papert (1993), who was the first to use the term. Wing (2006) defines CT as a human endeavor that "involves solving problems, designing systems, and understanding human behavior by drawing on the concepts fundamental to computer science" (p. 33). CT is a cognitive skill that all students are expected to use across disciplines and in multiple contexts (Mouza et al., 2016; Weintrop et al., 2016; Yadav et al., 2014). Weintrop et al. (2016) developed a taxonomy to incorporate the computational nature of mathematics and science into more recent educational endeavors. Their taxonomy for CT in mathematics and science includes the following practices: using data, modeling and simulation, computational problem solving, and systematic thinking. We use this definition of CT because it aligns well with both the computer science and engineering aspects of our study.

Methodology

Participants and Setting

Thirty teachers participated in most aspects of the Year-3 study. However, we limit this report to 19 teachers new to the project and 4 teachers, who chose to continue from the Year-2 study; these 23 teachers fully participated in the Year-3 study.

All of the teachers received the same training, which consisted of two logistics meetings and an eight-week course on game design and robotics. Game design was taught by a computer scientist, who was a member of the research team. Lesson plans focused primarily on creating mazes, Frogger, and PacMan games using SGD. Robotics was taught by a science educator, who was also a member of the research team. Lesson plans focused on using LEGO® EV3 robotics kits to make the basic car, gyro boy, and sumo bot. MINDSTORMS® programming controlled the robot's movements and use of ultrasonic, color, and touch sensors.

Data Analyses and Data Sources

Mixed methods were used to analyze data in the Year-3 study. Quantitative data were used to examine changes in teachers' CT beliefs (Yadav et al., 2014) and engineering practices (i.e., Engineering Education Beliefs and Expectations Instrument for Teachers (EEBEI-T) (Nathan et al., 2010). Internal reliability of the survey instruments revealed Cronbach alphas were in the acceptable range (Black, 1999): CT survey ($\alpha = 0.76$); EEBEI-T ($\alpha \geq 0.70$). The CT survey consists of 21 items related to understanding CT (i.e., definition), dispositions (i.e., comfort, interest, and classroom practices), and future careers. Examples of these items included: "*Computational thinking involves thinking logically to solve problems; Knowledge of computing will allow me to get a better job.* Using a 4-point Likert scale, scores ranged from 1 (strongly disagree) to 4 (strongly agree). Two constructs on the EEBEI-T were analyzed for this report: STEM Integration of Content and Applied Knowledge (13 items) and Engineering Cycle (5 items) to query teachers about the use of curriculum and engineering practices. Since these two constructs were specifically developed for use with teachers in Project Lead the Way (PLTW), reliability parameters are not available. However, the EEBEI-T survey in its entirety and inclusive of these two constructs does possess established content validity (Nathan et al., 2011). An example of these items included: "*I use curriculum activities that rely on application and design activities as a way to introduce students to basic laws in math and science.*" Using a 5-point Likert scale, scores ranged from (0-never; 1-almost never; 2-sometimes; 3-often; and 4-almost always). The *T*-statistic was used to analyze pre-post scores on each of these surveys. We set the confidence interval at 0.90 since the sample size was small (Quinn & Keough, 2002).

Qualitative data included using the Dimensions of Success (DoS) instrument to collect field notes and rate teachers' practices (Noam et al., 2014). Factor analysis revealed the 12 DoS dimensions can be aggregated into two groups: student learning and learning environment (Gitomer, 2014). Student learning may be further divided into three domains: (a) Activity Engagement; (b) STEM & Knowledge Practices; and (c) Youth Development. A score of 3 constitutes what researchers document as reasonable evidence while a score of 4 constitutes compelling evidence. Ratings were shared with teachers at the end of the study year as member checks. Additionally, three teachers agreed to serve as focal participants to share their experiences and reflections on the project.

Results

Computational Thinking (CT) and EEBEI-T Surveys

Twenty-three teachers completed pre-post CT surveys. The data (see Table 1) reveal significant differences on the CT survey from pre-post: (t = -2.173; p = 0.041; Cohen's d = 0.34). Cohen's d shows a small effect size for this increase. Two teachers who completed the CT did not complete the EEBEI-T. Table 1 also reveals significant differences from pre-post on the modified EEBEI-T (t = -2.882; p = 0.009; Cohen's d = 0.78). Cohen's d shows a large effect size for this increase. Interpretation of the data reveal teachers ranged from 'almost never' to 'sometimes' on the STEM Integration of Content and Applied Knowledge and Engineering Cycle constructs on the pre-survey but tended toward 'sometimes' on these constructs on the post-survey. Thus, teachers made progress on use of engineering practices in the Year-3 study.

Table 1: Results of Teachers' Survey

Construct	Pre-Survey Mean	Standard Deviation	Post-Survey	Standard Deviation
CT Survey (n=23)	3.34	0.30	3.45*	0.30
EEBEI (n=21)	1.88	0.57	2.40*	0.77

* $p < 0.05$

Dimensions of Success

We observed 23 teachers during the Year-3 study using the DoS instrument. However, three teachers were observed only once and, therefore, removed from the sample. Mean ratings (M_1) of the first observation were used as a baseline, and mean ratings (M_2) of the last observation were used to show growth. The interval between teaching episodes was typically four to six weeks unless the teacher participated in multiple years. Four teachers engaged in co-teaching and 12 taught lessons individually. Thus, 16 total observations were analyzed by lesson type to show trends on the three domains of interest (see Table 2). Teachers' ratings were slightly higher on robotics than game design at the baseline. Teachers' ratings showed compelling evidence ($M_2 = 4.0$) for Purposeful Activities, STEM Engagement, STEM Content Learning, and Relationships on final observations in the game design context. Scores increased from $M_1 = 2.9$ to $M_2 = 3.6$ for game design. However, the lowest rating was on the Relevance dimension ($M_2 = 2.8$). DoS scores dropped from $M_1 = 3.3$ to $M_2 =$ 3.2 in the robotics context, with the most substantial drop on STEM & Knowledge Practices. Overall, teachers' practice improved from $M_1 = 3.1$ to $M_2 = 3.3$ regardless of context. The strongest domain at the end of the study was Activity Engagement, which was followed by Youth Development and STEM & Knowledge Practices. Relevance ($M_2 = 2.4$) was the weakest dimension in Year 3 as well as in prior years (Leonard et al., under review).

Table 2: Analysis of DoS Observation

Observation	ACTIVITY ENGAGEMENT			STEM & KNOWLEDGE PRACTICES			YOUTH DEVELOPMENT			Mean
	Student Participation	Purposeful Activities	STEM Engagement	STEM Content Learning	Inquiry	Reflection	Relationship	Relevance	Student Voice	
Baseline – Gaming (n=9)	2.9	3.2	3.6	2.8	2.9	2.4	3.8	1.8	3.0	2.9
Final – Gaming (*n* =6)	**3.7**	**4.0**	**4.0**	**4.0**	**3.2**	**3.2**	**4.0**	**2.8**	**3.3**	**3.6**
Baseline – Robotics (*n*=7)	3.1	3.7	3.9	3.4	3.3	3.0	3.9	2.4	3.1	3.3
Final – Robotics (*n*= 10)	**3.3**	**3.4**	**3.7**	**3.0**	**3.1**	**2.8**	**3.8**	**2.2**	**3.1**	**3.2**
Total Baseline (*n*=16)	3.0	3.4	3.7	3.1	3.1	2.7	3.8	2.1	3.1	3.1
Total Final (n=16)	**3.4**	**3.6**	**3.8**	**3.4**	**3.1**	**2.9**	**3.9**	**2.4**	**3.2**	**3.3**

Case Narratives

The cases of three focal teachers allowed us to examine the complexity of enabling CT and engineering practices during OST. Each of the teachers worked with students during before- or afterschool clubs teaching game design and robotics for 20 hours in each context for a total of 40 hours or more. These teachers — one White female (rural), one African American female (urban), and one White male (urban) — wrote extensive narratives about their teaching experiences during this project. Pseudonyms were used for anonymity.

Annette. A technology specialist in rural Wyoming, Annette worked with nine White fourth- and fifth-grade students for two hours per day, four days per week, before and after school. Class sizes in rural Wyoming are typically 10-14 students. As a facilitator, Annette allowed her students to develop agency by having "deeper experiences, more meaningful conversations with peers, and ownership of the project" while "working towards a common goal." During game design, her students faced a challenge creating PacMan games. "The students understood that the chasers needed to chase their main character, but were not quite sure how to accomplish this. Now, I was able to teach them about hill climbing and diffusion because it was finally relevant. The main character had to emit something that would attract the chaser…[like] 'stinky feet.' Tracking the scent, is called a hill climb, and the scent being detected [is] diffusion, not to be confused with osmosis, which requires water. The video game needed to be programmed to take the main character's scent and diffuse it throughout the game. The chaser needed to be able to 'sniff' in all four directions and start to move in the direction

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

that the scent was strongest. I got lots of nods, and 'Oh, that makes sense.'" Thus, Annette provided students with opportunities to use more advanced CT strategies (Repenning et al., 2015) within a constructivist paradigm, which aligned with Saxe's (1999) cognitive theory and supported the finding for high Activity Engagement and STEM & Knowledge Practices on the DoS. Her ratings across the three DoS domains were stable M_1 = 3.6 (game design) to M_2 = 3.6 (game design).

Charles. As a teacher-leader in science and technology at an urban elementary school in Pennsylvania, Charles worked with 20 predominantly African American students for 90 minutes two times per week. None of his students had participated in robotics or game design before. He described how he laid the foundation for learning: "…I explained that computers are machines that are programmed to perform computations or actions based on the input they receive. We discussed the inputs that the human body might receive through senses, as well as the concept of involuntary reflexes." Then he distributed cards with conditional statements to help students understand how commands are implemented. "For example, if they heard someone say 'thank you,' someone else would respond, 'you're welcome.' I explained that computer programming was binary 'if/then' or 'yes/no' language that we needed to understand before creating a program." This activity revealed that cognitive and physical activity may be intertwined in computer science as well as robotics (Sullivan & Heffernan, 2016). However, Charles found that challenges to programming "involved rudimentary debugging. Having learned from their own mistakes, students became more effective in identifying problems in their classmates' games. Students were able to create increasingly intricate programs. Some chose to create multiple levels, multiple games, or multiple rules on a single worksheet." These activities showed how cognitive processes shifted as students engaged in social interactions (Nasir, 2005) and communal practices (Coleman et al., 2016). It also supported reasonable evidence of STEM Knowledge & Practices and Youth Development. Charles' ratings across the three DoS domains increased from M_1 = 3.0 (game design) to M_2 = 3.4 (game design).

L'wanda. As the school librarian, L'wanda was able to work with 20 students for 90 minutes a week at two different urban schools in Pennsylvania. None of the 40 predominantly African American students had participated in robotics before, and only one had previously experienced game design. L'Wanda was able to use "iPads to reinforce the skills students needed to develop CT skills. There were 10 robotics kits so students were able to work in groups of two. However, some students preferred to work in groups of three or four. Students used iPads to access templates on the LEGO website that helped them learn to build the robots. Students developed their research skills as they searched for information to improve their designs. Students used the scientific method to develop programs that would allow their robots to go longer distances and avoid obstacles. As they raced their robots against other teams, they constantly revised their designs and made changes to the programs to build better bots that would help them achieve their goals. Participation in the club was a turning point that changed attitudes toward learning. My students were able to see themselves as capable, intelligent leaders, who were able to be producers and not just consumers. Students used iPads to create videos that explained the steps they took to create their robots. These videos demonstrated the depth of their understanding of the material and showed that they had a real sense of purpose." Thus, L'Wanda's students engaged in learning progressions as they used RCKs to develop CT. L'Wanda's ratings across the three DoS domains increased from M_1 = 3.0 (robotics) to M_2 = 3.2 (robotics).

Discussion

The findings of our Year-3 study are promising. Teachers' pre-post surveys on computational thinking beliefs and engineering practices improved significantly, and overall, teachers' ability to implement STEM during OST showed improvement when baseline data were compared to final observations. Trends in terms of the overall DoS show teachers met the threshold of 3 on all domains during the last observation. The narratives were rich and provided examples of the learning that occurred during OST. Teachers were facilitators that provided guided inquiry when necessary to help students engage in independent learning. Both game design and robotics provided students with opportunities to engage in highly cognitive tasks without sacrificing cultural norms (Nasir, 2005). For example, students engaged in communal learning in urban settings (Coleman et al., 2016). Teachers encouraged student collaboration, and some children became peer coaches at times. Children were successful in designing digital games and performing robotics challenges (Leonard et al., 2016; Repenning et al., 2015; Sullivan & Heffernan, 2016). Nevertheless, some teachers did not promote cultural relevance or career awareness during their lessons. Although we encouraged culturally relevant pedagogy (CRP) (Ladson-Billings, 2009), some teachers paid cursory attention to students' culture and were less likely to link computer science and engineering tasks to STEM careers. We will address the issue of CRP in future studies through teacher reflection, case studies, and co-generative dialogue.

Significance

In this study, robotics and game design were used to broaden STEM participation in rural and urban communities. We learned that game design facilitated co-generative dialogue that allowed learners to take familiar concepts and apply them to a range of complex tasks including the creation of representations and models. We strongly believed that the research design not only facilitated CT applicable to STEM but also promoted the kinds of social engagement and collaboration that will build 21st century skills and normative habits that allow students to develop computer science and ICT skills.

Limitations

The results of this study are limited to the participants and settings where the study took place and should not be generalized to teachers in other contexts. One limitation was the smaller number of robotics clubs for baseline observations, which may have skewed the DoS tabulations. Another limitation associated with the quantitative data is the absence of reliability parameters for the two EEBEI-T constructs used in this study.

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INTEGRATING INTERACTIVE SIMULATIONS INTO THE MATHEMATICS CLASSROOM: SUPPLEMENTING, ENHANCING, OR DRIVING?

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High-tech tools can be integrated to serve a number of purposes in the mathematics classroom, with different purposes being appropriate for different learning goals. We focus specifically on the various purposes for interactive simulations (sims). This study followed three experienced middle-school mathematics teachers integrating PhET sims into their classrooms for the first time. Using both our data and literature about high-tech tool integration, we offer a framework defining three categories of purpose for sims in the classroom and describe how the teacher positioned the sim to meet that purpose. We also touch on each teachers' beliefs about high-tech tools in the classroom and the link between their pedagogical beliefs and sim integration practices. We believe this framework contributes to the field by defining varying categories of integration for a tool with growing utilization in the mathematics classroom.

Keywords: Teacher Beliefs, Technology, Modeling

The last two decades have seen an upsurge in the number of high-tech tools being introduced in classrooms. However, effective instruction is not guaranteed by mere inclusion of such tools (U.S. Department of Education, 2010). Despite easy accessibility to computers, teachers often integrate them to complement teacher-centered practices, rather than to transform the classroom environment to one that is more student-centered (Cuban, Kirkpatrick, & Peck, 2001). Even when a teacher holds positive views towards the advantages of technology, their teaching practices may be hindered by various classroom constraints (e.g., high-stakes testing, packed curriculum, limited planning time) (Ertmer, 1999; Hew & Brush, 2007).

We focus on one particular high-tech tool in mathematics: interactive simulations. We define interactive simulations (sims) for mathematics as dynamic environments that model a mathematical concept, relationship, system, or phenomenon and allow users to interact with the model within that environment. Sims may facilitate the use of multiple representations, support students' efforts to construct their own knowledge, focus student attention on conceptual ideas, and allow immediate feedback (D'Angelo et al., 2014; Hensberry, Moore, & Perkins, 2015).

Though sims offer great potential to benefit mathematics classrooms, it is how the teacher integrates this tool that will determine its effectiveness. In this article, we define a framework categorizing three different purposes sims serve and describing how the teacher may position the sim to meet those purposes. We also investigate how teachers' pedagogical beliefs influence how they position sims in their instruction. Specifically, we examine how teachers' views on whether sims offered affordances or created constraints to their teaching affected the sim's purpose and positioning in the lesson. We find this research pivotal as we consider a path for technology integration beyond mere tool inclusion, but rather a path where such tools offer opportunities to drive the lesson in new, transformative ways. Our research questions are as follows: (1) What purposes do sims serve in mathematics lessons, and how do teachers position sims to meet those purposes? (2) How do teachers' positioning of sims relate to their pedagogical beliefs about integrating high-tech tools and meeting content standards?

Conceptual Framework

High-Tech Tool Integration

In lessons involving high-tech tools, teachers assign various roles to such tools. We define this assignment of roles as *positioning*. Harré and Van Langenhove (1998) discuss positioning as the dynamic roles between members of a group. These positions hold varying levels of authority (e.g., leader, follower) that determine who is driving any particular activity in the classroom. Wagner and Herbel-Eisenmann (2009) note that positioning is “immanent,” meaning that actors do not hold a permanent role, but instead hold roles that vary depending on the other actors surrounding them. They also describe positioning as “reciprocal,” meaning that when one actor takes a leader role, other actors are positioned as followers. In applying positioning theory to sims, the sim’s role is dependent on the tasks the teacher has set up for the students. The same sim could be positioned to drive the lesson in one task or to supplement lecture-style instruction in another. Student positioning will be reciprocated by the sim’s positioning.

With these two aspects of positioning in mind, we examined integration frameworks from the literature with an eye for how high-tech tools like sims may be positioned in multiple ways and how other actors’ positioning is affected. Such frameworks are beneficial in understanding what characterizes expert use of a high-tech tool, as well as the pedagogical beliefs associated with such use. Three frameworks, each outlining multiple categories of integration, guide our understanding of the purposes sims may serve in a lesson and how teachers position sims in the classroom to meet those purposes. The Technology Integration and Curriculum framework focuses on the relationship between high-tech tools and lesson content (Ertmer et al., 1999). The SAMR Model is concerned with the influence of high-tech tools on lesson tasks and instruction (Puentedura, 2010). And a tool-specific framework (featuring interactive whiteboards (IAWs)) provides an example of how specific, high-tech tool features can be considered when describing sophistication in integration practices (Miller, Glover, & Averis, 2005). We have built our framework with each of these three lenses in mind.

Pedagogical Beliefs

Teachers’ beliefs about how students learn are manifested alongside their beliefs about how various classroom constraints (e.g., time, resources, testing, student behavior) are appropriately managed (Skott, 2001). These constraints often have both an intrinsic and extrinsic component; the teacher cannot influence their existence, but she can decide how they should best be managed (Philipp, 2007). For the purposes of this paper, we define *pedagogical beliefs* as comprising teachers’ beliefs both about how students learn and beliefs about the balancing and resolution of these classroom constraints.

Research on successful technology integration finds successful teachers are confident in their ability to use high-tech tools (Ertmer, Ottenbreit-Leftwich, Sadik, Sendurur, & Sendurur, 2012). These teachers overcome internal barriers by having a well-defined classroom vision that focuses on the affordances such tools offer (Ottenbreit-Leftwich, 2007). Teachers must be motivated for change by taking “positive approaches”—actively and autonomously *choosing* change in their classrooms with high-tech tools (Kershaw, 2016).

Research Methods

Setting

This study took place at a large, public charter school with a diverse student population reflecting state demographics. We focus on three middle-school mathematics teachers new to using PhET sims in their instruction who agreed to integrate PhET sims centrally in some of their lessons. All teachers were female and had at least seven years' experience teaching mathematics. Two members of the PhET research team introduced the teachers to the project through two introductory workshops. The three teachers taught a total of 11 lessons involving sims, each of which spanned one or two class sessions. Nine of the 11 lessons were observed and recorded by the researchers. Ten out of 11 lessons were authored by the teachers (with feedback provided from the researchers) and one was authored by one of the researchers.

Data Sources

Data sources consist of videos from end-of-year reflective interviews with the teachers, group meetings involving the three teachers and the researchers, nine lessons involving PhET sims, and six "business-as-usual" lessons in which the teachers did not use sims. We also collected the worksheets and lesson plans from the sim lessons. Finally, email correspondence was collected as supplemental data that enabled us to capture the interaction between the teachers and researchers concerning the teachers' lesson planning progress.

Methods of Analysis

To answer Research Question 1, we first took a grounded approach to define different categories of sim positioning. We began by investigating the recorded sim-based and business-as-usual lessons through a process of open coding in which we charted the lesson structure and noted various teacher instructional methods within each lesson (Savin-Baden & Major, 2013). We also inspected the worksheets teachers provided to students during each lesson and discussed how the various questions and directions contained on the worksheets positioned the sim in the tasks students completed. Through this preliminary descriptive analysis of the lesson videos and worksheets, we were able to define an initial set of sim positioning categories based on the kinds of tasks the students were asked to complete and the instructional methods implemented.

After this grounded analysis, we then took a more theory-driven approach by drawing on extant high-tech tool integration frameworks (Ertmer et al., 1999; Miller et al., 2005; Puentedura, 2010) to situate our framework in the current literature. We used this combination of data-centered and theory-driven analysis to iteratively revise our sim positioning categories. This process resulted in a three-tiered framework that defines three general purposes sims serve in mathematics lessons and describes how the teacher positions the sim to meet those purposes.

To answer Research Question 2, we used multiple-case study analysis (Yin, 2009) to create a profile for each teacher that described the affordances and constraints teachers faced in relation to integrating sims. These profiles were grounded in the interview data and email correspondence and were informed by themes and barriers highlighted in the literature. We completed a cross-case synthesis by identifying similar themes and concerns in the teachers' remarks about sims in the classroom. We also used pattern matching and explanation building to unpack each teacher's pedagogical beliefs and relate those beliefs to her sim positioning tendencies. The first author presented the teacher profiles to the second and third authors for evaluation and further revision. We noted distinct characteristics about each teacher's beliefs and categorized them under cross-cutting themes that seemed consistent across all three (see Table 1). Finally, the researchers mapped the teacher profiles to the framework categories, giving careful attention to how the profiles connected to kinds of sim positioning enacted in the classroom. The results are thus grounded in the available data and represent the consensus of the authors.

Table 1: Cross-Cutting Themes

Vision	Alignment	Barriers
Teacher's description of ideal classroom environment and the role that high-tech tools/sims should fill in this environment	Remarks about success or difficulty in adapting a sim to meet certain content standards and whether sim integration competed with meeting content standards	Remarks about other difficulties involving sim lessons, such lack of adequate planning time, low self-efficacy, and other concerns about high-tech tools

Results

Supplement, Enhance, Drive (SED) Sim Positioning Framework

Based on our analysis, we created a framework to define different categories of purpose for sims and the respective roles sims are assigned in these categories. Those categories are as follows: Sims positioned to supplement the lesson, enhance the lesson, or drive the lesson. We unpack each category by providing a general description and an example from our data.

Supplement. In this category, the sim aids the teacher in implementing a lesson with no critical differences from the lesson that the teacher might otherwise administer. The sim supplements the learning goals by making the lesson more precise or time-efficient, but the content is unchanged in terms of both depth and scope, and the tasks students complete are fundamentally the same. Sims in this role may act as a direct tool substitute or lesson add-on.

As an example of this category, Arlene used the "Graphing Lines" sim to supplement her lesson. Graphing Lines allows users to create lines on a Cartesian coordinate plane by moving points to define the line and seeing the slope equation adjust automatically. Arlene's two-day lesson had students first complete several rate problems and graph the ratio pairs on a Cartesian coordinate plane. After completing several rate problems on paper, students plugged in the values from the tables into the sim by creating a line with points matching the coordinate pairs they had recorded by hand. Students then recorded the slope values resulting from these lines.

While the sim afforded more precise graphs and verifications of slope calculations, Arlene did not position the sim to expand the range of content addressed or to enrich the presentation of the mathematics beyond what a paper or whiteboard drawing could accomplish. Students' attention was not drawn to the sim's dynamic features; rather, they focused on the static result after having inputted the numbers on their worksheet. The task that students completed with the sim was an extension of the task completed by hand to verify answers. For these reasons, we conclude the sim was positioned to supplement this lesson.

Enhance. In this category, the teacher integrates exclusive sim features to enhance lesson tasks beyond saving time or increasing precision. Opportunities for pattern-noticing and sense-making are made available through the sim's dynamic, interactive environment to enrich the content. Learning goals are influenced by this enrichment but are ultimately accomplished afterwards with the sim serving as a launching vehicle to meeting learning goals. Unique sim features are integrated but do not reinvent the tasks students complete. The content addressed is enriched without being expanded or reoriented to a new perspective.

Becky's sim lesson using "Equation Grapher" fit the enhance description. Equation Grapher allows students to observe dynamic changes in the graph of a linear or quadratic equation as the coefficient sliders are adjusted. Becky's learning goal focused on having students explain how changes to the parameters of a linear equation are reflected in its graph. Becky used this learning goal as an opportunity to emphasize the use of correct vocabulary when describing shifts in the graph. The lesson alternated between timed episodes of worksheet-directed sim use and episodes in which selected students responded to closed-ended prompts from Becky.

Even though Becky's lesson incorporated unique sim features (e.g., sliding coefficient scales resulting in dynamic changes to graph), these features served as a vehicle to enhance the lesson's procedural learning goal. The learning goal was addressed during the teacher-centered dialogue between Becky and the students and involved the sim as a common reference point. This task of applying vocabulary was certainly influenced by students' interaction with the sim, but this lesson was not set up for students to accomplish the learning goal solely through their interaction with the sim.

Drive. For this category, the sim serves as an impetus for lesson transformation as learning goals are accomplished in students' meaningful exploration and interaction with the sim. The range of content is expanded and/or an innovative perspective is achieved.

In Carmen's lesson involving the "Graphing Lines" sim, students were instructed to create various systems examples and comment on them. For example, "Do you think these lines will ever cross again? Why do you think that?" At one point, students were asked to create two lines they thought would never intersect. Carmen positioned the sim as a setting for exploration and discovery. As students moved through the worksheet, they collaborated with their neighbors and tried to identify patterns and generalize results. Carmen circulated around the room, challenging students to use precise language and supporting students who were struggling. During a summary discussion, Carmen had students share their answers and pushed for explanations.

Carmen did not position herself to introduce mathematical ideas; she instead created opportunities for students to share sense-making moments they had experienced while working with the sim. Carmen took a facilitating role that encouraged students to articulate their own connections more clearly. This contrasts with an enhance positioning in which the learning goals are accomplished away from the sim and depend on heavy teacher guidance.

Table 2 presents the SED Framework, which builds on the foundational lenses discussed in the Conceptual Framework to elaborate on each category as it applies to sim use.

Table 2: SED Framework

	Supplement	**Enhance**	**Drive**
Curriculum	Sim supplements presentation of mathematics by making the lesson more precise or time-efficient.	Sim enhances and enriches presentation of mathematics through engaging graphics and interactive features.	Sim drives presentation of mathematics by facilitating innovative perspectives and/or making new content accessible.
Nature of the Lesson Tasks	Students use sim to check and verify work completed by hand or to complete an unrelated, non-critical task.	Students complete a task that is influenced by their engagement with unique sim features. Sim is common reference point	Students complete task directly with the sim that directly addresses learning goals. Post-sim discussion is student-centered.
Unique Features	Interactive, dynamic features are largely absent from the lesson.	Interactive, dynamic features are present and tangentially influence the completion of the learning goals.	Interactive, dynamic features are present in the lesson and are used by students to directly accomplish the learning goals.

Sim Positioning and Pedagogical Beliefs

The teachers in our study demonstrated a range of approaches to positioning PhET sims into their instruction. Our findings indicated that sim purposes and positioning were influenced by each teacher's beliefs about whether the sim would bring affordances or create constraints for her teaching. We captured each teacher's perspective regarding sim affordances and constraints by coding her interviews for three emergent, cross-cutting themes: Classroom vision with sim, alignment between sim and content standards, and barriers to implementation. We next present each teacher's profile and connect their profiles to their sim positioning tendencies.

Arlene's profile: Sims as a tradeoff. Arlene's sim lessons consistently positioned sims in a supplementary role. Her vision for high-tech tools was constrained, most notably, by her concern that having students behind screens would detract from teacher–student interaction. She viewed these tools as creating hindrances within her vision of a normal classroom environment. Arlene stated that sims often worked best toward the end of the unit after students are familiar with the content; otherwise students might be confused and fail to engage with the sim productively.

Arlene's views regarding alignment between the sims and the content standards were partly demonstrated in her selectivity with choosing sims. She frequently mentioned that there were not a lot of options on the PhET website for sims that could be used in 6th grade math. She struggled to find PhET sims that she saw as aligned with the content standards or to find a way to adapt the available sims to her needs. Arlene described sim lesson planning as an "awaking moment" where "things just pop up," suggesting that she perceived creating a sim lesson as being more like waiting for a stroke of brilliance rather than systematic planning that she could control.

In summary, Arlene viewed sims as something to accommodate into her lesson. While she frequently talked about being excited to learn more about using high-tech tools and about liking sims, that optimism did not translate into a smooth, sophisticated integration. Arlene's lack of an established vision for how sim lessons could enhance or drive her lessons left her to dwell on concerns about what she might lose by integrating sims. We believe this lack of vision explains her frequent positioning of sims as an extension to traditional, business-as-usual tasks.

Becky's profile: Sims as a visual aid. Becky's classroom vision had sims positioned to enhance the lesson, allowing students to visualize the mathematics they were working on. She remarked that student understanding was facilitated by "seeing" rather than "memorizing." She was excited during the interview as she recounted how students would commonly reference the sim as they motioned and verbally described the shapes and shifts of various graphs. She viewed sims as a means to launch the class into various mathematical tasks later on.

Becky initially struggled to balance PhET recommendations of facilitating student discovery with her perceived need to move quickly through the "packed" Algebra I curriculum. In response to that tension, she felt the need to cut discovery time short and fall back on more traditional, teacher-centered instruction, even though she saw value in letting students discover.

Becky demonstrated a compromise positioning in both of her sim lessons that included opportunities for sim-driven student exploration, but ultimately sidelined the sim as a reference tool during teacher-led discussions. She used students' experience with the sim as a reference point from which to draw from as she directed the class' pivotal discussion times.

Carmen's profile: Sims as an advantage. Carmen, who positioned the sim to drive the lesson in 2 of her 4 recorded sim lessons, articulated a vision that focused predominately on what sims could afford. She described high-tech tools as a central focus of her lessons. She cited the necessity of sims and similar tools for fostering increased student engagement, facilitating opportunity to do mathematics as students might in a real job, and creating space for discovery. She also had strong opinions on sims being best incorporated at the beginning of a unit: “I mean it’s not discovery learning if you’ve already told them all the rules.”

Carmen valued student discovery while also acknowledging students’ tendency to get off track if left to work unsupervised for too long. She referred to the moments where she brought the entire class together as “checkpoints” to ensure that they were still heading in the right direction and to better understand what they were gathering from their activities with the sim.

Like Arlene, Carmen mentioned limitations to the PhET math sims available at the time; however, she was able find science sims that could be linked to relevant mathematics. Carmen believed that her sim lessons appropriately prepared students for standardized testing. She saw making connections between the standards and available sims as an intriguing challenge. Rather than feel constrained by the lack of sims clearly linked to 7th grade mathematics, Carmen seemed to approach sims with an open mind and found creative ways to relate them to the standards.

While Arlene and Becky both focused on external barriers or constraints related to integrating sims, Carmen primarily focused on the affordances. Barriers were things to be overcome. Her classroom vision for high-tech tools clearly articulated the many affordances of sims and provided motivation for sim-centered lessons. Like Becky, Carmen valued the importance of creating space for student discoveries, but for Carmen, the importance of discovery was not diminished by the need to meet standards. She avoided that tension through her flexible integration of both mathematics and science sims and her ability to facilitate student discussion toward learning goals. Carmen’s comprehensive technology vision combined with her flexible sim integration practices explain how she positioned sims to drive her lessons.

Discussion and Conclusion

The SED framework identifies three distinct roles in which sims may be positioned in math lessons and describes the purposes of each role—supplementing, enhancing, or driving. We see this framework serving the research community by applying knowledge of high-tech tool positioning to a growing collection of sims with great potential to drive classroom learning environments.

Although the SED framework consists of defined categories, we do not see the categories as describing a hierarchy of less to more effective. We believe that all three sim roles can be appropriate in different situations. For example, Carmen administered two sim lessons at the driving level, but toward the end of the year, she decided to position the “Graphing Lines” sim as a demonstration tool to introduce the Pythagorean Theorem. That does not mean that Carmen’s pedagogical beliefs changed or that she had a fluke lesson; she simply saw an opportunity where one particular sim supplemented her lesson by saving time and aiding in the teaching of a mathematical idea.

In classifying these three teachers as consistently embodying one of our categories, we have chosen to highlight certain moments that we believe will help readers understand each category. But we also recognize that each teacher is complex and aggregately reflects characteristics of all three categories. No teacher fits a perfect pedagogical stereotype, but instead exemplifies both traditional and reform-based teaching practices (Crespo, 2016). Similarly, we believe teachers may articulate pedagogical beliefs from multiple categories of our positioning framework.

There were fundamental differences in how each teacher chose to position sims and in their related pedagogical beliefs. Constraining views about what sims could and could not do narrowed what Arlene and Becky considered to be in the realm of possibility. Enabling a teacher to envision a purpose for sims across the spectrum—from supplementing to driving the lesson—and likewise position the sim to accomplish that purpose seems inextricably linked to whether she sees sims as tools to be administered under constraints or tools that unleash possibilities.

Endnote

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MATHVISION: A MOBILE VIDEO APPLICATION FOR MATH TEACHER NOTICING OF LEARNING PROGRESSIONS

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We report on the development and evaluation of MathVision, a mobile-application designed to develop Virtual Professional Learning Communities through asynchronous discussion about 2nd, 3rd, 4th, and 5th grade students' mathematical thinking. MathVision allows teachers to upload videos of problems solving sessions using Cognition Based Assessment tasks and foster discussion aligning those strategies to research-based learning progressions for Length and Measurement. Our findings indicate that while it was possible to develop such an interface, sparking productive online discussion was difficult. The application served as a tool for enhancing physical teacher meetings and drawing attention to student thinking consistent with conducting task-based interviews, rather than actually facilitating this talk entirely.

Keywords: Learning Progressions, Technology, Elementary School Education, Geometry and Geometrical and Spatial Thinking

MathVision is a mobile application learning technology designed to help 2nd, 3rd, 4th, and 5th-grade elementary teachers develop a virtual professional mathematics community in their schools around noticing student's mathematical thinking. Specifically, teachers use MathVision as an all-inclusive tool to capture short video of their students solving validated mathematical tasks and then engage in chat room-like asynchronous, online discussions about what they notice in these videos. We built MathVision upon the extensive research on mathematics teacher noticing (Jacobs, Lamb, & Philipp, 2010; Sherin, Jacobs, & Philipp, 2010), teacher video clubs (Van Es & Sherin, 2008, 2010), learning progressions (Battista, 2011, 2012), and professional learning communities (DuFour, 2004).

In building and piloting MathVision in an elementary school, we explored the following research questions: 1) How should a Virtual Professional Learning Community (VPLC) environment be created that enables elementary mathematics teachers to base their teaching on research-based mathematical Learning Progressions, without requiring timely and resource-heavy physical interactions between teachers thus making teacher learning more accessible? 2) What affordances of MathVision will teachers utilize (i.e. commenting on each other's videos, noting each other's learning progressions ratings, a database of mathematical tasks) within their classroom practice? 3) How might using these features affect their teaching practice? 4) How will physical team-meetings be altered or augmented through the use of this technology, which allows teachers to view each other's videos before these meetings? And 5) How will teachers interact and talk to each other in reference to the technology?

Background and Rationale

Mathematics Teacher Noticing

Emphasizing how a teacher specifically listens to and responds to what a student says or does is one of the core tenants of modern mathematics education reform, often referred to a "professional noticing" (Jacobs et al., 2010). The very act of listening itself opens up space for a student to share his or her mathematical strategy, thereby positively impacting a student's mathematical growth (Empson & Jacobs, 2008). However, learning how to listen to and respond to student's mathematical thinking is a complex process that takes years to develop (Jacobs et al., 2010). There has been some evidence that video clubs, in which teachers watch video of each other's classroom teaching, can help teachers in the same school develop these crucial noticing skills (Van Es & Sherin, 2010). However, video clubs require busy teachers to find a common space and time to meet. Yet, when teachers watch and analyze student's mathematical thinking on their personal devices (e.g. smartphones, tablets), they can more better focus on the nuances of a student's thinking in much more detailed ways than video clubs (Author).

Virtual Professional Learning Communities

One research-based solution for supporting teacher learning without requiring high demands of support and time involves using Professional Learning Communities (PLC), which revolve around learning, collaboration, and instructional action (DuFour, 2004). Teachers working within PLC structures often have better instructional approaches, are more satisfied with their careers, and are more likely to remain in teaching long enough to become accomplished educators (Fulton & Britton, 2011). Yet because of a lack of shared spaces, meeting times, and the small number of teachers with common mathematics grade-levels in the same building, elementary school culture in the U.S. often prevents effective PLC organization. To solve this problem, we developed a learning technology environment that fosters the creation of a Virtual Professional Learning Community (VPLC) that can go beyond school boundaries so that elementary mathematics teachers can connect with each other without having to find a common meeting time and space (McConnell, Parker, Eberhardt, Koehler, & Lundeberg, 2012).

Research-Based Learning Progressions to Understand Students' Mathematical Thinking

Often missing from most mathematics teacher support models is assessment practices utilizing research-based Learning Progressions (LP), which help teachers understand the complexities involved in students' evolving mathematical thinking. The Cognition Based Assessment series provides research-based assessment tasks for teachers to identify students' Learning Progressions and then follow-up with relevant instructional tasks. However, the successful implementation of CBA-based mathematics instruction requires teachers to regularly meet to watch video from each other's classroom and discuss how they would interpret individual student's thinking on the learning progressions. As stated earlier, however, modern elementary school culture in the U.S. makes it difficult for regular teacher meetings to come together to focus on mathematical thinking or pedagogy (Horn, Garner, Kane, & Brasel, 2017).

Building upon these various literatures on mathematics teaching, we built a mobile application, MathVision, to see what would happen if teachers could capture and upload videos of students working on CBA tasks and then use the same application to discuss these students' strategies and how those strategies aligned to the established learning progressions. Key to this research was building technology that met teachers where they are: the technology had to work on multiple devices (e.g., laptops, phones, tablets), had to be intuitive to use for capturing and commenting on video, and had to contain all necessary supports and materials (i.e., the CBA tasks, the learning progressions framework, etc.).

Methods

Research Design and Measurement/Instruments

Since the main purpose of this study involved building and piloting a new learning technology, the research design followed a case study methodology of teachers using the actual technology, videos and discussion captured from within the MathVision application, observations from teacher team meetings after they had used MathVision in their teaching, and interviews with teachers. Teachers used MathVision as a grade-based team, meaning that all the 4th-grade teachers viewed and commented on each other’s videos.

During this 4-week intervention, teachers regularly used this learning technology to assign CBA research-based mathematics tasks about Length and Measurement to their students (Battista, 2012). Teachers then selectively captured video of their students’ strategies as they solved these tasks, focusing on particular on strategies that might elicit conversation among their professional learning teams. Within each team, teachers uploaded, commented and categorize on each other’s videos according to the CBA-framework and selected where on the Learning Progression scale they feel the student was in their mathematical thinking (Figure 1 & Figure 2). Teachers then engaged in chatting with each other about the specific videos through these online discussions. In particular we were interested in how teachers came to consensus with respect to student comments on particular corresponding CBA levels and the means by which they recommend instructional support to scaffold students through their thinking.

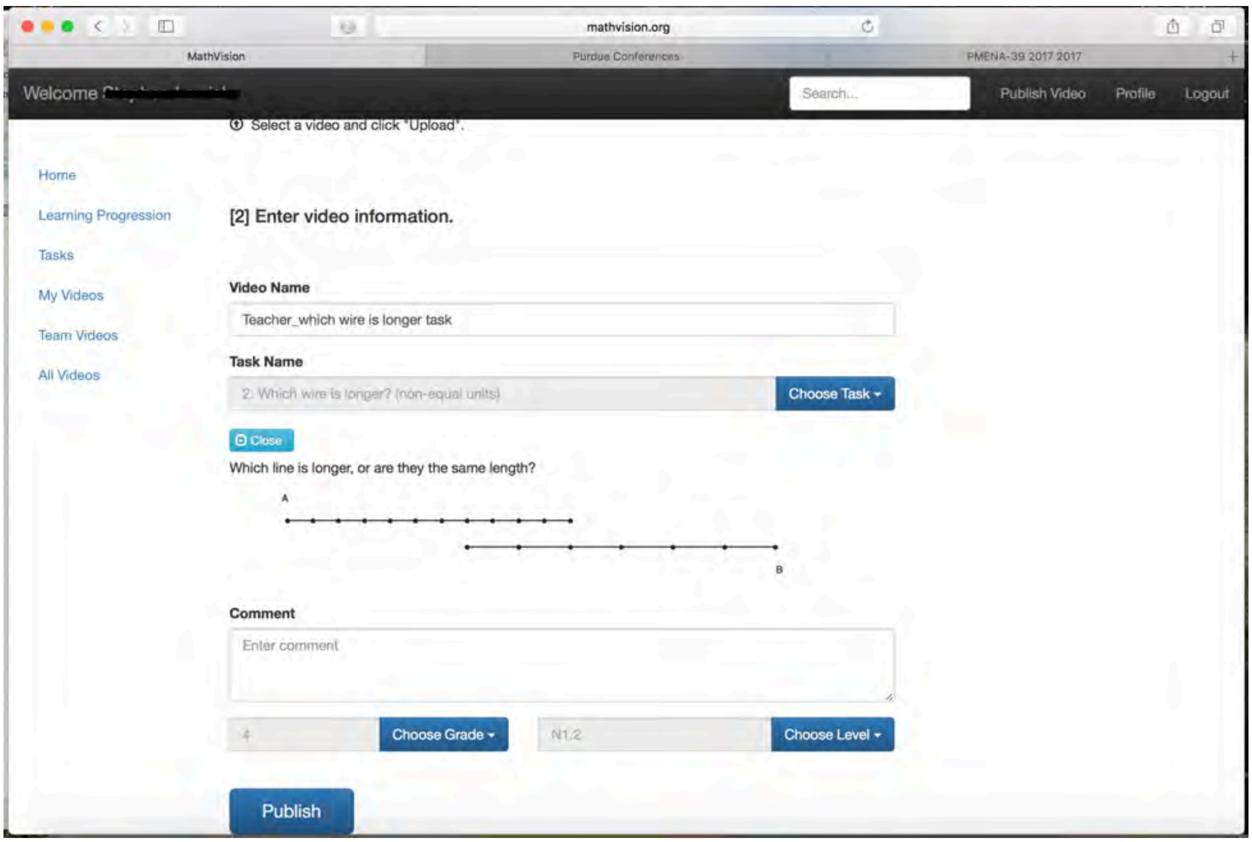

Figure 1. Video upload page where teachers enter metadata for video files including grade level, initial alignment to CBA level, corresponding task and initial thoughts or comments on problem solving strategies.

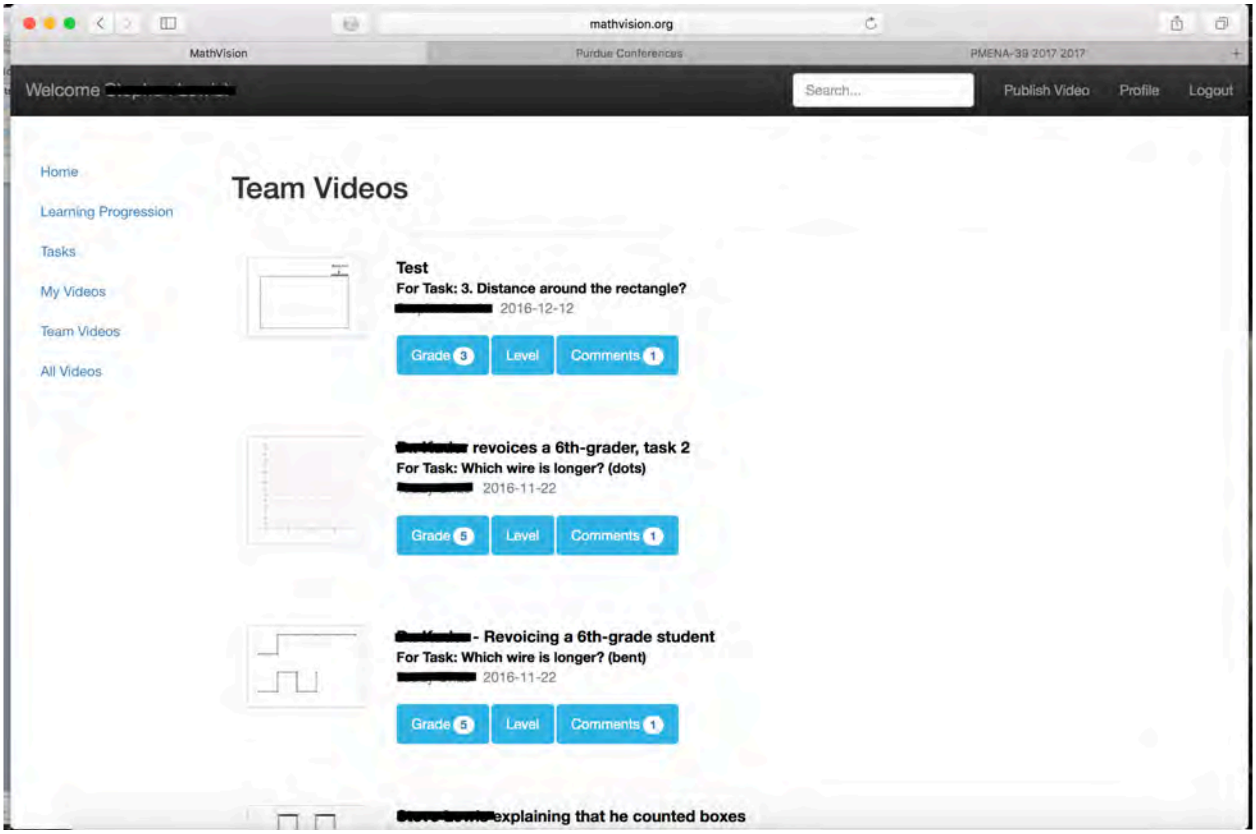

Figure 2. Repository of team videos designating grade, Learning Progression level, and comments where teachers can organize videos by team, individual videos or all videos.

Classroom Observation and Teacher Generated Video

During this 4-week intervention, our first point of data involved visiting the partnering classrooms to observe and capture video of how teachers and students used and interacted with the technology. This involved a research team member capturing video of the mathematics classroom, following the teacher as he or she used the MathVision technology with students.

Additionally, while the teachers used MathVision, they continually generated video and text data through uploading videos of their students and commenting on their own and other teachers' videos (Figure 3). Our second point of data involved our research team monitored this data collection as it unfolded, noting the progression of the teacher discussion and sophistication of the students' mathematical strategies.

Finally, at the end of the 4-week intervention, our research team met up with all the teachers, the mathematics instructional team, and the administrators who used MathVision for a reflective debrief. For our third point of data, our research team conducted follow-up focus group interviews with the teachers, instructional leaders, and administrators about their experience with the technology and how they used it. The focus of these interviews was to learn how the MathVision application helped in their professional practice, what particular changes or features might be beneficial to add, and how MathVision might fit into their school culture.

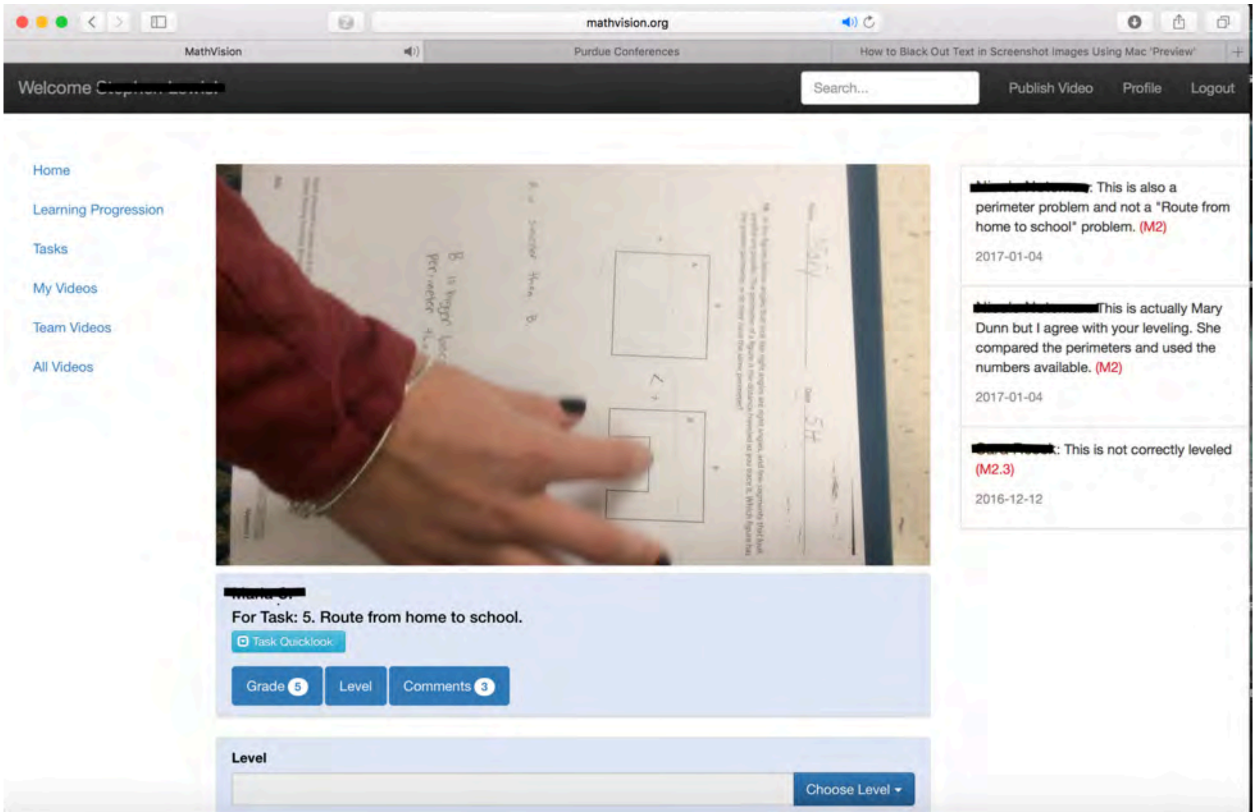

Figure 3. Video watching and commenting page allowing teachers to look at student strategies and engage in asynchronous chatting with other team members on aligning those strategies to CBA levels. Each comment (on the right) is tagged with a CBA level as a means for calling attention to particular aspects of the learning progression.

Sample

The Sample for this study consisted of the 2nd, 3rd, 4th, and 5th-grade teacher teams, the mathematics instructional team, and the administrators of a public elementary school in a high-socioeconomic neighborhood in a mid-sized city in Midwestern United States. Each math team consisted of 3 teachers and their 20 to 25 students. The mathematics instructional team consisted of the schools permanent mathematics instructional coach who also taught enrichment mathematics classes, and the administration group consisted of the head principal and the instructional leader for the school.

Internal Validity

While this research study utilized a case study methodology based around observation of participants using the technology, we also focused on the building and usability of the technology along with observing its use by teachers. Study bias was mitigated by not utilizing the developers of the technology to be involved in the data collection or the observation. Additionally, we began continual conversation with the research participants and presented our analysis to them for a form of "member checking" to ensure that our conclusions mirrored their own experiences with the technology.

Data Analysis

The primary data used for this study were the uploaded videos and comments generated from participants within the MathVision application. These were analyzed by the research team to understand the evolution of the professional knowledge that teachers exhibited through and with the application. This analysis involved continual watching and reading of this data, research team meetings to discuss our interpretations, and member-checks back to the teachers in the final interview. The secondary data involved video observations from the classroom. This data was analyzed through a grounded theory methodology (Corbin & Strauss, 2008) in order to notice any emergent themes that inform us as to how teachers and students utilized (or did not utilize) the technology.

Results

Our analysis uncovered the following results to our research questions. First, to answer how a VPLC environment might be created that enables elementary mathematics teachers to base their teaching on research-based mathematical Learning Progressions, without requiring timely and resource-heavy physical interactions between teachers, we found that this was possible through mobile technology. However, a larger finding was that teachers did not actually use this technology to house their discussions. For instance, teachers primarily focused on the novel ideas associated with doing individual task-based interviews with their students rather than having collective discussions about their respective strategies and aligning them to the learning progressions. The discussions themselves that emerged as a result of these task-based interviews were primarily done physically rather than through the web-based application. The technology itself was mainly used as a place to sore and retrieve the videos in an easily accessible way.

Second, to answer the question of what affordances of MathVision teachers utilized and how these features affected their teaching practice, we found that teachers did not actually engage in watching each others videos prior to their physical meetings, but rather used the meeting time to watch and discuss each others videos. It seemed that the physical meeting space afforded teachers the ability to meet, think about and discuss the video data itself. We ponder as to whether that had teachers a specific timeframe to upload and view/comment on each others videos might have been beneficial in accomplishing prior commenting.

Finally, in asking how will physical team-meetings be altered or augmented through the use of this technology, which allows teachers to view each other's videos before these meetings as well as how will teachers interact and talk to each other in reference to the technology, we found that teachers collaborated within this VPLC environment, teachers seemed to support each other through the process, but that seemed to be primarily due to the collaborative infrastructure that previously existed in the school. However, teachers indicated that the videos unveiled a level of depth into each student's individual thinking that they had previously not seen.

A number of results also emerged with respect to enhancing student learning. In particular we found that the instructional decisions and discussion that teachers had with each other about supporting the needs of their students, based on the strategies they saw, were important as they uncovered student conceptions related to solving particular length and measurement problems. In response, the teachers hoped that more instructional tasks could come about from this experience to support learning in this area.

As a result of this study we also observed that the teachers did engage in significant conversations about student learning process and their alignment to the research based learning progressions. This is one area that we feel may be appropriate for future inquiry.

Discussion

Overall in looking across these findings we offer the following overarching results. First in consideration of the proof of concept of this study we observed that it was possible to build a functional interface for teachers to have meaningful discussions about student strategies and align them to the CBA framework. However, with respect to this mobile application, we found it still very difficult for teachers to utilize for a number of reasons. First, the selection of length and measurement as a focus caused some difficulty, as this was not aligned to the current content they were teaching and treated more as an "add-on" to their already busy schedule. Additionally because the technology platform was still in a "beta" phase of development, the platform did not always perform as it was envisioned, particularly as teachers access to it over the school's WiFi wavered during the course of the study.

Additionally, the small-group meetings where the learning progression alignment was discussed were not consistent across teachers. When teachers were in the same physical space, there was seemingly less motivation to challenge each other's lines of reasoning and agree on a particular level after only a short explanation. With the drive of the interface as being to produce a space for disagreement and reconciliation of those disagreements through productive chat, this is something that we had hoped would arise through the use of the application. Furthermore there was a considerable number of interviews conducted by a relatively small number of teachers with one teacher in particular conducting a much larger number than others due to the busy schedule of the participating teachers. We question how our observed interviews would have been different with additional teacher voices conducting the interviews.

One other result was that the teacher who was presenting her own students' strategy often felt a sense of pride and ownership in the student's mathematical thinking. Teachers reflected that when their peers critiqued or commented on their students' thinking, their first reaction was often to take it as a personal critique and not a learning opportunity. Perhaps the immediacy of watching video on one's own mobile device or tablet instills a sense of ownership or connection to the student.

While teachers were excited and motivated to pilot this web-based application, we wonder as to the extent that which VPLC's were actually established, as there was initially little dialogue that occurred online. Rather simply participating in the study and having a home for the relevant video data sparked physical discussion on students' levels of reasoning and actually was able to bring out instances where students who were deemed as being high performing fell apart on tasks due to the nature of the CBA tasks. In one particular example, a young male student "Dylan", a pseudonym, responded incorrectly on a measurement task with an incorrect line of reasoning. Initially the teachers had wanted to dismiss this as a poor interview, but later realized that they were uncovering quite a bit of detail on the students' line of mathematical reasoning.

We did uncover some positive results with respect to using this interface during the course of our study. Teachers reported that this experience was helpful to their teaching practice, in particular in being able to watch and talk about each other's recorded videos. Teachers also felt this project connected strongly to the existing practices they see in other educational roles, in particular with their Reading Recovery program, in which they present student reading and work to the team of teachers for discussion. Further, through conducting these interviews, teachers were able to get a better understand of their students' lines of thinking as well as increasing familiarity with other teachers' students. Finally, and arguably most important finding, the teachers felt they gained significant knowledge in mathematical interviews and learned much about the learning progressions. As a result of this, our teachers reported that they had to question their own assumptions of mathematics learning and achievement and would be better equipped to use the learning progressions in their own teaching.

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OPPORTUNITIES TO POSE PROBLEMS USING DIGITAL TECHNOLOGY IN PROBLEM SOLVING ENVIRONMENTS

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This article reports and analyzes different types of problems that nine students in a Master's Program in Mathematics Education posed during a course on problem solving. What opportunities (affordances) can a dynamic geometry system (GeoGebra) offer to allow in-service and in-training teachers to formulate and solve problems, and what type of heuristics and strategies do they exhibit during this process? Results show that combining semi-structured problems with the use of GeoGebra can be useful in motivating and involving teachers in various episodes of problem formulation. In this context, important strategies included analyses of the variation in the attributes of figures using dynamic points and loci.

Keywords: Problem Solving, Technology, Geometry and Geometrical Spatial Thinking, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development

1 Introduction

It is widely recognized today that formulating, or posing, problems is a central activity in the practice of professional mathematics and a fundamental component of mathematical thinking (Cai et al., 2013). In this regard, in the past two decades problem formulation and problem solving have been identified as central topics in mathematics education (Rosli et al., 2015). On this theme, Osana and Pelczer (2015, p. 470) commented that:

> A growing movement in mathematics education that placed problema solving at the center of school mathematics further contributed to researchers' focus on problem posing, particulary its role in teaching and learning. (p. 470).

In this perspective, and in educational contexts, mathematical activity is conceived as a form of thinking in which a community (teacher and students) formulates questions and new problems to give meaning to, and resolve, problematic situations. In this scenario, the community recognizes the importance of seeking different means of supporting their responses. Santos-Trigo, Reyes-Martínez and Ortega-Moreno (2015) observe that one objective of mathematical activity is to identify and contrast diverse approaches to representing, exploring, conjecturing, resolving and formulating new problems. In communities of this kind, the role of teachers is determinant for students' learning because they are responsible for choosing and presenting the tasks that will allow learners to develop their ability to formulate and resolve problems. However, some researchers recognize that, in general, in-service and in-training teachers experience serious difficulties when confronting the tasks involved in preparing and posing problems (Rosli et al., 2015; Lavy, 2015).

What role does the use of digital technologies play in learning communities that promote and value problem posing and problem solving? In mathematics education digital technologies can provide an effective way of developing mathematical knowledge and transforming teaching scenarios by orienting them towards the formulation and resolution of problems (Aguilar-Magallón & Reyes-Martínez, 2016). To date, however, little research has been conducted on the role of technology in designing and implementing tasks whose goal is to enhance the ability to formulate and resolve problems (Abramovich & Cho, 2015).

In light of the foregoing, the principal objective of this study consisted in analyzing how the systematic use of a Dynamic Geometry System (DGS) by in-service and in-training teachers can contribute to the processes of problem formulation and problem solving. Thus, our research is oriented by the following questions: what opportunities (affordances) can the *GeoGebra* Dynamic Geometry System offer current and future math teachers in relation to problem formulation and problem solving, and what kinds of heuristic resources and strategies are exhibited in this process?

2 Conceptual Framework

Posing Problems and the Use of Digital Technologies

The literature sustains that the process of posing problems centers on two fundamental activities: formulation and reformulation. The formulation consists in generating new problems based on certain information, situations or contexts, while the latter entails elaborating new problems by modifying the conditions and/or objectives of an earlier, given problem (Silver, 1994). Reformulation activity also occurs when a problem that is in the process of resolution is transformed or re-posed in order to simplify it (Silver, Mamona-Downs, Leung & Kenney, 1996). According to this characterization, formulation and reformulation activities may take place before (the formulation or understanding of a statement), during, or after (reformulation) the resolving problems (Silver, 1994).

Following these ideas, Stoyanova and Ellerton (1996) presented a typology of problems that specifies the following three categories: open, semi-structured, and structured, as a function of the formulation or reformulation activities they involve. In open problems, individuals must posit problems based on information presented in the form of figures, tables, numbers, etc. The statement of problems of this kind do not include any specific requirements or objectives. In semi-structured problems, individuals are required to generate and/or add conditions in order to reach a solution; that is, the statement of this type of problem contains only partial information or conditions. Structured problems, finally, stipulate both the objective and all the information and conditions necessary to resolve them. Thus, open problems entail primarily formulation activities, while structured problems involve reformulation activities. Semi-structured problems can propitiate both formulation and reformulation activities. Silver (1997) holds that open or semi-structured problems can be useful in propitiating episodes of problem formulation.

Santos-Trigo, Reyes-Martínez and Aguilar-Magallón (2015) underscores the importance of the systematic utilization of various digital tools in environments of problem formulation and problem solving. Here, the goal is to have individuals constantly identify and examine distinct types of relations, posit conjectures, determine and analyze patterns, employ different systems of representation, establish connections, apply distinct arguments, generalize and extend initial problems, communicate their results, and posit their own problems. Some research has focused on examining the processes involved in posing problems using specific digital tools, such as DGS (Leikin, 2015; Lavy, 2015). According to Lavy (2015), a DGS constitutes a cognitive visual support based on immediate interactions between the tool and its user that can facilitate the processes involved in posing problems. Imaoka, Shimomura and Kanno (2015) recommend that the design of problem formulation activities utilizing a DGS entail exploring variable attributes of such figures as areas, perimeters, lengths, and angles, among others. They further advise designing problems that can be represented and solved in distinct ways; that is, they underline the importance of posing problems that are not made trivial once a DGS is applied. Leikin (2015), finally, argues that an important strategy for designing activities related to posing problems consists in transforming structured problems into open or semi-structured ones; *i.e.*, eliminating the specific conditions or objectives of structured problems to encourage exploration and research with the aid of a DGS.

3 Methodology

Participants

The participants in this study were nine students enrolled in a Master's Program in Mathematics Education. The study design consisted of fourteen weekly sessions, each with duration of three and a half hours. The group included six in-service and three in-training math teachers, all of whom had formal academic training in the field of mathematics.

Design of Activities

A total of five activities were implemented during the study, taking into account the ideas proposed by Imaoka et al. (2015) and Leikin (2015); that is, we began with a series of structured problems related to the area of figures that were then transformed into semi-structured research topics. In this report, we analyze the results of one of those problems. That problem emerged when the structured problem used by Schoenfeld (1985) was transforming by modifying its conditions (Table 1).

Table 1: Transformation of a Structured Problem Into a Semi-Structured One

Structured Problem	***Semi-Structured Problem***
You are given a fixed triangle T with base B. Show that it is always possible to construct, whit straightedge and compass, a straight line that is parallel to B and divides triangle T into two parts with equal area. Can you similarly divide the triangle into five parts of equal area? Schoenfeld (1985, p. 16).	***P.1.*** *Given any triangle, divide it into two regions with the same area.*

Implementation of the Activity and Data Collection

The development of the activity can be characterized as including three phases: 1) individual or pair work; 2) plenary discussions; and, 3) on-line discussions. The first phase consisted in three (weekly) in person sessions of three hours each. They took place in a computer laboratory so that each student had access to a personal computer with internet. During the plenary discussions, participants presented their ideas or advances in resolving the activity to the whole group. The on-line discussions utilized a digital wall (*Padlet*) that allowed participants to continue the discussion outside and beyond the in-person sessions. Study data were collected by video taping the in person sessions, recording the participations in the digital wall, *GeoGebra* worksheets, individual written reports, and interviews.

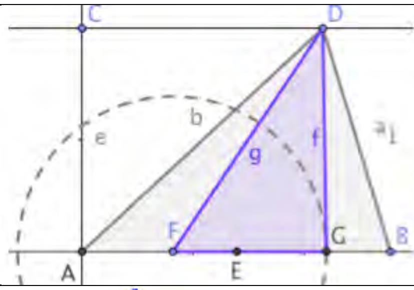
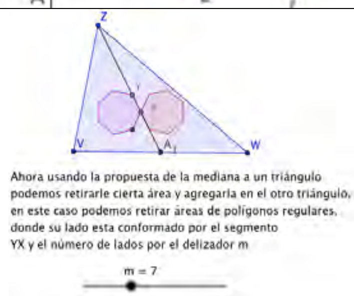
4 Results

In this section, we discuss the resources, heuristics and strategies that were presented in participants' efforts to solve the problem (P.1). Special emphasis is placed on the episodes involving problem formulation propitiated by the use of *GeoGebra*.

Initial Solutions

In a first instance, participants solved the problem using two basic ideas: 1) bisecting the area of the triangle by means of a median (*i.e.*, dividing the base in two equal parts while maintaining the height); and, 2) bisecting the area by dividing the height into two equal parts but conserving the initial base. Thus, participants used both static and dynamic solution strategies. Some of the initial dynamic solutions are shown in Table 2. One essential aspect of these approaches was the search for diverse ways to identify regions with the same area.

Table 2: Some Initial Dynamic Solutions to the Problem

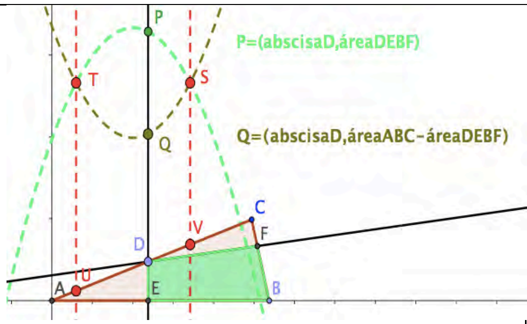
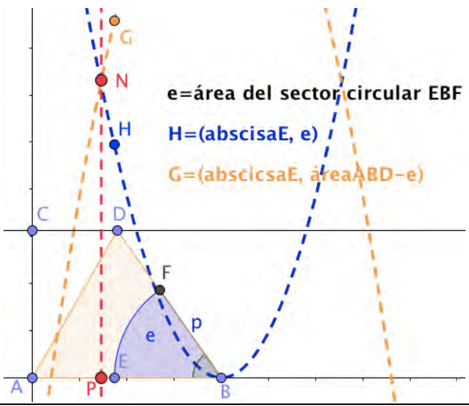
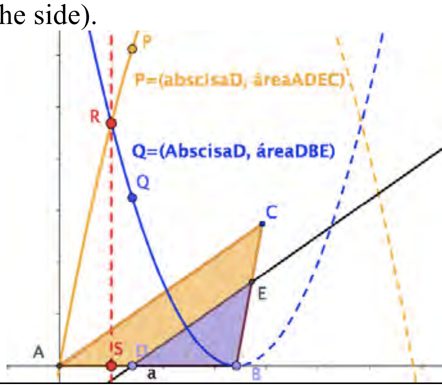
Solution	Resources and Strategies
	*Resources*: circumference to transfer measurements, mobile point on the segment, mid-point, triangle. *Dynamic strategy:* point F on AB. Construct a dynamic triangle, FGD, with the same height as triangle ABD, but a movable base, FG, of constant length equal to AE where $AE = \frac{1}{2}AB$ (infinite solutions).
	*Resources*: Median, regular polygon, mobile point on the segment. *Dynamic strategy*: use a slider, "m", to draw a regular polygon of "m" sides and reflect it by means of the median; add and subtract dynamic polygons of the same area on both sides of the median (infinite solutions).
	*Resources*: parallel mean, mid-point, triangle, mobile point on the segment, Euclidean proposition 37. *Dynamic strategy*: divide the height in two parts by the parallel mean, ED, and construct two triangles with the same base (equal to half of side AC) and mobile vertices, G and H, on the parallel mean (infinite solutions). The area of the green region is equal to that of the blue region.

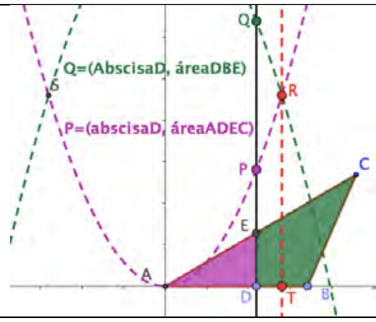
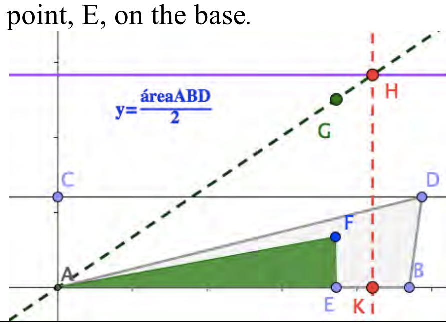
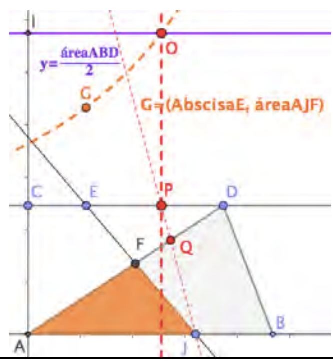
Problems Posed by Participants

After presenting their initial solutions in a plenary discussion, participants proposed new ways to find regions with the same area as the given triangle, motivated by the dynamic exploration of elements inside the configuration (Table 3). For example, one participant suggested using a circular sector to divide the triangle in two sections of the same area. Another focused attention on a construction that involved a quadrilateral. All approaches were based on a graphic representation of the variation of the area of the figures involved (Table 3).

Table 3: Problems Posed with Exploration and Solution Strategies

Problem Posed	*Resources and Solution Strategies*
1.1. Divide the triangle with a quadrilateral whose sides are perpendicular to two sides of the triangle (variation of a point D on side AC).	***Exploration***. Constructing a family of quadrilaterals based on mobile point D and with sides perpendicular to those of the triangle.

	***Important question***: where should point D be placed on side AC so that quadrilateral EDFB has the same area as the sum of the areas of triangles AED and DFC? ***Solution***. Construct the dynamic points *P = (x(D), areaDEBF) and* *Q = (x(D), areaABC - areaDEBF).* The intersections T and S of the loci described by P and Q upon moving D determine solutions U and V.
1.2. Use a circular sector to divide the triangle (variation of a point E on side AB). 	***Exploration***. Constructing a family of circular sectors of variable area "*e*" from mobile point E. ***Important question***: where to place point E on side AB so that circular sector BEF has the same area as section AEFD? ***Solution***. Create the points *H = (x(E), e)* and *G = (x(E), areaABD – e).* The intersection N of the loci described by H and G upon moving E determines solution P.
1.3. Division by means of a straight line parallel to one of the sides (variation of a point D on the side). 	***Exploration***. Constructing a family of triangles DBE with D mobile on AB and side DE parallel to side AC of the initial triangle ABC. ***Important question***: where to place point D on side AB so that quadrilateral EDAC and triangle DEB have the same area? ***Solution***. Create the points *P = (x(D), areaADEC)* and *Q = (x(D), areaDBE).* The intersection R of the loci described by P and Q upon moving D determines solution
1.4. Division by means of a straight line perpendicular to one of the sides (variation of a mobile point D on the side).	***Exploration***. Constructing a family of right-angled triangles ADE from mobile point D on side AB and with side DE perpendicular to side AB of the initial triangle ABC. ***Important question***: where to place point D on side AB so that quadrangle DBCE and triangle ADE have the same area? ***Solution***. Create the points *P = (x(D), areaADE)* and *Q = (x(D), areaDBCE)*

<table>
<tr>
<td>
</td>
<td>The intersection R of the loci
described by P and Q upon moving D determines solution T.</td>
</tr>
<tr>
<td>1.5. Bisect the area by means of a free mobile point, F, inside the initial triangle and another
mobile point, E, on the base.

</td>
<td>Exploration. Constructing a family of triangles AEF from mobile points F and E. Point F moves freely inside triangle ABD. Point E moves freely on side AB.
Important question: where to place points E and F such that triangle AEF has half the area of triangle ABD?
Solution. Create point G = (x(E), areaAEF)
The intersection H of the loci described by G (upon moving E) and the straight $y = \frac{áreaABD}{2}$ determine solution K.</td>
</tr>
<tr>
<td>1.6. Use any straight line to divide the triangle.

</td>
<td>Exploration. Constructing a family of triangles AFJ from mobile points J and E.
Important question: where to place points E and J so that triangle AJF has half the area of triangle ABD?
Solution. Create point G = (x(E), areaAFJ).
The intersection O of the locus described by G (upon moving E) and the straight line $y = \frac{áreaABD}{2}$ determine solution Q.</td>
</tr>
</table>

The exploration strategy applied in these problems consisted in constructing dynamic sections (quadrilaterals, circular sectors, triangles) inside the initial triangle and then visualizing the change in the area of those sections by dragging points until the section had half of the initial area. Visualization of the change in area was performed using dynamic points and their respective loci. Solutions were determined in terms of intersection points between those loci (Table 3).

New Problems Posed After Solving the Original Problem

The solutions reached by participants were reviewed in a plenary discussion. Those solutions involved using such loci as parabolas and hyperbolas. New problems emerged as a product of this discussion, and participants then attempted to resolve them by: i) determining the important elements geometrically (focus, directrix, axis of symmetry, vertex, etc.) of the conic sections utilized to resolve the problem; and, ii) finding the equations of those conic sections and obtaining a general algebraic solution of the problem. To find the important elements of the conics sections, participants had to review their geometric properties (in different on-line resources); for example, to find the focus of parabolas they used their reflexive property, while to find the equations, they generally used parametrization of the attributes of the triangle and extreme cases (Figure 1). Finally, participants pondered extending the initial problem by considering how to divide a triangle in three and more sections of the same area.

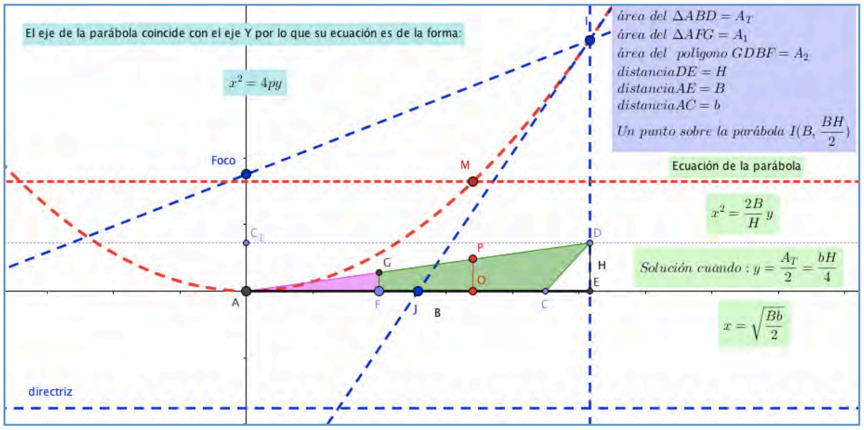


Figure 1. Parametrization and use of extreme cases to find the equation of the parabola used to resolve problem 1.4.

5 Discussion of Results

The results shown suggest that using the Dynamic Geometry System (*GeoGebra*) makes it possible to generate processes for posing problems by transforming a traditional (structured) problem into a semi-structured research problem. This transformation is achieved by *not making* certain objectives or conditions explicit. In the initial problem, *not making* the condition of dividing the triangle by means of a straight line parallel to one of the sides explicit proved to be determinant in leading the participants to formulate and resolve diverse problems.

Thanks to the ability to drag objects inside the dynamic configurations, participants were able to resolve the problem by applying dynamic approaches. These approaches allowed them to find infinite solutions that would be very difficult to visualize using traditional static tools like pencil and paper. Moreover, the dynamic exploration of the task motivated participants to pose a series of problems whose solution required analyzing variations in the areas of the figures. The use of dynamic points and their respective loci was crucial in this analysis. Later, another phase of posing problems emerged as participants explored the loci (conic sections) obtained to determine their important elements (focus, directrix, vertex, etcetera), their equations and, finally, algebraic solutions to the problems.

6 Conclusions

Any attempt to include the posing and resolution of problems in teaching and learning contexts in mathematics education depends, first and foremost, on the teacher(s) involved. In this regard, posing or formulating problems is important for teachers both in terms of their own training in the discipline and for their teaching practice. On the one hand, formulating problems allows both in-service and in-training teachers to develop their creativity and construct or strengthen their knowledge of mathematics. On the other, formulating problems is a fundamental pedagogical ability, because it is always necessary to formulate or reformulate problems as a function of students' needs, resources, ideas or errors. In this sense, it is necessary to address two key issues: 1) teacher training; and, 2) designing tasks that require formulating problems.

This study presented an example of the design and implementation of such a problem-formulation task. The use of a DGS was fundamental because it made it possible to transform a traditional problem in an activity that required exploration and research. We can conclude that the DGS can motivate processes of exploration and research that will eventually lead to the formulation and resolution of distinct problems. This idea could well become an essential element in the design of teacher training programs based on posing and resolving problems with the aid of digital technologies.

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THE EFFECTS OF TWO SIMULATIONS ON CONCEPTIONS OF RATE OF CHANGE

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The focus of the current proposal is to examine the effect of two dynamic simulations on the participants' conceptions of rate of change. Conceptions of rate of change were measured according to Carlson et al.'s (2002) Mental Actions framework and how the participants related the physical simulations to the graphical representations (Heid, et al., 2006). Results indicate that the simulations increased participants' covariational understanding, but did not help the students create a more accurate understanding of rate of change.

Keywords: Technology, Algebra and Algebraic Thinking

Students who do not understand the concept of rate of change are unlikely to develop a conceptual understanding of algebra (Roschelle, Kaput, & Stroup, 2000) or calculus (Thompson, 2008). It has been suggested that dynamic simulations could help students, particularly in the middle grades, develop a better understanding of rate of change (Rochelle, et al., 2007). However, new simulations are created continuously, and it is unclear how these simulations affect the cognition of the individuals who interact with them. Thus, the focus of the current paper is the following question: How did two dynamic simulations affect middle school, high school, and undergraduate students' understandings of rate of change?

Significance

Past research has shown that visualizations are important in: developing an understanding of rate of change (Roschelle et al, 2007), and the historical mathematical development of rate of change (Struik, 1969). Roschelle, Kaput, and Stroup (2000) propose the inclusion of technology as a necessary aspect of introducing rate of change to students before algebra or calculus.

However, visualizations and simulations change the way that students interact with and develop an understanding of various concepts (Hegadus, 2005). Even when educational experts have designed visualizations, novices notice different features or interpret the features differently than the experts intended (Roschelle, 1991). Further, the introduction of dynamic simulations may result in the *development* of different or undocumented cognitive obstacles. For instance, in a study of preservice teachers' understanding of the definition of limit in interactive geometry environments, Cory and Garofalo (2011) found that their participants became unsure of which variable is dependent on which (amongst N, epsilon, and delta). Because of the structure of the technology, some of the well documented cognitive obstacles disappeared (i.e. what N, epsilon, and delta represent physically), but the misunderstanding of dependence appeared as a new cognitive obstacle.

Other work provides evidence of what tasks or teaching practices would be important in using dynamic simulations as part of rate of change instruction (i.e. Roschelle et al, 2007). However, it does not address how, separate from instruction, simulations may impact an individual's cognition. It is essential to examine how simulations affect the students' conceptions of rate of change. This information will allow for informed implementation of dynamic simulations centered on rate of change into a learning environment and into future research on rate of change.

Background

Development of an understanding of covariation has been linked to an improved understanding of rate of change (Thompson & Thompson, 1996; Confrey & Smith, 1994). An understanding of covariation refers to the understanding that as one variable changes continuously, the other dependent variable changes simultaneously. Thompson and Thompson (1996), in documenting a teaching experiment with one 6th grade student, indicated that students tend to initially conceptualize speed as a compound unit called *speed-lengths*: the time it takes to travel a given distance. That is, the participant was only able to understand discrete parts of the variation, rather than describe how continuous variation in time affects the continuous variation in distance. The authors reasoned that school learners first understand “speed as a distance and time as a ratio (total length/speed-length)” (Thompson & Thompson, 1996, p. 3).

In a study of secondary teachers’ creation of a graph of a bottle that is similar to the boiling flask shown in Figure 4, the researchers used two lenses to examine the participants’ work: “use and coordination of macro-perspective and micro-perspective; and coordination of mathematical entities and their features” (Heid, et al., 2006, p. 4). The study postulated that a central theme in students’ reasoning about the bottle problem was the “macro-perspective” (examining the overall view) and the “micro-perspective” (examining a smaller part such as small changes in the height to consider what change in the volume that would cause) (p. 5). A key factor in how successful their participants were was whether or not the participants were conscious of both perspectives and whether or not they could shift between them to overcome obstacles. The second key factor in the participants’ success was how the individuals related the mathematical entities (the graph) and the physical entities (the bottle). For example, sometimes participants would be unable to coordinate the mathematical and the physical entities or other times the participants would fixate on a particular connection and use only that connection to generalize.

Thus, the current literature indicates that students will likely have difficulty thinking about how the variables in rate of change tasks are related. In addition to this, participants will struggle to understand when gestalt views of the objects/graphs or piece-wise views of the objects/graphs will be helpful to their reasoning.

Research Design

The current qualitative study included two tasks that were part of a larger effort to document students’ conceptions of rate of change (Tague, 2015). Each participant took part in a task-based based interview (Goldin, 2010) lasting, on average, 70 minutes. The goal of the interviews was not to design instruction nor to teach the participants, but instead, to document the participants’ conceptions of rate of change before and after use of a dynamic visualization. As such, the interviewer did not push the participants toward a correct solution; however, follow-up questions were asked to clarify the participants’ conceptions.

Each interview was video recorded, and transcribed verbatim. The video recordings captured the participants written work, their hand movements, and their interactions with the visualizations. The transcripts included gestures where the participants did not possess the vocabulary to articulate their full understanding of rate of change (Roorda, Vos, & Goedhart, 2009). For example, one of the participants showed, using her hands, that when the bottle narrowed, the graph would increase in slope, by tilting her hands in and then out because she could not articulate the vocabulary for narrow/widen.

The transcripts were then analyzed according to the Mental Action Framework (Carlson, et al., 2002) shown in Figure 1 to determine the level of the participants’ understanding of covariation. The framework was developed through studying second year calculus students’ understanding of average rate. The authors argued that determining level of understanding of covariation involved examining many mental actions that might be elicited by a task, and that an individual should not attain higher levels of mental actions without mastering the lower levels. We also examined what other features or conceptions were important in completing the task (Heid et al., 2006), in interacting with the simulations, and in completing the task a second time. For example, if a participant matched bottles to graphs in the Water Filling task by iconic translation (how closely the shape of the graph matched the physical shape of the bottle) (Monk, 1992), that individual was not actually using any kind of covariation to complete the task.

Mental Action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g. y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the rate of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Construction contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Figure 1. Mental actions and indicators of the covariation framework (Carlson, et al., 2002, p. 357).

Participants

The participants (Table 1) were chosen purposefully to represent students at specific educational levels - before algebra (middle school students), after algebra (high school students and students taking calculus courses), and after calculus (students enrolled in differential equations courses). Algebra (Saldanha & Thompson, 1998) and calculus (Thompson, 2008) have been shown to be key places where a robust understanding of rate of change is necessary.

Middle school and high school participants were recruited through letters sent to parents from teachers in a large professional development program. In the case of the undergraduate students, participants were recruited through Calculus and Differential Equations courses at a large Midwestern University. Participants were chosen from the volunteers to maximize variation amongst the participants. When possible, the participants were chosen from different parts of two Midwestern states, or different courses.

Table 1: Participants and Pseudonyms

Participant Grade Level	Pseudonym
Middle School – 6th grade	Forrest
Middle School – 8th grade	Amy
High School – Precalculus	Sarah
High School – Precalculus	Kristi
Undergraduate - Calculus	Kyle
Undergraduate - Calculus	Angela
Undergraduate - Calculus	Brian
Undergraduate - Calculus	Amanda

Task Design and Choice of Simulation

Two simulations were used, and in both cases, the participants were asked to complete a task before the simulation, to interact with the simulation, and then to complete the same task again. During the second time through the task, the participants' original work was put away, and they had the option of continuing to use and test options in the simulations while working.

The first dynamic simulation was a Java applet called "The Moving Man" shown in Figure 1 (PhET). The applet has the image of a man that begins in the middle of a horizontal axis. The man can be dragged using the mouse or he can be programed to move in a particular way by choosing an initial position, velocity, and acceleration. If the user moves the man manually, the position, velocity, and acceleration change simultaneously.

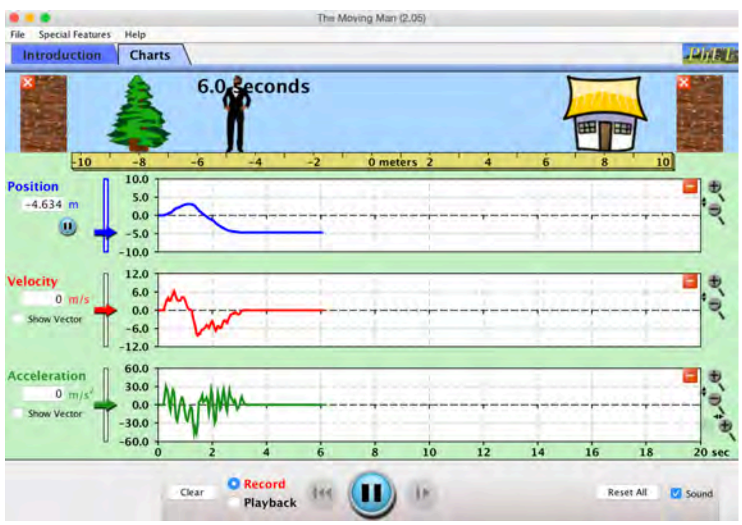

Figure 2. Screenshot of The Moving Man (applet by PhET).

The task associated with this dynamic simulation is the following: *Draw a picture of what you think the position, velocity, and acceleration graphs will look like if the man starts at the tree, realizes he is hungry, and then goes home to eat*. The task was deliberately left vague in order to allow participants to connect with their intuitive knowledge of how people move and how that motion affects their velocity and acceleration. Participants were also asked if they understood the terms position, velocity, and acceleration, and were given explanations if necessary.

The second dynamic simulation was a screenshot video of an individual playing with Wolfram Alpha's Bottle Filling simulation shown in Figure 3. In the online environment, the user can drag the outside points of the bottle, and then drag the fluid height level up. As the bottle is filled on the left side, a simultaneous graph of volume versus height is created on the right side. The participants could pause the video at any time, drag the action backwards or forwards, and watch as many times as they wanted to while they completed the task for the second time.

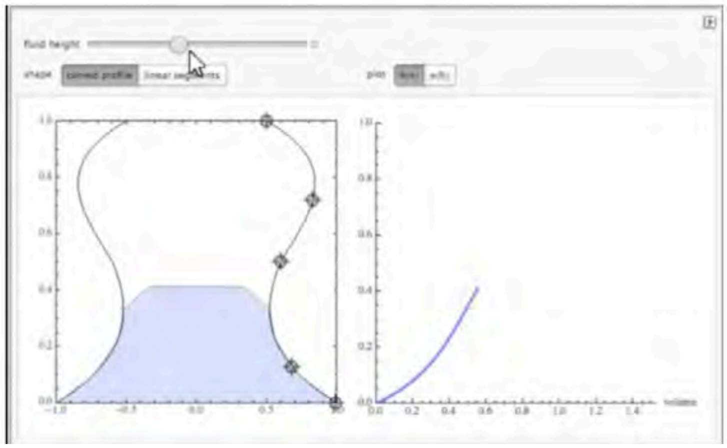

Figure 3. Screenshot of water filling simulation applet (Wolfram Alpha).

The task associated with the water filling simulation stated, "*Imagine filling each of the six bottles below (Figure 4), pouring water in at a constant rate. For each bottle, choose the correct graph, relating the height of the water to the volume of water that's been poured in*" (Annenberg Learner). Note that the graphs of C, G, and H do not match with any of the bottles, but their bottles would look like those shown in Figure 5. After the participants matched the bottles to graphs, we asked them to choose any graph they had leftover and sketch what the associated bottle would be.

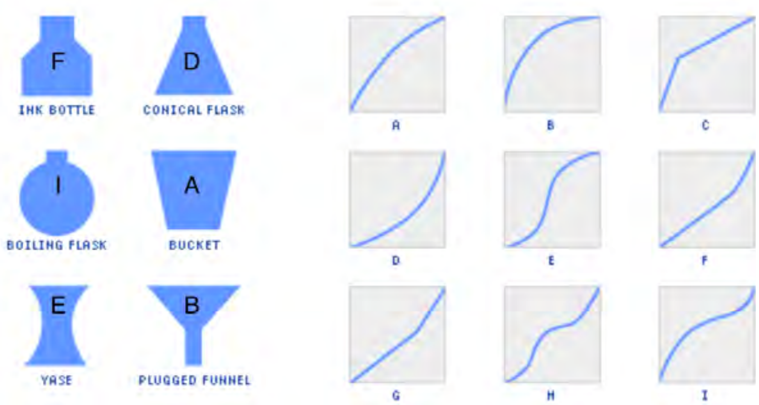

Figure 4. Task associated with the bottle filling simulation with the intended matches marked. (Annenberg Learner).

Figure 5. Bottles matching graphs C, G, and H from Figure 4 (Annenberg Learner).

The two dynamic simulations were chosen purposefully to be accessible, yet challenging to all participants from middle school through differential equations students. Both simulations were also chosen because they represent physical activities that the participants were likely to have experienced. The Moving Man represents a graphing of position, velocity, and acceleration, which is a paradigmatic type of rate of change problem that many individuals come to equate with their

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definition of rate of change (Zandieh, 1997). The bottle filling task is one that has been used by many researchers examining rate of change, and so would allow for comparisons with previous literature (Carlson, et al., 2002; Heid, et al., 2006).

Results and Discussion

Table 2 illustrates the mental actions of the participants associated with covariation before and after interacting with the simulations. As the tables illustrate, the participants generally moved toward a more covariational view of rate of change or maintained their current level. However, more covariational mental actions did not always coordinate with a more accurate physical understanding, as explained further.

Table 2: Mental Actions Before and After the Dynamic Simulations

	Water Filling		Moving Man	
	Before	After	Before	After
Forrest	None	None	MA1	None
Amy	None	None	None	None
Sarah	MA5	MA5	MA5	MA5
Kristi	MA3	MA5	MA1	MA3
Kyle	MA2	MA3	MA3	MA4
Angela	MA4	MA5	MA4	MA5
Amanda	MA5	MA5	MA4	MA5
Brian	MA5	MA5	MA5	MA5

The middle school participants, Forrest and Amy, matched bottles both before and after the simulation, using Monk's (1992) description of *iconic translation*. Monk (1992) described how students sometimes create graphs that replicated the physical features of a problem. For example, when asked to create a rate graph of someone biking across a flat surface and then biking up a hill, students are likely to create a horizontal line attached to a positive sloping line. Forrest and Amy matched the bottles with the graphs based on the physical features of the bottle that matched the physical features of the graphs. For example, they both matched graph D with the vase, and Amy explained, "because I think I was just trying to match the shape of it and not the actual amount of liquid it can be filled with." Further, Amy's explanation indicated that she was not even considering either of the variables involved in the task, and rather looked at the overall shapes to match. Neither one attempted to draw a bottle from one of their leftover graphs.

The rest of the participants were either at the highest covariational understanding of rate of change (for the water filling problem), or moved towards a better understanding (Table 2). Still, as before, improvement in understanding of covariation did not necessarily indicate a more accurate response. For example, Kyle's matches were based on a generalization of one physical feature of the bottles – corners. His reasoning was similar to that of the participants in Heid and colleagues' (2006) study, in that, although he was considering different uniform changes in volume and how that would correlate to height, he based those changes around relating the physical features of corners to physical corners in the graphs. However, he was ranked at MA3 afterwards because he could describe that wider parts of the bottle would results in more volume, but less height whereas before the simulation he could only describe as more water was added, the height would increase.

The Moving Man interaction seemed to have none or a negative effect on the rate of change conceptions of the middle school participants. Forrest actually moved from creating graphs where he considered the change with respect to time to creating graphs with iconic translation (Monk, 1992), or using the physical motion of the man to create the shape of the graph. Note, from Figure 6 that all of his graphs were horizontal and the man's movement can only be horizontal. Forrest had no cognitive dissonance about the fact that his graphs differed from those on the simulation. He was insistent that the graphs must look "just like the man moved." In Amy's case, she persisted in creating three discrete points for the graphs: one at (18,8) on the position, one at (18,8) on the velocity, and one at (8,6) for the acceleration. Like Forrest, she was undisturbed by the difference between her graph and the simulation. For the rest of the participants, the simulation moved them towards a more covariational understanding of rate of change. However, all of the other participants also copied what the simulation created, whether or not they understood it. For example, Kristi originally created a linear position function, and after interacting with the simulation, she changed it to curved. When she was asked why, she was unable to provide reasoning until the interviewer asked her what would happen if the acceleration were set to 0.

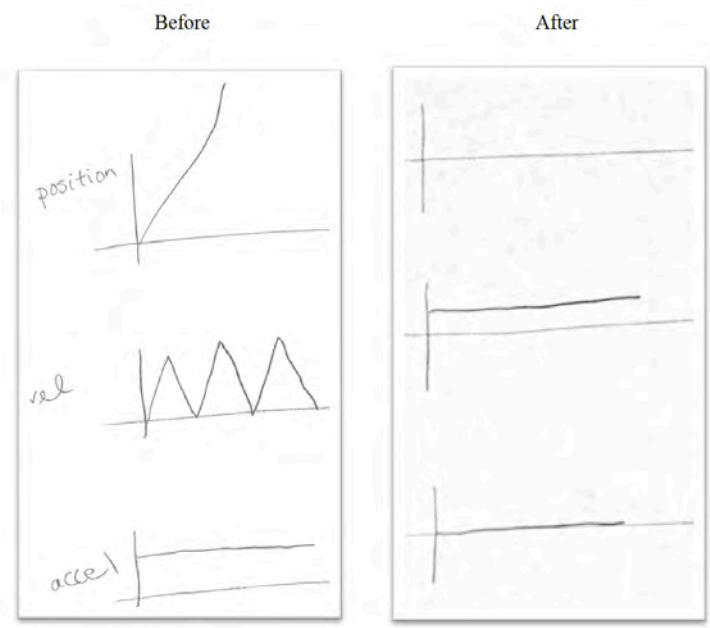

Figure 6. Forrest's graphs before and after the Moving Man simulation.

Conclusion

In summary, the dynamic simulations moved the participants' conceptions of rate of change towards a move covariational understanding. However, a better covariational understanding did not directly mean that they produced a more accurate graph/bottle/etc. It's possible that in addition to developing an understanding of how the variables co-vary, individuals also have prior experiences that cause them to focus only on one aspect of the physical situation or to use iconic translation. A better understanding of the relationship between the prior experiences and how they relate to covariational understanding is necessary to be able to describe fully individuals' understandings of rate of change.

It is clear from the current study, that questioning is essential because exposure to the simulations, in some cases, caused new misconceptions, led to less use of covariational reasoning, or did not address underlying misconceptions. Furthermore, technology is essential in studying understanding of rate of change, but it also transforms understandings, and as such, requires attention to changes caused by the technology, and further study on what kinds of questions and tasks would help students maximize their understandings.

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USING A VIRTUAL MANIPULATIVE ENVIRONMENT TO SUPPORT STUDENTS' ORGANIZATIONAL STRUCTURING OF VOLUME UNITS

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In this study, we investigated how Grade 3 and 4 students' organizational structure for volume units develops through repeated experiences with a virtual manipulative for building prisms. Our data consist of taped clinical interviews within a micro-genetic experiment. We report on student strategy development using a virtual manipulative for counting cubes as a measure of prism volume. A descriptive case of one student, Jim, is included as an example of how students developed increasingly efficient counting strategies built on understanding of structuring, composite units, multiplicative thinking and an understanding of the number of cubes along an edge. We found students were able to develop structure for volume and advance their level of thinking along a learning trajectory for volume measure.

Keywords: Geometry and Geometrical and Spatial Thinking, Learning Trajectories, Measurement, Technology.

Introduction and Theoretical Framework

Volume measurement is an important component of elementary school mathematics; however, geometric and spatial structuring is a topic that presents students with significant challenge (Barrett, Clements, & Sarama, 2017). Unlike length or area measurement, students must coordinate the measurements from three dimensions to measure volume. Spatial thinking is related to enumeration strategies as children measure volume (Battista & Clements, 1996). Thus, it is important to characterize the development of enumeration strategies. We have established hypothetical learning trajectories to characterize such development and to support the development of curriculum and enhance teacher knowledge (Barrett et al., 2017). A trajectory includes the mathematical learning goal, the thinking and learning in which students might engage and the pertinent learning activities to support growth from one level to the next (Simon, 1995). We employed a trajectory to classify student growth patterns and to design and test a learning activity. In this study, we designed and tested a virtual manipulative intended to support organizational structuring of volume units.

Students initially attempt to organize and structure volume units by working without coordinating the set of cube faces on the prism. We now describe this level of thinking as *volume unit repeater relater* (VURR), a fourth of seven hypothetical levels in that trajectory (Barrett et al., 2017). As children gain capability for coordinating spatial components, they see the array as space filling (i.e., a child may rightly predict an entire collection of rows to fill a layer given only one visible row). This level of thinking is described as *volume initial composite 3D structurer* (VICS), the fifth level. Once children completely integrate a set of locally coordinated structures within a global structure, they use layering strategies to enumerate the volume of a prism. This level is called *volume 3D row and* *column structure*r (VRCS), the sixth level of the trajectory. There is still a need to expand our knowledge of effective instructional interventions for any given level in this trajectory. We developed an intervention intended to bring students up to the VRCS level. Battista and Clements (1998) found that most grade five students, but only 20% of grade three students, characterized a rectangular prism as a series of layers of rows and columns of cubes. Thus, we focused on grades three and four.

We developed and used a computer manipulative, our intervention, to help students recognize that edge lengths can be used to predict the number of cubes along an edge, and develop their use of composite units or units of units. We hypothesized that the manipulative would highlight the efficiency of enumerating composite units by constraining how students interact with increasingly complex units: first single cubes, then row collections of cubes, then layered collections of rows of cubes. Further, we expected to prompt for a correspondence between an edge length and the number of an appropriate unit along that edge. The following research questions guided our investigation:

- How does a student's organizational structure of volume units change through repeated experiences with a virtual manipulative for building prisms?
- What are the critical features of the treatment that supported student development?

Methodology

In this study we investigated the thinking of 31 participants in Grade 3 (14 students) and Grade 4 (17 students) at a private school in the Midwest. We used a microgenetic method to focus on growth. Three aspects of this provide insight on growth: (a) observations that span the whole period of rapidly changing competence, (b) the density of observation within this period is high, relative to the rate of change; and (c) observations of changing performance are analyzed intensively to indicate the processes that give rise to them (Siegler & Sventina, 2006, p. 1000). All 31 students participated in five interviews with one or two researchers. During each interview (taking approximately 15 minutes), the students completed three trials. A trial included first a paper version of a prism to measure, and then a virtual manipulative of the same prism.

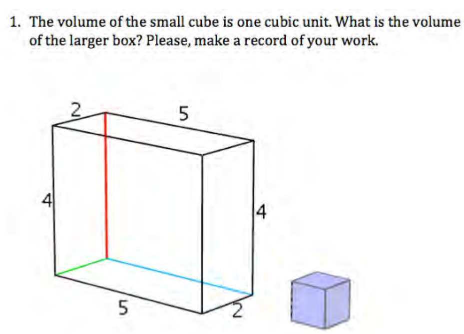


Figure 1. Paper portion of trial**.**

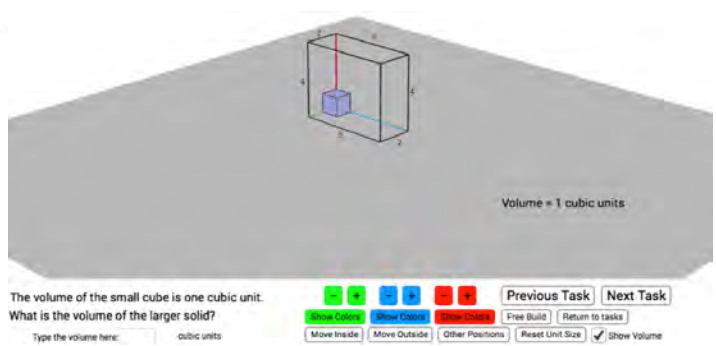
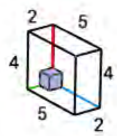
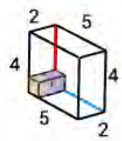
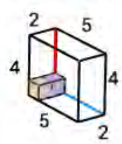
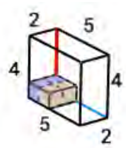
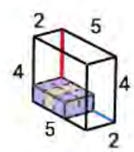
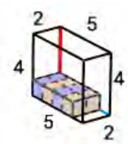
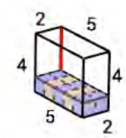
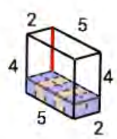
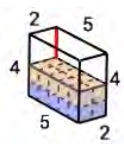
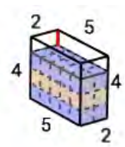
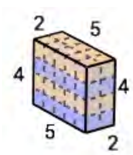


Figure 2. Virtual manipulative of trial**.**

For each trial, students had to find the volume of a rectangular prism (see Figure 1) using a pencil and a paper showing a prism with some edges labeled for length (in cm). The interviewer asked, "The volume of the small cube is one cubic unit. What is the volume of the larger solid?" After the student completed the paper task, they were asked to predict the volume of the same rectangular prism using a virtual manipulative. Each student was prompted to predict the next outcome of a button sequence, first for width, then length and finally altitude. They were asked, "How many cubes will you have when you are done pressing the green button [the first button sequence, width of prism]?" The green button sequence produced one complete row. Likewise, we asked students to predict for the second button sequence (the blue button; forming one layer of rows, now width and length), and for the third button sequence (red button; the total volume) (see Figure 2) if they continued making correct predictions about the accumulation of cubes and groups of cubes. If not, the interviewer directed them to press the relevant button so they would accumulate the cubes along that current dimension; this produced a report of the number of cubes associated with the button sequence they had been predicting. Our approach merely hinted that the prediction had been incorrect. Working thus, in three stages, students gradually filled the rectangular prism shown on a computer screen (see Table 1). (The virtual manipulative can be found at: https://www.geogebra.org/m/FgQVdDTb).

Table 1: Virtual Manipulative Program

	Click 0	Click 1	Click 2	Click 3	Click 4
First button sequence (green)			NA	NA	NA
Second button sequence (blue)					
Third button sequence (red)					NA

Results and Discussion

Our research question focuses on changes in students' organizational structure of volume units, so we categorized our participants into three categories to select participants who had changed their structuring. Of the 31 participants, eight (two Grade 3 and six Grade 4) showed prior, adequate knowledge of the structuring of volume and thus were not positioned to benefit during the study. Of the remaining 23 participants, eight (five Grade 3 and three Grade 4) did not display any meaningful changes in their ability to leverage the structure of 3D arrays to find volume. This left us with 15 (seven Grade 3 and eight Grade 4) participants out of a possible 23 (65%) who demonstrated changes in their organizational structuring of volume. The fact that 65% of the participants who could have benefited from this study suggests the virtual manipulative helped them measure prism volume by emphasizing and portraying structured sequences of units. To find the nature of the changes and relate them to their experiences, we describe a case study of one student. We anticipated finding a correspondence between changes in structuring the groups of units and the salient features of the virtual manipulative.

Case Study: Jim

Jim, a fourth-grade student, demonstrated the VICS and VRCS levels across the eleven trials. We trace his development of increasingly sophisticated use of units and understanding of length labels. We begin the case study by sketching our own model of his strategic interaction with the interviewer and the tasks and the given tools: initially, Jim's structuring included rows as units but the number of units in a row was not connected to the labels. Trial 4 was the first time he incorporated the length labels to guide his structuring of units. Next, he extended his structuring to include rows as units guided by the length labels, and then he extended his structuring to include layers as units. Finally, Jim's skip counting evolved into multiplication as a scheme for finding volume of rectangular prisms. Next, we offer a detailed interpretation across four trials. Lastly, we interweave observations and interpretations to present an ongoing model of Jim's thinking in keeping with our theoretical perspective drawn from the learning trajectory of volume measurement.

On the paper portion of Trial 1 (2 by 5 by 4 rectangular prism; see Figure 1), Jim referred to the unit cube as one, pointing to it. Next, he mentioned five, "it was five long, right?" dragging his finger across the bottom front edge (edge labeled 5). Jim then dragged his finger up the image of the prism skip counting, "five, ten, fifteen, twenty, twenty-five." He then said, "twenty-five of those" pointing back to the unit cube. We note that on the first trial Jim was already skip counting, an indication that he was treating a collection of five as a repeatable unit. However, he counted five sets of five, which was inconsistent with the length labels of 5 and 4. Additionally, he appears to have only dealt with two of the dimensions to arrive at his final answer of 25.

On the computer portion of Trial 1 Jim moved the unit cube inside the prism. He pointed with the cursor to the cube and said, "one". He then pointed to the next place he expected a cube would fit and said "two". He paused and continued in a regular pattern: "3, 4,…, 5, 6, …, 7, 8, …, 9, 10". Each pause included a motion to the next position on the base of the prism. We take this pattern in his counting and motion as an indication that although he was still counting single units. He was also attending to groups of two units to fill the bottom layer of the prism.

Still, prior to pushing any buttons (i.e., he could see just one cube inside the prism corner), he moved the cursor along the bottom front edge and the bottom back edge (edge labeled 5) saying, "so five, five". Then he took the cursor as he had moved his finger earlier on the paper portion and moved up successively on the front face. This time he skip counted by tens instead of fives and said, "So it would be 50 not 25. Ah, that is 25 times 2." His actions and statements indicate to us his attention to five countable entities during this trial: a unit cube, a row of two cubes, and a row of 5 cubes, horizontal layers of 10 cubes and vertical layers of 25 cubes. Although he exhibited an understanding of composite units, he did not mention or make use of the three distinct length labels. He treated the height as if it were five units high, even though that edge was labeled "4". We believe he was counting by visual estimation and repeated pointing gestures to find the height. This seems to indicate the VICS level because he attended to unit cubes as parts of rows, and rows as parts of layers, yet he was not using all three dimension labels to structure his groups (as he would if he operated at the VRCS level).

Continuing with the computer portion of Trial 1, the researcher asked Jim to predict how many cubes he would have when he finished with the first button sequence (colored green). He answered correctly, "two". Next the researcher asked him to predict the number of cubes after pressing the second button sequence (blue). Note that the second button was designed to add additional rows along the second dimension of the prism. Jim's response did not match that of the computer environment, "There will be five (motioning along the blue line) plus one is six. There is going to be six blocks." Because he used the cursor to touch five points along the back edge, and then one in the front left corner, he counted six cubes that included a row plus one cube next to that row. Next, Jim clicked through the first button sequence (green), and stopped with two cubes (to form a row). Thus, his prediction of two matched the number of cubes displayed. At this moment, Jim said, "wait no it adds two times five. The blue is going to add four more." We believe he meant four more groups of two cubes each. The interviewer asks him how many he would have when he finished using the blue button (which is the second button sequence) to add cubes. He said, "ten" (the correct number of cubes following the blue button sequence). We infer that he was creating a more sophisticated approach to counting composite collections of units that incorporated multiplicative operations. He was counting not only single cubes, but by coordinating the two dimension labels, of 2 and 5 along the base edges, he was able to use multiplication to group 10 cubes as five sets of two.

The researcher then asked him how many cubes he thought he would have when he finished with the red button (the third button sequence that builds vertically). Jim said he thought that it would take 30 for the whole box to be full. If he was thinking he would have to add 30 more, he was correct, but 30 cubes was not the total. Now the researcher asked him to use the blue button (the second sequence). While clicking along the blue line, Jim stated that he was wrong because it only added two and he thought he would click it once to add eight. We believe Jim was making sense of the program and the structuring of volume. Once he finished pressing the blue buttons (the second sequence), Jim was asked to predict how many cubes there would be when he was finished with the red button (the third button sequence). Jim said there would be fifty, which is incorrect. Nevertheless, after one click of the red button with 20 cubes showing, Jim said, "no, it only adds the ten more." Jim then went on to say, "now I get it, each one adds one more, like times two." We think this statement influenced his later work and solution on the paper portion of Trial 3. Once Jim filled the box, he said that the answer was forty but said he did not know for sure. Recall his work on the paper portion and his first two predictions on this computer trial were 25, and 50 cubes, rather than 40. To sum up, we think Jim relied on visual estimates to find the number of cubes in the rows during trial 1. Alternatively, he may have reported the number of square faces on the front of the prism.

Moving on to trial 2, we expected improved performance from Jim, particularly on the computer portion, as he had now practiced using the sequence of three buttons. Also, we anticipated that Jim would build on his use of composite units evidenced by his skip counting and multiplication strategy for counting row groups in Trial 1. On the paper portion of Trial 2 (5 by 4 by 3 rectangular prism), Jim drew a cube inside the rectangular prism, mimicking the virtual manipulative. He then said that it was four across, three up, consistent with the length labels. He counted four across and then said twenty; we take this as evidence that he was treating a row of five as a composite unit, and four of these rows would be consistent with the bottom layer of the rectangular prism. Next, he said "twenty times three, twenty times two, forty. I think it is forty." We notice a shift in his use of units, as he is now counting groups of twenty. The researcher then told Jim that if he needed help with calculations he could help. Jim responded, "Then you go one, two, three, four, five, that is four times five that is twenty. Then you already have one floor done, then you times two, that is forty. That fills the full box." Here we take his counting, one, two, three, four, five as another instance of his counting composites, five sets of fours, which gives him twenty; we also interpret his actions as a way of unitizing the twenty as "one floor".

On the computer portion of Trial 2, Jim correctly predicted the first button sequence and the second button sequence (he finds 20 on the floor layer), but not the third (for the whole prism). He said, "There will be thirty [in the whole prism]." Next, Jim used the virtual manipulative to fill the rectangular prism (the computer screen showed: *volume = 60 cubic units*). When he finished, he said, "No, 60…. I know why I am wrong." The interviewer asked Jim what he thought the actual volume was. After a pause, Jim said, "I don't know. I am thinking 60 cubic units and I did 40 cubic units on the last one (he refers to his answer on paper). I am thinking 60 but I don't know. It might be, hum… I think it is 60." We believe Jim is still operating at the VICS level because he has now begun coordinating spatial components more widely, but has not yet completely integrated the set of local structures into a global structure. His ability to identify a layer without finishing the structure to find 60 cubes indicates a transitional state of thinking.

Next, on the paper portion of Trial 3 (3 by 3 by 4 rectangular prism), Jim said, “4, 3, 3… now I get it.” While he said that he pointed to the length labels of each number. Next, he said, “that would be nine on the bottom… nine, nine plus nine is 18… 18 plus 9 is …” The researcher told him that was 27. Then Jim said, “I have to add 27 more. Twenty-seven plus 27.” He said, “44, …, no, 54.” Then he said, “I am going with 54. I hope I get this right.” The correct answer was 36. During, this portion of Trial 3, we believe Jim showed a development in composite units; he found the bottom layer of nine and then used that number of cubes to accumulate another unit of nine, perhaps another layer. He appeared to skip count, adding 9’s, but when he reached the third layer he doubled that quantity of 27, reaching 54. Why did he do this? We believe he followed a pattern he had set in his work on the virtual manipulative for Trial 1 when he had explained moving from one layer to two as a doubling. Because it had been the first layer, a doubling process was equivalent to adding one layer. But he did not try this doubling approach with higher levels until this instance, and here he doubles the third layer sum of 27 to get 54. He appears to intend this 54 as the accumulation of the fourth and final layer.

For the computer portion of Trial 3, Jim correctly predicted the first button sequence and the second button sequence, giving a correct prediction for the number of cubes in the bottom layer, similar to his paper task work. Next, Jim was asked how many cubes there would be when he was finished with the red button (third button sequence). He said, “it goes up 9, 18, wait… 27, 27 plus 9… I know it is 54.” As Jim filled the rectangular prism with the red button sequence, he skip counted by nine. After the prism was completely filled with cubes, Jim explained, “I went too many high because for 54. I added 18 more at 27 because 27 plus 9 is 36 … I know why I keep screwing it up.” Next, he said that he thought the box (pointing to the unit cube) was smaller and he “did tiny ones”.

We believe this experience with trial 3 was transitional for Jim and this is where he made a connection between the length label for the height of the rectangular prism and the number of layers. When Jim was asked what the actual volume was, he said it was 36 cubic units without hesitation as before. We believe Jim was transitioning toward VRCS in Trial 3, but still operating at the VICS level. Next we describe trial 4 which took place at the beginning of the second interview.

On the paper portion of Trial 4 (3 by 5 by 4 rectangular prism), Jim paused his work on paper to comment how he thought the virtual manipulative (computer task) worked. He said (pointing to the left bottom of his paper prism), “There are three here” and he made three marks with his pen (see Figure 3). While pointing to the computer, he said, “You hit the green, you would get one more, no wait, you would get two more.” Next, he wrote down 3 units along the green line. Following he said, “The blue, you would get three more” and he wrote that down on the page along the blue line. Next, he said, “The red goes up one more but adds three” and wrote that on his page along the red line. Then he said, “So you get three (pointing along the green line) and then three more and three more, no that is six, six more and then you will have twelve.” The researcher asked him if he thought 12 was the total volume and Jim responded, “No, no, you have twelve altogether right there in that little square” pointing to the 3x4 vertical face. The researcher asked, “What happens after that?” Jim responded, “You go, 3 and 5 (pointing to the base of the prism) that is 15, 15 + 15 is 30, wait no… 15 so, 15 (drawing one mark under the prism), 30,… two (drawing another mark below his prior), um 45 is three (drawing another mark), and then 50, no, 60 is four, yeah. So there is going to [be] 60 blocks in all.” On this trial, Jim integrated a set of locally coordinated structures within a global structure, and used a layering strategy to enumerate the volume of a prism. We interpret this incident to mean Jim is operating at the VRCS level for measuring volume. He coordinates sets of cubes as rows and groups of rows as layers.

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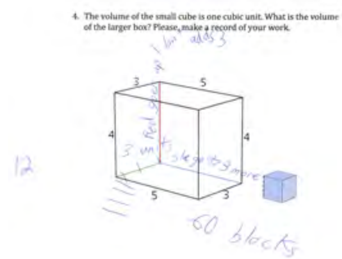

Figure 3. Jim's paper portion of Trial 4.

On the computer portion of Trial 4, Jim correctly predicted the number of cubes for each button sequence. After he filled the prism, Jim appeared very excited, "Got that one right!" Jim is demonstrating thinking consistent with the VRCS level. On the rest of the Trials (5-11), Jim completed the paper and computer predictions correctly. As he worked on the paper portions, he talked about how the computer program would work. At the end of Trial 5, Jim said he could use multiplication. He explained that you just do the six times the five, times the two. Jim showed another advancement of thinking. He shifted from skip counting to multiplying to predict the number of cubes. However, on the Trial 6 Jim used repeated addition again to find the answer. He said he was surprised he had the right answer because, "threes and sixes are hard for multiplication." Beyond Trial 6, Jim mostly used multiplication to multiply three edge lengths.

Conclusions and Implications

Measuring volume is a challenge for students (Barrett et al., 2017). We found that only six of the 31 participants in the study exhibited prior knowledge of spatial structuring for measuring the volume of rectangular prisms. Fifteen of the 23 participants who could have benefited from the experiences in this study developed more effective strategies and answered with increasingly correct measures. We take this as evidence that the treatment in this study was effective in guiding students to build a more structured understanding of volume.

We have used Jim's case to represent many students who did not initially use the length labels to predict the number of units fitting an edge. It was not until Trial 4 that Jim first used length labels to predict the number of units along an edge. Specifically, students developed more efficient enumeration strategies as well as a meaningful interpretation of the length labels as a way to predict the number of volume units fitting along an edge. The effectiveness of this treatment suggests an intervention to complement the volume learning trajectory we used to design the treatment (Barrett et al., 2017). The features of the treatment that supported student growth were those that prompted students to associate length measure labels on the prism edges with 3D arrays and those that promoted the flexible use of unit groups. Thus, the repeated pairing of length measurements (length labels) with the corresponding number of volume units was an important feature in helping students discover the predictive power of the length labels. We emphasized the pairings of labels to edge length by the continual, predict-and-check questions about accumulating quantity, through various unit groups (i.e., "How many will there be when you are done with the blue button?"). Second, students operated the three sequences of buttons independently, to meet goals of filling along three different but related dimensions of the prism. Their actions resulted in predictable but different numbers of additional cubes filling out various prism cases. We believe this was an important feature in guiding students to develop more efficient enumeration strategies. Students were expected to cope with a single button press resulting in three possibilities: the addition of a single cube, a row of cubes, or a layer of cubes, depending on the sequence. The pairing of a single action (a button press) with the appearance of either a 1D, 2D or 3D array of units suggested to students the value and utility of grouping units, and of the global coordination of those units.

In summary, we claim that changes in students' organizational structure of volume units followed from their experience with particular aspects of the virtual manipulative we employed. Secondly, we found students learned organizational structure of volume units through developing a flexibility of single units and composite units (i.e., single unit cubes, rows as units, and layers as units). Lastly, students enhanced their strategies for calculating volume by transitioning from repeated addition to multiplicative thinking.

Acknowledgments

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A PRELIMINARY ANALYSIS OF USERS' INTERACTIONS WITH AN ARTIFACT: STUDYING LINEAR RELATIONSHIPS WITH TECHNOLOGY

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Technology as a cognitive tool helps students to externalize their internal representations and take actions on them in dynamic, interactive environments. In this study, the instrumental approach provided a conceptual lens to analyze the students' introductory interactions with an artifact in a technological environment. The results highlighted various aspects of the instrumental approach such as various instrumentation and instrumentalization instances via different utilization and instrumented action schemes.

Keywords: Technology, Middle School Education, Algebra and Algebraic Thinking

In *Principles to Actions*, the National Council of Teachers of Mathematics (NCTM, 2014) promotes the effective use of mathematical action technologies: "the ability to shift between different representations of a problem…can help students develop a deeper understanding of mathematical concepts" (p. 84). Changes in individuals' mathematical thinking as a result of an experience in an environment supported with technology is a main theme Heid and Blume (2008) synthesized as emerging from the research. Mathematical action technologies allow "students [to] learn mathematics by taking mathematical actions…on mathematical objects… observing the mathematical consequences of those actions, and reflecting on their meanings" (Dick, 2008, p. 334). This study aims to understand how students interact with these technologies and how their understanding shapes and is shaped these experiences as they decide what actions to take on different mathematical objects and reflect on the consequences of their actions. Once we learn more about individual students' ways of using the technology, we then are better equipped to enhance the teaching and learning environment when technology is used consistently in a classroom setting (Özgün-Koca, 2016).

Theoretical Framework

The instrumental approach provides a lens to analyze "the learning process in technological environments of increasing complexity" (Drijvers & Trouche, 2008, p. 366). An artifact is the bare tool available to the user with some constraints and possibilities. The instrumental approach argues that an artifact (which contains external representations) mediates the activity and influences the mental processes. Only after the user becomes aware of how the artifact can extend one's cognitive processes, it becomes an instrument; so instrument=artifact + scheme. The bidirectional relationship and interaction exists between the artifact and the user. Possibilities and constraints of the artifact shape the conceptual understanding. This process is called *instrumentation*. "The conceptions and preferences of the user change the ways in which he or she uses the artifact" (Drijvers & Trouche, 2008, p. 369) and even allow him or her to shape the artifact, "loading it progressively with potentialities, and eventually transforming it for specific uses" (Artigue, 2002, p. 250). This process is called *instrumentalization*.

Trouche and Drijvers (2010) also distinguish techniques and schemes. Since we cannot directly observe students' mental schemes, we are restricted to observing their actions, what they describe as instrumented techniques which are "more or less stable sequences of interactions between the user and the artifact with a particular goal" (p. 673). Instrumented techniques are results of utilization and instrumented action schemes. Utilization schemes are elementary schemes directly linked to the artifact. Instrumented action schemes are global schemes built up from utilization schemes. Both utilization and instrumented action schemes are mental schemes involving both technical and conceptual aspects of an activity (Drijvers & Trouche, 2008, Trouche & Drijvers, 2010).

Data Collection Methods

The participants were eighth grade students (*n*=44) from three middle schools enrolled either in pre-algebra, remedial algebra, or algebra classes and were chosen by convenience sampling. Students participated in 30-45 minute clinical interviews during which they did a mathematics activity using the computer software version of TI-Nspire. The interviews were digitally recorded to save the computer screen, the students' interactions with it, and surrounding audio. This way the main visual and audio cues were recorded for further detailed analysis of participants' instrumented actions as they interacted with an artifact. This kind of recording is defined as "observational recording" by Penn-Edwards (2004) where "a researcher follows subjects engaged in an activity" (p. 268). During the clinical interviews, students were encouraged to think-a-loud when working with the program.

The activity using TI-Nspire was designed to help students to explore the effects of dynamically-linked representations when completing tasks on linear relationships. The software was a novel instrument for most of them, and most had previously studied linear functions. The larger design of this study had the main goal of studying the effects of multiple linked dynamic representations. Therefore, there were different representations available to different users and two of four tasks in the activity were different for different users. This was a purposeful design decision to see how and when the utilization of various capabilities (potentialities) of an artifact would allow students to make connections between different representations of a linear relationship. Hence, we can observe students' utilization and instrumented action schemes.

Ratcliff's (2003) variation of Erickson's microanalysis approach was adapted and followed when analyzing the digital recordings. Following a five-step approach (Ratcliff, 2003), each digital recording was viewed in its entirety without pausing or rewinding while taking notes. Next, playing and replaying the video identified major events/segments. Linkages between major events/segments were sought. Vital statements and nonverbal behaviors from the major events/segments were transcribed. Finally, the analysis of major events/segments was compared to the remainder of the video data. Member checks, thick descriptions, and considering rival explanations were used to ensure trustworthiness of this study (Guba & Lincoln, 1989).

Results

A preliminary analysis of all interviews highlighted various aspects of the instrumental approach such as various instrumentation and instrumentalization instances. In this paper, detailed evidence for only chosen results will be shared: Deciding the algebraic form from a line graph—(i) as students enter the y-intercept and slope at the same time versus enter the slope and y-intercept separately with many attempts and (ii) as students use informed versus uninformed guess and check.

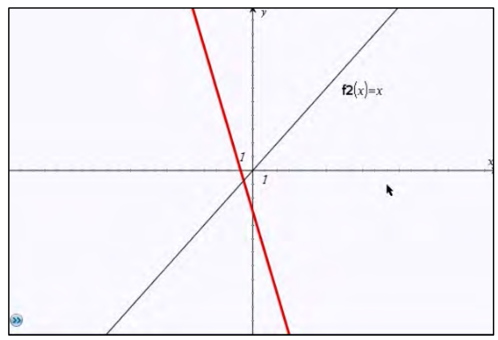
A part of the activity called Match Game asked participants to decide on the algebraic form of a line graph. The user's task was to match a bold red graph on the screen by making changes to the algebraic form $f_2(x) = x$. Four algebraic forms were used: $4x$, x-3, $2x$+4, and -$3x$-3. Schematic aspects of the techniques for deciding the algebraic form from graphical and/or tabular representations included both utilization and instrumented action schemes:

- How to change the algebraic form, the label notation, entry line, or the notation cell in the table (if the table representation was accessible)
- How to enter the slope and y-intercept into the algebraic form

    - Some students entered signs of the slope and y-intercept such as $4x$+-3 which was accepted by the software. But when +$4x$+-3 was entered, the software did not produce a graph due to its syntax rules.
- Noticing, relating, and using the algebraic, graphical, and tabular representations during the decision making.
- Recognizing the effects of changes in the algebraic, graphical, or tabular representations.
- Using feedback from previous changes effectively in subsequent rounds of changes.

While the first two schemes are utilization schemes, the last three are instrumented action schemes built on a combination of the first two utilization schemes and mental schemes. Table 1 shows one user's interaction with the software during the last task of the match game.

Table 1: Match Game With Informed Guesses

User's Comments	Screenshots
All right, this one's negative (see the figure to the right). So, the slope is going to be a negative number. So, it is going to be negative 2. *Int:* How did you decide that number? Well, you figure that if the number is too high, then it is going to be too steep. But if it is not high enough, it is going to be too horizontal. So, I am just going to pick a number and if it's wrong, then the reference it to be steeper or…And then, with the y-intercept, it'll probably be negative 2…A negative 3… *Enters -2x-3.* Probably the slope is going to be negative 3. *Enters -3x-3.*	

He recognized that he needed to change both the slope and y-intercept and he could think about the changes to the slope and y-intercept simultaneously. He could immediately differentiate the negative slope and determine the y-intercept. Although he was not able to determine the slope in one step, he had a strategy that depended on feedback from the software. So, this affected how he used the artifact according to his needs or wants; hence influencing the instrumentalization process. Finally, he was able recognize the effect of his changes in the algebraic representation and use the feedback from his previous change to inform his next moves. Knowing how to interpret the graphical representation and how to make changes to the algebraic form while using this software positively affected the whole process.

Some users were more dependent on the feedback from the software. Trying to match the red line (y=2x+4), a student's first choice for the y-intercept was -2. Immediately, she recognized that a negative y-intercept was not what she wanted. Even though she made multiple rounds of entries to match the y-intercept, she could recognize the effect of her changes and used the feedback from her changes effectively in her next rounds of changes. Although she could not show the y-intercept of the red line on the graph initially, she knew when she had matched them at the end. Her thinking was shaped by the artifact; the instrumentation process was influenced.

Both students noticed and used the algebraic and graphical representations in the artifact during the decision making. While the first student was immediately able to detect the type of slope or the exact y-intercept from a graphical representation, the second student tried negative slope and y-intercept to match 2x+4. Both students were able to recognize the effects of changes in the graphical representation as they make changes to the algebraic representation. How they used the feedback from previous changes in subsequent rounds of changes differed. Then the question becomes: how can we structure the task accompanying this artifact or the artifact itself to benefit (these) different students?

Discussion and Implications

We hear of more and more schools adapting technologies (with multiple linked dynamic representations) to their mathematics curriculum daily. Making changes in one representation and observing the results in another representation is complex work. Via more structured tasks we can ask users to record the effects of their changes in one representation on all other visible representations. We can also direct users' attention to the specific representation(s) by arranging what representations are visible. Another instructional move would be asking users to write down their next entry and explain why in order to help them use the feedback from previous changes to guide the next round of change. As Trouche and Drijvers (2010) state no tool is "'ready to do' computing, graphing, investigation, problem solving, learning or teaching. Doing requires appropriating a given tool" (p. 680). The accompanying task is a crucial part of the process of appropriating the tool for effective use and to foster the instrumentation and instrumentalization processes. Therefore, the task itself (e.g. sequence of the sub-tasks and available representations in each sub-task) is one of the important constructs that influences and completes the whole experience and the interaction between the user and the tool.

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CREATING A SOCIAL ECOLOGICAL MODEL FOR ELEMENTARY MATHEMATICS HOMEWORK

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Research suggests that it is not simply time spent on task, independently completing mathematics homework, but rather purposefully designed, engaging homework activities, completed in specific social contexts, that helps students to achieve academic gains. In this study, two urban teachers share their use of blogs and discussion forums to communicate with parents and to help facilitate mathematics discourse with their students. The teachers highlight challenges with the age group ten to twelve and their access or use of technology in the home, or with the support of their parents.

Keywords: Middle School Education

Introduction

Homework is utilized by many professionals to communicate with parents, to increase student time on task and is believed to improve student achievement (Trautwein, 2007); despite a lack of consistent empirical evidence (Rønning, 2011). The National Council of Teachers of Mathematics (NCTM, 1980) suggests the sole purpose of mathematics homework is to productively extend student engagement (Landers, 2013); therefore, educators and administrators must plan homework tasks carefully to ensure students are actively engaged. Although the completion of homework has been shown to increase understanding and retention of academic material (Zimmerman, & Kitsantas, 2005) and a relationship exists between time on task and academic achievement (Trautwein, Köller, Schmitz, & Baumert, 2002), there is still a lack of strong empirical support. Considerations regarding instructional design (Epstein & Van Voorhis, 2001; Marzano, & Pickering, 2007; Van Voorhis, 2010), in addition to the social context in which students complete their homework (Landers, 2013), provide insight towards a homework-achievement relation and provide a rationale for contradictory results in homework gains. If research can suggest that it is not simply time spent on task, independently completely mathematics homework, but rather purposefully designed, engaging homework activities, completed in specific social contexts, results may prove to be more consistent.

The purpose of this study is to investigate the potential impact computer- supported collaborative learning (CSCL) environments have in developing mathematical discourse. The present study is guided by the following questions: (1) How do teachers' of students (aged 10-12 years) use online asynchronous communication tools in mathematics to facilitate discourse? (2) What evidence exists to support the argument online asynchronous communication tools in mathematics increase students' (aged 10-12 years) engagement in mathematical processes?

Theoretical Framework

Research on homework interventions in mathematics have considered multiple variables regarding the homework achievement equation such as: using selected homework strategies (i.e. real life contexts, homework planners) (Bryan, & Burstein, 1998); frequency of homework assignments (Trautwein, Köller, Schmitz, & Baumert, 2002); student perception of the purpose of homework (i.e. practice, preparation and extension) (Rosário, Núñez, Vallejo, Cunha, Nunes, Mourão, & Pinto, 2015); mother's level of self-efficacy in mathematics (Hyde, Else-Quest, Alibali, Knuth, & Romberg, 2006); student self-regulation (Zimmerman, & Kitsantas, 2005); family involvement (Van Voorhis, 2010) and parental monitoring (Sirvani, 2007).

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

School-family partnership programs are one approach for educators to address inequalities amongst their mathematics students (Hyde, Else-Quest, Alibali, Knuth, & Romberg, 2006; Patall, Cooper, & Robinson, 2008; Sirvani, 2007; Van Voorhis, 2010). Based on earlier beliefs regarding the utility of homework, homework frequency and time on task, it is generally accepted that homework begins in elementary school and grows in difficulty, based on a student's ability to work independently and regulate their time (Zimmerman & Kitsantas, 2005). Younger students are less likely to be efficient or successful when working independently (Rønning, 2011). Whereas middle school students perform better on homework tasks when they perceive a sense of control and autonomy, away from the scrutiny of their parents (Núñez, Suárez, Rosário, Vallejo, Valle, & Epstein, 2015). Therefore, a small window of opportunity exists in the elementary grades for a school-family partnership to yield positive results related to homework and academic gains.

When students perceive a purpose for their homework they are more inclined to become engaged. Research suggests, a purposeful design of engaging homework is more effective than simply increasing time on task (Van Voorhis, 2010; Epstein & Van Voorhis, 2001). In a recent study of 638 grade 6 students, Rosário et. al., (2015) examined the effect of homework purposes on mathematics achievement. Of the three homework purposes, extension was found to have a positive impact on achievement, while practice and preparation did not. Purposeful design of homework, that ensures student engagement and extension in mathematics, is of great importance for teachers and school administrators (Rosário et. al., 2015).

It is of the utmost importance that schools approach families in a targeted attempt to improve "children's exposure to math-relevant experiences" (Galindo & Sonnenschein, 2015, p. 25). Without this intervention, children from low SES families are unlikely to "develop sufficient math skills to be competitive in today's technological world" (Galindo & Sonnenschein, 2015, p. 25). Furthermore, Jorgensen, Gates, & Roper, (2014) strenuously point out that it is important to understand the "wider set of social practices" (p. 221) in mathematics education by considering cultural backgrounds, dispositions of learners and learning environments. In many aspects, a child's ability in mathematics, sometimes evident as early as kindergarten, is not under the control of the teacher or the school (Jorgensen, Gates, & Roper, 2014), but rather, is shaped by their social background (Jorgensen, Gates, & Roper, 2014).

Background Information

As the primary investigator, and in my role as a Grade 7 teacher, I had daily contact with 75 grade 7 students for a period of 4 months between March 2016-June 2016. The students participated in a daily math journaling activity over the course of 8 weeks between April 2016- June 2016, along with 6 other teachers in the school who were also engaged in a math journaling activity with their respective classes. The purpose of the journaling activity was to increase mathematics discourse and time devoted to developing the mathematical processes. At the start of the 8-week period, the students were invited to participate in a goal setting activity related to mathematics and academic gains. Of the 75 participants, 26 completed the goal setting activity (0.347) and together submitted 66 goals for the 8-week period. Content analysis of the goals identified by the participants provided evidence that 14 of the 26 students (0.538) specifically referenced the word "homework" in their goal setting and perceived mathematics achievement gains to the timely completion and utilization of mathematics homework. Although other goals were cited (staying on task and asking more questions during class), homework is identified as the only activity related to academic gains in mathematics outside of the classroom.

Discussion

The idea to consider the relevance of homework within the parameters of the study, using online asynchronous communication tools to communicate with parents and students regarding homework (problems of the week, discussion topics) emerged as a promising focal point for the second year. At the close of the first year of study, and after analyzing the results of the previous mathematics journaling activity, five discoveries were made that impacted the direction of the study:

- Grade 7 students believed utilization of homework was related to academic gains in mathematics
- Grade 7 students believed improving their homework habits would produce academic gains in mathematics
- Grade 7 students believed they were in control of their academic gains in mathematics
- Grade 7 students believed academic gains in mathematics were directly related to their ability to self-regulate their time and attention
- Grade 7 students believed their home environment was not a barrier to their academic gains in mathematics

In the fall semester of 2016, semi-structured interviews were used to investigate six urban mathematics teachers who used technology to engage their mathematics students in discourse. The initial interviews lasted 60 minutes in length and provided a baseline to understand the teacher's background, familiarity with the technology and the teacher's motivation and intent for using CSCL to communicate with parents in the home and facilitate mathematics discourse with their students outside of the classroom. Of those six interviewed, two were selected based on their use of technology with their students, the age of their students and their motivation for participating in the study.

Veronica was chosen to participate in the second year of the study because she teaches mathematics to grades 3 and 4 with a heavy emphasis on asynchronous technology to communicate with parents and uses parent involvement to productively extend and engage her students in homework. Although Veronica is still struggling to think outside of the box for questions about mathematics and how to generate discussions about concepts students are learning, her acknowledgement and willingness to develop literacy or understanding in in Mathematics makes her an ideal candidate. Veronica has pre-established routines and a solid parent-school relationship that was developed before her participation in the study. The addition of a mathematics focus will be challenging for her but beneficial. Jeff is an ideal candidate because he teaches grade 6 students, who are preparing for a provincial assessment test in June. The additional pressure of performing well on an achievement test will increase his students' motivation and the motivation of their parents to be involved in additional practice and homework in mathematics. At my suggestion, Jeff has included a discussion focus for the PATs where students share concerns about the upcoming test, questions they have, and as a general resource for websites or materials other students have found helpful.

Follow up interviews are ongoing throughout the current school year to discuss challenges faced this year. It is hoped in the second year of the doctoral study, specific findings will be made related to the use of online asynchronous communication tools for engaging both parents and students in mathematics discourse, by identifying challenges related to:

- Engaging students in mathematics
- Communicating with parents about mathematics learning
- Assigning mathematics homework at the elementary level

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EXTENSION OF INTERACTIONS BASED ON TECHNOLOGY: BRIDGING ELEMENTARY MATHEMATICS CLASSROOMS IN KOREA

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This paper describes how communicative technology between two Korean classrooms located in different sociocultural contexts was used to support mathematics instruction. We analyzed what interactions emerged using the technology, how sociocultural differences were leveraged to construct mathematical knowledge, and how students built this knowledge together across urban and rural area classrooms. The results show that reciprocal interactions emerged. Teachers co-designed lesson plans and tasks with consideration of the different classrooms' social contexts. Through teacher's interaction, students justified their ideas and prepared answers before the connected discussions, and various ideas were synthesized as collaborative knowledge. These findings suggest that communicative technology has the potential to enhance learning opportunities for students across different social contexts.

Keywords: Technology, Elementary School Education, Equity and Diversity

According to the theory of social constructivism, learning can be seen as taking place in the social interactions between different contexts (Vygotsky, 1978). The interactions are mutual adjustments among teachers, students and content in environments (Cohen, Raudenbush, & Ball, 2003; Herbst & Chazan, 2012; Lampert, 2001). Interactive relationships can be used to support what has been called *knowledge building* (Scardamalia & Bereiter, 2006), a process that involves creative and sustained work with ideas. In knowledge building, students work collaboratively to improve shared ideas and to extend the frontiers of public knowledge. Network technology can play a significant role in providing an environment for students to engage in knowledge creation and collaborative idea improvement (Moss & Beatty, 2006).

The goal of the present study is to explore the interaction in mathematical instruction made possible when knowledge-building strategies are supported by Bridging Mathematics Classrooms via Skype [BMCS], a widely used computer-mediated communication technology. In particular, this study sought to examine the following questions: (a) What interactions emerged by using communication technology in mathematics classrooms? (b) How might sociocultural differences between schools be leveraged in BMCS to enhance learning opportunities?; (c) How do students build knowledge across classrooms through communication technology? Through this paper, we bring attention to a process in which students draw on their different social contexts to build mathematical knowledge via communication technology.

Theoretical Framework

Lampert (2001) described a teacher, students, and content as three key components in the mathematics classroom, and analyzed the complexity of teaching mathematics by characterizing the relationship between these elements. She defined teaching as an interactive process of managing the connections between students and content. Cohen et al. (2003) modified this model to include the situated environments in which teaching occurs, proposing a new view of mathematics instructional effects and resources. The instructional triangle diagram (see Figure 1. left) conceives instruction as a stream affected by environments such as teachers, students, and a local district. One question we sought to address in this study is how this framing of instruction might look different with the inclusion of multiple classes connected through networking technologies.

In particular, connected classes might have potential to impact the nature of student-student interactions, by supporting opportunities for students to engage in "knowledge building. Knowledge building, introduced and studied primarily in the domain of science education, is a process by which people not only create knowledge through social interactions, but also utilize knowledge collaboratively (Scardamalia & Bereiter, 2006). A key principle of knowledge building is 'epistemic agency,' which describes the power students have to set forth ideas and negotiate a fit between their ideas and the ideas of their peers. In mathematics classroom, this agency is related to students' own ideas and strategies of problem-solving. Students and groups need to justify their own mathematical conclusions with evidence, and, finally, a classroom negotiates its own generalized and consented conclusion through discussion (Moss & Beatty, 2006). Knowledge building discourse - refined and transformed knowledge through the discursive practices of communities - includes constructive and collaborative argument (Bereiter, 2002). Another core principle in knowledge building is "improvable ideas" (Scardamalia, 2002). Individuals' ideas are refined through collaborative interactions, which could employ network technology, which can lead to sustained improvement of these ideas (Scardamalia, 2002).

Methodology

In this exploratory case study, we used a methodological approach called *netnography*, a form of participant-observational research situated in online fieldwork (Kozinets, 2010). The case in this study was a cooperation between Mr. YH and Mr. KJ (pseudonyms), who are both 6th-grade teachers in Korea. In particular, Mr. YH worked in an urban area (CS school) and Mr. KJ did in a rural area (YG school). To conduct the netnography, we collected three types of data: elicited, archival, and field notes (Kozinets, 2010). As a first step, eight lesson episodes, already recorded and preserved on Mr. YH's YouTube channel, were selected and documented through field notes. Second, elicited data, co-created through personal and communal interactions, included three interviews with Mr. YH (overall, pre and post about Episode 3). Third, archival data, directly copied from pre-existing online material, consisted of students' activity results, lesson plans, and Mr. YH's reflection reports.

The procedures of data analysis are consistent with Charmaz's (2014) grounded theory approach. Based on literature review, we gained three framing codes: instructional triangle, knowledge-building, and context. We created line-by-line initial codes from the oldest episode, and added additional codes as needed in subsequent episodes. Then, we converged focused codes from initial codes. To confirm the categories, the teacher's reflection reports were compared and contrasted after implementing the lesson plans. We utilized the focused codes between interview data and filed notes back and forth and compared the teacher's planning with implementing.

Findings

To explore the interactions and knowledge building in Mr. YH's classroom, we concentrated on emerging patterns and how interactions were constituted through the incorporation of the communication technology. On account of the limitation of length in this paper, we are only reporting on the analysis of Episode 3. The episode was the last day of the 9-day project in which students planned, promoted and surveyed about which sport was appropriate to be selected for an after school sports club shared between CS and YG schools. In the episode, students were tasked with using survey data from different population sizes (CS: 327 students, YG: 73 students). They transformed the raw data into percentage graphs, such as pie chart or bar chart with proportions for each item, and then discussed their findings with the other classroom via Skype to make a final decision regarding the selection of common sports club. The following section presents the findings related to co-planning lessons and tasks to show how the interaction was adapted by applying communication technology and related to how social agreements worked to support knowledge building practices across classrooms.

Teacher-Teacher Interaction within Environments

Mathematical task. To make an interactive task suited to the context of each classroom, the two teachers co-designed the mathematics task. First, they justified why different populations were necessary. Mr. YH described how the teachers devised the mathematical task to consider the different sociocultural contexts with such instance in pre-interview:

> For example, students could compare the proportion between who like spring (50%) and summer (25%). In one class, this comparison could be not much different from using the bar graph. On the other hand, the comparison of the same favorite season, such as spring, should utilize the concept of proportion in the context of two different classes. In other words, the reason why the proportion of students who like spring in A school is different from the proportion in B school is connected to the relationship between the part and the whole, and I do want to teach this kind of lesson to my students on that day.

The two teachers devised the task for students to experience comparing and analyzing percentage graphs with the different populations by using actual survey data. When students dealt with the small population (rural school) and big population (urban school) simultaneously, the teachers reasoned that it would be possible to develop more sophisticated proportional thinking, because the total number of students was different. For example, a variation in the small sample can change the proportion more significantly than the same variation in the larger sample, and the teachers wanted a task that would make this apparent to students. They believed the sports club task would accomplish this because the different population sizes could provide an opportunity to deal with the discrepancy mathematically.

Students-Students Interaction within Environments

Social agreement. Part of the rationale behind BMCS is that when students in one classroom reach consensus with another demographic group, their ideas could be more meaningful to a greater number of people. In the field note, the following example emerged from the final connection via Skype:

> To share the comprehensive conclusion, Sujin (CS school) says, "We have the conclusion to choose dodgeball. Dodgeball is the largest in school YG and second largest in our school. The gap is only 7%. Therefore, we will select dodgeball." Dongju (YG school) also answers, "The sum of each percentage is highest in dodgeball. Therefore, we want to choose dodgeball, too." Finally, teacher YH comments, "Our conclusion is same. As a result, dodgeball is tentatively selected as the sports club."

In the first connection, they exchanged information between the two schools. After the activities and discussions about the percentage graph, they debated with each other using the data and providing a rationale for their ideas. This provided opportunities to learn not only the application of a percentage graph, but also how to solve an everyday-life problem using mathematic knowledge of statistical surveys and graphs. Each class' students approached the task using a percentage operation, not the raw numerical values. CS School used subtraction ("the gap is only 7%") and YG used addition ("the sum of each percentage") to make a reasonable decision. Both schools' students consistently had the opportunity to think about how the students in the other schools' idea beyond their classroom were related to their own idea. Consequently, individual students' ideas created a foundation for group discussion, and this discussion extended across both classrooms involved in BCMS. This example suggests that mathematical knowledge could be collaboratively synthesized through the same learning content with communicative technology.

Discussion

The findings of this study demonstrate that what interactions among teacher, students, and content in environments could be arisen and how communicative technology could support knowledge building in mathematics instruction. Based on our analysis, we propose a revised interaction framework for BMCS (see Figure 1. right). In the case reported here, utilizing Skype provided opportunities to result in the emergence of extended interactions among components and to connect student's interactions with teaching practices. Such opportunities included preparing a discussion that bridged classrooms and helped students consider the sociocultural context of another school that had different opinions. In addition, BMCS provided opportunities for students to focus on social contexts, particularly those that differed across schools, as well as mathematical meaning. For knowledge building, students engaged in mathematical tasks to construct and improve collaborative knowledge with networking technology rather than to transmit unchangeable mathematical truth. Consequently, the result of this exploratory study suggests that the efficiency of technology can support bridges between classrooms for extended mathematics learning experiences. BMCS is still in the beginning phase and needs further development with following studies.

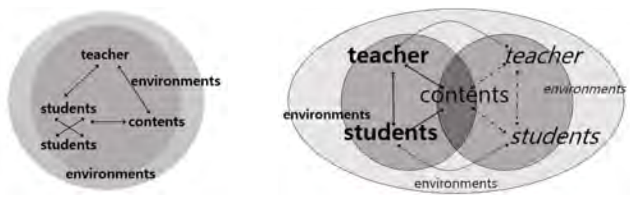

Figure 1. Revised interactional framework for BMCS environments (left: Cohen et al., 2003 (p. 124), right: revised).

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FRACTIONS, MENTAL OPERATIONS, AND A UNIQUE DIGITAL CONTEXT

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Following the PME-NA 2017 metaphor of crossroads, and acting with(in) the intersection point between digital media and mathematics education, I present the mental operations that middle school students engaged in while playing my fractions console game, Rolly's Adventure*. I found that participants engaged in iterating, partitioning, and re-unitizing quantitative representations of fractions with varying levels of units, and that these individual mental actions composed six distinct complex mental operations: splitting; iterating with units; iterating with pre-partitions; iterating with mental partitions (that is, not pre-partitioned by the game); reverse iterating with pre-partitions; and reverse iterating with mental partitions.*

Keywords: Technology, Cognition, Informal Education, Instructional Activities and Practices

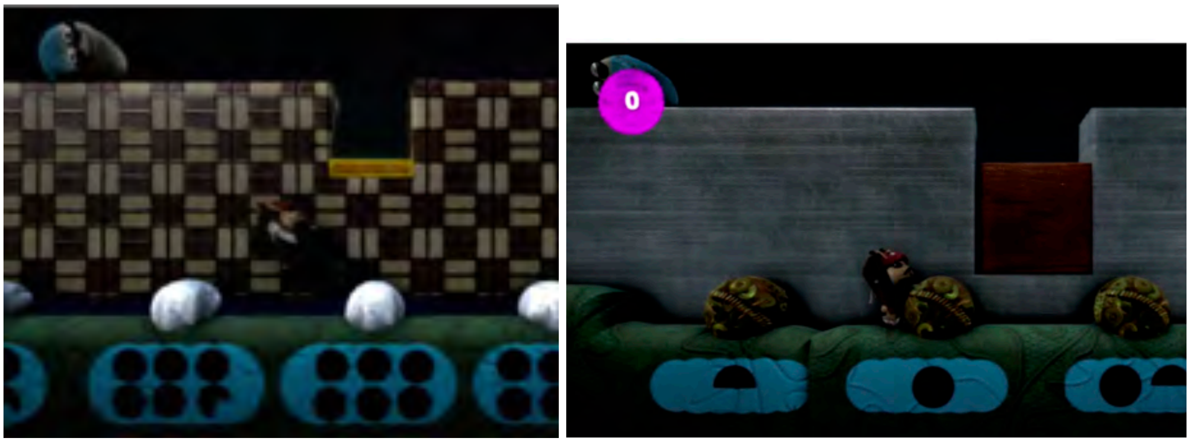
Digital media such as videogames provide the opportunity to investigate how new representations and interactions can contribute to and push back against current research on learning mathematics (Williams-Pierce, 2016a), and provide a useful and intriguing platform for engaging people in mathematical learning and play (e.g., Gresalfi & Barnes, 2016; Steffe & Wiegel, 1994; Williams-Pierce, 2016c). Following the PME-NA 2017 metaphor of *crossroads*, and acting with(in) this *intersection point* between digital media and mathematics education, I designed *Rolly's Adventure* (*RA*; Williams, 2015; Williams-Pierce, 2016a, 2016b) to take full advantage of the learning affordances of videogames (e.g., Salen & Zimmerman, 2004) and the ways that students best learn particular mathematics concepts (in this case, fractions), as identified by education researchers (e.g., Brousseau, 1997; Hackenberg 2007; Steffe & Olive, 2010).

In the following manuscript, I examine how players of *RA* engage in fractions reasoning with a particular focus on how individual mental operations (such as *partitioning*) compose more complex mental operations (such as *iterating with partitions*). In particular, structures within the game supported certain mental operations, and participants experienced and used those game structures differently within their mathematical activity. I found that participants engaged in iterating, partitioning, and re-unitizing quantitative representations of fractions with varying levels of units, and that these individual mental actions composed six distinct complex mental operations: splitting; iterating with units; iterating with pre-partitions; iterating with mental partitions (that is, not pre-partitioned by the game); reverse iterating with pre-partitions; and reverse iterating with mental partitions.

Theoretical Framework

I subscribe to the constructivist view of learning (von Glasersfeld, 1995), developed in detail within the field of fractions by scholars such as Steffe and Olive (2010) and Hackenberg (2007). I follow Thompson's (1995) view of quantity as a person's conception of measurable attributes of objects prior to any actual measuring, and rely upon my intentional design of *RA* to present fractions as quantities that support players in constructing their own understanding of the game and the underlying mathematical patterns directing the game's behavior. Although I generally follow Steffe and Olive's (2010) and Hackenberg's (2007) definitions of iterating, partitioning, and disembedding, I split from them (pun intended) with Confrey's (1994) characterization of splitting as an instantaneous and intuitive act of duplication.

Important notes: First, the game provides puzzles that each have a hole that the player must fill perfectly by operating upon blocks that have been preset in the hole (see Figure 1; see also Williams-Pierce, 2016a). Second, since I have insufficient evidence to hypothesize about interiorization with this data, I focus on *coordinating* levels of units, and my *levels of units* codes universally refer to coordinating and not interiorization.



(a) (b)

Figure 7: (a) Puzzle 7, which introduces the countability textures; and (b) Puzzle 4.

In Figure 1a above, the provided block in gold is exactly the height of the square texturing on the left of the hole. Since this texturing supports counting the numbers of squares, and concluding that is how many blocks fit in the hole, I termed this *countability textures*. In Figure 1b above, there are no countability textures provided, and unlike the gold block in 1a, duplication of the brown block will not perfectly fill the hole. Rather, the brown block and one-half of itself is the quantity that will fill the hole, so participants must compare the empty space to the brown block, and determine that one-half of the block will fill it. When participants disembedded one half of the brown block and iterated it to fill the hole, I termed that *mentally partitioning and iterating*, despite the fact that all iterating and partitioning is inherently mental (Steffe & Olive, 2010; Hackenberg, 2007). In short, when I use the word *mentally* in front of a word or phrase that already refers to a mental operation, it indicates my way of operationalizing the distinction between that mental operation with game supports, and that mental operation without.

Data Collection Methodology

Sixteen middle school students were recruited in dyads and solos, resulting in 11 males and 5 females from eleven to fourteen years who lived in a small Midwestern city. Each session included a playthrough of *RA*. The playthroughs were captured through video upon the participants' bodies and screen capture of their gameplay. Participants were asked to play the game as they would play any game at home, and told that I would likely not answer any questions they had while playing. Participants were asked to talk to me or each other as they played, and stopped them occasionally and ask questions. It is important to note that I never indicated that the game was about fractions – or indeed, mathematics of any sort – and during gameplay, never gave mathematical advice.

I used MaxQDA, a qualitative data analysis software, and iteratively open-coded gameplay transcripts. I then re-organized and finalized the coding scheme through the process of constant-comparison. The final coding scheme included two umbrella codes that will be examined in this paper: Individual Mental Operations, and Complex Mental Operations.

Results

The subcodes of Individual Mental Operations (Table 1) served to identify what mental operations were occurring. The goal to provide a reasonably smooth and natural game playing experience precluded me from asking the types of questions that can lead to more rigorous second-order models of thinking. The subcodes of Complex Mental Operations are presented in Table 2 with brief definitions.

Table 1: Individual Mental Operations Subcodes and Definitions

Subcode	Nested Subcode	Definition
Levels of Units	Two Levels of Units	When participants perceive the preset block(s) and the hole as units.
	Three Levels of Units	When participants perceive the preset block(s), a partition of those preset block(s) (mental or pre-partitioned), and the hole as units.
	Four Levels of Units	When participants perceive the preset block(s), a partitions of those preset block(s) (mental or pre-partitioned), a mental grouping of those preset blocks, and the hole as units.
Re-Unitizing		When participants re-unitize from perceiving one unit as the primary unit that all other units stand in reference to, to another unit as the primary unit.
Partitioning	Mentally Partitioning Blocks	When participants mentally create a partitioned block that is not offered by the game. The new partitioned block must be considered as maintaining a relationship to the original block or the hole.
	Mentally Partitioning Hole: Instant	When participants mentally partition the hole into block-sized partitions instantly, without counting or using countability textures provided by the game.
Mentally Iterating (Countability or Perfect Presets)		When participants are iterating the block in the hole, but the game is overtly supporting that mental operation, such as through countability textures.

Table 2: Complex Mental Operations Subcodes and Definitions

Subcode	Definition
Splitting	When participants instantaneously visually perceive the preset block(s) as a unit that can be duplicated to fill the hole, and manipulates their mental images of the block(s) to do so.
Iterating with Units	When participants mentally iterate preset block(s) as a unit to fill the hole. This mental iteration is not instantaneous, as with *Splitting*.
Iterating with Pre-Partitions	When participants disembed and iterate a single partition of preset pre-partitioned blocks to fill the hole. May be co-coded with *Iterating with Units*, if the participants view the disembedded partition as a unit.
Iterating with Mental Partitions	When participants mentally partition the preset block(s), then disembed and iterate that mental partition to fill the hole.
Reverse Iterating with Pre-Partitions	When participants disembed and reverse iterate a single partition of preset pre-partitioned blocks to fill the hole.
Reverse Iterating with Mental Partitions	When participants mentally partition the preset block(s), then disembed and reverse iterate that mental partition to fill the hole.

I considered the individual mental operations to *compose* the complex mental operations based upon co-occurrence rates, which also depended upon the game structures available for each puzzle. Due to space constraints, I cannot present those composition relationships here.

Discussion and Conclusion

The goal of this paper was to illustrate the mathematical reasoning that emerged when participants played *RA*, and to situate those findings within the field of fractions learning. Different patterns of Individual Mental Operations codes appear to compose different Complex Mental Operations, depending in large part upon which game structures were available for each puzzle. During my presentation, I will share more detailed accounts of the individual and complex mental operations described above, and how *RA* provokes fractions learning and reasoning differently from other digital contexts.

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INVESTIGATING SECONDARY MATHEMATICS PRE-SERVICE TEACHERS' TECHNOLOGY INTEGRATED LESSON PLANS

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The purpose of the study is to investigate secondary mathematics PSTs' technology selection and the intended way of utilizing technology in their lesson plans. In this multi-case study, six PSTs in their senior year were interviewed and their lesson plans were collected. What types of technology PSTs select and in what purposes the selected technologies have been used to support students' mathematical thinking were explored. The findings show that some PSTs selected cognitive technologies due to their dynamic features, ease of use, and providing different visual representations. On the other hand, some of them did not integrate these types of technologies because of their lack of technology knowledge, availability of technology in context, students' familiarity with technology and time considerations. Their way of integrating technologies vary and while some PSTs integrated as reorganizer, others planned to use technologies as amplifier.

Keywords: Technology, Teacher Education-Preservice

In the last decade, teacher education programs have more emphasized preparing teachers with the necessary knowledge and skills to teach with technologies to meet the needs of 21st century learners. Researchers argued that courses in teacher education programs and schools in which pre-service teachers (PSTs) complete field experiences and student teaching, significantly impact their learning (Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004). Therefore, teacher education programs offer technology-specific content and methods courses for PSTs to learn about available technologies and experience integrating these into mathematics teaching and learning. For instance, Niess (2005) required PSTs not only to integrate technologies into their mathematics teaching but also to think deeply about how they would use it and why they considered it critical in teaching the target mathematics concept.

Developing technology-integrated lesson plans help PSTs not only to demonstrate their competencies in technology integration but also to reflect on the process of selecting appropriate technologies to enhance teaching and learning. Although PSTs are required to develop technology-integrated lesson plans during their teacher education programs, little is yet known about their practice. Therefore, the purpose of this study is to investigate secondary mathematics PSTs' technology selection and the intended way of utilizing technology to support students' mathematical thinking in their lesson plans.

Conceptual Framework

In the literature, there are many categorizations of technology used in mathematics teaching and learning. However, this study employs Dick and Hollebrands's (2011) categorization of technologies, "conveyance technologies" and "mathematical action tools" and Pea's (1985, 1987) earlier categorization of cognitive uses of technologies as amplifiers or reorganizers (see Figure 1). This framework was considered as a useful one for discussing the uses of digital technologies in instruction to support and promote students' learning and mathematical thinking.

Digital Technologies as Conveyance Tools

Conveyance technologies are used to convey information and include tools for presentation, communication, sharing/collaboration, and assessment/monitoring/ distribution (Dick & Hollebrands, 2011). This type of technology has been the most commonly used in classrooms. For instance, presentation tools enable the display of documents, videos, or computer screens on a board for group viewing of the same information. Jonassen (1995) emphasized that such technologies as "conveyors of information… [are used] to 'teach' students by presenting prescribed information to them which they are obligated to 'learn' (p. 1). When conveyance technology is in use, the role of learners is to perceive the information as presented by the tool rather than interact with the tool to construct their own knowledge.

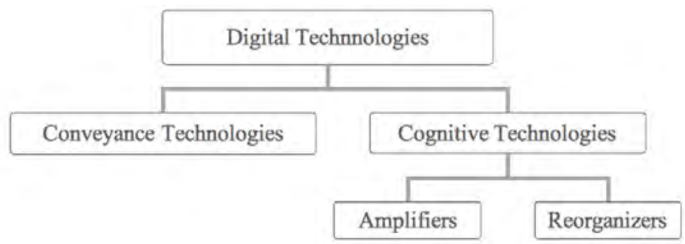

Figure 1. The categorization of digital technologies.

Digital Technologies as Cognitive Tools

Pea (1987) defined "cognitive technologies" as those which help users "transcend the limitations of the mind…in thinking, learning, and problem-solving activities" (p. 91). In this environment, the role of users is not passive receivers but active agents. Cognitive technologies can stimulate or amplify cognitive processes (Kozma, 1987). Learners actively interact with cognitive tools, which activate their strategic and critical thinking (Jonassen, 1995). Under this broad categorization, Pea (1985) stated that cognitive technologies in mathematics could be any tool to make students' thinking visible in order to promote analyzing, reflecting and discussing. The interest in this study is limited to digital cognitive technologies that may create environments for mathematical activity to support students' mathematical thinking.

Researchers have used the terms "cognitive technologies" and "cognitive tools" are used interchangeably. Peressini and Knuth (2005) identified technology as a cognitive tool that allows students "to represent and explore a variety of mathematics procedures and concepts so that they can be examined from a conceptual perspective" (p. 280). A cognitive tool allows students to investigate many cases of similar situations in a short time while directing their attention to their own actions and enabling them to make and test their own conjectures. Using cognitive technologies in mathematics education facilitates "technical or conceptual dimensions of mathematical activity" (Zbiek, Heid, Blume, & Dick, 2007, p. 1171). In terms of technical dimensions of mathematical activity, a cognitive tool must allow the user the means to act on mathematical objects or representations of those objects. To facilitate the conceptual dimension of mathematical activity, a cognitive tool must react in response to the user's actions by providing clearly observable evidence of their consequences. Similarly, Dick and Hollebrands (2011) defined mathematical action tools as those that enable users to interact and receive feedback during the performance of mathematical tasks. Thus, the term of "mathematical action tools" can be considered synonymous with cognitive tools that allow students to perform actions, receive immediate feedback, and so investigate mathematical ideas.

Cognitive Technologies as Amplifiers or Reorganizers

Pea (1985) explained cognitive technologies with the metaphors of amplifier and reorganizer of mental activity. Many researchers have adopted his notion to describe technology use in mathematics teaching and learning (Ben-Zvi, 2000; Goos, Galbraith, Renshaw, & Geiger, 2003; Lee & Hollebrands, 2008; Zbiek, Heid, Blume, & Dick, 2007). In this study, his metaphors of using technology as amplifier and reorganizer is employed when exploring PSTs intended way of using technology in their lesson plans.

When technology is served as an amplifier, it enables users to accomplish a task more efficiently and accurately with significantly less time. As Lee and Hollebrands (2008) stated, as an amplifier, a tool "expedites a process that could be completed *without* its use" (p. 329). Use of technologies as an amplifier increase what students can do without converting their actions but does not change the nature of what they think (Barrera-Mora & Reyers-Rodriguez, 2013; Sherman, 2014). For instance, performing arithmetic computations with a calculator can be considered using technology as an amplifier. With that, students can complete the task more efficiently and more accurately without any arithmetic errors. Their cognitive focus is on performing computations, and it does not change their actions and thinking (Sherman, 2014).

When technology is served as a reorganizer, technology transforms users' actions and enables changes in their thinking that otherwise would be difficult or impossible (Sherman & Cayton, 2015). For instance, with the use of dynamic geometry software, students can construct a triangle and its medians in order to make and test conjectures about the relationships between the medians of a triangle. Through their actions in this environment, students' mathematical thinking and behaviors might change. While the technology performs calculations, stores, and retrieve information, students are responsible for recognizing and judging patterns of information, and organizing it accordingly. This environment provides opportunities for students to interact with tool and so they may develop their own understanding as a result of their interactions.

Method

The following research questions are addressed in this study; (1) What types of technologies do secondary mathematics PSTs select in their lesson plans? and (2) In what ways do secondary mathematics PSTs plan to use technology in their lessons?

In this study, a qualitative multi-case study (Merriam, 2009) approach is employed. The participants of the study are six secondary mathematics PSTs who were enrolled in a method course in the Mathematics Education department in a large Midwestern state university and have completed all required content and methods courses. The site is selected intentionally because the objectives of the course is for students to develop knowledge of appropriate uses of technology and gain experience of using technology for mathematics learning and teaching. These PSTs created technology-integrated lesson plans during a semester.

The data sources of the study include these PSTs' lesson plans and semi-structured interviews. Their lesson plans provide what technologies these PSTs selected and how they plan to integrate those to support students' mathematical thinking. On the other hand, one-to-one interviews with each PST helps to clarify in what purposes they intended to use selected technologies. Interview transcriptions and lesson plans were analyzed by coding based on the conceptual framework to identify and categorize the technologies. Then, two researchers performed check-coding on each lesson plan by independently coding and then comparing the codes. Then, they discussed their analysis and came to consensus. Finally, these PSTs' intended way of technology use were identified based on Pea's metaphor of amplifier and reorganizer.

Preliminary Findings

The results of analyzing lesson plans of these PSTs revealed that each lesson plan consists of at least a task with utilizing a technology. These PSTs mostly intended to use cognitive technologies including dynamic geometry software (GSP and GeoGebra), and online or hand-held graphing calculator (Desmos and TI-Nspire). Their reasons of selecting cognitive technologies include dynamic features of the tool, ease of use, and providing different visual representations. The most frequently selected cognitive technology used in these lesson plans is GSP. In addition to cognitive technologies, some of PSTs preferred to use conveyance technologies including iPad and interactive whiteboard in their lesson plans. It was recognized that their lack of technology knowledge, availability of technology in context, students' familiarity with technology and time considerations were their main reasons to not use cognitive technologies. The intended way of technology use varied. While some PSTs integrated as reorganizer, others planned to use technologies as amplifier. In the presentation, detailed description for each case will be provided and evidence will be shared with the audience.

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ONE TEACHER'S IMPLEMENTATION OF PROFESSIONAL DEVELOPMENT AROUND THE USE OF TECHNOLOGY

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There is an opportunity at the crossroads of research, policy, and practice to translate calls for the use of technology into professional development (PD) that is relevant to teachers. In this study, I examine how one teacher translated a sequence of PD meetings into the design and implementation of two lessons using GeoGebra. Three overarching themes emerged from the teacher's work: bidirectional decision making in the selection of technology and mathematics content; adapting existing lessons to accommodate a new technology; and allowing "productive struggle" among students with the technology. This study suggests that the use of technology tools may help teachers conceive of new ways in which students may experience mathematics, but this new perspective can sometimes be at odds with efforts to adapt existing resources.

Keywords: Technology, Middle School Education, Teacher Education-Inservice/Professional

Discrepancies between policy recommendations for the purposeful use of technology in mathematics classes (NCTM, 2000; NGAC, 2010) and rates of implementation among teachers (Banilower et al., 2013), there is an opportunity at the crossroads of research, policy, and practice to support teachers to implement calls for the use of technology in mathematics teaching. This study addresses the issue of how professional development (PD) that is conceptually framed around the use of technology to support students' reasoning and sense making translated into practice for one eighth-grade mathematics teacher. I pose the question: *How does a mathematics teacher apply professional development in the use of mathematics technologies to design and implement technology-based lessons?* Through a six-month partnership in which I introduced the teacher to two technology tools and collaborated with her in planning two lessons, I investigated how the theoretical components of the PD translated into instructional decisions that supported the teacher's goals for her students and her content.

Conceptual Framework

The PD discussed in this study was framed through the intersection of teacher learning and student use of technology for mathematical reasoning. This paper addresses specifically the component of teacher learning, with regards to how a teacher applied PD in the design and implementation of technology-based lessons. The PURIA (play, use, recommend, incorporate, assess) model has been introduced by Beaudin and Bowers (1997), and elaborated by Zbiek and Hollebrands (2008), to describe different modes of technology integration by teachers. In the first two modes, teachers become familiar with the use of a technology, first in a non-directed way and then to engage with some specific mathematics problem or idea. In the recommend mode, teachers begin to work within small groups to suggest uses of a technology. In the incorporate mode, teachers integrate the technology into their classroom instruction; and in the assess mode they examine students' use of the technology in terms of their mathematical learning. The PURIA model provides an outline for the implementation and analysis of PD around the use of a mathematics technology.

Data and Methods

The PD program was implemented in a large suburban middle school in the Midwest with a strong technology initiative and a 1:1 ratio of students to laptop computers. I proposed the PD opportunity to a teacher whom I refer to as Ms. S as an opportunity to learn about new mathematics technologies, and then to select the technology she saw as most relevant to her content and to design two lessons incorporating it. Ms. S had three full years of middle school mathematics teaching experience at the start of the project. Upon agreeing to participate in the project, she chose to use her 8^{th}-grade honors-track mathematics class as the focal class for which she would design and plan two lessons.

Our work began in the fall, during which time Ms. S and I met approximately on a weekly basis for 1.5 hours after school. I first introduced her to two technologies that were new to her. The first, SageMath, is an open-source computing software that allows users to program through a Python-based language. The second, GeoGebra, is an open-source dynamic geometry environment that is more typically used in middle and secondary mathematics classrooms for representing geometric figures and relationships. The purpose of introducing two different technologies was for Ms. S to have the opportunity to *play* and *use* (Beaudin & Bowers, 1997) the technologies and to compare which may be more relevant to the content she would teach in the following quarters. For each technology, I designed two prototype lessons that Ms. S completed before she ultimately selected GeoGebra as the environment she would incorporate with her students. In collaboration, we designed one lesson on the topic of geometric transformations, and one on the properties of the perpendicular and angle bisectors of a triangle.

Data for this study come from video recordings of the PD sessions, as well as video recordings of the classroom implementation of the two lessons. The purpose of the study was to characterize some of the ways that Ms. S translated her work during the PD sessions into the design and implementation of lessons for her students. As such, I used a constant comparative method (Strauss & Corbin, 1998) to identify categories of phenomena that emerged from passes through transcripts from both the PD sessions and in-class implementation. Specifically, I looked for evidence in the transcripts of our after-school meetings of how Ms. S implemented the ideas that surfaced during her use of the technology, which could be corroborated or disputed by Ms. S's actions in the classroom.

Findings

Three overarching themes emerged that characterize Ms. S's work of designing technology-based lessons: bidirectional decision-making, adapting new lessons, and productive struggle.

Bidirectional Decision Making in the Selection of Technology and Content

In conceptualizing the PD program, the purpose of introducing two alternative technology tools was so that Ms. S could determine which technology would be most relevant to her content. In practice, however, the process of alignment between the technology tool and the content to be taught was bi-directional. Ms. S's comments about her goals for the project, which I asked her about during our initial meeting together, encapsulate this tension:

> *Ms S:* My goal would be to learn something new and to see if I can integrate it…There's five of us that teach 8^{th}-grade math, and we work very closely together and plan together, and feel like over the years we've developed some really good things and I'm now kind of like, I want to get to keep using this I like it. So I don't do as much as I did in my first two years just looking around and seeing what other tools or activities are out there.

With her comments above, Ms. S indicated two sentiments. First, she saw her participation in the project as an opportunity to learn something new that she could integrate into her mathematics teaching. Throughout her experience with the two new technology tools, Ms. S often noted discoveries of novel mathematical ideas that she proposed as interesting areas of exploration for her students. For example, working on a lesson about triangle constructions using GeoGebra, Ms. S recognized that the intersection of angle bisectors of a triangle is the center of that triangle's inscribed circle. This was not a topic that was currently embedded within the course she taught, but Ms. S was inspired by the ways in which the use of GeoGebra might open up new a new area of discovery during students' study of triangles.

In contrast to the excitement over identifying new mathematical ideas that could be explored through the use of technology, Ms. S also expressed a desire to find a tool that would align with her current curriculum. As she noted, the group of 8th-grade mathematics teachers had been working together for multiple years to develop a stable set of lessons and resources that were aligned with the state and district learning standards for the course. Ultimately, Ms. S selected the use of GeoGebra to incorporate into her class because she felt it better aligned with the existing content of the course.

Adapting Existing Lessons to Accommodate New Technology

A common theme during the "play" and "use" segments of the PD sessions was for Ms. S to recall a lesson she had taught in the past and to consider how it might be adapted or modified through the use of GeoGebra:

> *Ms. S:* Just last year we started doing constructions with them…And we, we did it by hand; we agreed that they could do it by hand. But, doing it on something like this would be pretty nice. Because it, then, because the measurements are already there, so it takes out that sort of error. Just measuring lines for them, sometimes, you know?

When Ms. S worked on the prototype lessons, she recalled materials she had used in the past and how they could be improved or adapted with the technology tool. In these cases, the use of technology did not open doors to novel mathematical ideas as much as it served to streamline existing activities.

Allowing "Productive Struggle" Among Students with the Technology

The use of the term "productive struggle" in Ms. S's conversations was of note especially because of how her use of the term in relation to technology compared to the way it is typically used in mathematics education research. Productive struggle in mathematics education most typically refers to extending "effort to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p. 387). Ms. S indicated that the concept of productive struggle had surfaced frequently during the math department meetings at her school, as a goal for the type of activity students would engage in. When working on an early lesson using GeoGebra, Ms. S was struggling with the menu options to construct an angle of a given measure. When I noted that the software could sometimes be fickle, Ms. S noted with regards to her students, "I guess it's okay to allow that productive struggle." Although the struggle Ms. S was having would more typically be considered troubleshooting with the technology, she identified it as an opportunity for productive struggle for her students.

Following Ms. S's introduction of the term productive struggle, we regularly discussed the lessons and students' use of GeoGebra with respect to this idea. Towards the end of our lesson planning, I asked Ms. S to describe how she defined productive struggle for her students. In her response, she specifically addressed the role of technology:

> *Ms. S*: That's where I think GeoGebra is, again, I think it will be more straightforward for

them, um, a productive mathematical struggle rather than a logistical, you know, 'what did I type in that's wrong?' type of struggle.

Ms. S's ideas of productive struggle with regards to technology evolved throughout our time working together. Initially, she seemed to use this phrase interchangeably to discuss students' mathematical work and the learning curve associated with a new technology tool. Throughout the planning and implementation process, Ms. S developed a more nuanced way of discussing struggle, making a clear distinction about how the technology could promote "productive mathematical struggle" without creating undue burden for students trying to learn the tool.

Discussion and Conclusion

When experiencing two new technology tools, and then deciding about how to incorporate one of these tools into her teaching, Ms. S articulated a clear tension: There was opportunity to use new technology to foster mathematical discoveries that had not previously been part of the curriculum. At the same time, Ms. S saw opportunity to modify, and marginally improve, existing lessons by using GeoGebra in place of other materials. There is support from research to justify either position, and ultimately a teacher must determine which is the most appropriate for a given class. In the context of this study, Ms. S. aligned her use of GeoGebra to the content of existing lessons, but she leveraged the tools of GeoGebra to create more opportunities for reasoning and proof than had previously been a part of students' work.

This study also suggests how the discourse of mathematics teaching and learning translates to the introduction of a technology tool. Ms. S, although she spoke fluently about productive struggle in the context of her students' mathematical learning, initially applied this term to the use of GeoGebra in ways that referenced fairly superficial challenges with the tool. Throughout her work on the project, Ms. S became increasingly sophisticated in how she distinguished between different types of struggle, and this was reflected through the types of support she gave students during their work on the lessons. In sum, Ms. S's practical experience using and teaching with GeoGebra informed her understanding of the theory and research around struggle in mathematics, which in turn helped her to refine her practice of teaching with GeoGebra.

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RESOLUCIÓN DE PROBLEMAS Y USO DE TECNOLOGÍAS DIGITALES EN UNA PLATAFORMA EN LÍNEA
MATHEMATICAL PROBLEM SOLVING AND DIGITAL TECHNOLOGIES IN A MASSIVE ONLINE COURSE

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La conectividad y potencial que ofrece la tecnología digital está generando nuevas oportunidades para aprender y compartir conocimiento matemático. ¿Cómo diseñar e implementar un ambiente de resolución de problemas y uso de tecnologías digitales en un escenario de aprendizaje masivo (MOOC) que promueva en los participantes una discusión matemática hacia el entendimiento de conceptos y resolución de problemas? Los resultados del estudio indican que el diseño de las actividades y el uso coordinado de GeoGebra, Wikipedia, KhanAcademy, WolframAlpha, Open edX y foros virtuales, permiten y favorecen la creación de un ambiente de colaboración en la resolución de problemas. Los participantes trabajaron colaborativamente y transitaron desde soluciones visuales y empíricas hasta la presentación de argumentos geométricos y algebraicos en la validación de las conjeturas formuladas.

Palabras clave: Resolución de Problemas, Tecnología, Actividades y Prácticas de Enseñanza

Introducción

Las tecnologías digitales abren nuevas rutas en el proceso de aprendizaje, no solo para obtener o compartir información, sino también son un medio para que los estudiantes compartan ideas, discutan, critiquen y se involucren en actividades matemáticas (Santos-Trigo, Moreno-Armella & Camacho-Machín, 2016). La disponibilidad de diversas tecnologías digitales abre nuevas interrogantes sobre qué transformaciones son necesarias en el sistema educativo y cómo incorporarlas en los ambientes de aprendizaje. Los cambios tecnológicos demandan una transformación en la práctica educacional, se requieren cambios en el proceso de enseñanza donde los estudiantes sean el centro de toda actividad y el profesor un apoyo para el desarrollo de habilidades y destrezas en la resolución de problemas (Churchill, King, & Fox, 2016).

Un MOOC (por sus siglas en inglés), es un Curso Masivo Abierto en Línea diseñado e implementado por una institución educativa a través de un equipo de expertos en el tema. En éste, se puede inscribir un gran número de participantes sin importar su nivel de estudios, edad o lugar geográfico en que encuentren. En este estudio interesa analizar y documentar el diseño y los resultados de implementar un MOOC basado en el modelo de diseño de ambientes aprendizaje de Churchill et al. (2016); donde las tareas matemáticas, basadas en resolución de problemas, promuevan la exploración de atributos y relaciones entre los objetos matemáticos dentro de una representación dinámica del problema creada en el Sistema de Geometría Dinámica (SGD) GeoGebra. El movimiento de objetos, la medición y los lugares geométricos son estrategias que fomentan la formulación de conjeturas y promueven la búsqueda de argumentos que las sustenten.

Marco Conceptual

Churchill et al. (2016) proponen un marco para el diseño de ambientes de aprendizaje en línea llamado RASE (Resources-Activities-Support-Evaluation), basado en la premisa de que todo ambiente de aprendizaje debe incluir e integrar esos cuatro componentes. Los *Recursos,* se refieren a los materiales disponibles a los estudiantes: videos, imágenes, documentos digitales, calculadoras, software, etc. Las *Actividades* tienen como objetivo involucrar a los estudiantes en el proceso de aprendizaje a través del uso de Recursos en tareas tales como experimentos y resolución de problemas. El *Soporte* indica que es necesario contemplar medios para proporcionar ayuda a los estudiantes en el momento en se presente alguna interrogante relacionada con la tarea que están realizando. La *Evaluación* enfatiza que los estudiantes deben recibir retroalimentación que les permita reflexionar sobre su aprendizaje.

Santos-Trigo (2014) afirma que la resolución de problemas es una actividad esencial en el aprendizaje de las matemáticas ya que es un medio que permite identificar, explorar, probar y comunicar los procesos de solución. Cuando se incorpora el uso coordinado de tecnologías digitales en los procesos que intervienen en la resolución de problemas, se ofrece a los estudiantes oportunidades para representar, explorar, compartir y discutir los conceptos y la resolución de problemas. GeoGebra favorece la exploración de situaciones matemáticas desde distintas perspectivas permitiendo a los estudiantes tener nuevas formas de visualización de los conceptos y objetos de estudio y analizar, de una forma más precisa, los elementos matemáticos que cuando se utiliza solo papel y lápiz (Santos-Trigo & Camacho-Machín, 2016; Aguilar-Magallón & Reyes-Martínez, 2015).

Metodología

El curso se diseñó como parte de la plataforma digital MéxicoX que utiliza *Open edX*. Se enfocó en la construcción del conocimiento matemático a partir de la resolución de problemas y el uso de tecnologías digitales, es decir, no se abordó de manera puntual una serie de contenidos específicos como generalmente se presentan en un ambiente tradicional de enseñanza.

Las tareas o problemas matemáticos incluyen diversos *Recursos*: representaciones dinámicas del problema elaboradas en GeoGebra, videos de KhanAcademy y vínculos a fuentes de información: Wikipedia, KhanAcademy y WolframAlpha.

Un aspecto fundamental en el diseño de las *Actividades* fue la idea de darle movimiento a figuras simples como triángulos, rectángulos, etc., por medio de representaciones dinámicas. Las tareas o problemas matemáticos guiaron al participante en su trabajo y fomentaron sus procesos de construcción o desarrollo del pensamiento matemático, mediante un método inquisitivo. El objetivo fue que los participantes al observar el comportamiento o variación de algunos objetos o atributos (medida de ángulos, áreas, perímetros, etc.) propusieran algunas conjeturas que den cuenta de su comportamiento y las sustentaran con argumentos. Este proceso les permitió explorar y buscar varios caminos de solución y extender y generalizar los resultados.

Se utilizaron diversas herramientas digitales para dar *Soporte* a los participantes: Wikipedia, KhanAcademy y WolframAlpha permitieron a los participantes consultar en línea conceptos o relaciones matemáticas y, los foros fueron un medio de comunicación para que los integrantes del curso exhibieran sus ideas, formularan interrogantes, conocieran otros puntos de vista y recibieran retroalimentación como una ruta para comprender ideas matemáticas.

La *Evaluación* estuvo presente durante todo el curso. A través del Foro de cada *Actividad*, los participantes desarrollaron y produjeron evidencias de su aprendizaje mediante la exploración de las propiedades de los objetos representados en un modelo dinámico y reflexionaron sobre éstas como producto de la retroalimentación que recibieron por parte de otros integrantes del curso.

La investigación es de carácter cualitativo. La unidad de análisis fueron las conversaciones desarrolladas por los participantes en cada Actividad. Estas permitieron obtener información sobre los comportamientos de los participantes, formas de razonamiento que exhiben durante el proceso de resolución de un problema y cómo éstas pueden ser modificadas a partir de la interacción con otros participantes. En el MOOC se inscribieron 2491 personas. Debido a los contenidos abordados en las tareas matemáticas, el único requisito solicitado a los interesados fue poseer estudios correspondientes al grado 12. Los datos se recolectaron a través de las conversaciones que se desarrollaron en el Foro de cada Actividad. En total hubo 35 foros que incluyeron 7573 comentarios de los participantes.

Presentación y Discusión de Resultados

Se discute una de las *Actividades* del MOOC referente al uso de lugar geométrico como estrategia en la resolución de problemas. En la primera sesión de trabajo se presentó a los participantes un modelo dinámico que incluía un segmento AB, su recta mediatriz n y un punto móvil C sobre $n,$ esto fue la base para que ellos construyeran triángulos isósceles y encontraran alguna posición de C donde también el triángulo fuera equilátero. En la segunda sesión, se proporcionó a los participantes una familia de triángulos isósceles FGC, cuyo lado FG estaba sobre AB y sus lados congruentes tenían una longitud dada r. También se incluyó, la gráfica que modela la variación del área de la familia de triángulos isósceles como resultado de mover el punto C. El punto R, relaciona la longitud de la altura con el área del triángulo (Figura 1).

En las conversaciones, un participante mencionó que el lugar geométrico que describe el punto R cuando C se mueve correspondía a una parábola, otros le indicaron que tal afirmación era falsa argumentando que su ecuación era $A(h) = h\sqrt{r^2 - h^2}$. Los participantes aseguraron que al mover el punto C, el valor aproximado del ángulo β debía ser 90°. Ante esto, algunos preguntaron en el Foro: "*¿Cómo justificar que el área máxima se obtiene cuando $\beta = 90°$?*". Los participantes argumentaron la existencia de un triángulo de área máxima a partir de la observación de la gráfica, ya que el valor del área aumentaba desde cero y después disminuía para volver a ser cero. También, indicaron la existencia de dos posiciones, una sobre y la otra bajo el eje X, para el punto C donde era posible obtener un triángulo de área máxima. Posteriormente, se proporcionó a los participantes un nuevo modelo dinámico en donde el triángulo FGC fue colocado de tal manera que el punto C coincidiera con el origen del plano cartesiano y el lado CG estuviera sobre el eje X; además, se construyó la altura h_1 sobre CG (Figura 2).

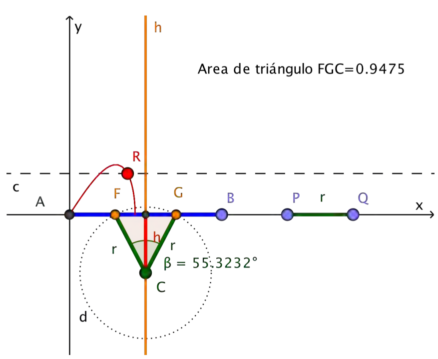

Figura 1. Representación dinámica de una familia de triángulos isósceles.

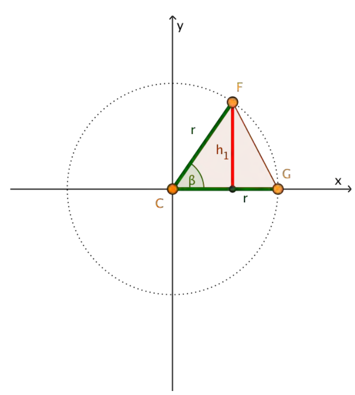

Figura 2. Otra vista del Triángulo $\boldsymbol{FGC}$.

En las conversaciones los participantes coincidieron en que la nueva representación del triángulo FGC les permitió observar la relación que existe entre el punto máximo del lugar geométrico que describe el punto R cuando C y el ángulo β y, así, concluir que de todos los triángulos isósceles de lados congruentes r, el de área máxima es también triángulo rectángulo. Observaron que h_1 maximiza el área del triángulo FGC, pues la base CG permanece constante, y h_1 es máxima cuando mide lo mismo que el radio de la circunferencia, es decir, cuando está sobre el eje Y. Después de la solución geométrica, dos participantes plantearon dos justificaciones adicionales utilizando desigualdades y trigonometría.

Las diversas representaciones que ofrece GeoGebra para visualizar de manera instantánea la variación en los atributos (longitudes, ángulos, áreas, etc.) de objetos geométricos ayudaron a los participantes a establecer conjeturas que posteriormente justificaron. Una estrategia visual y empírica importante fue el uso de un lugar geométrico para modelar la variación del área de una familia de triángulos como un recurso adicional para resolver el problema.

El diseño de las actividades permitió a los participantes, mediante el uso de los foros de discusión, comunicar y contrastar sus ideas en una comunidad virtual, lo cual favoreció la construcción o refinamiento de conceptos e ideas matemáticas ampliando sus recursos matemáticos y estrategias en la resolución de problemas. En los foros, la moderación que realizó el equipo de diseño del MOOC en las conversaciones, orientó la discusión de los participantes en la resolución del problema y, en conjunto con el rol que asumieron varios participantes de proporcionar retroalimentación a otros, promovieron la comprensión de conceptos e ideas matemáticas.

Conclusiones

El uso coordinado de diversas herramientas digitales ofreció los medios para crear un ambiente de aprendizaje MOOC en una plataforma en línea. En este escenario, los participantes trabajaron en un ambiente de colaboración que les permitió compartir sus ideas, a través del Foro de discusión, durante el proceso de resolución de problemas. Todos los participantes tuvieron la oportunidad de explorar representaciones dinámicas que les permitieron identificar conceptos, buscar conjeturas y diversas maneras o argumentos para sustentarlas. En este proceso, los participantes mostraron heurísticas asociadas con el uso de las herramientas como movimiento ordenado de objetos dentro de la configuración dinámica, la cuantificación de atributos (longitudes, ángulos, áreas) y la generación de lugares geométricos.

En el Foro de la *Actividad* varios participantes respondían dudas y daban seguimiento puntual a sus propios comentarios, lo cual favoreció el refinamiento de las ideas matemáticas propuestas inicialmente. Es importante reconocer que el diseño e implementación del MOOC representa un gran reto relacionado con los niveles de compromiso y responsabilidad de los participantes para que ellos mismos monitoreen sus avances en la comprensión y uso de las ideas matemáticas en la resolución de problemas.

The connectivity and potential offered by the use of digital technologies opens up novel opportunities for learners to construct and share mathematical knowledge. How could we design and implement an online learning scenario (MOOC) that fosters the use of digital technologies to engage the participants in a continuous mathematical discussion to understand concepts and to solve problems? The results showed that the design of interactive activities and the coordinated use of digital technologies (GeoGebra, Wikipedia, KhanAcademy, WolframAlpha, Open edx and virtual forums) became important for the participants to formulate conjectures, to look for different ways to validate them and to communicate results. To this end, the participants work collaboratively and transited from the use of visual and empirical arguments to the presentation of geometric and algebraic validation.

Keywords: Problem Solving, Technology, Instructional Activities and Practices

Introduction

Digital technologies are opening new pathways in learning processes, not only in terms of obtaining or sharing information, but also as means for students to share, discuss and criticize ideas while becoming more involved in mathematics activities (Santos-Trigo, Moreno-Armella &

Camacho-Machín, 2016). The availability of diverse digital technologies raises new questions as to what transformations of educational systems are necessary, and how to incorporate them into learning environments. Technological changes demand a transformation of educational practices and require modifying teaching processes to make students the center of all activity while teachers act as supports for developing abilities and skills related to problem-solving (Churchill, King & Fox, 2016).

A Massive Open On-line Course (MOOC) is designed and implemented by educational institutions through the efforts of a team of experts in the field. Large numbers of participants can be enrolled, regardless of their level of schooling, age, or place of residence. The present study is concerned with analyzing and documenting the design and results of the implementation of a MOOC based on the design model of learning environments presented by Churchill et al. (2016), in which mathematics tasks, based in problem-solving, promote exploring the attributes of, and relations among, mathematics objects in a dynamic representation of a problem created in the GeoGebra Dynamic Geometry System (DGS). Object movement, measuring, and locus are three strategies that foster the formulation of conjectures while also promoting the search for arguments to sustain them.

Conceptual Framework

Churchill et al. (2016) proposed a framework for designing on-line learning environments called RASE (Resources-Activities-Support-Evaluation). Their system is based on the premise that all learning environments should include and integrate these four components. Resources refer to the materials available to students, including videos, images, digital documents, calculators and software, etc. The objective of the Activities is to involve students in the learning process by applying Resources to tasks like experiments and problem-solving. Support indicates the need to contemplate the means that will provide students with the help they require when a question or doubt arises in relation to the task they are asked to perform. Evaluation, finally, emphasizes that students need to receive feedback that will allow them to reflect on their learning.

Santos-Trigo (2014) sustains that problem-solving is an essential activity in learning math because it is a medium that leads students to identify, explore, test and communicate solution processes. When the coordinated use of digital technologies is incorporated into the processes that intervene in problem-solving, students are provided with opportunities to represent, explore, share and discuss both concepts and the process of problem-solving itself. GeoGebra fosters the exploration of math situations from distinct perspectives and offers students new ways of visualizing concepts and objects of study while also analyzing elements of mathematics with greater precision than is possible using only pencil and paper (Santos-Trigo & Camacho-Machín, 2016; Aguilar-Magallón & Reyes-Martínez, 2015).

Methodology

This course was designed as part of the MéxicoX[i] digital platform that utilizes Open edX[ii]. It does not address a series of specific contents as is usually the case in traditional learning environments. Instead, it focuses on constructing mathematical knowledge through the process of problem-solving supported by using digital technologies. The math tasks or problems presented include diverse resources: dynamic representations of the problem elaborated in GeoGebra, videos from KhanAcademy and links to sources of information such as Wikipedia, KhanAcademy and WolframAlpha.

One fundamental aspect in the design of the Activities was the idea of endowing simple figures – such as triangles and rectangles, among others– with movement using dynamic models. The math tasks or problems guided participants in their work and fostered their processes of constructing or developing mathematical thinking through an inquisitive method. The objective was for participants to observe the behavior or variation of certain objects or attributes (measuring angles, areas, perimeters, etc.), propose conjectures that might account for their behavior, and then sustain or refute those conjectures through argumentation. This process allowed students to explore and search for various means of solution while extending and generalizing results.

Several digital tools were used to provide Support to participants: Wikipedia, KhanAcademy and WolframAlpha allowed them to consult mathematics concepts or relations on-line; while forums provided a means of communication where they could present their ideas, formulate questions, and give or receive feedback to improve their understanding of mathematics ideas.

Evaluation was conducted throughout the course in the Forums associated with each Activity. Participants developed and produced evidence of their learning by exploring the properties of the objects represented in a dynamic model and reflecting on them as products of the feedback they received from other students enrolled in the course.

A total of 2,491 people registered in this MOOC. Given the contents of the math tasks involved, the only requirement for registration was that interested individuals had a minimum schooling level equivalent to grade 12.

The research involved was qualitative in nature. The units of analysis were the conversations developed by the participants themselves in each Activity, which allowed researchers to gather information on participants' behaviors, the forms of reasoning they exhibited during the problem-solving process, and how these could be modified through interaction with other participants. Data were collected from the conversations held in the Forum for each Activity, and included a total of 7,573 comments by participants in 35 forums.

Presentation and Discussion of Results

Only one of this MOOC's Activities is discussed. It refers to the use of locus as a problem-solving strategy. In the first work session, participants were presented with a dynamic model that included a segment, AB, its straight perpendicular, n, and a movable point, C, on n. From this base, they were told to construct isosceles and equilateral triangles. The next Activity involved a family of isosceles triangles, FGC, whose side FG was on AB and whose congruent sides had a given length of r. This task included a graph that modeled variations in the area of this family of isosceles triangles that resulted from moving point C. Point R related the length of the height to the area of the triangle (Figure 1).

In the conversations, one participant mentioned that the locus of point R when point C moves along the n, corresponded to a parabola, but another replied that this was false, arguing that the corresponding equation was $A(h) = h\sqrt{r^2 - h^2}$. The participants affirmed that upon moving point C, the approximate value of angle β should be 90°. At that moment, some students in the Forum asked: "How can we justify that the maximum area is obtained when $\beta = 90°$?". Participants argued for the existence of a triangle with maximum area based on observing the graph, since the value of the area increased from zero and then decreased to return to zero. Moreover, they sustained that there were two positions for point C –one on, the other under, the X-axis– that indicated where it was possible to obtain a triangle with maximum area.

Later, students were given a new dynamic model in which the triangle FGC was positioned in such a way that point C coincided with the origin of the Cartesian plane, side CG was on the X-axis, and the height, h_1, was constructed on CG (Figure 2).

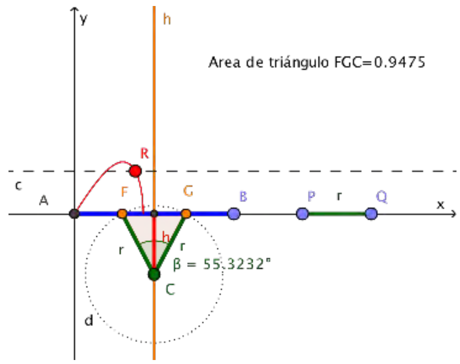

Figure 1. A family of isosceles triangles Dynamic representation.

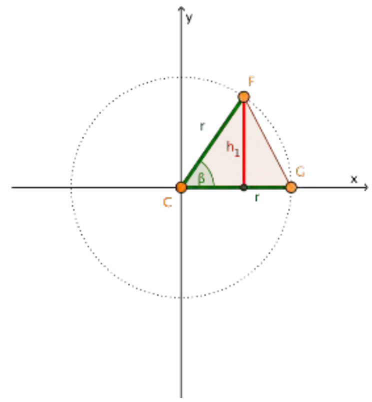

Figure 2. Another view of triangle $\boldsymbol{FGC}$.

In their forums, participants coincided in that the new representation of triangle FGC allowed them to observe the relation that existed between the maximum point of the locus of point C when point E moves along n, and angle β and, therefore, reach the conclusion that of all the isosceles triangles with congruent sides, r, the one with the maximum area is also a right-angle triangle. They observed that h_1 maximizes the area of triangle FGC, since the base, CG, remains constant, and h_1 is maximal when it measures the same as the radius of the circumference; that is, when it is on the Y-axis. After arriving at this geometric solution, two participants presented two additional justifications utilizing inequalities and trigonometry.

The diverse representations that GeoGebra offers for instantaneously visualizing variations in the attributes (lengths, angles, areas, etc.) of geometric objects helped participants establish conjectures that were justified later in the problem. One important visual and empirical strategy was the use of locus to model variations in the area of a family of isosceles triangles as an additional resource for resolving the problem.

The design of the activities allowed the participants, through the use of discussion forums, to communicate and contrast their ideas in a virtual community. This fostered the construction or refinement of mathematics concepts and ideas by broadening access to mathematical resources and strategies for resolving problems. The moderating of the forums performed by the team that designed the MOOC oriented participants' discussions towards a search for responses. Various participants took on the role of providing feedback to others through comments offered in the forums.

Conclusions

The coordinated use of several digital tools provided the means for creating a MOOC learning scenario in an on-line platform. In this scenario, participants worked in a collaborative environment that allowed them to share their ideas through discussion forums during the problem-solving process. All participants had the opportunity to explore the dynamic models and conceptualize them as a starting point for identifying concepts and looking for relations, and then devising diverse means or arguments to sustain them. During this process, participants showed heuristics associated with the use of such tools as the ordered movement of objects inside the configuration, the quantification of attributes (lengths, angles, areas), and loci.

Through the Activities of this MOOC, a participative environment was generated in the forums as participants adopted different roles, including some who responded to questions, others who resolved doubts, and individuals who offered timely follow-up to comments. All of this propitiated the refinement of the mathematics ideas proposed initially. It is important to recognize that the design and implementation of MOOC represents a great challenge in terms of the levels of commitment and responsibility required of participants so that they can monitor their advances in the understanding and use of mathematics ideas in problem-solving.

Endnotes

[i] More information at http://mx.mexicox.gob.mx/about

[ii] More information at https://open.edx.org/about-open-edx

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VARIATIONS IN TEACHING PRESENCE: FACTORS CONTRIBUTING TO SOCIAL PRESENCE AND EFFECTIVE ONLINE DISCUSSION

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The purpose of this study is to identify factors that impact discussion in asynchronous online learning environments. Various facets of teaching presence related to the design and facilitation of online discussion activities are considered in conjunction with common indices from social network analysis. Levels of in-degree and out-degree centrality and betweenness speak to the social presence within a given forum while average degree, density, and connectedness are representative of the volume and diversity of connections comprised within that forum. Findings indicate that having students initiate their own thread within a forum leads to a more balanced discussion, while required forums tend to have both a higher volume of communication and a greater diversity of connections than optional forums. The information gained from this study will inform practices of online, discussion-based courses offered at the post-secondary level.

Keywords: Technology, Post-Secondary Education, Instructional Activities and Practices

Introduction

Programs in higher education across the country are at a crossroads as traditionally seated courses are being transitioned to online alternatives. A common feature of most online courses is the use of asynchronous discussion forums as a means for students to interact with the instructor, each other, and the course content. Indeed, research has shown that online discussion forums have the potential to promote skills like knowledge construction, critical thinking, and problem solving in students (An, Shin, & Lim, 2009) but only do so when the activities are properly designed and facilitated by the instructor in such a way as to incite high student engagement (Jo, Park, & Lee, 2017). While general consideration has been given to the role of the instructor in online settings, An et al. (2009) recognized that little attention is often devoted to specific strategies that can increase the effectiveness of online discussion. Therefore, our research focuses on those facets of design and facilitation that impact student interaction in online discussion.

According to Garrison, Anderson, and Archer (1999), students and teachers in online courses must form a community of inquiry (COI) in order to generate worthwhile educational experiences. Teaching presence, social presence, and cognitive presence are the three core elements in a COI, and learning is said to occur as a result of the interaction between these three elements. We focus specifically on the interaction between teaching presence and social presence and its impact on asynchronous online discussion.

Teaching presence is "the design, facilitation, and direction of cognitive and social processes for the purpose of realizing personally meaningful and educationally worthwhile learning outcomes" (Anderson, Rourke, Garrison, & Archer, 2001, p. 5). Akyol and Garrison (2008) identify three categories of teaching presence in a COI: (a) design and organization; (b) facilitating discourse; and (c) direct instruction. We consider the first two categories. For an online discussion forum, design and organization refers to the structure of the forum, the topic of discussion, and the nature of the prompt used to generate discussion. Facilitation of discourse is related to the instructor's level of involvement in the forum, which should decrease over time as students become able to sustain discussion on their own (An, Shin, & Lim, 2009).

Social presence, the second element in a COI, refers to "the ability of participants to identify with the group or course of study, communicate purposefully in a trusting environment, and develop personal and affective relationships progressively by way of projecting their individual personalities" (Garrison & Akyol, 2013). The categories of social presence as listed by Joksimović et al. (2015) include interpersonal communication, open communication, and cohesive communication. Social presence is fostered through teaching presence (Joksimović et al., 2015; Rogers & Lea, 2005); so, the choices an instructor makes regarding the design and facilitation of online discussion can directly influence the levels of interpersonal, open, and cohesive communication that are evident in a discussion forum.

The purpose of this study is to explore variations in teaching presence within and across two virtual COIs to determine if and how those variations influence the students' social presence and quality of generated discussion. The following research questions guided the study: (1) How does teaching presence (as evidenced by the design and facilitation of discussion board activities) impact students' social presence (as measured by interpersonal, open, and cohesive communication) in online discussion forums? (2) How does teaching presence impact the quality of asynchronous online discussions?

Methodology

The data were collected from two sections (Section 1 and Section 2) of an online, asynchronous course that served as an introduction to doctoral studies in a department focused on preK-16 education at a public research university in the northeast U.S. Students' grades in the course were determined by the number of points they compiled on required and optional assignments. Assignments required posting to discussion forums that related to the weekly reading assignments. At the beginning of the course, students were given a general directive to contemplate, extend, explore and/or push back on ideas raised by peers or in the readings to mimic a conversation that would occur in a face-to-face learning environment. Each reading also had a specific prompt. Readings from the textbook tended to have short instructional prompts for discussion, since this was already built into the text's readings. Other readings tended to have slightly more explicit directions on how to start the discussion but were rather vague.

The course instructors for the two sections had co-planned the course and implemented almost identical instructional interaction on the forums. Each section instructor responded to every participant and on every thread at least once in the first discussion forum, to about a third of the threads for the second forum, about a fourth of the threads for the third forum, and then only to a few (zero to three) posts on all subsequent forums. For each forum, both instructors would summarize the discussion and provide additional resources via an email sent to the students after the discussion forum closed. The instructors purposefully refrained from always responding to posts and would share information as needed via a separate e-mail communication. To conclude, by design each instructor was a heavy poster at the beginning of the course as students introduced themselves but quickly weaned away from participating in the discussion forums intentionally to ensure that student-student relationships were being built.

Social network analysis (Borgatti, Everett, & Johnson, 2013; Scott, 2000) via UCINET was used to examine person-to-person interactions in each forum at both the individual and group levels. The following common indices from social network analysis were used to measure social presence at the individual level: *in-degree centrality* or IDC (the number of responses a participant receives), *out-degree centrality* or ODC (the number of responses a participant sends), and *betweeness* or BTW (a measure of the degree to which a participant serves as a bridge connecting others in discussion). Each index correlates to one of the three categories of social presence identified above. IDC relates to interpersonal communication and is an indicator of a participant's prominence in the forum. ODC represents open communication and speaks to the influence a participant has on the discussion. BTW is indicative of cohesive communication and signifies the control a participant holds over others and their connections (Scott, 2000).

Three additional indices were used to explore interactions at the group level: *average degree* (mean number of connections, both IDC and ODC, per participant per forum), *density* (extent to which all possible connections between each participant are present), and *connectedness* (percentage of pairs of participants who were linked in some way across the network of possible connections). These help to assess the quality of online discussions (Jo, Park, & He, 2017), as they speak to the volume of communication and the diversity of connections within a forum.

Results & Discussion

Teaching presence is evident in the structure of discussion board activities, which varied across the 13 forums. First, some forums were required and some were optional. Next, for many of the forums (six optional and two required) it was not mandated that each participant initiate a thread. On those forums, a student could opt to forgo drafting an initial post and instead make all posts as responses to others. On other forums students were expected to initiate a thread as well as craft varying amounts of response posts to their peers.

The distribution of prominence (IDC), influence (ODC), and control (BTW) within a forum tends to be determined by whether or not initiating a thread is required for that forum. In Figures 1a and 1b below, the sociograms for two forums from section one are shown. Forum 11 was a forum in which students did not have to initiate a thread. The discussion centered on a single student participant, as was the trend for forums in this category; one or two participants held most of the power. In contrast, the sociogram of Forum 10 depicts a discussion in which power was dispersed across four participants. The distribution of power in Forum 10 is representative of others where initiating a thread was mandatory. This implies that discussions are more balanced in forums in which all participants devote at least one of their posts to initiating a thread; otherwise, the discussion tends to be dominated by one or two participants.

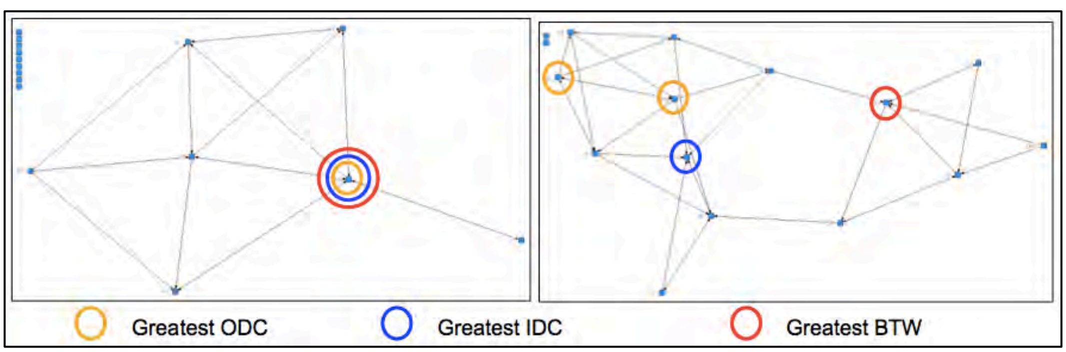

Figure 1a: Sociogram of Forum 11. **Figure 1b:** Sociogram of Forum 10.

Average degree, density, and connectedness are indices concerned with a network as a whole (Jo, Park, & He, 2017). Values for these indices were calculated for each forum and compared based on variations in teaching presence. Whether participation in a forum was required or optional was the first consideration. Independent *t*-tests found that the mean for the 14 required forums was significantly higher than the mean for the 12 optional forums across all three indices (average degree: $t_{13.7} = 2.338, p = 0.035$; density: $t_{13.0} = 2.510, p = 0.026$; connectedness: $t_{16.4} = 2.161, p = 0.046$). Therefore, required forums generated more posts and were more complete than optional forums and participants in required forums were more connected to each other than in optional forums. This is not surprising.

Variations in the minimum number of posts required for a forum did not impact social presence or the quality of discussion. The forums depicted in Figures 1a and 1b above each required participants to make at least four posts, but the distribution of prominence, influence, and control varied greatly. There were forums with fewer required posts that matched both types of distributions. Consider two forums from section 2: Forum 8 had a higher volume of communication and a greater diversity of connections than Forum 9 (average degree: 4.45 vs. 2.25; density: 0.234 vs. 0.118; connectedness: 0.671 vs. 0.289), although Forum 8 only required three posts per participant as opposed to the four required for Forum 9. This shows that, in terms of teaching presence, factors beyond a required minimum amount of posts lend to the development of social presence and quality of discussion.

Conclusions and Future Research

Our work has revealed that variations in teaching presence can impact both social presence and the overall effectiveness of asynchronous online discussions within a COI. When the design of a discussion forum requires participants to initiate their own thread, the discussion tends to be more balanced and is not dominated by one particular student. This helps create a more collaborative environment in which social presence can develop. In settings in which both required and optional forums are present, required forums have both higher volumes of communication and a greater diversity of connections. Requiring a minimum number of posts per participant, another facet of teaching presence, did not have an impact on social presence.

Moving forward, it will be useful to examine the impact other facets of teaching presence (i.e., topic and timing of posts) have on asynchronous online discussion. An examination of individual participants' habits in online discussion forums could also help to explain variations in social presence. This information will be crucial for practitioners in every field of higher education who want to ensure knowledge construction, critical thinking, and problem solving in their students as they transition into online learning spaces.

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VISUAL-SPATIAL REASONING, DESIGN ENGINEERING, AND 3D PRINTING

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Design engineering is potentially a very useful way of developing visual-spatial reasoning. In this research, seventh grade students designed and constructed cube puzzles and then printed these models using a 3D printer. The task provided for many opportunities for students to both develop and use their visual-spatial reasoning. Most challenging for the students was using the technology to visualize mental rotations and, in this respect, the physical model was essential to the task. Also essential to the task was the inclusion of the 3D printer which inspired significant engagement and also further learning. Recommendations for future directions for research are provided.

Keywords: Geometry and Geometrical and Spatial Thinking, Elementary School Education

Introduction

Visual-spatial reasoning has been identified as being essential to many STEM-careers (Wai, Lubinski, & Benbow, 2009). Moreover, there are numerous studies that report both a link between visual-spatial reasoning to overall mathematical performance and the predictive capacity of it to future mathematical achievement (Frick, Möhring, & Newcombe, 2015). In this paper, we explore pedagogy aimed at providing opportunities to support the development of visual-spatial reasoning through a design engineering task that involved 3D printing.

Design engineering, as used in this context, refers to "any engagement in a systematic practice of design to achieve solutions" (National Research Council, 2012, p. 11). Design engineering tasks that make use of 3D printing have been found to increase STEM engagement (Buehler, Comrie, Hofmann, McDonald, & Hurst, 2016; Wendt & Wendt, 2015). In this research, students were challenged to design a three-dimensional interlocking cube puzzle (see Figure 1). The cube puzzle was to include at least five different interlocking puzzle segments. Students first produced a physical model and then developed the renderings (digital plans) using Tinkercad (https://www.tinkercad.com/) to execute the 3D print (see Figure 2). The mathematical strands explored were geometry and spatial sense and measurement.

Figure 1. Cube puzzle.

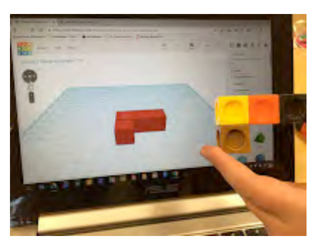

Figure 2. Tinkercad.

The overarching aim of integrating 3D printing in this classroom stems from a commitment to provide design-based opportunities for students to develop their visual-spatial reasoning whilst exploring the pedagogical affordances of virtual 3D learning environments. This approach encourages "a more visual endeavour and connects with what 'real' mathematicians do when they are exploring patterns in the world and making discoveries"(Ontario Ministry of Education, 2014, p. 3). We sought to answer the following question: What are the challenges and benefits associated with attempting to develop visual-spatial reasoning through a design engineering task which also uses 3D printing? There is a dearth of research on the affordances of 3D printing in mathematics learning; this is a contribution of this research.

Theoretical Framework

This research draws on the framework for visual-spatial reasoning proposed by (Uttal et al., 2013). According to Uttal and colleagues, there are four dimensions to visual-spatial reasoning and these are intrinsic (object specific), extrinsic (relation of an object to a frame of reference), static (object is stationary), and dynamic (object is moving). Building 3D models involves both mental rotations and the relation of objects to a frame of reference (or other aspects of the model). The interlocking 3D puzzle inspiring this project is an example of both intrinsic-dynamic and both extrinsic-static visual-spatial reasoning. Mental rotation is involved in the construction but then to rebuild the puzzle, pieces are rotated in reference to one stationary component of the puzzle. Mental rotation is proposed to be an important indicator of visual-spatial ability, with males tending to outperform females (Jordan, Wüstenberg, Heinze, Peters, & Jäncke, 2002; Thompson, Nuerk, Moeller, & Cohen Kadosh, 2013).

Block and puzzle play are good examples of both intrinsic-dynamic and extrinsic-static visual-spatial reasoning. The majority of the available research on block and puzzle play considers young children. Block play and talk about blocks, particularly in young children, have been identified as an important visual-spatial activity leading to more advanced visual-spatial ability (Dearing et al., 2012; Kersh, Casey, & Young, 2008). Puzzle play has also been found to be very important in building visual-spatial ability (Cannon, Levine, & Huttenlocher, 2007; Levine, Ratliff, Huttenlocher, & Cannon, 2012).

Methods

Participants

This research took place in an interdisciplinary seventh grade class in a large urban setting. Participants included the classroom teacher (lead author) and 22 students (14 females, 8 males). The 3D printer was purchased for the classroom (by the second author) and was present in the classroom from the onset of the school year. The students and the classroom teacher did not have any direct prior exposure to 3D printing. The research was part of a broader study exploring the development of visual-spatial reasoning in children.

Data Sources and Procedures

The first six weeks of school centered on learning how to use the 3D printer and the related software. To construct the cube puzzle, first students individually created a physical model using interlocking 2 cm by 2 cm by 2 cm linking cubes (see Figure 2). Then, working in small groups, students selected one of the models created by one of the group members to print a scaled down version of that model. To increase the cognitive challenge of the task, the side lengths of the individual cubes in the scaled version could either be 1 cm or 1.5 cm. In pairs, students within the group developed the renderings to build their segment of the model. The final step involved joining all the segments to recreate a scaled version of the physical cube model. Data sources included teacher generated reflections as well as reflections captured in student blogs.

Results and Discussion

All the groups were successful in printing their scaled model although some of the models did not fit perfectly together. There were four main findings. First, developing the renderings proved to be the most challenging for students, particularly engaging in visual rotations to make the pieces fit correctly. An understanding of geometry was central to the process, as also reported by others using

3D printing (Buehler et al., 2016). This is reflected in this student blog:

> Our piece we had to design was one that had 5 cubes on the bottom and 1 cube on top. It was very easy to construct the bottom but it took a little more trial and error to do the one piece on the top because we had to figure out how to put the cube in the air and make sure it wasn't floating.

Students had challenges visualizing how cubes could be stacked and then in the stacking of the cubes recognized the need for precision. Students problem-solved by duplicating cubes and by making better use of the visualization capabilities of the software which occurred as a result of questions emerging from manipulations with the physical model. For example, students would rotate their cube segments 90° in their hands and explore the tools necessary to make the same rotation on the screen so that the image and the model matched (see Figure 2).

Consequently, and second, the physical model played a very important role in developing the renderings and assisting with the visualization. Building models has been found to be helpful in supporting learning (Lazarowitz & Naim, 2013), and this was true in our study too. The model was a true partner to the software and allowed the students to calibrate the rotations that were occurring with the models in their hands with the software. Knowledge of transformational geometry also proved to be very instrumental in that students relied on concepts such as flips, rotations, and translations to position the virtual blocks of their cube segments.

Third, the affordances of the 3D printer posed additional visual-spatial challenges in that mental rotations were again required by students from those used in the renderings and in the model for a more efficient print (e.g., laying the piece flat rather than standing as it may have been in the actual puzzle). Finally, scaling proved to be equally challenging. Measurements needed to be precise and this was made abundantly evident through the rendering process – particularly for those students that opted to maximize the volume of their object versus those that chose a more efficient unit of measurement (i.e., side lengths of 1 cm in length). We see in this quote from a student's blogs that the tensions associated with choosing the best measurements was present but the optimal options were not perhaps initially:

> Decided on creating a rather small puzzle cube (10 mm by 10 mm by 10 mm) unlike some other groups who did 15 mm by 15 mm by 15 mm and I am excited to see if it works and if not then to see what went wrong. Also I wonder which measurement is more appropriate and whether the larger one will be much easier to use.

Conclusions

This design engineering task provided students with opportunities to develop visual-spatial reasoning – particularly around mental rotations. The need for physical models was an important component of the task as was the 3D printing. The geometry and the measurement required for the task was a challenge in that a fair amount of geometric knowledge was necessary to do the task but this can be learned while engaged in the design process. The possible affordances associated with the use of 3D printers and visual-spatial learning cannot be under estimated. This field of inquiry is truly at its infancy – particularly studies where students are responsible for the printing rather than teachers (cf. Wendt & Wendt, 2015). In our own jurisdiction and likely across others, teachers can be found who are using 3D printers in their classes to inspire learning and numerous blogs are available from teachers using this technology; however, very little systematic research exists and more understanding about effective pedagogy is needed to expand use to the potentially more reluctant or novice teacher. The 3D printing component and the required rendering was crucial for the design engineering process in this project. It allowed for a robust exploration of different types of visual-spatial reasoning, as defined by Uttal et al. (2013). Like others have reported, the entire design engineering process was made more impactful because of the 3D printing (Murray, 2013). Our view is that this project would not have inspired as many opportunities for visual-spatial reasoning if the end-game did not include 3D printing. Knowing how teachers can include design engineering with 3D printing in their classrooms will be an important future direction for research. Understanding the cognitive affordances of this approach for visual spatial reasoning is also a necessary future direction for research.

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CAN I MEASURE THAT WITH MY PHONE? MOBILE MEASUREMENT APPS FOR LONG LENGTHS

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This study investigates how three 4th-grade children apply their prior knowledge of measurement when using mobile measurement apps to measure real objects that are difficult to estimate. We show mobile measurement apps as a potential learning tool to explore concepts of length and measurement and further improve children's estimation of length measurement.

Traditional mathematics instruction in measurement usually emphasizes "the procedures of measuring rather than the concepts underlying them" (Stephan & Clement, p.3, 2003). This procedural focus does little to help children conceptualize length and measurement, since numerous studies have found that focusing on concepts rather than procedure connects much better with the way children think about measurement (e.g., Clements & Sarama, 2009). Yet, despite this research, the teaching and learning of length measurement and estimation continues to be problematic and procedural (Joram, Subrahmanyam, and Gelman, 1998). Therefore, we explored innovative ways for children to conceptualize length measurement and estimation beyond learning measurement procedures, utilizing mobile app technology as a tool for interacting with and measuring the real world.

The four sessions in this study involved three 4th-graders exploring mobile apps in a teaching experiment facilitated by the first author, a 10-year veteran elementary mathematics and science teacher. The three participants were enrolled in a gifted program in their elementary school and historically performed well in mathematics. This mathematical background meant they had prior knowledge or at least exposure to concepts involving length and measurement. In the first two sessions, the children explored length and measurement without the use of mobile technology. In the last two sessions, the children explored more challenging length and measurement tasks with access to three mobile measurement apps. While the first three sessions lasted 30 minutes, the fourth session which focused heavily on children's individual exploration, lasted an hour.

Our results show that the mobile apps allowed children to measure objects they previously had difficulty estimating. The children also exhibited their conceptual knowledge and misunderstandings of length and measurement during the tasks. This exploratory study leads us towards larger research investigating the specific use of mobile apps in teaching and learning measurement.

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EXAMINING HOW GRADUATE STUDENTS WITH DIFFERENT SCHOOL MATH EXPERIENCES INTERACT WITH *ROLLY'S ADVENTURE*

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Keywords: Technology, Number Concepts and Operations, Cognition

In the most recent National Educational Technology Plan, the U.S. Department of Education (2017) encourages incorporating games in education and more educational research with digital games. In this preliminary study, I investigated how graduate students with different views of math and school math learning experiences interacted with *Rolly's Adventure*, a videogame designed to support fractional knowledge and reversible multiplicative reasoning (Williams-Pierce, 2016). The study adopts von Glaserfeld's (1995) constructivist view that learning is the process of constructing and adapting one's mental models or schemes while interacting with one's environment. Thus, knowledge is situated in the activity and the context in which learning takes place (Brown, Collins & Duguid, 1989). Meanwhile, learning involves grasping specific subject matter as well as developing attitudes, dispositions and understanding of cultural practices (Greeno, 2006).

As a multiple-case study, there were three graduate student participants. Data sources included pre- and post-game interviews and game playing (audio and screen recordings). First I open-coded data with content analysis (Hsieh & Shannon, 2005). After organizing these codes based on constructs from previous literature, I coordinated and refined codes across three cases to make clearer comparisons.

This study's main findings are as follows. First, each participant's view of math and school math experiences corresponded to his/her gameplay patterns (e.g., performance- vs. understanding-oriented). Second, participants with advanced fractional understanding and adequate reversible multiplicative reasoning (RMR, Hackenberg, 2010) developed more sophisticated ideas about the fractional problems in *Rolly's Adventure*. More interestingly, Jamie, a low math performer at school, demonstrated insufficient RMR when solving a word problem but successfully made sense of and solved the problems in the videogame. The findings imply that *Rolly's Adventure* has potential in supporting learners to reason about reversible multiplicative relationships and to demonstrate RMR in action.

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ITERATIVE (RE)VISIONING: AN IMPROVEMENT SCIENCE APPROACH TO ONLINE PROFESSIONAL DEVELOPMENT DESIGN AND IMPLEMENTATION

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Introduction

The proposed poster presentation is situated within a larger project, a four-year NSF/DRK-12 research grant focused on developing middle-school mathematics teachers' formative assessment strategies. A central component of the project involves design and delivery of a multi-year professional development (PD) program for participating teachers. In Year 3 of the project, an online platform was developed to provide a flexible, accessible environment for delivery of synchronous PD and formative assessment resources. Improvement Science was adopted as a framework to guide implementation, testing, and revision of the platform to address the emerging needs of participating teachers as well as the broader research goals of the project. This poster presents a summary key design and implementation issues addressed.

Design Framework

Improvement Science (IS) produces knowledge that informs the use of practice in real, practical, and varied settings (Bryk et. al., 2016). It utilizes tools of disciplined inquiry that are user- and implementation- focused. Most importantly, it provides a nimble cycle of design-implement-test-revise that privileges the realities of education settings. Several main principles drive IS: the first set focuses on problem definition, analysis, and specification; the next set focuses on iterative prototyping and testing; the last principle focuses on organizing networks to drive sustainability.

Results

Challenges associated with implementation of the platform design included teacher responsiveness and attrition, logistical issues, and interface interactions in the online environment. Specifically, Table 1 presents a few of the curricular, pedagogical, and technological problems we will unpack along with the specific prototypes developed, evidence collected, and revisions implemented to address them. We also discuss lessons learned using an IS approach to design and the overall impact of the approach on outcomes related to teachers' formative assessment practices.

Table 1: Problems, Data, and Adjustments of the Online PD

Issue	Evidence	Revision
Reflection fatigue	Contribution patterns	Reduce session length
Reflection takes longer than anticipated	Unable to complete full cycle in time allotted	Assign as homework; re-tool as multi-year cycles of PD
Responsive and concurrent display of teacher comments	Teacher feedback; Facilitator feedback	Streamlining the submission and display process
Inequitable participation and communication	Teacher feedback	Scaffold online interactions with in-person opportunities

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A MULTIDISCIPLINARY TEAM'S EMERGING VIEWS OF MATHEMATICS LEARNING: DEVELOPING A DIGITAL MATHEMATICS GAME

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Theoretical Perspective

While existing research has identified different approaches to the development of digital educational games for mathematics teaching and learning, still unexplored are the ways in which design teams' priorities and views of learning influence the design process. As with the development of curricula, digital games involve constructing opportunities to learn based on the views of mathematics learning of the designers. Biddlecomb (1994) described the central importance of views of mathematics learning and game operations as influenced by existing models of children's mathematics. In our research, we draw from Biddlecomb's view that children's ways of operating must inform the operations allowed in the game environment to argue that design teams should begin by identifying views of learning and development.

Research Questions and Design

Using grounded theory (Strauss & Corbin, 1994) to investigate archived data from the project, we asked: How does a multidisciplinary research team with different epistemologies work together to design a digital educational game/program to support children's mathematics learning? Specifically, how do team-members' ways of knowing and views of mathematics emerge in their work and affect the final product?

Data Collection Techniques and Analysis

We used constant comparative analysis (Strauss & Corbin, 1994) to analyze team meeting notes with initial categories of aesthetic, children's mathematical thinking, and technology. Our disciplines included technology, programming, graphic design, special education, mathematics education, early childhood education, and English education.

Summary of Findings

Preliminary findings indicate that, although members of the team seldom made their models of how children learn mathematics explicit, team members did operate with explicit models (Biddlecomb, 1994). As we complete further analysis, we will generate theory on how these explicit models influenced the development of the digital educational game/program.

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MATHEMATICS COMPUTER-ASSISTED INSTRUCTIONS FOR STUDENTS WITH LEARNING DIFFICULTIES: A SYSTEMIC REVIEW

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Students with or at risk for learning disabilities in mathematics (LDM) experience considerable difficulties learning mathematics. According to Author et al. (2016), the gap between students with disabilities and their same-age peers in mathematics seems becoming wider, rather than closing. To close this gap, computer-assisted instruction (CAI) has been widely used in practice providing additional supports for students with LDM. However, the latest meta-study of Seo and Bryant (2009) demonstrated that CAI did not show conclusive effectiveness on improving the mathematic achievement of students with LDM. Therefore, further investigation into the effects on CAI is necessary. The purpose of this study is to conduct a systemic review and analysis of empirical research where CAI is used to facilitate mathematics learning of elementary and secondary students with LDM.

Method

Research articles published between 2008 and 2016 were identified from electronic databases, major journals, and ancestral searches. We set the inclusion criteria as follows: 1) studies involved students with LDM in elementary or secondary schools, 2) studies focused on CAI as their independent variable, 3) studies considered the mathematics achievement as their dependent variable, 4) studies applied group comparison or single-case design, 5) peer-reviewed studies conducted in the US. Giving above selection criteria, we identified 13 articles. Following variables were coded: 1) participants, 2) research design, 3) setting and duration, 4) intervention agent, 5) the nature of the CAI program, 6) the nature of tasks, 7) instructional strategies, 8) software developer, 9) targeted math concept/skills, 9) dependent measure used, 10) social validity.

Results and Discussion

In the 12 studies, CAIs were implemented for an average 16.9 sessions with a total of 698 of students. The majority (67%) of the participants were at risk for learning disability. Five studies implemented the researcher-developed CAI and seven studies used commercial software. Six of the CAIs were designed on the basis of learning theories. The majority of the studies targeted basic operation and problem-solving skill. The results of these studies indicated that CAI was effective with varying degree in promoting mathematics performance for students with LDM. In addition, the results from group comparison studies indicate that there was no statistically significant difference between outcomes of CAI and teacher-delivered instruction. Overall findings from this systemic review suggest that CAI is promising in improving the mathematics performance of students with LDM. For future research and practice, it is important to design and implement CAI programs that taking into consideration of the characteristics of students with LDM.

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DEVELOPING DIGITAL INSCRIPTIONAL RESOURCES: CONNECTING DESIGN, CLASSROOM ENACTMENT, AND STUDENT THINKING

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Representations often refer to the information in material form on paper or the computer screen such as written text, graphical displays, tables, equations, diagrams, maps, and charts. Representation can also refer to a learner's internal thought. In our work, we use the term *inscription* as "a potential change in route" where meanings of external representations of thinking are developed in social settings. Research shows that student capacities to represent knowledge and make sense of their thinking is efficiently developed in social settings where meanings are publicly shared and negotiated among students (e.g. Medina & Suthers, 2013).

As part of a four-year project we are developing and studying digital inscriptional resources that will allow students new ways to work collaboratively to develop mathematical thinking. Building on the intersection of the literature in mathematics educational technology (e.g. calculators, dynamic geometry software, computer algebra systems) and information communication technology (e.g. collaborative whiteboards, real-time editing and sharing), the digital inscriptional resources support students to collaboratively create representations of their mathematics thinking, incorporate ideas from other students, and share their work with the class.

Our research is guided by the following questions: (1) What features of the digital learning environment help students to produce and refine inscriptions of their thinking as they explore mathematics problems? and (2) How does the construction, manipulation, and interpretation of inscriptions change over time? An iterative design research process incorporates multiple phases of development, testing and revision, to study student use of the digital learning space and related inscriptional resources. Data includes classroom observations and artifacts, student and teacher interviews and surveys, student assessment data, and analytics from the digital resources. The project team embraces a goal of working at the crossroads of research and practice, progressively testing and refining theories over multiple phases of design and enactment.

Our findings will inform the work in collaborative mathematics education technology by connecting design principles, classroom enactment of the developed resources, and student outcomes. The project will report on how evidence of student thinking is made visible through the use of digital inscriptional resources and the ways student inscriptions are documented, discussed, and manipulated in collaborative settings.

Acknowledgements

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USING ELECTRONIC JOURNALS WITH PRESERVICE TEACHERS TO INFORM MATHEMATICS TEACHER EDUCATOR INSTRUCTIONAL DECISIONS

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Keywords: Teacher Education-Preservice, Technology, Instructional Activities and Practices

The benefits of using journal writing to both teachers and students in mathematics class are numerous. For teachers, they can use journals to assess their students' progress in a course. They can also allow teachers to reflect upon and improve their teaching as well as provide better individualized instruction for their students (Borasi & Rose, 1989). However, providing feedback on journals can be a very time consuming process for teachers (Baxter et al., 2005). Thus, teachers need to carefully determine how often to collect the journals and how quickly they can feasibly provide helpful feedback. New technology is emerging that has the potential to address these issues with journal writing in mathematics. In particular, electronic journals where students can write by hand and upload their journal pages to the cloud (e.g. Google Drive) are making their way to the marketplace.

Purpose and Methods

While there have been many studies on the impact of journal writing in mathematics classrooms, there are few, if any, studies that explore the use of electronic journals with preservice teachers. The goal of this exploratory project is to determine the impact on a mathematics teacher educator's ability to adjust her instruction to fit the needs of her students as a result of better and more consistent access to her students' journals. Each student enrolled in a mathematical problem solving course for elementary preservice teachers received an electronic journal. They were required to use this journal every class day throughout one semester to record their mathematical thinking as they worked on high-level mathematics tasks. At the end of the semester, the preservice teachers also completed an open-ended survey regarding their experience using these journals in the course. Additionally, the instructor of the course kept a journal to reflect on the benefits and challenges of using these journals. Qualitative study methodology was employed for this study. Data from all data sources were analyzed using open coding to look for patterns and themes in the responses.

Results and Discussion

The results of this study suggest that although it was initially a struggle to find a productive method of providing feedback to the preservice teachers on their electronic journals, there were benefits to having students use them. For instance, the instructor was able to easily access the students' journal entries when planning for whole class discussions on the in-progress problems they were trying to solve. The results also suggest that there may need to be more accountability measures to ensure students read the instructor feedback. More research needs to be done regarding how electronic journals can be used to the benefit of teachers' instructional decisions.

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ATTITUDES OF THE TEACHER TO THE ADOPTION OF ACTIVITIES THAT INCLUDE MODELING, GRAPHICS AND TECHNOLOGY

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The training courses for math teachers have the purpose of providing them with knowledge that translates into changes in the classroom to improve the learning of their students. However, changes in how teachers organize their students' learning do not occur immediately. The complexity of how the diverse types of teacher knowledge are interconnected (Pena-Morales & Pelton, 2016) explains why a new knowledge requires time and intentionality to insert it into the educational system. To the extent that the knowledge required by the teacher involves several elements there are more adaptations in the organization of the learning activities that the teacher should perform. In this report, we describe the transformations that teachers introduce in a kind of mathematical tasks that involve the work of modeling with technology. Training courses for mathematics teachers should involve a more active participation, courses with technology should not include mainly technical questions of the use of hardware, software or applications, but in relation to the potential of mathematical learning that can be get. For the purposes of this research, a teacher training workshop was designed with the knowledge of modeling integration, graphing and technology (Suárez, 2014), consisting of four elements: 1) reflection, 2) mathematical work, 3) use of technology and 4) creating new learning situations. The question of research is: Which of these four elements the teacher adopts better? The workshop was designed with the purpose of working mathematical situations of the movement, analyzed with graphics and technological tools to generate a teacher reflection on the nature of school modeling in high school.

The workshop had four moments: (1) teachers reflected on the modeling and use of technology; (2) in a second moment they were presented with a mathematical modeling activity that required a graphical (pencil and paper) response, 3) the third time integrated modeling, plotting and technology, a tutorial Tracker was presented as an aid to generate and analyze graphs of motion and 4) were asked that they propose a new mathematical task capable of integrating learning experience, with the second and third moments. Among the main conclusions is seen that teachers have an open, reasoned and immediate attitude to reflect about the integration, however, in the sections of real integration in the resolution of tasks with technology and implementation of new learning situations, participation is lower and longer response times.

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FROM PLAYERS TO CREATORS: TEACHING COMPUTATIONAL THINKING THROUGH PLAYING AND CREATING EMBODIED MATH GAMES

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With the growing number of jobs within STEM fields that require technology and abstract thinking, it is beneficial for mathematics and CS education to consider computational thinking as a basic and necessary skill. Computational thinking, a concept typically reserved for undergraduates in CS, involves an iterative design and problem solving process that engages learners in multiple levels of abstraction (Guzdial, 2008; Wing, 2006). Finite-State-Machines (FSM) are a relatively simple model that can be used to represent and specify technology programs by breaking down computer behavior into a series of states, namely actions and reactions. Our prior research in embodied cognition and mathematics education has shown that using games that use FSM and wearable technology in the classroom can improve performance and motivation for math (Arroyo et. al, 2011; Arroyo et. al, 2016). Teaching students about FSM within a game-based mathematics activity may encourage students to move past surface-level features of technology, engage in a higher level of abstraction, and be a pedagogically-sound method to promote computational thinking and higher-level mathematics.

In this study, we explore whether the act of playing and creating games, explained through a FSM framework, could serve as a vehicle to teach computational and mathematical thinking. In the play activity, students played Estimate It!, an active math game that uses cell phones to set up a scavenger hunt for students to practice estimating the size of objects. In the create activity, students created multi-player math games for 4th-6th grade-level students that involved physical activity. In the adapt activity, students received an introductory lesson on the state-based technology behind Estimate It!, and then were asked to redesign their games to fit a FSM framework and to draw a state-based diagram. We were interested in the ordering effects of whether students played or created the games first. In this poster, we will present examples of the students' games, describe how order affected the nature of discourse and representations of the students' games, and explain how this FSM and games method provides insight into methods for teaching computational thinking in the mathematics classroom.

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TEACHERS' FACILITATION OF PLAY WITH PHET INTERACTIVE SIMULATIONS IN MIDDLE-SCHOOL MATHEMATICS LESSONS

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U.S. mathematics teachers face pressures to keep up with pacing guides and prepare students for standardized tests (Roach, 2014). At the same time, they are called upon to engage students in innovative exploratory activities that incorporate new technologies (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2014). Amidst these competing priorities, we investigated how mathematics teachers facilitated play in lessons with interactive simulations (sims).

We worked with 4 middle-school mathematics teachers at a public school in the Southeastern United States. Each teacher had at least 7 years of experience but was new to using PhET sims (phet.colorado.edu). Previous research found advantages to including an initial play period in sim-based lessons, during which students explore a sim prior to more structured activities (Podolefsky, Rehn, & Perkins, 2013; Moore, Herzog, & Perkins, 2013). Play offers students time to freely explore the sim, manipulate controls, ask questions, discover relationships, and generate interest in the topic (Podolefsky et al., 2013). Given this recommendation, the teachers planned and taught lessons that included play prior to more structured activities.

We conceptualize play in sim lessons in terms of Sicart's (2014) ecological theory of play—as contextual, carnivalesque, appropriative, disruptive, creative, personal, and autotelic. Some of these characteristics are conducive to learning in a classroom environment, whereas others may not be. Our analysis of 15 mathematics lessons involving play led to the identification of 4 characteristics that distinguish the play phases of these lessons. Based on combinations of these characteristics, we identified 3 profiles of play, which lie at different points along a continuum of priorities from foregrounding students' ideas to keeping pace. In the profile *A Disciplined March*, play serves the purpose of familiarizing students with the sim. In the profile *Wandering Exploration*, play serves the additional purpose that students make and share discoveries. In *Hiking with a Guide*, students make and share relevant discoveries. We discuss the implications associated with each profile of the play phase, and we begin to develop a theory that frames teaching with play as a matter of balancing divergent and convergent modes of activity.

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USING DYNAMIC TOYS TO EXPLORE CONTINUOUS THINKING IN PROPORTIONAL SITUATIONS

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Research has shown that teachers and students struggle to conceive of covariational relationships in continuous ways (e.g., Thompson & Carlson, 2017). That is, many people conceive of the relationship in chunks, wherein a relationship is conceived of as a set of instances in time that can be, for example, plotted on a coordinate plane then connected together by drawing a line between them. While this is certainly a practical way to construct graphs, it becomes problematic when trying to understand continuous variation and other concepts relevant in higher-level mathematics.

In our own prior work on proportional reasoning, we tried to address this "chunky" thinking (Thompson & Carlson, 2017) through the use of sketches made in Geometers' SketchPad that allowed the user to perceive the dilation of sketches of animals as continuous movement. In this environment, we asked practicing middle grades teachers whether the relationship between two sketches is always there or whether it is only there when the slider stops and the images are a fixed size. We noted that some teachers believed that two images could only be similar when they were both frozen in time and space whereas other teachers understood the continuous nature of the relationship.

Building from these foundations, we are now developing a virtual "toy box" that is intended to engage middle school teachers in examining the continuity of proportional relationships by engaging them with toys that allow them to play (Bruner, 1972) with the ideas of continuity in proportional situations. For example, one of our toys examines the growth patterns of an alien plant designed to collect and hold water while another toy is a picture dragger that allows resizing of images in both similar and nonsimilar ways. These tools move participants toward continuous thinking when combined with tasks that challenge teachers to think about whether situations stay similar/proportional as they change or only at particular points.

In this poster, we will bring examples of our toys (both images and an iPad with the toys actively running) and the ways in which they promote continuous reasoning in proportional situations. We will explain our goals for this research and projected outcomes from our pilot work to date.

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VIRTUAL REALITY IN THE MATHEMATICS EDUCATION OF CHILDREN WITH CEREBRAL PALSY

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Keywords: Technology, Equity and Diversity, Measurement

Affecting 1 in every 323 children, cerebral palsy (CP) is the most common motor disability in the world (Christensen et al, 2014). Few studies have regarded the mathematical learning of children with CP. These studies have uncovered severe disparities in students' problem-solving skills when compared to typically developing children (Jenks et al, 2012). Measurement is a special challenge for children with CP due to the traditional use of manipulatives for tactile reasoning. This study investigates: What effects do immersive virtual reality have on mathematical learning, physiological health, and psychosocial well-being for children with CP? What kinds of unit schemes and reasoning do children exhibit as they encounter measuring tasks in an immersive virtual reality setting? How do children's strategies compare to paper/pencil tasks and known unit measurement trajectories?

Nickels and Cullen's (2017) analytical model of Brousseau's (1997) Theory of Didactical Situations of Mathematics and Sarama & Clement's measurement trajectories (2009) are used as theoretical frames for the design and analysis of the study. We report on a single case study participating in six weeks of task based interviews utilizing previously validated unit measurement tasks (e.g., Barrett et al., 2011). Data collected included: (1) baseline measurement knowledge; (2) VR use and measurement knowledge; and (3) socio-emotional and physiological well-being. As this is an ongoing study, results are preliminary at this time. A positive correlation was found with the use of VR with the child's socio-emotional and physiological well-being. Data regarding the student's measurement content knowledge after using the VR system is still being analyzed.

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Chapter 13

Theory and Research Methods

Research Reports

Brief Research Reports

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Posters

ANALYZING CLAIMS ABOUT COGNITIVE DEMAND AND STUDENT LEARNING

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A commonly accepted claim in mathematics education is that there is a relationship between the cognitive demand of mathematical task enactments and students' subsequent learning. One study often cited to support this claim is Stein and Lane (1996), and in 44% of those citations, Stein and Lane (1996) is the sole reference provided. Citation analysis reveals that many of these claims go beyond the warrants provided by the Stein and Lane study, either by granting more confidence in the relationship than the study design allows or by phrasing the claim causally. A few other studies are occasionally cited in conjunction with Stein and Lane (1996), but there remains a need for replication studies to provide better empirical support for claims about cognitive demand and student learning and to refine our shared understanding.

Keywords: Cognition, Instructional Activities and Practices, Research Methods

Replication studies are rarer in education research than in other fields, leading Makel and Plucker (2014) to call for more replications because such studies can both identify and remedy methodological biases and can be instrumental in buttressing robust findings or clarifying inconsistent findings. In these ways, replications can play a role in the field's systematic accumulation of knowledge (National Research Council, 2002). An area ripe for replication would be a testable, widely-held belief that is resting on a relatively inadequate empirical foundation. One belief seemingly shared by most scholars in mathematics education is that the cognitive demand of mathematical tasks and task enactments (Stein, Grover, & Henningsen, 1996) is important with respect to student learning outcomes. It may be that experiences with cognitively demanding tasks lead to positive learning outcomes or it may be that having students experience cognitively demanding tasks is an end itself. The latter is a philosophical position based on values, whereas the former is a testable position that currently rests on some supporting evidence, but what is the extent of that evidence? In our own past work related to cognitive demand (e.g., de Araujo, 2012; Otten & Soria, 2014), we noticed an extensive literature base on the nature of cognitive demand (Doyle, 1983; Stein, Smith, Henningsen, & Silver, 2009) and factors influencing cognitive demand throughout mathematical task implementations (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Wilhelm, 2014), but a weaker empirical foundation for the direct link between cognitive demand and student learning.

The hypothesis motivating this study was that a single reference—Stein and Lane (1996)—constituted a large portion of the warrants for claims in the mathematics education literature about the link between cognitive demand and learning. If this hypothesis were true, then it would become imperative to critically analyze the research design, evidence, and claims made in Stein and Lane (1996) and to consider possibilities of replication. We examined the claims for which Stein and Lane (1996) was included as a citation and we identified other references, if they existed, that were also cited for those same claims. In the following sections, we briefly summarize the Stein and Lane (1996) study, describe our method for compiling and analyzing citations to Stein and Lane (1996), and then share our key results.

Summary of Stein and Lane (1996)

Stein and Lane (1996) stems from a project well known in mathematics education—Quantitative Understanding: Amplifying Student Achievement and Reasoning (QUASAR). QUASAR involved a university partnership with six urban middle schools with the overall goals of promoting reform-oriented mathematics instruction and investigating the feasibility of such instruction in schools with a history of poor mathematics performance (see Silver & Stein, 1996, for an overview). Within that context, Stein and Lane (1996) sought to “present evidence regarding the degree to which the presence of reform features of instruction are linked to increases in student understanding of mathematics” (p. 51). Their study focused on 4 of the 6 middle schools from the QUASAR project over a three-year period. Data consisted of narrative summary field notes and video recordings of three three-day observation cycles in three teachers’ classrooms in each school each year, a classroom observation instrument completed based on the field notes and video recordings, and Fall and Spring administrations of a project-developed assessment instrument (Lane, 1993). Mathematical tasks were identified and of the 620 main tasks, a stratified random sample of 144 was drawn and the task set-up and task implementation of these 144 tasks were coded for cognitive demand. Levels of cognitive demand were collapsed to 2 (high and low). A 25% sample of the 144 tasks was double coded with 79% agreement.

The assessment instrument consisted of 36 open-ended tasks distributed into four forms (9 questions per form) and a 5-point scoring rubric (0–4) for each task. Analysis focused on 11 of the tasks and used not the scores themselves but “the average percentage of student responses across tasks that were scored at the two most proficient score levels (3 or 4)” (p. 68) and how this average percentage shifted between Fall Year 1 and Spring Year 3.

To generate their findings, Stein and Lane rank ordered the four schools based on their gains in percentage of students at the top two levels of proficiency and then focused on Site A, which had gained the most (36%), and Site D, which had gained the least (17%). They compared these learning gain rankings with the school profile for task enactments and noted the following:

> Site D’s profile can be seen as embodying a more conventional mathematics program in which many or most tasks lent themselves to being solved with a single strategy, using only one representation (usually symbolic), and without much explanation and/or discussion. Site A’s profile, on the other hand, suggests a well-functioning reform program that is successfully utilizing tasks that invite and support students’ use of multiple solution strategies and multiple representations, along with student discussion of their work. (p. 71)

In other words, the tasks in Site D classrooms were often set-up and implemented at low levels of cognitive demand whereas tasks in Site A classrooms were often set-up and implemented at high levels, and Site D had the lowest proficiency gains whereas Site A had the highest.

The fact that the sites conformed to a positive relationship between cognitive demand and gain scores seems, on the surface, to be compelling. The number of sites, however, is quite low for making even correlational claims about the relationship between task features and student achievement. Moreover, the use of only 11 items to measure learning over a three-year period gives pause. In addition, sites were only compared in relation to each other, not to a standardized score, raising questions about the size of the differences between the outcomes of the different sites. The study also did not account for differences among teachers within the same site. The authors acknowledged “the possibility that the findings of this study may partially reflect differences in school-level variables in addition to the documented differences in instructional practices” (p. 75), but in other instances the authors shifted from discussing the results in terms of correlations to suggesting causation. For example, the authors drew the following conclusions:

> [S]tudents appear to **benefit more from** inconsistently implemented tasks that began with the encouragement to use multiple solution strategies, multiple connected representations, and explanations, than they do from tasks that—from the start—required only single solution

strategies, single representations, and little or no mathematical communication (p. 74, emphasis added)

The authors appropriately hedged this statement with the word "appear" but the language in bold suggests not just that Site A used more high demand tasks and performed better, but that the use of high demand tasks was a primary reason for higher student achievement. This claim seems to extend beyond the actual warrants of the study given its limitations. Yet, as we show below, the hedged claims appear to have been accepted widely by several researchers and national organizations alike, frequently in an unhedged fashion. Although Stein and Lane recommended that a replication of their study be carried out, to our knowledge this has not occurred except a substantially modified replication (Otten, 2012) involving 12 middle school classrooms in which cognitive demand was decidedly not correlated with measured student learning.

We wish to emphasize that by pointing out some of the limitations of the study, we do not mean to criticize the study itself but rather to raise caution about the claims that can be made—by Stein, Lane, or others—based on this single study's evidence. These cautions led us to investigate the claims that have been made based on Stein and Lane's (1996) work.

Method

Compiling Citations of Stein and Lane (1996)

Similar to Leatham and Winiecke (2014), we used the "cited by" tool on Google Scholar to obtain a list of articles in which Stein and Lane (1996) was cited. This initial search yielded 298 sources. Because we were interested in how the field of mathematics education specifically draws upon this reference, we retained only those articles that were published in mathematics education journals that received an A-grade in Toerner and Arzarello (2012, December) and also appeared as a top-five journal in the rankings compiled by Nivens and Otten (in press). These journals were *Educational Studies in Mathematics*, *Journal for Research in Mathematics Education*, *Mathematical Thinking and Learning*, *Journal of Mathematical Behavior*, *Journal of Mathematics Teacher Education*, and *ZDM – The International Journal of Mathematics Education*. This constraint yielded 26 articles. We then expanded our search beyond the Google Scholar results to include other forms of codified mathematics education literature—namely, the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007) and two major policy documents from NCTM (2000, 2014). This additional search led to the identification of 4 handbook chapters and one policy document (NCTM, 2014) that cited Stein and Lane (1996). The 31 analyzed resources are marked with * in the reference list.

Analyzing Citations of Stein and Lane (1996)

The goal of our analysis was to understand the claims for which authors cited the Stein and Lane (1996) study. Within the 31 sources, we located the Stein and Lane (1996) citations in text, yielding 60 excerpts. We then used analytic memos to briefly describe the claim that was being supported by the Stein and Lane citation and met to identify the recurring themes and further refine them into codes. We came to a consensus on four codes (see Table 1), which we used to code all but one of the excerpts.

To gain a more nuanced understanding of the ways in which authors used the Stein and Lane reference, we subdivided the learning claim excerpts into two groups based on whether the claim was causal or non-causal. For example, the following excerpt was coded causal because the authors' state that tasks result in an increase in student understanding:

> As the research conducted by the QUASAR project indicates, when teachers choose tasks that require a high-level of cognitive demand, set them up and implement them in ways that maintain

> a high-level of cognitive demand, **the result is** an increase in student understanding and reasoning (Stein & Lane, 1996). (Arbaugh & Brown, 2005, p. 527, emphasis added)

In the same article, we coded an earlier reference as non-causal:

> The relationship between the types of tasks students engage in when learning mathematics and the mathematics they learn has been a subject of research for many years (see, for example, Hiebert & Wearne, 1993; Marx & Walsh, 1988; Stein & Lane, 1996). These research studies indicate that **a relationship exists** between the level of student thinking required by a mathematical task and the nature of students' understanding of mathematics. (Arbaugh & Brown, 2005, p. 505, emphasis added)

Table 1: Coding Scheme for Citations of Stein and Lane (1996)

Code Descriptions	Examples
Claim about a relationship between tasks and student learning [**learning claim**]	"There is evidence that solving a task of high cognitive demand or a cognitively demanding task (CDT) has a positive impact on students' conceptual understanding (Stein & Lane, 1996)." (Wilhelm, 2014, p. 637)
Claim that tasks (or task implementations) and learning have been studied [**study claim**]	"Research has focused on instructor questions across the K–16 spectrum and examined … the value of questions to student learning (Silver, 1996; Stein & Lane, 1996)." (Fukawa-Connelly, 2012, p. 332)
Claim about the levels of cognitive demand (but no connection to student learning) [**cognitive demand claim**]	"[Stein and Lane] use four categories: Memorization, Procedures without Connections, Procedures with Connections, and Doing Mathematics." (White & Mesa, 2014, p. 678)
Claim about a research method used [**method claim**]	"Of particular note are the procedures developed by Stein and Lane (1996) and Stein et al. (1996) for sampling and coding mathematical tasks and linking those findings to student outcomes." (Gearhart et al., 1999, p. 309)

We also distinguished four levels of attribution within the learning claims, from weak to strong—(1) the author(s) state that others have made claims about the relationship but the authors do not necessarily endorse the claims themselves, (2) the authors state the relationship with an explicit hedge (e.g., it is "suggested" or there is "possibly" a relationship) or as being found specifically in the context of the cited study, (3) the relationship is stated as something found in past studies and the authors explicitly or implicitly endorse the findings beyond the cited study's context, and (4) the relationship is stated as a generalized fact. We independently assigned each of the learning claim excerpts a level of attribution and met to discuss any differences until we came to a consensus.

Results

Table 2 contains frequencies for the codes described in Table 1. The most common claim for which Stein and Lane (1996) was used as support was the notion that the cognitive demand of mathematical tasks or task implementations is somehow related to student learning. More than three-quarters (77.4%) of the resources made claims of this sort, encompassing 60% of the total Stein and Lane citations analyzed. Of the 36 learning-claim citations of Stein and Lane (1996), 16 (44.4%) cited only Stein and Lane. This supports our hypothesis that Stein and Lane (1996) often stands alone as the empirical basis for such claims. A substantial minority (25%) of the claims (spanning 41.9% of the resources) did not involve student learning but instead focused solely on the construct of cognitive demand or task implementation. Six claims (10%) simply acknowledged that studies such as Stein and Lane (1996) existed and two (3.3%) referenced the type of study design that Stein and Lane employed. Overall, the citations of Stein and Lane (1996) predominantly involved the relationship between cognitive demand and student learning.

Table 2: Frequencies of Claims for which Stein and Lane (1996) was Cited

Code	Number of Excerpts	Number of Sources
Learning claims	36 (60.0%)	24 (77.4%)
Study claims	6 (10.0%)	6 (19.4%)
Cognitive demand claims	15 (25.0%)	13 (41.9%)
Method claims	2 (3.3%)	1 (3.2%)
Other	1 (1.7%)	1 (3.2%)

Of the 36 learning claims, 25 (69.4%) were correlational claims and 11 (30.6%) were causal. Although correlational claims are more justified than causal claims, few alluded to limitations of the Stein and Lane (1996) study. Furthermore, we found 11 citations that made causal claims. For example, Wilhelm (2014) stated, "There is evidence that solving a task of high cognitive demand or a cognitively demanding task (CDT) **has a positive impact** on students' conceptual understanding (Stein & Lane, 1996)" (p. 639, emphasis added).

Returning to the 36 learning claims overall, Table 3 summarizes their attribution level. More than half (58.3%) of the claims were Level 2—that is, rephrasings of Stein and Lane (1996) or hedged claims that stated what Stein and Lane (and possibly others) had found, without implying that it was a generalized result. For example, Otten and Soria (2014) wrote that "**Stein and Lane (1996) argued** that maintaining high cognitive demand has positive benefits with respect to student learning" (p. 816, emphasis added). Such instances are defensible because it is true that Stein and Lane argued for the benefits of high cognitive demand tasks.

Table 3: Attribution Levels for Learning Claims Supported by Stein and Lane (1996)

Attribution Level	Number of Excerpts	Number of Sources
1 (non-endorsed)	4 (11.1%)	4 (12.9%)
2 (hedged or context-specific)	17 (47.2%)	10 (32.3%)
3 (study-supported)	9 (25.0%)	8 (25.8%)
4 (general fact)	6 (16.7%)	5 (16.1%)

Yet, 41.7% of the learning claims went further by referring in some sense to a general relationship (level 3 or 4 attribution). For example, NCTM (2014) wrote about the relationship in a general fashion, as a matter of fact:

> Learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature (Boaler & Staples, 2008; Hiebert & Wearne, 1993; Stein & Lane, 1996). (p. 17)

In this case, NCTM did not mention cognitive demand specifically but instead mentioned key features related to cognitive demand. They also cited studies in addition to Stein and Lane (1996) as support for their claim, but due to space limitations, we will not describe those studies here.

Discussion

Our citation analysis revealed that, across primary journals in our field, Stein and Lane (1996) was used to support a substantial number of claims about the link between cognitive demand of mathematical tasks and students' mathematical learning. Of those claims, 55.6% also involved additional references beyond Stein and Lane, but these were often policy or practitioner works rather than empirical research, perhaps aligning with a philosophical stance on cognitive demand but not an empirical one. The U.S. Department of Education and the National Science Foundation (2013) described six types of education research: foundational, exploratory, design and development, efficacy, effectiveness, and scale-up. Exploratory research "examines relationships among important constructs in education and learning to establish logical connections that may form the basis for future interventions or strategies to improve education outcomes" (p. 9). They indicate that exploratory "connections are usually correlational rather than causal" (p. 9). Certainly, the work of Stein and Lane (1996) and others (e.g., Hiebert and Wearne, 1993; Tarr et al., 2008) provides valuable information about a potential relationship between cognitive demand and student learning, but based on the evidence, we judge this work to be at no higher than the exploratory level. Replications are needed to test whether this relationship holds under various conditions (design and development research), and a significant amount of work would need to be done in order to make causal claims that could estimate the average impact of using high cognitive demand tasks. As Stein and Lane said in 1996, "the analyses discussed herein should be replicated" (p. 75), but rather than answering this call, the field has instead justified the belief that cognitively demanding tasks relate with (or cause) higher student learning outcomes by drawing on studies in the initial stages of the research progression.

Though we argue for the need of later-stage research studies to examine the link between cognitively demanding tasks and student learning, we do not dismiss the value of smaller-scale studies. We recognize the importance of a wide array of research; however, certain types of claims (e.g., factors that generally relate to measureable student outcomes) are best supported empirically by larger-scale study designs. If we include such claims in our work, we should be cognizant of the nature of the empirical support, and modify claims accordingly. And if claims are value-based rather than empirical, that is appropriate, but they should be written as such.

At the center of this article is a deep question about the relationship between cognitively demanding mathematical tasks and students' mathematical learning, broadly construed. Although we have critiqued some of the sources of evidence for claims made about this relationship, it is not our intention to cast doubt on the relationship itself. In fact, we believe it is highly likely that a positive relationship does exist. Yet, as a field, we should not be satisfied with shared beliefs based on insufficient evidence. Instead, we should strive for a body of evidence that would convince not only someone who is already predisposed to value cognitively demanding experiences but that would convince a skeptic. Therefore, we join Makel and Plucker (2014) and Warne (2014) in encouraging editors and agencies to open the door to replications, even—or perhaps especially—if it investigates claims that many already take to be true.

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DETECTING MATH ANXIETY WITH A MIXTURE PARTIAL CREDIT MODEL

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The purpose of this study was to investigate a new methodology for detection of differences in middle grades students' math anxiety. A mixture partial credit model analysis revealed two distinct latent classes based on homogeneities in response patterns within each latent class. Students in Class 1 had less anxiety about apprehension of math lessons and use of mathematics in daily life, and more self-efficacy for mathematics than students in Class 2. Moreover, students in Class 1 were found to be more successful in mathematics, mostly like mathematics and mathematics teachers, and have better educated mothers in comparison to students in Class 2. However, gender, attending private or public schools, and education levels of fathers did not appear to differ between the classes. Capturing such fine-grained information extends recent advances in measuring math anxiety.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Middle School Education, Research Methods

Identifying affective characteristics, such as anxiety and depression, that students experience in school settings and dealing with these characteristics are significant challenges for educators. Math anxiety, as one such characteristic, can be defined as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972, p. 551).

Math anxiety has been shown to cause low academic performance (Ashcraft, 2002), reduced cognitive information-processing (Young, Wu, & Menon, 2012), and low perceptions of one's own mathematics abilities (Hembree, 1990). Low math abilities and low working memory, as well as non-supportive teachers can also be considered as important risk factors for the existence of math anxiety (Ashcraft, Krause, & Hopko, 2007). As a result, math anxiety can lead to avoidance of selecting career paths involving mathematics (Ashcraft & Moore, 2009). Previous research on mixture item response theory (IRT) models (e.g., Mislevy & Verhelst, 1990; Rost, 1990) has suggested that these models may be useful in detecting latent classes of individuals that differ along one or more cognitive or affective characteristics. Latent classes are statistically determined groupings of individuals who are homogeneous on such characteristics. Latent classes are latent because they are not directly observable as gender or ethnic groups. Previous research has demonstrated that latent classes in a population may differ on multiple kinds of characteristics including problem solving (Bolt, Cohen, & Wollack, 2001), test speededness (Cohen & Bolt, 2005), mathematical knowledge (Izsák, Jacobson, de Araujo, & Orrill, 2012), reading comprehension (Baghaei & Carstensen, 2013) and on personality traits such as depression (Hong & Min, 2007). In view of the negative, long-term impacts of math anxiety, it would be useful to distinguish latent classes of students who differ in their math anxiety. Such an identification of the latent classes would potentially help teachers improve the affective environment in school settings by applying specific interventions based on the needs of students in each latent class.

The purpose of this study was to investigate the utility of a mixture IRT methodology for detection of latent classes of middle grades students' math anxiety. The following research questions were addressed in this study:

1. Are there distinct latent classes of middle grades students that differ in their math anxiety?
2. What does the existence of these latent classes imply about the different response patterns of math anxiety that exist in this population?

3. What are the effects of manifest variables such as mathematics achievement, gender, liking mathematics, liking mathematics teachers, attending to private or public schools, education levels of mothers and fathers on latent class membership?

The present study makes at least two contributions. First, past research has attempted to identify students' math anxiety levels based on their total scores on a math anxiety scale. Results from the present study suggest that relying on use of total scores may miss important qualitative and quantitative differences in students' math anxiety and for understanding the structure of math anxiety. Second, past research has traditionally focused on explaining math anxiety by measuring its dimensions through exploratory and confirmatory factor analysis (e.g., Baloğlu & Zelhart, 2007; Kazelskis, 1998) and on the structural equation modeling of the relationship between math anxiety and variables such as mathematics achievement (e.g., Harari, Vukovic, & Bailey, 2013; Meece, Wigfield, & Eccles, 1990). However, to our knowledge, there has been no study yet reported in the literature on the detection of different latent classes of the math anxiety population by using relatively new psychometric models, such as mixture IRT models. Therefore, the present study demonstrates that a mixture IRT model can be useful for identifying characteristics of latent classes and for obtaining fine-grained information about particular strengths and weaknesses of middle grades students' math anxiety. Results of this study suggest one potentially useful route for mathematics education research in the future by providing a unique approach on identifying math anxious students in school settings.

Theoretical Framework

The theoretical framework for this study is based on the mixture Rasch model (MRM; Rost, 1990), which is a combination of a latent class model and a Rasch model. Unlike the standard Rasch model, which assumes that the same Rasch model applies to all examinees in the population, the MRM assumes that distinct latent classes exist in the population and that a different Rasch model applies to each. In the MRM, the relative difficulty of ordering the items is determined by a class membership parameter, and the number of items which the examinee is expected to answer or endorse is influenced by a continuous latent ability variable specific to the latent class. For each item, the MRM specifies a separate item difficulty for each latent class and for each examinee, a probability of being a member of a particular latent class.

In contrast to the dichotomous form of the MRM with scoring of an item in two categories such as agree or disagree, the polytomous form of the model can be used when items are scored with more than two categories such as strongly agree, agree, disagree, or strongly disagree, and this form is called a partial credit model (PCM; Masters, 1982). The probability of an answer for the mixture form of this model, the mixture partial credit model (MixPCM), can be written as

$$P(x_{ij}=k|\theta_{jg}) = \frac{\exp[\sum_{r=1}^{k}(\theta_{jg}-\delta_{irg})]}{\sum_{t=0}^{m_i}[\exp\sum_{r=1}^{t}(\theta_{jg}-\delta_{irg})]} \qquad (1)$$

where P is the probability that examinee j gives a response in category k of item i, θ_{jg} is a latent trait of examinee j, and δ_{irg} is a threshold parameter indicating the intersection of adjacent category response curves.

The MixPCM enables one to detect homogeneities in the ways examinees in different latent classes respond to items on a scale. As in equation (1), the relationship between the probability of selecting a response category and the latent trait varies across latent classes. The differences in response patterns to each item of a scale reflect homogeneities in characteristics of members of each latent class. In the MixPCM, the relative difficulty of the ordering of a particular response category among the ordered categories is determined by a class membership parameter, and the number of items answered. In this way, the MixPCM could assign two examinees with similar scores on a scale to different latent classes as a result of the differences in their response patterns.

Methods

Participants

The sample consisted of 244 Turkish 6th and 7th grade students attending public and private schools in Turkey. While the number of male and female students is similar (N=128 and N=116 for males and females, respectively), their range of age was around 13-14 years. A written consent form was obtained from one of the parents of each student before the study.

Instruments

The *Math Anxiety Scale* (MANX; Erol, 1989), is a four-point Likert type scale written in Turkish with options for each item ranging from "never" to "always." There were 45 items yielding minimum and maximum scores of 45 and 180, respectively. Higher scores demonstrate a higher math anxiety level. An internal consistency reliability estimate of .90 was obtained in this study. This was consistent with previous results of .92 on a sample of 754 middle school and high school students (Erktin, Dönmez, & Özel, 2006). Erktin et al. detected four factors, which were *test and evaluation anxiety*, *apprehension of math lessons*, *use of mathematics in daily life*, and *self-efficacy for mathematics*.

Demographic information was also obtained regarding students' mathematics grade at the end of the previous semester, their gender, whether or not they liked mathematics and mathematics teacher, the type of school they attended, and parents' education levels.

Data Analysis

The data were analyzed using the MixPCM as implemented in the computer program WINMIRA (von Davier, 2001). First, different numbers of latent classes were estimated in separate models to determine the relative fit of each model. That is, the MixPCM was estimated with one class, two classes, three classes, and four classes. Second, three indices for each model were compared to select the best fitting model: the Akaike's information criterion (AIC; Akaike, 1974), the Bayesian information criterion (BIC; Schwarz, 1978), and the consistent AIC (CAIC; Bozdogan, 1987). These indices are defined as AIC = $-2 \log L + 2p$; BIC = $-2 \log L + p (\log N)$, and CAIC = $-2 \log L + p (\log N + 1)$ where L is the maximum likelihood value, p is the number of estimated parameters, and N is the sample size. AIC, BIC, and CAIC all include penalty functions to modify the -2 log likelihood for either the number of parameters or the sample size or both. BIC has been found to more accurately select the best fitting model for dichotomous mixture IRT models (Li, Cohen, Kim, & Cho, 2009). In this study, the model with the smallest BIC value was selected as the best fitting model. Next, we analyzed the characteristics of each latent class by focusing on places where item locations differed significantly by latent class and places where members of one latent class considered items to be easier or harder to endorse than other latent classes. In addition, independent sample t-tests and chi-square tests were conducted to examine the relationships between manifest variables and latent class membership.

Results

Unidimensionality for the Scale

An exploratory factor analysis using maximum likelihood estimation as implemented in the SPSS 16.0 software (SPSS Inc., 2007) indicated eigenvalues of the first three factors as 14.1, 2.6, and 2.5.

The total variance explained by the first factor was 31.4%. Reckase (1979) reports that if the amount of variance explained by the first factor is 20% or more, then the scale can be considered as essentially unidimensional. Based on these results, the MANX was considered to be essentially unidimensional.

Number of Latent Classes

The information indices for model selection are given in Table 1. Minimum values for AIC, BIC, and CAIC of 12883.82, 13705.72, and 13978.72, respectively, all suggested a two-class solution in the data. Class 1 had 126 students (51.5%) and Class 2 had 118 students (48.5%).

Table 1: Model Fit Indices of the Mixture Rasch Model

Model	AIC	BIC	CAIC
One class	13757.02	14166.47	14302.47
Two classes	**12883.82**	**13705.72**	**13978.72**
Three classes	13091.45	14325.81	14735.81
Four classes	13335.07	14981.88	15528.88

Note. AIC = Akaike information criterion; BIC = Bayesian information criterion; CAIC = Consistent Akaike information criterion; the smallest information criterion index is bold.

Item thresholds indicate the point on the trait scale between each adjacent score category and indicate the relative ease of endorsing each item in each latent class. Item thresholds for each class are plotted in Figure 1 and Figure 2. Thresholds lower on the scale (e.g., -3, -2) indicate that examinees had a greater propensity to endorse that response category. Similarly, thresholds higher on the scale (e.g., 2, 3) indicate that examinees had a greater propensity to endorse a higher response category. Thresholds may differ by latent class, meaning that relative propensity for endorsing a category of an item is specific to each latent class. Because the MANX has four response categories ranging from "never" to "always," there are three possible thresholds that can be used to interpret the math anxiety level for each item as follows:

Categories: "never" "sometimes" "usually" "always"
Scores: (1) (2) (3) (4)
Thresholds: |--------T1--------|--------T2--------|--------T3--------|

For example, if an examinee's trait level is smaller than T1 (i.e., the first threshold), then the response is expected to be "never." If an examinee's trait level is smaller than T2 (i.e., the second threshold) but larger than T1, then the response is expected to be "sometimes."

Figure 1 and Figure 2 present plots of the item thresholds for Class 1 and Class 2. It is clear that students in Class 1 were more variable in endorsing or agreeing than students in Class 2, with thresholds ranging from -7.41 to 9.00. Students in Class 1 also had lower tendency to endorse items above threshold 1, and greater tendency to endorse items above threshold 3 than students in Class 2. On the other hand, students in Class 2 were more constrained in endorsing items with the range of thresholds from -2.186 to 2.249.

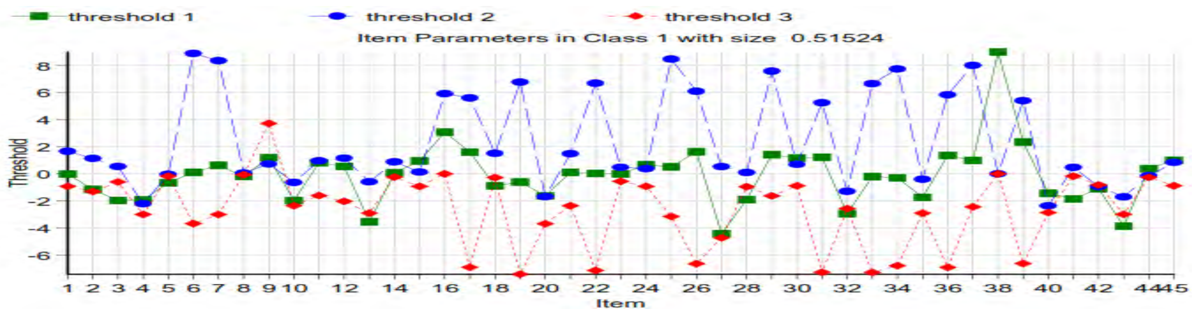

Figure 1. Item thresholds for class 1.

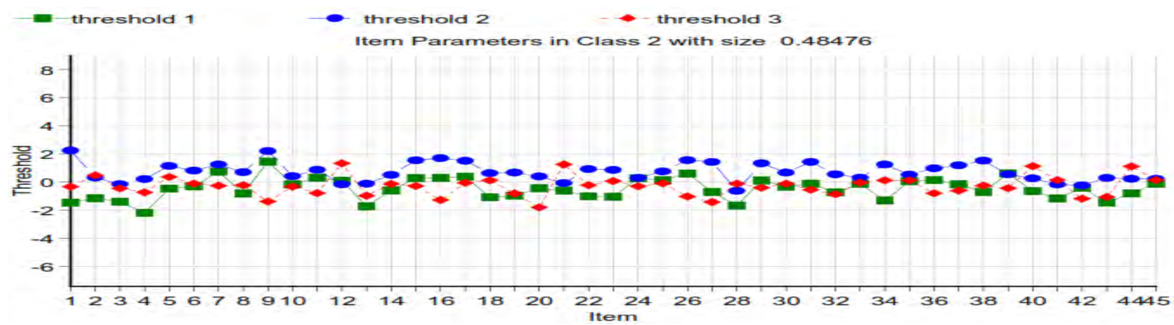

Figure 2. Item thresholds for class 2.

Analyses of Item Locations and Item Response Distributions

The item location is the mean of all item thresholds for an item. Higher mean thresholds indicate lower propensities of endorsement (Masters, 1982). Thus, item locations suggest which items cause differences in response patterns between latent classes.

In addition to the analysis of item locations, item response distributions between the two latent classes were compared to examine similarities and differences in item responses for each latent class. Analyses of item locations and item response distributions led to three main results: (1) Students in Class 1 were less anxious than students in Class 2 in terms of having anxiety about *apprehension of math lessons* and *use of mathematics in daily life*, (2) students in Class 1 had more *self-efficacy for mathematics*, and (3) students in both latent classes had similar levels of *test and evaluation anxiety*.

Figure 3 presents item locations for each latent class. Based on the figure, it can clearly be said that Class 1 had different and more variable item locations than Class 2.

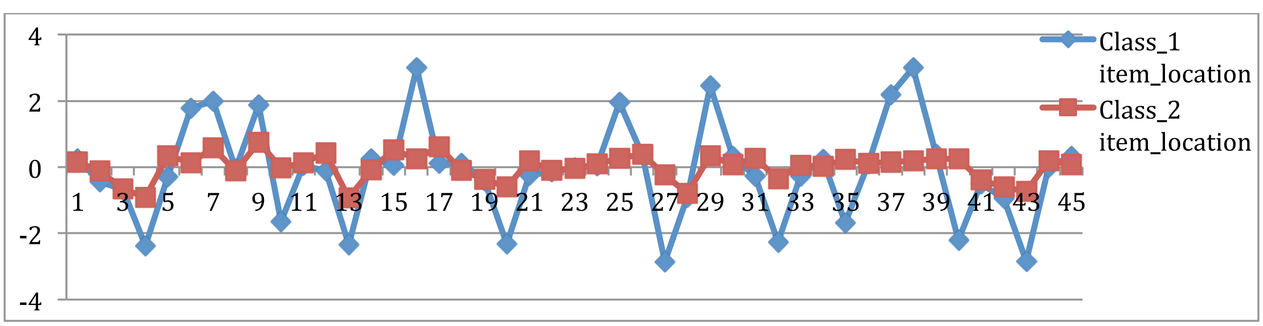

Figure 3. Item locations for class 1 and class 2.

Items with a difference on the scale of 1 logit or more were considered as indicating differences between the two latent classes. Based on the item locations for the two latent classes (see Figure 3), items 4, 6, 7, 9, 10, 13, 16, 20, 25, 27, 29, 32, 35, 37, 38, 40, and 43 appeared to have different response propensities for Class 1 and Class 2.

Items reflecting anxiety about *apprehension of math lessons* (i.e., Items 6, 7, 16, and 37) were more difficult to endorse for Class 1 than Class 2 (see Figure 3). For example, for Item 16, "Math book bothers me," the item location for Class 1 was 3 but for Class 2, it was .25. The proportions selecting the options of "never" and "always" in Class 1 were 98.7% and 0% respectively, in contrast to 53.3% and 15.1% in Class 2, respectively. On items such as Item 6 and Item 7, which asked students to identify how much they panic when a lot of mathematics problems are given as homework and how uncomfortable they feel when studying a hard mathematics topic (For Item 6 and Item 7, Class 1 item locations were 1.78 and 2; Class 2 item locations were .14 and .58), students in Class 1 mostly agreed with the option "never" (80.2% for Item 6 and 87.1% for Item 7) than students in Class 2 (38.6% for Item 6 and 64.2% for Item 7).

On the other hand, students in Class 1 endorsed more easily items with positive statements such as enjoying numbers (i.e., Items 4, 10, 13, 20, 32, 35, and 40). For example, for Item 40, "Opening any book on math and looking at one of its pages full of mathematics problems makes me happy," (Class 1 and Class 2 item locations were -2.21 and .25, respectively), the proportion selecting "always" was 70.3% in Class 1 as opposed to 7.8% in Class 2.

Items focusing on anxiety about *use of mathematics in daily life* (i.e., Items 9, 29, and 38) were harder for students in Class 1 to endorse than for students in Class 2 (see Figure 3). For Item 29, "When I am asked to help a primary school student with his/her homework, I may refuse to help because I feel afraid that there may be some problems that I could not solve" (Class 1 item location was 2.46; Class 2 item location was .35), almost all students in Class 1 (93.5%) and half of the students in Class 2 (50.5%) selected the option "never." On items that asked students to rate their ideas about *test and evaluation anxiety* (i.e., Items 2, 3, 8, 11, 14, 18, 19, 21, 22, 24, 25, 28, 30, 33, 41, 42, and 44), item locations as well as the distributions of responses were similar across choices for both classes.

Finally, on items involving *self-efficacy for mathematics* (i.e., Items 27, and 43), item locations and the distribution of responses were different for the two classes. For Item 43, "When I think I succeeded at a math exam, I feel relaxed and peaceful while waiting for the announcement of the results" (Class 1 item location was -2.84; Class 2 item location was -.73), the proportion selecting "always" in Class 1 was 72.8% as opposed to 39.5% in Class 2.

The Relationships Between Manifest Variables and Latent Class Membership

To obtain detailed information about the characteristics of each latent class, we examined the relationships between manifest variables and latent class membership using independent sample t-tests and chi-square tests. Regarding mathematics achievement, students in Class 1 were significantly more successful than students in Class 2 (t (df = 111) = 3.71, $p < .01$). In terms of gender, there was no significant association between the two classes ($\chi^2(1) = .98$, $p = .32$). The associations between students' liking mathematics and liking their mathematics teachers, and latent class membership were significant ($\chi^2(1) = 11.83$, $p < .01$ and $\chi^2(1) = 6.30$, $p < .01$, respectively). However, there was no significant association between the type of school attended and latent class membership ($\chi^2(1) = .57$, $p = .45$). Finally, education level of mothers was higher for students in Class 1 than Class 2 (t (df = 136) = 2.36, $p < .02$), but there was no significant difference in terms of education levels of fathers (t (df= 136) = 1.07, $p = .29$).

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Discussion

In this study, we examined the utility of a mixture IRT methodology, named MixPCM, for detecting latent classes of middle grades students' math anxiety. With respect to the first research question, two latent classes were detected with distinct patterns of math anxiety. With respect to the second research question, Class 1 consisted of students who were reported being less anxious about *apprehension of math lessons* and *use of mathematics in daily life*, and as having more *self-efficacy for mathematics* than students in Class 2. However, there did not exist any difference between Class 1 and Class 2 in terms of *test and evaluation anxiety*. With respect to the third research question, students in Class 1 were found to be more successful in mathematics, mostly like mathematics and mathematics teachers, and have better educated mothers in comparison to students in Class 2. Moreover, there was no significant association between the two classes in terms of gender, attending private or public schools, or education levels of fathers.

The results reported here on the relationships between math anxiety and the manifest variables were consistent with the findings in the literature. Similar to the previous findings indicating that math anxiety was negatively related to mathematics achievement (e.g., Hembree, 1990), students in Class 1, in the present study, were reported being less anxious and more successful in mathematics. Previous research on the effects of positive attitudes and education levels of mothers on math anxiety has led to a consensus that positive attitudes towards mathematics and education levels of mothers were negatively associated with math anxiety (e.g., Engelhard, 1990; Meece, Wigfield, & Eccles, 1990). In this study, students in Class 1 were found to be less anxious but be more likely to have positive attitudes such as enjoying mathematics and liking their mathematics teachers, and to have mothers with higher education levels than students in Class 2. Including the analysis of manifest variables along with results from the MixPCM and obtaining consistent results with previous research strengthen the validity of the interpretations about the characteristics of each latent class reported in this study.

The results of this study have important implications for teachers and researchers. First, it may be misleading to compare all students based on their total scores on a scale of math anxiety. Rather, within a population of students, there exist latent classes that differ in their math anxiety. Relying on only single total score, therefore, might hinder gaining insight about particular characteristics of students. In this regard, the MixPCM was found to be a useful tool for identifying those students with different patterns of math anxiety in classroom settings. This, in turn, could help teachers make interventions specific to the needs of each student. For example, they can focus on reducing some particular students' anxiety levels towards mathematics lessons by not calling on these students to solve a problem at the board; engage some students with more mathematics related activities in daily life by presenting simulated real-life situations and asking word problems in a real-life context; and help some students build self-confidence for mathematics through asking mathematical problems from simple to more complex.

In conclusion, the present study was the first study that examined the utility of the MixPCM for detection of distinct latent classes based on different patterns of math anxiety. The results reported here provide initial evidence that the MixPCM, when applied to a scale like the MANX, can provide fine-grained information about latent classes of middle grades students population and their characteristics of math anxiety. Future studies should continue on conducting similar studies with other popular math anxiety scales in different populations.

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"THEORY AT THE CROSSROADS": MAPPING MOMENTS OF MATHEMATICS EDUCATION RESEARCH ONTO PARADIGMS OF INQUIRY[1]

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In this essay, traveling through the past half-century, the authors illustrate how mathematics education research shifted, theoretical, beyond its psychological and mathematical roots. Mapping four historical moments of mathematics education research onto broader paradigms of inquiry, the authors make a case for the field to take up a theoretical "identity" that refutes closure and keeps the possibilities of mathematics teaching and learning open to multiple and uncertain interpretations and analyses.

Keywords: Equity and Diversity, Research Methods

Introduction

Over the past half-century, mathematics education research could be characterized as shifting from searches for certainty to acknowledgments of doubt (cf. Skovsmose, 2009). Discussions about theory during this time have grown from being nearly nonexistent in the 1960s to filling a visible and frequently contested space for productive scholarly debate in more recent times. For instance, in the mid-to-late 1990s, Steffe, Kieren, Thompson, and Lerman debated the often-dichotomized theoretical traditions of radical constructivism and social constructivism (Lerman, 1996, 2000a; Steffe & Kieren, 1994; Steffe & Thompson, 2000). In the late 2000s, Gutiérrez and Lubienski debated the uses (or not) of broad socio-cultural and -political theories (and methods) when reporting on the mathematics "achievement gap" (Gutiérrez, 2008; Lubienski, 2008; Lubienski & Gutiérrez, 2008). And Confrey and Battista (Battista, 2010; Confrey, 2010), individually, in the early 2010s, responded to Martin, Gholson, and Leonard's (2010) rejoinder to the assumptive question: "Where's the math (in mathematics education research)?" (Heid, 2010, p. 102) These debates hinged largely on the theoretical traditions taken up by the researchers, which, in turn, determined what questions might be asked and how data might be collected, analyzed, and represented (see Lester & Wiliam, 2000; Valero, 2004).

The productive theoretical debates that have engaged mathematics education researchers since the 1990s are in stark contrast to the debates (or lack thereof) from the 1960s and 1970s. In those early developmental years of mathematics education research, the chief method of establishing legitimacy for the field was for researchers to align themselves with the existing epistemologies of mathematics and the developing theories of psychology (Kilpatrick, 1992). This allegiance was formally instituted in 1976 when the International Group for the Psychology of Mathematics Education was founded during the 3rd International Congress for Mathematical Education (ICME-3). Overall, in research reporting during these early developmental years, theoretical considerations were merely implicit. When researchers discussed theory, it was most often in the context of developing a single theory or theoretical network specific to research on mathematics teaching and learning (e.g., Becker, 1970).

The Emergence of Theoretical Discussions

The allegiance to "traditional" psychology waned in the late 1970s and early 1980s, as theories (and methodologies) began to be adapted from the disciplines of anthropology, cultural and social psychology, history, philosophy, and sociology (Lester & Lambdin, 2003). Over three decades ago, Higginson (1980) proposed that mathematics education be informed not simply by mathematics and

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psychology, but also by sociology and philosophy. He noted that allegiance to mathematics is self-evident and that the "battle for the recognition of a psychological dimension in mathematics education has been won, for almost all purposes, for some time now" (p. 4). Higginson then made a two-pronged argument for the recognition of a sociological dimension: (a) the need to more fully understand the social role of schooling and the interpersonal and intrapersonal dynamics among teachers, students, and the mathematics being taught and learned; and (b) the need to more fully understand the influences of cultural values, economic conditions, social structures, and emerging technologies on schools generally and on teaching and learning specifically.

In arguing for the inclusion of philosophy, Higginson (1980) cautiously noted that with the inclusion of sociology, "it might appear [to some] that the gates have been open too far already" (p. 4). But for Higginson, the inclusion of philosophical considerations in mathematics education (research or otherwise) was important because all human "intellectual activity is based on a set of assumptions of a philosophical type" (p. 4). These assumptions, he argued—

will vary from discipline to discipline and between individuals and groups within a discipline. They may be explicitly acknowledged or only tacitly so, but they will always exist. Reduced to their essence these assumptions deal with concerns such as the nature of "knowledge", "being", "good", "beauty", "purpose" and "value". More formally we have, respectively, the fields of epistemology, ontology, ethics, aesthetics, teleology and axiology. More generally we have the issues of truth, certainty and logical consistency. (p. 4)

Higginson's (1980) point was soon taken up. For example, in 1984 a new Topic Study Group on Theory in Mathematics Education [TME] was formed at ICME-5. The purpose of the group, as Steiner (1985) summarized, was "to give mathematics education *a higher degree of self-reflectedness* and *self-assertiveness,* to *promote another way of thinking* and of *looking* at the problems and their interrelations" (p. 16; emphasis in original). Steiner also provided a list of 10 topics that the TME Group might explore in the future; these topics included (among others): definitions of mathematics education as a discipline; use of models, paradigms, theories in mathematics education research; relationships between theory and practice; and explorations of ethical, societal, and political aspects of mathematics education.

Mathematics education research of the 1990s and beyond certainly reflects this list of topics, broadening not only possible theoretical traditions that might be taken up but also expanding the very identity of mathematics education as a research domain (see Sierpinska & Kilpatrick, 1998). By way of example, conferences held since the mid-1980s include Political Dimensions of Mathematics Education (1990, 1993, 1995); Critical Mathematics Education: Toward a Plan for Cultural Power and Social Change (1990); Mathematics Education and Society (1998, 2000, 2002, 2004, 2008, 2010, 2013, 2015, 2017); and Mathematics Education and Contemporary Theory (2011, 2013, 2016). Furthermore, edited books published since that time include *Equity in Mathematics Education*: *Influences of Feminism and Culture* (Rogers & Kaiser, 1995); *Ethnomathematics*: *Challenging Eurocentrism in Mathematics Education* (Powell & Frankenstein, 1997); *Sociocultural Research on Mathematics Education* (Atweh, Forgasz, & Nebres, 2001); *Which Way Social Justice in Mathematics Education* (Burton, 2003); *Mathematics Education within the Postmodern* (Walshaw, 2004); and *Culturally Responsive Mathematics Education* (Greer, Mukhopadhyay, Powell, & Nelson-Barber, 2009). These listings are by no means exhaustive but merely illustrative of the conferences and books that have assisted in shifting mathematics education research beyond its psychological and mathematical roots.

Theory Defined

Conferences and books like those mentioned have indeed contributed to the broadening of theoretical (and methodological) traditions within mathematics education research. However, we have not yet defined or described what we mean by *theoretical tradition* or *theory* here. This is an intentional omission. In our view, theory often conveys different meanings and assumes different purposes. For instance, Sriraman and English (2010) have drawn attention to the notion of a "grand" theory sought by some researchers (e.g., Becker, 1970; Silver & Herbst, 2007), while Lester (2005) has suggested that mathematics education researchers adapt theoretical concepts and ideas from a range of perspectives. Brown and Walshaw (2012) have argued that mathematics education researchers use "theory as a vehicle for new productive possibilities in mathematics education" (p. 3).

These different purposes signal that theory is being conceptualized at different levels. To that end, E. A. St. Pierre (personal communication, June 2000) has proposed a three-tier structure for discussing theory: high-level, mid-level, and ground-level theories. High-level theories are the larger philosophical traditions in which a researcher might position her or his science (e.g., analytic philosophy or continental philosophy). These traditions rest on a set of assumptions about epistemology, ontology, ethics, aesthetics, teleology, and axiology or, more simply stated, about truth, certainty, and logical consistency (Higginson, 1980). Mid-level theories are the various theoretical traditions and ideas that might be derived from one or more broader philosophical traditions (e.g., activity theory, cognitive theory, constructivist theory, critical theory, poststructural theory, sociocultural theory). Ground-level theories, not to be confused with grounded theory (Glaser & Strauss, 1967), are the theories or models developed or used to make sense of the data collected during data analysis; that is, the theory that is on the ground, closest to the data (e.g., cognitively guided instruction; see, e.g., Fennema et al., 1996). It is important to note, however, that a specific ground-level theory is only possible through the set of philosophical and theoretical assumptions, beliefs, values, and perspectives operating in the context of the high- and mid-level theories taken up by the researcher (E. A. St. Pierre, personal communication, June 2014). The danger in too much of the existing mathematics education research, however, is that researchers often do not acknowledge the philosophical assumptions present in the high- and mid-level theories that make the ground-level theories they develop or use possible.

For our purposes here, our focus is on high- and mid-level theories or what taken together could be called the *paradigm of inquiry* in which the researcher resides. That is, when using the word theory or the phrase theoretical tradition we are concerned about the epistemological stance of the researcher as she or he conducts research within a set of assumptions about truth, certainty, and logical consistency, being mindful that science, social or otherwise, is always already entangled with and in these broader concerns of philosophy (St. Pierre, 2011).

The Paradigm Wars and Education Research

Generally speaking, the broadening of theoretical traditions in mathematics education has been played out in the larger paradigm wars of education social science (see Gage, 1989; Guba & Lincoln, 1994; Lather, 2006; St. Pierre, 2006). The use of Kuhn's (1962/1996) concept *paradigm* is meant to describe shifts in the traditions of "normal science" (i.e., firmly based historical traditions of science) that are differentiated not by failure of one method to another but rather by the "incommensurable ways of seeing the world differently and of practicing science in it" (p. 4). Although the use of the term paradigm in social science research has been contested (see Donmoyer, 2006), Guba and Lincoln (1994) have pointed out that inquiry paradigms highlight for researchers "what it is they are about, and what falls within and outside the limits of legitimate inquiry" (p. 108). Inquiry paradigms, they have argued, are defined by responses to three fundamental and interconnected questions—the ontological question, the epistemological question, and the methodological question. The three questions are interconnected "because the answer given to any one question, taken in any order, [more times than not] constrains how the others may be answered" (p. 108).

Much has been written in the past 50 years or so about the beginning (early 1960s), the aftermath (late 1980s), and the resurgence (early 2000s) of the paradigm wars (see Lather, 2006; St. Pierre, 2006). Writing in 1989, in a futuristic account of education research at the turn of the 21st century, Gage proposed (and hoped for) an armistice of sorts as "researchers came to a new realization that paradigm differences do not require paradigm conflict" (p. 7). But rather than an armistice, paradigm conflicts have seen a resurgence as both experimental and quasi-experimental research have been hailed as the "gold" standard in educational research (see, e.g., National Research Council, 2002). St. Pierre (2006), underscoring the gravity of the resurgence, has argued—

> The stakes are high because the very nature of science and scientific evidence and therefore the nature of knowledge itself is being contested by scholars and researchers who think and work from different epistemological, ontological, and methodological positions as well as by those postmodernists who challenge the metaphysical project altogether. If one believes that different theoretical frameworks are grounded in and structured by different and, perhaps, incommensurable assumptions about the nature of knowledge, truth, reality, reason, power, science, evidence, and so forth, then one can see why educators are taking sides in this debate that is already organizing the limits and possibilities of what we can think and know and, thus, how we can live in the complex and tangled world of educational theory, research, policy, and practice. (pp. 239–240)

Within the complex and tangled world of U.S. mathematics education research, this resurgence of paradigm conflicts is visible within the pages of *Foundations of Success*: *The Final Report of the National Mathematics Advisory Panel* (NMAP; 2008) and in a special issue of the *Educational Researcher* (Kelly, 2008) published in response. Throughout the pages of both the final report and the response special issue it is often noted, explicitly and implicitly, that supporting certain theoretical and methodological traditions does not mean complete abandonment of others. The authoring committee of the NMAP final report, however, included only experimental and quasi-experimental research to make evidential knowledge claims about mathematics teaching and learning. So as politics took the place of scientific inquiry (Boaler, 2008), the authoring committee took direct aim at some epistemological possibilities, and thus theoretical and methodological possibilities. For instance, they erased "race" from the conversation on mathematics teaching and learning altogether (Martin, 2008). In the end, as a proliferation of paradigms to think about and do science became possible within the decades of the 1980s and 1990s (Lather, 2006), both education research in general and mathematics education research in particular experienced a backlash in the early 2000s and beyond. The war rages on as the battles over the nature of knowledge, truth, reality, reason, power, science, evidence, and so forth, continue.

Mapping Moments to Paradigms of Inquiry

In an attempt to make sense of the proliferation of theoretical traditions used in mathematics education research since the 1970s, Stinson and Bullock (2012) have identified four distinct yet overlapping and simultaneously operating shifts or historical moments: (a) the process–product moment (beginning in the 1970s); (b) the interpretivist–constructivist moment (beginning in the 1980s); (c) the social-turn moment (beginning in the mid-1980s); and (d) the sociopolitical-turn moment (beginning in the 2000s). These moments of mathematics education research are not intended to suggest that movement among the moments occurs in some linear fashion, arriving at a "best" or "better" place across a continuum. Rather, the moments are merely arranged in loose historical chronological order. As a case in point, Frankenstein (1983) and Skovsmose (1985) began exploring the sociopolitical implications of critical mathematics education several years before the sociopolitical-turn moment identified as beginning in the 2000s. Table 1 maps the four moments of mathematics education research onto one and, in some cases, two paradigms of inquiry. Representing an adaptation of a conceptualization offered by Lather and St. Pierre (see Lather, 2006), four broad paradigms are singularly worded by their general intentions: prediction, understanding, emancipation, and deconstruction (see Stinson & Bullock, 2015).

Table 1: Mapping Moments of Mathematics Education Research to Paradigms of Inquiry

- Process–Product Moment (1970s–)→***Predict***
- Interpretivist–Constructivist Moment (1980s–)→***Understand***
- Social-turn Moment (mid 1980s–)→***Understand*** (albeit, contextualized understanding) or ***Emancipate*** (or oscillate between the two)
- Sociopolitical-turn Moment (2000s–)→***Emancipate*** or ***Deconstruct*** (or oscillate between the two)

Paradigms of Inquiry

Predict	***Understand***	***Emancipate***		***Deconstruct***
*Positivist Experimental Quasi-experimental Mixed Methods>	*Interpretivist Social Constructivist Radical Constructivist Sociocultural> Phenomenological Ethnographic Symbolic Interaction	*Critical <Feminist> Critical Race Theory> Latino/a Critical Race Theory> Critical Theories of Race> <Participatory Action Research Critical Ethnography	*BREAK*	*Poststructural/ Postmodern Postcritical Postcolonial Posthumanist Post-Freudian <Discourse Analysis

*Indicates the term most commonly used < or > Indicates cross-paradigm movement

The Break in the original Lather and St. Pierre table indicated a shift from the Enlightenment humanist paradigms on the left to the post-Enlightenment, posthumanist paradigm on the right. Here it indicates a hybrid, in-between space where the researcher might adopt a critical postmodern theoretical tradition (see Stinson & Bullock, 2012, 2015).

Paradigms of Inquiry adapted from Lather and St. Pierre in Lather, 2006, p. 37.

The purpose in mapping the moments to larger inquiry paradigms is to illustrate the different theoretical and methodological possibilities within each moment. Although the table does not exhaust all possibilities, it does provide an expansive list of the kinds of research that might be undertaken within mathematics education. For instance, Table 1 illustrates that research in the process–product moment (beginning in the 1970s) is marked by attempts to predict "good" mathematics teaching by linking mathematics teachers' classroom practices (process) to student outcomes (product). Grounded both theoretically and methodologically in positivist inferential statistics, cognitive and behavioral theories derived from experimental psychology and behaviorism are the primary theoretical traditions (e.g., Good & Grouws, 1979). The interpretivist–constructivist moment (beginning in the 1980s) attempts to understand mathematics teaching and learning rather than to predict it; interpretivist and constructivist theories derived principally out of sociology and developmental psychology are the primary theoretical traditions (e.g., Steffe & Tzur, 1994; Thompson, 1984). The acknowledgement that meaning, thinking, and reasoning are products of social activity in contexts marks the social-turn moment (beginning in the mid-1980s; see Lerman, 2000b); theories drawn from disciplines such as cultural and social psychology, anthropology, and cultural sociology are the primary theoretical traditions (e.g., Boaler, 1999; Zevenbergen, 2000). And a shift toward recognizing knowledge, power, and identity as interwoven and arising from and constituted with and in sociocultural and sociopolitical discourses distinguishes the sociopolitical-turn moment (beginning in the 2000s; see Gutiérrez, 2013); here critical and poststructural theories are the primary traditions (e.g., Gutstein, 2003; Walshaw, 2001).

Closing Thoughts

In the foreword to Mathematics Education as a Research Domain: A Search for Identity, Sierpinska and Kilpatrick (1998) wrote—

> The theme of the ICMI Study reported in this book was formulated as a question: 'What Is Research in Mathematics Education and What Are Its Results?' No single agreed-upon and definite answer to the question, however, is to be found in these pages. What the reader will find instead is a multitude of answers, various analyses of the actual directions of research in mathematics education in different countries, and a number of visions for the future of that research. (p. x)

These multiple answers and various analyses are clearly visible within the moments of mathematics education research as depicted in Table 1. Indeed, similar to researchers in education generally, researchers in mathematics education have experienced a proliferation of paradigms to think with when conducting research on the teaching and learning of mathematics. Similar to Lather (2006), we believe that this proliferation of paradigms is a good thing. Is it possible, then, to characterize mathematics education research? What can we say about its identity? In our view, an identity for mathematics education research is one that is fragmented, incomplete, and continually reconstituted within sociopolitical relations of power. Such a perspective refutes closure and keeps the possibilities for mathematics teaching and learning open to multiple and uncertain interpretations and analyses.

Endnotes

1. The text in this essay was extracted and revised from Stinson and Walshaw (in press).

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ANALYTICAL FRAMEWORK FOR STUDYING INDUCTIVE REASONING IN MATHEMATICS TEACHERS WHEN SOLVING TASKS

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An analytical framework is presented in order to characterize and analyze the evolution of inductive reasoning in mathematics teachers in relation to mathematical conceptualization. It is made up of four components: heuristic strategies, mathematical content, representations and stages of reasoning. It was constructed in order to analyze the written productions of secondary school teachers when solving generalization tasks by means of induction, implemented in a professional teaching development program focusing on the development of inductive reasoning. The use and scope of this analytical framework is discussed herein.

Keywords: Reasoning and Proof, Teacher Education-Inservice/Professional Development, Middle School Education

Introduction

In general, it is accepted that the development of inductive reasoning maximizes mathematical learning processes, for example, in the recognition of patterns and regularities in the solution of non-routine problems, by establishing conjectures, carrying out generalizations, developing arguments and mathematical tests (NCTM, 2000; Common Core State Standards Initiative, 2010). Empirical research supports the idea that argumentation based on inductive reasoning contributes to and supports the justification of conjectures and the construction of deductive tests (Papageorgiou, 2009; Conner, Singletary, Smith, Wagner & Francisco, 2014; Martinez & Pedemonte, 2014).

Various studies have examined inductive reasoning from a cognitive point of view in relation to the understanding of mathematical knowledge in elemental education students, with some adhering to the definition of Klauer, Willmes and Phye (2002), who conceive this type of reasoning as an analytical skill focused on the detection of regularities and irregularities between the attributes and relationships of objects (e.g. Barkl, Porter & Ginns, 2012; Csapó, 1997). Other studies also in a cognitive framework have been based on Peirce's notion of reasoning (e.g. Christou & Papageorgiou, 2007; Papageorgiou, 2009). Some research (Cañadas & Castro, 2007: Cañadas, Castro & Castro, 2008; Castro, Cañadas & Molina, 2010) has transcended to a wider, more complex vision of that which is inductive as a process inherent to the construction of mathematical knowledge and has placed emphasis on the formulation of a theoretical model of inductive reasoning by means of the identification of stages in its development in secondary school students. This paper is based on that vision.

Within the framework of the professional development of teachers, on which the paper reported herein focuses, some have questioned the relationship between mathematical reasoning (inductive and deductive) and learning styles (Arslan, Ilkörücü and Seden, 2009), identifying types of reasoning in a mathematics class for trainee teachers (Soler-Álvarez and Manrique, 2014), as well as recognizing the connection between abductive – inductive reasoning in the generalization of patterns (Rivera & Becker, 2007). Very few studies have examined how to incorporate the inductive nature of reasoning associated with the construction of mathematical knowledge in experiences of the professional development of mathematics teachers. As a first step, in order to advance in this direction, we sought to characterize the inductive reasoning of working mathematics teachers in relation to their mathematical conceptualization. Bearing this in mind, the following analytical framework is proposed.

Analytical Framework of Inductive Reasoning in Their Solution of Tasks

The analytical framework is based on a conceptual positioning of inductive reasoning or induction based on the epistemology of mathematics and science. Historically, the construction of mathematical knowledge has been socioculturally located in empirical inductive practices, as a way to move from the observation of concrete or individual realities to general abstract realities. Induction has been inherent in social and cognitive processes of construction of knowledge, which primarily begin with the study of specific cases and local patterns, which are rationalized in order to carry out a generalization. In this respect, the vision of inductive reasoning discussed in Castro et al. (2010) is shared.

The framework is an adaptation of the model proposed by Cañadas y Castro (2007), in so far as it incorporates components that allow us to not only understand the reasoning, but also recognize the epistemic nature of mathematical concepts by the teacher. This is made up of four components. The first consists of the seven *Stages of inductive reasoning* of the aforementioned model, in which three components are incorporated as analysis tools: the heuristic resolution strategies, the mathematical content that is mobilized and the semiotic representations that allow us to recognize cognitive processes of reasoning and elements of mathematical conceptualization. The unit of analysis is comprised of each solution of tasks that involve inductive generalization processes for their solution. As far as methodical matters are concerned, the analysis components are:

A. **Stages of Inductive Reasoning**. These do not necessarily occur in a linear form and some may not be necessary or may not appear in the solution. The stages of the model are (Cañadas y Castro, 2007, p. 69): *Work with specific cases, organization of specific cases, identification of patterns, formulation of conjectures, justification of the conjectures, generalization and demonstration.*
B. **Heuristic Strategies**. According to Guzman (2007), the resolution of mathematical problems includes a heuristic component in relation to thinking processes and another component relating to the specific content of mathematical thinking. The analysis of heuristics is not only useful for elucidating the extent to which both concepts are linked, but also allows us to recognize the specific inductive strategies used by teachers and thus situate their level of reasoning. Cañadas, Castro and Castro (2008) define them as “the type of strategies that can be described in problems where induction can be used as heuristic” (p. 139).
C. **Mathematical Content**. It is assumed that the mathematical conceptualization in teachers consists of signification and resignification processes based on the recognition of the epistemic nature of the concepts in a triadic relationship: procedural, conceptual and structural (Aparicio, Gómez & Sosa, 2017). All mathematical concept has a double complexity, the first is its dual nature: process – object (Gray & Tall, 1994). The idea of *process* refers to the “*operational*” or procedural quality that any mathematical object must have in order to act as an instrument, to be able to manipulate reality and express it in mathematical terms (signs, symbols and operations). The idea of *object* refers to the cognitive part (the notion, the idea, the thought, in general, that which is conceptual), which allows us to conceptualize a reality and look at it systemically. Such aspects need to be viewed as a more complex *mathematical structures* up to grade of using, explaining, and connecting relationships between objects in an aware way. Mason, Stephen and Watson (2009) call it structural thinking.
D. **External Representations**. The cognitive processes of reasoning can be expressed and described using representations (e.g. Cañadas et al. 2008). Likewise, they are means of objectification for generalization (e.g. Radford, 2010).

Design of the Tasks

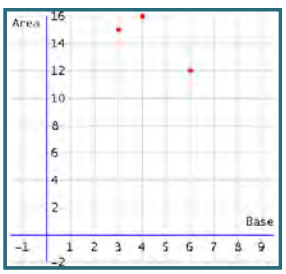
Inductive reasoning involves the process of generalization oriented to the perception of the general in the particular (Kodnik y Manfreda, 2015). For the analysis, three tasks (T) were designed and structured so that the solution would lead to a process of *generalization by induction*. The epistemic elements of mathematics in designed tasks are associated with quadratic behavior between quantities and/or variables and imply notions such as: quadratic functions for modeling the area of a family of rectangles (T1), successions with quadratic behavior presented using geometric dot configurations (T2), and squared binomials as expressions of algebraic generalization of numerical relationships (T3). They focus on the establishing and justification of a conjecture, explicitly requested only in the third.

Two examples of these tasks are shown in Figure 1, which were solved individually using a pencil and paper by two groups of secondary school mathematics teachers in Mexico.

<table>
<tr>
<td>

T1. With regards a family of rectangles, information has been provided in relation to the measurements of the base and area of three rectangles in graph form, as shown in the following illustration.

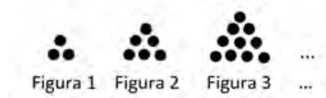
.



Generate an algebraic expression permitting the calculation of the value of the area of any rectangle in that family. Provide a detailed argument of the solution process.

</td>
<td>

T2. Establish an algebraic expression in order to calculate the amount of dots needed for the i-th figure of the following sequence:



Provide a detailed argument of the solution process.

</td>
</tr>
</table>

Figure 1. Examples of tasks for generalization by induction.

In T1, for example, the identification of the quality or regularity between the lengths of the base and the height of a family of rectangles is established. The conjecture relates to the detection of a regularity on the semi-perimeter of the rectangles. The generalization of the conjecture may be established using the algebraic expression of a quadratic function. Said generalization allows for empirical justifications or analytical procedures for their demonstration.

Discussion and Reflections

In experiences of teacher professionalization in Mexico, it has been detected that the inductive reasoning prevails absent in the middle school mathematics teachers' reflections when they propose school treatment of mathematical content and also when they interpret and solve problems. The analytical framework presented is a methodological tool of a study regarding the role of induction as a way to encourage processes of professional development of mathematics teachers.

It has been formulated in order to carry out cognitive analyses (individual and group), which permit the description of inductive reasoning in teachers in relation to mathematical conceptualization based on the external representations used in the solution of generalization tasks by induction. If you are interested in a social analysis in order to identify the variables of the context that influence or optimize the reasoning, then an analysis of the associated arguments would be appropriate. The integration of *mathematical reasoning-content* within the framework is important in order to recognize the aspects of the mathematical thinking of teachers that need to be enhanced, as well as to understand the epistemic nature of mathematics.

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DEVELOPING A CONCEPTUAL FRAMEWORK FOR STUDENTS' UNDERSTANDING OF CROSS PRODUCT

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This study uses data from 434 students to explore how they think about vectors, and cross products in particular, by analyzing student responses to open-ended questions from an online, conceptually-oriented, multivariable calculus cross product activity. We identify several categories that could outline a conceptual framework of student understanding of vectors and cross product. The analysis also identifies common transitional conceptualizations evidenced in student responses.

Keywords: Technology, Post-Secondary Education, Advanced Mathematical Thinking

Vectors play a foundational role in advanced mathematics; as a topic, they are firmly situated at the crossroads from secondary to post-secondary mathematics. Yet, most research regarding students' understanding of vectors has been related to topics in the physics and engineering curriculum (e.g., Barniol & Zavala, 2014; Flores, Kanim, & Kautz, 2003; Nguyen & Metzler, 2003). There is little research on student understanding of vectors in post-secondary mathematics and what does exist tends to focus on linear algebra or geometry. Stewart and Thomas (2009) considered vectors in linear algebra by combining Dubinsky's (1991) APOS (action-process-object) theory with Tall's (2004) categorization of mathematical ways of thinking. Kwon (2013) worked in college geometry to identify three representations of a vector: vector as a translation, vector as a point and point as a vector, and geometrical vector sum. While basic vector concepts and vector arithmetic are often presented in secondary mathematics, students often do not experience vector dot and cross products until their post-secondary calculus courses.

Previous research, some of which is cited above, suggests that students' concept images of vectors develop in a rather fragmented manner, and students do not develop the rich conceptual knowledge required to relate alternate representations of vectors (Van Deventer & Wittmann, 2007). Despite the regular occurrence throughout the curriculum, students continue to have significant conceptual difficulties with vector concepts and manipulations. For instance, Knight (1995) found ~ 40% of students in a college, calculus-based physics course had no idea what a vector was and not one was able to evaluate a vector cross product.

The purpose of this study is to investigate students' understanding of the cross product of vectors as they work through an online exploration activity with embedded questions.

Theoretical Framing

Conceptual understanding has been reported as "what is known (knowledge of concepts)...[and] the way that concepts can be known" (Star, 2005, p. 408). Many studies work from a deficit model concentrating on what is not known and highlighting misconceptions. However, a more productive way to study student understanding is to focus on how it develops. *Transitional conceptions* relate to students' current notions of a concept that are cued by the task at hand and that may include what some would call misconceptions. Transitional conceptions are often not fully integrated in a coherent manner and tend to be in flux. They represent developing understanding and result from a sense-making activity. They may only address some (but not all) aspects of a concept and may be productive in some (but not all) contexts. The study of transitional conceptions in post-secondary mathematics is gaining traction (Chiu, Kessel, Moschkovich, & Muñoz-Nuñez, 2001; Cho & Moore-Russo, 2014; Wolbert, Moore-Russo, & Son, 2016) and stands to provide a more nuanced view of student understanding (Moschkovich, 1999). To understand meaning-making processes, it is imperative that mathematics educators consider the transitional conceptions that occur when students are learning new concepts.

Past research has considered the many "notions" or "components" associated with a single, particular concept in mathematics (e.g., work on slope by Moore-Russo, Conner, & Rugg, 2011; Nagle & Moore-Russo, 2013). Because there is scant research on transitional conceptions specific to vectors in multivariable calculus classes, this study aims to add to the existing body of knowledge, focusing, in particular, on students' transitional conceptual understanding of the vector cross product. More specifically, this study looks to use student responses to:

1. Identify key notions of vector cross product
2. Determine common transitional conceptions for each key notion identified

Data Collection and Analysis

CalcPlot3D (Seeburger, 2016) is an online, freely available 3D-graphing applet that allows students to visualize and manipulate concepts, including vectors. It also offers discovery-learning activities for students to explore concepts. The data analyzed were the electronic responses of 434 college-level multivariable calculus students to a sequence of four open-ended questions listed in Table 1 from an online activity related to vector cross products. In this activity, students were directed to a visual applet that contained two vectors (one red and one blue) along with their cross product. The two vectors were graphed with initial points situated at the origin in the *xy*-plane. Students could manipulate the length and direction of the two vectors on the *xy*-plane, and based on the students' input, the applet automatically redrew the cross product, computed the magnitude of the cross product, and indicated the angle between the two given vectors. The data were collected over four years from students from a variety of U.S. post-secondary institutions including community colleges, four-year private colleges, and four-year public colleges.

Table 1: Sequence of Open-Ended Data Collection Questions

Question	Description
1	What is the geometric relationship between the cross product vector and the two vectors that form it?
2	How is the cross product vector geometrically related to the two vectors that form it?
3	For vectors of fixed length, but varying the direction of one of the vectors, when is the magnitude of the cross product at a maximum?
4	For vectors of fixed length, but varying the direction of one of the vectors, when is the magnitude of the cross product at a minimum?

A general inductive analysis was used for the data. The research team examined the data with no preconceptions allowing categories to emerge naturally after multiple passes through the data set based on their observations (Thomas, 2006). The first two authors read through the data set at least five times and then created categories for the different notions associated with vector cross product. They consulted with the other two authors to confirm the notion categories that were developed and to refine the category descriptions. The first and second authors coded all responses. With the exception of the miscellaneous category, all codes resulted in Krippendorff's alpha above 0.80. The first and second authors then came to a consensus on the few instances in which they were in disagreement and then further refined the category descriptions with input from the other two authors. Once the notions were identified, the research team then followed a similar process to identify transitional conceptions associated with each identified notion.

Results and Discussion

Table 2 displays the identified categories for seven notions that outline a conceptual framework for student understanding of vector cross product. The analysis also identified common transitional conceptualizations evidenced in student responses associated with these notions. Students also made other statements not directly related to the identified notions that were irrelevant or incoherent and involved x and y coordinates, quadrants, and vector addition. Student responses to the different questions often provided evidence of them having more than one notion about cross product. This would not have been apparent if only one or two open-ended questions had been used for data collection.

Table 2: Notions Associated with Vector Cross Product

Notion	Common Transitional Conceptions
The angle between two vectors must be between 0° and 180°.	Students reported angles between the two vectors that were: • negative • greater than 180° (often a multiple of 180° or π)
The cross product vector is orthogonal (perpendicular or 90°) from the two vectors that form it.	Students made statements that while incorrect, lacking precision, or vague hinted at a consideration of orthogonality, such as the cross product is: • *vertical to the two vectors that form it* • *perpendicular to the intersection of the initial two vectors* • *also in the plane that the vectors do not lie in*
The right hand rule or a statement that correctly addresses the need to attend to orientation of one vector relative to another.	Students made statements that were either not precise or incorrect but suggested some consideration of the right hand rule, such as *you curl your fingers in and the vector goes down*.
The magnitude (length) of cross product is equal to the area of parallelogram formed by the two vectors.	Students wrote statements like the notion description but missing reference to: • the "area" of the parallelogram • the "length" or "magnitude" of the cross product. The frequency of the above cases suggests that this is more than just careless wording. Other students wrote less coherent statements that included the word parallelogram, such as *the cross product forms a parallelogram* or that referenced triangles rather than parallelograms.
The magnitude of cross product equals $\|\|a \times b\|\|=\|a\|\|b\| \sin\theta$, or some related statement that involves a relation between the sine and the two vectors.	Student responded with incorrect statements: • involving multiplication (often related to dot product) • that confused sine and cosine in the formula
The angle between the vectors influences the magnitude of the cross product.	Students made vague statements about how changing the angle between two vectors will affect the length of the cross product, such as *the cross product depends on the angle between the two vectors*.
The length of the two vectors influences the magnitude of the cross product.	Students made statements about the length of the cross product that were dependent on the specific vectors that they had been working with in the online learning activity, such as *the maximum a cross product can be is 4*.

It is hoped that the students develop a more sophisticated conceptualization beyond the transitional understanding evidenced in the data through further instructional intervention, but that was not the focus of this study.

Limitation and Suggestions for Future Work

Future work is needed to continue exploring student understanding of vectors. This should consist of further investigation of student's transitional conceptions. More study is also needed to determine if and how students make connections between the various notions of vector cross product. Continued experimentation should include the study of instructional methods that have been successfully leveraged, either within the classroom or in an online platform, to help students develop a better understanding of vector cross product.

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LEARNING TO BECOME A RESEARCHER IN AN ONGOING RESEARCH PROJECT: A COMMUNITIES OF PRACTICE PERSPECTIVE

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We apply Wenger's (1998) communities of practice ideas to the process of incorporating new researchers into an ongoing mathematics education research project. We illustrate this application by describing how the Leveraging MOSTs research project coding team can be viewed as a community of practice. We describe how we have used this particular community of practice to bring new researchers into the project, and new researchers reflect on their experiences with the coding team. Mutual engagement in project work with experienced researchers and having a rich shared repertoire to draw on led to the new researchers developing a shared understanding of the project and being successfully incorporated into the MOST community. This work speaks to the importance of deliberately creating communities of practice for new mathematics researchers to participate within.

Keywords: Research Methods, Informal Education

Researchers have found that participating in well-designed and effectively-implemented *communities of practice* (Wenger, 1998) supports preservice and in-service mathematics teachers' professional learning (e.g., Goos & Bennison, 2008; Hodges & Cady, 2012). Applying Wenger's ideas to research projects holds promise for increasing both the functionality of the project—by improving ways project staff work in community together—and the learning of new researchers. In this paper, we use Wenger's ideas to make sense of how to develop synergy at the crossroads that occur when new researchers become part of an ongoing research community.

Theoretical Framework: Applying Wenger's Social Theory of Learning

Wenger's (1998) social theory of learning identifies three essential dimensions of a *community of practice* (CoP): *mutual engagement*, *joint enterprise*, and *shared repertoire*. Mutual engagement refers to participants' regular interactions with others in a community; joint enterprise refers to participants' common understanding of, and desires to, achieve the purposes of that community; and shared repertoire refers to the shared ways of doing things developed during mutual engagements for the joint enterprise of that community (e.g., routines, tools, artifacts, stories). Participating in a CoP enables individuals to interact with the others in that CoP, which in turn enables them to acquire meanings from these interactions. Negotiating these meanings with others in the CoP contributes to their learning (Wenger, 1998). Therefore, participation, negotiation of meaning, CoP and learning are all interpreted as interrelated and essential dynamics of Wenger's social theory of learning.

The National Science Foundation Collaborative Research Project *Leveraging MOSTs* (LeveragingMOSTs.org) is an example of a CoP. Researchers from Brigham Young University, Michigan Technological University, and Western Michigan University came together to work on the joint enterprise of investigating secondary school mathematics teachers' productive use of student mathematical thinking during instruction. In line with this joint enterprise, researchers mutually engaged in practices (e.g., online and face-to-face meetings, data collection, data analysis), and these engagements produced tools, artifacts and shared ways of doing things (e.g., research reports, further research ideas, codebooks, meeting notes). As more researchers became involved and the scope of the work broadened, the MOST research project became a constellation. Constellation is a term used by Wenger (1998) to describe individual, but interacting CoPs in a system. For example, in the

MOST research project, teams work on analyzing the same data to answer different research questions; that is, they work on different but related parts of the joint enterprise. The interaction among these CoPs in the MOST constellation is primarily carried out by the principal investigators (PIs) of the project. Wenger (1998) defines such people—those who mutually engage in different CoPs in a constellation—as *brokers*. These brokers enable the joint enterprise and shared repertoire of the constellation as a whole.

Wenger has also likened CoPs to "black boxes." From the outside, it can be seen that there is a box, but since the box is opaque it is not possible to see what is inside it (Wenger, 1990). As newcomers' understanding of a CoP increases, the box becomes more transparent until eventually they can see through it to the workings of the CoP. When the box finally becomes invisible, as if there is no box, the newcomer has been fully integrated into the CoP (Wenger, 1990). The difficult process of making the box transparent may explain why Wenger (1998) claimed that becoming a part of a CoP is a challenging process for newcomers.

In longitudinal research projects, such as MOST, it is highly likely that new researchers will join the research team. Integrating these new researchers into a project CoP is essential for them to fully participate in pursuing the joint enterprise of the research team. Since this integration is a challenging process, there is a need to better understand productive ways for new researchers to join ongoing research projects. Toward this end, we explain our process for integrating new researchers into the MOST project and provide insights into how the opaque box of our CoP gradually became transparent for a recent group of researchers who joined the project.

The MOST Coding Team as a Community of Practice

Coding based on the MOST Analytic Framework (see Leatham, Peterson, Stockero, & Van Zoest, 2015) serves as the foundation for the vast majority of the CoPs in the MOST constellation. Thus, the *joint enterprise* of the coding CoP is to label and organize the data to enable further analysis by the other CoPs of the MOST constellation. As a result, the coding CoP is a logical place for new researchers to begin their participation in the MOST constellation. In the current coding CoP, there are five researchers who are new to the project and two PIs. Initially, all seven researchers met twice a week in online video meetings to work on the shared repertoire of the MOST constellation in general and of the coding CoP specifically. The main reifications in the *shared repertoire* of the MOST constellation for the coding CoP are publications related to the MOST Analytic Framework, the MOST codebook, and meeting notes. During the initial meetings, the PIs and the new researchers began coding a set of training data using the MOST Analytic Framework. After coding individually, the group met to reconcile their coding and discuss any discrepancies, connecting those discrepancies to the shared repertoire of the MOST project. Once the new researchers had established a basic working knowledge of the coding framework and its applications, the new researchers continued these online coding meetings on their own. The PIs occasionally rejoined the coding meetings to discuss issues that arose during the new researchers' discussions and to serve as brokers of knowledge for the CoPs in the MOST constellation. These different types of meetings were the main source of *mutual engagement* for the coding CoP, which enabled the new researchers to negotiate meanings about the coding process.

The Voices of New Researchers: Reflections on How the Opaque Box of the Coding CoP Became Transparent

The first step to participating in the MOST constellation and the coding CoP was to understand the constellation itself. In order to do this, we were given access to artifacts such as published articles, the NSF grant proposal, previous meeting notes, and the MOST codebook. Having this rich shared repertoire helped us to develop a general understanding of the constellation of the MOST project: why this research is being done, how this research is being done, and outcomes of the research. After this process, we had a general sense of the joint enterprise of the MOST research team and the coding CoP, so we tried coding the training data using the MOST codebook—what would become our most important artifact.

At this point, the black box of the coding CoP had only become slightly less opaque for us as new researchers, and we needed mutual engagement with brokers regularly to help make the box more transparent. During our initial coding CoP meetings, the PIs helped us to make sense of an important aspect of the joint enterprise of the CoP through the application and understanding of the coding framework. At first, we asked a lot of questions to help us determine what was in the opaque box, but after the box became a little more transparent, our questions changed to discussions with the brokers about how things should be coded. In other words, we could see into the box enough to discuss its inner workings. Through these mutual engagements, we gained a better understanding of the joint enterprise of the coding CoP.

The next step to becoming full participants in the coding CoP was for us to continue our discussions and try to reach agreements on coding the training data without the direct help of the brokers. At this point, the box had become transparent enough that we could discuss it using the codebook as our main resource. During this period, we felt that we were increasing the transparency of the box. However, our developing understanding did not mean that we had fully internalized the joint enterprise of the coding CoP and were using the shared repertoire in an authentic way. Therefore, occasional mutual engagement with the brokers was still crucial to maintain alignment with the MOST project's joint enterprise and shared repertoire. For example, as new researchers, we thought that it was always important to focus on very small details of what we were coding, but as the brokers explained, we could not do this at the expense of looking at the bigger picture. Eventually, we got to a point where we felt that the box had become transparent and the understanding of the CoP that had been captured by the codebook had been transformed back into understanding for us as new researchers. The brokers agreed that we were well on our way to becoming full participants in the coding CoP. This does not mean that the box was invisible—there were still aspects of the joint enterprise that we did not fully understand—but the box had become transparent enough that we were able to move to more central participation in the coding CoP by engaging with coding not-yet-coded data from the project.

Discussion & Conclusion

The reflections of new researchers in the MOST coding CoP revealed the process they went through during their integration into the MOST project. After developing a general understanding of the project through engaging with reifications of project work, the new researchers improved their understanding of the joint enterprise of the coding CoP through mutual engagements with each other and experienced researchers from the project. As a result of this process, these new researchers moved from being peripheral participants towards being central participants in the MOST research project. There were two significant contributors to their learning to become full participants in the MOST constellation: (1) having a rich shared repertoire, and (2) mutually engaging with the other new researchers and brokers. The rich shared repertoire in the MOST constellation helped the new researchers to gain a general understanding of the joint enterprise of the MOST constellation and the coding CoP. Their mutual engagement with the other new researchers in the coding CoP allowed them to develop their own thinking about the shared repertoire, while the mutual engagement with the brokers of the MOST constellation calibrated their shared repertoire with that of the larger project. Thus, our experiences integrating new researchers into the MOST project support Wenger's (1998) claim that participation and reification work as complementary pieces of learning. It appears that neither rich reifications nor mutual engagements of researchers are sufficient on their own to make the box completely transparent. Rather, making the box completely transparent for new researchers requires blending the shared repertoire of the research team with the mutual engagements of both new and experienced researchers.

Our experience is an example of how Wenger's (1998) *communities of practice* ideas can be used to make sense of how to develop synergy at the crossroads that occur when new researchers become part of an ongoing research community. Applying Wenger's ideas to the MOST research project helped to integrate new researchers into our project and prepared them to be central participants in the work. We encourage other research projects to carefully design and effectively implement CoPs. Specifically, we emphasize the importance of research projects both intentionally developing rich reifications of their shared repertoire to serve as the foundation for new researchers' participation in their projects and deliberately creating opportunities for new researchers to mutually engage with each other and experienced researchers.

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MULTILEVEL MODELING OF MATHEMATICS ACHIEVEMENT USING TIMSS 2011

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The purpose of this study was to investigate the effects of student-level and school-level factors on mathematics achievement in four countries including Finland, South Korea, Taiwan, and Turkey. Based on 8th grade students' responses to a student questionnaire and a mathematics test, and school principals' responses to a school questionnaire, the data of the Trends in International Mathematics and Science Study (TIMSS) 2011 were analyzed using Hierarchical Linear Modeling. The results revealed that students' having educational resources at home, their self-confidence in mathematics, and schools' emphasis on academic achievement were the common factors that influenced mathematics achievement in all four countries. However, no significant effects of education levels of fathers, fights or physical injuries to other students, and lack of resources for mathematics instruction on students' mathematics achievement were found.

Keywords: Research Methods, Middle School Education, Data Analysis and Statistics

Purpose and Background

Large-scale assessment programs such as the Trends in International Mathematics and Science Study (TIMSS) and the Program for International Student Assessment (PISA) are two main programs that provide countries opportunities to monitor their relative rankings and to shape their educational policies for the sake of improving their education systems. Because countries need citizens with high competence in mathematics and science for surviving in the global economy, it is important to analyze such data sets by detecting similarities and differences across the countries in terms of their students' mathematics and science achievement.

In the analyses of TIMSS and PISA to date, many statistical techniques have been used to explain mathematics and science achievement including analysis of variance (e.g., Pahlke, Hyde, & Mertz, 2013), multiple regression (e.g., Sulku & Abdioglu, 2015), structural equation modeling (SEM; e.g., Kalender & Berberoğlu, 2009), and hierarchical linear modeling (HLM; e.g., Wang, Osterlind, & Bergin, 2012). In this study, HLM was preferred as a statistical technique over other techniques because TIMSS and PISA data are nested in nature (i.e., students are nested within schools) and HLM is an appropriate technique to account for the nesting of students within schools.

Due to the complexity of mathematics achievement, past research has provided contradictory results about the effects of student, home-family background, and school related factors on students' mathematics achievement. For example, while some studies (e.g., Chiu, 2010) pointed out that school related factors have little role in contributing to students' mathematics achievement, some other studies (e.g., Edmonds, 1979) reported that school makes a difference on achievement. Similarly, while some studies (e.g., Papanastasiou, 2002) suggested that students' attitudes towards mathematics and their beliefs about success in mathematics did not influence their mathematics achievement, several other studies (e.g., Berberoğlu, Çelebi, Özdemir, Uysal, and Yayan, 2003) documented that factors such as students' perception of failure or success in mathematics greatly influenced students' mathematics achievement.

The purpose of this study was to compare the effects of student- and school-level factors on 8th grade students' mathematics achievement in Finland, South Korea, Taiwan, and Turkey using TIMSS 2011 data. The results of this study contribute to the literature by helping reconcile the contradictory findings reported so far on the effects of student- and school-level factors, and this will enable policy-makers to distinguish similarities and differences in the four countries and to shape their education systems. In this study, the following research questions were addressed:

1. Which student-level factors contribute to mathematics achievement of 8th grade students in Finland, South Korea, Taiwan, and Turkey?
2. Which school-level factors contribute to mathematics achievement of 8th grade students in Finland, South Korea, Taiwan, and Turkey?

The student-level factors of this study were *MotherEducation* (i.e., Education levels of mothers), *FatherEducation* (i.e., Education levels of fathers), *HomeResources* (i.e., Educational home resources), *Bullying* (i.e., Bullying at school), *LikeMath* (i.e., Like learning mathematics), *ValueMath* (i.e., Value learning mathematics), and *ConfidenceMath* (i.e., Self-confidence in mathematics). In addition, the school-level factors were *IncomeArea* (i.e., Schools' immediate area), *JobSatisfaction* (i.e., Teachers' job satisfaction), *Intimidation* (i.e., Intimidation or verbal abuse among students), *Fights* (i.e., Fights or physical injuries to other students), *LackofResources* (i.e., Lack of resources affecting instruction), *SchoolEmphasis* (i.e., Schools' emphasis on academic achievement), and *SchoolDiscipline* (i.e., School discipline and safety).

Methods

Participants and Instruments

A two-stage sampling design, with the selection of a sample of schools, and then the selection of randomly sampled one or two classes in each school was used in this study.

Table 1: Descriptive Statistics by Country

Country	number of schools	number of students	Rank	Mean Score
Finland	131	3626	8	514
South Korea	147	5015	1	613
Taiwan	149	4958	3	609
Turkey	236	6502	24	452

The data for this study consisted of 8th grade students' responses to the TIMSS 2011 Student Questionnaire about their home and school lives, and the Mathematics Test, and school principals' responses to the TIMSS 2011 School Questionnaire about school environment such as school resources, and roles of the principals (Mullis, Martin, Foy, & Arora, 2012).

Data Analysis

Model building approach using HLM (Raudenbush & Bryk, 2002) was followed in this study to study the incremental contribution of student-level and school-level factors in explaining mathematics achievement. As the first step, unconditional models, which do not contain any student- or school-level factors were estimated for each country. The second step estimated was the random coefficients models containing only student-level factors. Finally, the third step estimated was the full contextual models with both student- and school-level factors.

In each step, model building approach was pursued through model comparisons using Likelihood Ratio Test used to compare deviance values of two nested models. Deviance represents the badness of fit of a given model, and subtracting the deviance of the simpler model from the deviance of the more complex model demonstrates the change in the deviance values. HLM 7.00 software

(Raudenbush, Bryk, & Congdon, 2011) was used to build a two-level HLM model. To handle missing data, listwise deletion was performed before starting the analysis.

Results

HLM models

Unconditional models were found to be same for each country. Regarding intra-class correlation (ICC) values, the largest variance between schools was in Turkey (31%), and the smallest variance was in South Korea (9%), indicating another evidence for the use of HLM in this study. Fixed effects of final mathematics achievement models for each country are given in Table 2.

Table 2: Fixed Effects of Student- and School-Level Factors in the Final HLM Models

	Finland	South Korea	Taiwan	Turkey
Grand mean, γ_{00}	701.17**(10.62)	948.56**(7.38)	975.83**(17.03)	807.34**(17.50)
Student Level				
MotherEducation, γ_{10}	—	—	—	5.21**(0.79)
FatherEducation, γ_{20}	—	—	—	—
HomeResources, γ_{30}	20.13**(2.07)	40.05**(1.93)	35.55**(2.34)	30.27**(2.10)
Bullying, γ_{40}	-4.81**(1.41)	—	—	-3.72*(1.58)
LikeMath, γ_{50}	—	13.46**(1.96)	22.30**(2.26)	—
ValueMath, γ_{60}	2.54*(1.24)	17.95**(1.66)	10.60**(1.87)	—
ConfidenceMath, γ_{70}	47.47**(1.19)	57.77**(2.25)	44.87**(2.42)	63.72**(1.60)
School Level				
IncomeArea, γ_{01}	—	12.30**(2.28)	29.63**(6.92)	—
JobSatisfaction, γ_{02}	—	-1.69(2.22)	—	—
Intimidation, γ_{03}	-6.66(3.57)	—	—	—
Fights, γ_{04}	—	—	—	—
LackofResources, γ_{05}	—	—	—	—
SchoolEmphasis, γ_{06}	7.98*(3.45)	6.67*(2.71)	20.14**(5.82)	33.09**(5.48)
SchoolDiscipline, γ_{07}	13.75**(4.03)	—	—	12.51*(3.93)

— not statistically significant, and significantly worse model, therefore removed from the model
* $p < 0.01$; ** $p < 0.001$; otherwise, not significant; Round brackets indicate standard errors

Based on Table 2, *FatherEducation*, *Fights* and *LackofResources* had no effects on students' mathematics achievement in any of the four countries. While *MotherEducation* had an effect only in Turkey, *HomeResources*, *ConfidenceMath* and *SchoolEmphasis* were found to influence students' mathematics achievement in all four countries. Moreover, there was a significant negative relationship between *Bullying* and mathematics achievement in Finland and Turkey, but not in South Korea and Taiwan. Similarly, *LikeMath* and *IncomeArea* did not contribute to students' mathematics achievement in Finland and Turkey, but they did in South Korea and Taiwan. Finally, *SchoolDiscipline* was related to students' mathematics achievement in Finland and Turkey, but not in South Korea and Taiwan.

Conclusion and Discussion

Based on the results, in Finland, the final mathematics achievement model consisted of *HomeResources*, *Bullying*, *ValueMath*, and *ConfidenceMath* at the student level; *Intimidation*, *SchoolEmphasis*, and *SchoolDiscipline* at the school-level. Bullying at school and intimidation or verbal abuse among students such as texting might prevent Finnish students from spending less time with mathematics. In South Korea, the student-level factors were *HomeResources*, *LikeMath*, *ValueMath*, and *Confidence-Math*; and the school-level factors were *IncomeArea*, *JobSatisfaction*, and *SchoolEmphasis*. The results suggest that affective characteristics such as enjoyment of learning mathematics, value of learning mathematics, and self-confidence in mathematics are important factors for South Korean students to perform better in mathematics. In Taiwan, the student- and school level factors were found to be exactly the same as those of South Korea except *JobSatisfaction*. In Turkey, the final mathematics achievement model included *MotherEducation*, *HomeResources*, *Bullying*, and *ConfidenceMath* at the student-level; *SchoolEmphasis* and *SchoolDiscipline* at the school level. While the strongest predictors of mathematics achievement for Turkish students were *ConfidenceMath* and *SchoolEmphasis*, the weakest predictors were *Bullying* and *SchoolDiscipline*. Hence, more focus, in Turkey, should be given on placing high academic standards at schools and increasing students' confidence levels.

In conclusion, *HomeResources*, *ConfidenceMath*, and *SchoolEmphasis* were the common factors reported here regarding their contributions to students' mathematics achievement in all four countries. The more educational resources are available at home, the more self-confidence students have and the more emphasis on academic achievement at schools is, the better students' mathematics performances will be. Although there was no causal evidence in this study about the relationships among mathematics achievement and these factors, the consistent positive relationship among them in each country suggests that *HomeResources*, *ConfidenceMath*, and *SchoolEmphasis* are critical and necessary characteristics of mathematics achievement.

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PARTNERING FOR PROFESSIONAL DEVELOPMENT AT SCALE

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In this paper, we present a set of principles used to co-design a statewide professional development initiative to support the implementation of new high school mathematics standards. We describe the focus of our research-practice partnership, use of design-based implementation research to organize a collaborative work, theoretical perspective on learning, and design principles for learning environments and infrastructures to support learning at scale.

Keywords: Standards, Research Methods, Design Experiments

As mathematics educators, we believe that research in our field can improve mathematics teaching and lead to more equitable learning opportunities and outcomes for children in schools. Yet, the disconnect between evidence from the research community and its use in practice and policy remains a significant challenge to educational improvement (Battista, 2007). In recent years, there has been increased interest in collaborative research and development where researchers and practitioners work together on persistent problems of practice. For researchers, research-practice partnerships provide opportunities to design and research innovations that are both timely and useful in practice, thus promoting the development of theories of implementation (Penuel & Farrell, in press). For practitioners, partnerships provide opportunities to work closely with researchers to generate solutions to problems of practice that are timely and relevant. Evidence from implementation research suggests research-practice partnerships are a promising approach for improving practice (Coburn et al., 2013).

The goal of this paper is to present a set of principles that guided the initial design of a multi-year, statewide professional development initiative to implement new mathematics standards and promote more equitable learning opportunities and outcomes for each and every student in schools. As part of a design-based implementation research (DBIR) (Fishman et al., 2013) study to identify and create mechanisms to promote the use of research in practice, this paper describes our initial claims about organizing respectful and productive professional learning opportunities for mathematics teachers and leaders at scale. We begin by describing the background and context of our research-practice partnership. Next, we outline our theoretical perspectives on learning and implementation. We then offer a set of design principles that translate our theoretical perspective into an initial design and provide an example to illustrate the ways the principles reflect our overarching conjecture about learning.

Background

The partnership began when our state initiated a process of reviewing and revising the Common Core State Standards. A group of mathematics teachers, district and state leaders, mathematicians, and education researchers were convened to review data collected from multiple stakeholder groups, a set of recommendations from a legislatively appointed committee, and research on student learning to inform a new set of high school mathematics standards for three integrated and sequenced high school mathematics courses. During the process, the group voiced concerns over previous efforts to implement standards and shared their commitments to better supporting teachers, schools, and districts with timely instructional resources and quality professional learning opportunities. In June of 2016, the new standards were adopted and scheduled to be implemented the next academic year (August, 2016). After the official adoption, the state agency initiated a formal partnership with our institutions to co-design and study a statewide professional development initiative to support efforts to implement the new standards.

We elected to use DBIR to organize our work. As an approach to collaborative research and development, DBIR focuses not only on developing tools and environments for learning but also on creating structures and supports to scale and sustain them (Fishman et al., 2013). Partners agree to focus on a problem of practice and commit resources to co-design tools and learning environments to address the problem with attention to scale. During implementation, they seek to both improve the design and generate theories of learning and implementation through research. They also create supporting infrastructures to develop capacity and sustainability.

Through negotiation, we agreed to focus on designing professional learning opportunities to share research on mathematics learning and teaching as a means of supporting the standards implementation. To this shared problem of practice, partners brought a broad range of resources. Teachers, researchers, and leaders collectively brought expertise in mathematics content, research on teacher learning, mathematics education, educational leadership, state and district policy, and the contexts of classrooms. Some brought experience in establishing systems for communication, others brought knowledge of research underlying revised standards and skills at conducting research. In what follows, we provide an overview of our efforts at co-designing by first describing our theoretical perspective and overarching conjecture about learning.

Theoretical Perspectives

Our initial design follows from a theoretical perspective of learning as a transformation of participation in practice (Lave & Wenger, 1991; Wenger, 1998). Wenger (1998) outlines learning as social participation in the practices of communities and forming identities. He defines practice as representing social, contextual, and historical ways of belonging that reside within and among members of communities. He outlines three dimensions of practice that bring coherence to a community – mutual engagement, joint enterprise, and shared repertoire. Because practice is a defining characteristics of a community, communities are formed by collaboratively engaging with common resources (e.g. routines, tools, language) toward a common goal. As members participate in the practices of the community, they negotiate new meanings and refine their practice. From this perspective, learning occurs as an ongoing negotiation of meaning within the community.

Though boundaries of practice distinguish communities across the social landscape, they are also a source of new learning. In boundary encounters, members of distinct communities come together and negotiate meaning around boundary objects - artifacts that carry meaning in multiple communities and support knowledge exchange. In boundary encounters, as engagement around boundary objects occur, members from these distinct communities introduce, negotiate, and integrate elements of other practices with their own and over time form boundary practices that are distinct to the boundary community. Thus, learning also occurs through ongoing negotiations of meaning in boundary encounters which may result in the incorporation of new elements of practice in their respective communities.

Large enterprises, which may be too diverse to consider a single community or a small set of boundary communities, still share a number of related yet distinct practices with a similar goal. These “constellations of practices” (Wenger, 1998, p. 127), while distinct in some ways, are aligned toward some common focus and share commonalities (e.g. history, shared artifacts, geography). To enable connections of practice among the communities in the constellation, Wenger suggests infrastructures that enable coherence and alignment across the constellation as key components of any design for learning.

Wenger (1998) uses the idea of a "learning architecture" to describe necessary elements and decisions one makes when designing for learning. Central to a learning architecture are infrastructures that enable alignment and meaningful engagement across the design. Given this perspective, we view teachers, school districts, state leaders, and researchers as members of distinct communities of practice within a constellation of mathematics educators within the state focused on implementing new mathematics standards and promoting more equitable learning opportunities and outcomes for each and every student.

Principles for Designing Architecture for Learning

To translate this theoretical perspective into a learning architecture, we began by reviewing literature on learning, professional development, mathematics teacher learning, implementation, and infrastructures to identify key ideas about learning in boundaries. Based on this review and the shared experiences in professional development within the partnership, we articulated a set of design principles to connect our theoretical perspective on learning to the specific learning environments – including the tasks, tools, participation structures, and discursive practices – and the infrastructures we aimed to design. The principles that guide our efforts to co-design a learning architecture to promote and sustain learning across boundaries of practice are described below. Principles related to learning environments that promote professional learning:

1. Build from the consensus view of effective professional development (Desimone, 2009) by providing access to safe and respectful opportunities to learn, fostering professional relationships (Darling-Hammond et al., 2009), and eliciting and using teachers' and leaders' expertise, contexts, and histories as resources for learning (Bransford et al., 2000; Goldsmith et al., 2014).
2. Focus on the social, cultural, and mathematical resources students bring to instruction (Civil, 2002).
3. Balance immediate support with participation structures that foster engagement by elucidating limitations of previous practice (Spillane et al., 2002; Stein & Coburn, 2008).
4. Utilize tools designed to introduce findings from research on student learning in mathematics and mathematics instruction (Bell & Rhinehart, 2016).

Principles related to infrastructures that build capacity for sustained change:

5. Coordinate resources and tools across the architecture to increase coherence (Hopkins et al., 2013; Wenger, 1998).
6. Make tools available for teachers and leaders to enlist parents, families, and community members as partners in students' mathematics education (Dantec & Disalvo, 2013).

To illustrate our use of the principles, we share the design of a closed webspace created for teachers and leaders as part of the implementation efforts. The list below outlines key features of the webspace that follow from one or more of the principles. For example, design feature 3 follows from principles 2, 4, and 5.

1. Use space to focus on discussions of student work and a forthcoming research brief series on equity in mathematics education

2. Use space as a repository for resources and means for discussion
3. Use space to promote the use of research on student and teacher learning through attention to designed content and pedagogy briefs
4. Use space to maintain access to recordings of prior webinars and regional meetings
5. Use space to promote the use of resources to support parents and community in understanding the new standards.

Conclusion

The principles outlined in this paper guided the translation of our theoretical perspective on learning into our initial design. Currently, we are conducting ongoing analyses of teachers' and leaders' participation to refine aspects of the learning architecture, investigate ways of improving infrastructures, and identifying additional elements to promote learning at scale. As we consider the conference theme, "Synergy at the Crossroads", we see partnerships with practitioners, not only sitting at the intersection of theory and practice, but doing so in a way that has the potential of significantly affect change at scale.

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REFLECTED ABSTRACTION AND PEDAGOGICAL NEED: TEACHERS' INTERTWINED KNOWLEDGE AND MOTIVATION FOR INSTRUCTIONAL REPRESENTATIONS

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Preparing beginning teachers to use instructional representations remains a significant challenge, in part because both knowledge and motivation are required. In this conceptual paper, the author uses the distinction between the Piagetian constructs of reflecting versus reflected abstraction as a lens to examine the central problem of teachers' intertwined knowledge and motivation for using instructional representations. Learning to use instructional representations of mathematical concepts may require a different kind of cognitive process than does learning the concepts themselves. The author argues that conceptual knowledge decreases teachers' intellectual need for representing foundational ideas. Pedagogical need is introduced as an alternative source of motivation.

Keywords: Teacher Knowledge, Teacher Beliefs, Teacher Education-Preservice

Instructional representations such as base ten blocks and number lines are an essential component of effective mathematics instruction in the elementary grades. Unfortunately, preparing beginning teachers to use instructional representations remains a significant challenge (Inoue, 2009; Kinach, 2002). Effective use requires knowledge (Izsák, 2008; Mitchell, Charalambos, & Hill, 2014), but knowledge is not enough. In a national sample of middle grade teachers—many with elementary certification—Jacobson and Izsák (2015) found knowledge was correlated with representation use in the classroom only to the extent that teachers were also motivated to use instructional representations. Without understanding how teachers' knowledge and motivation are intertwined, the problem of preparing effective elementary teachers remains.

I describe an initial theoretical account of both (1) why it is hard for teachers to develop knowledge and motivation to use instructional representations and (2) how such development might occur. I contrast an example of a student who is using a representation to develop new understanding with a preservice teacher (PST) who is using a representation to explain a concept they already understand. The examples emphasize a key difference between students who are learning mathematical concepts and PSTs who already understand them, a difference that I explicate using the Piagetian construct of reflecting abstraction. Finally, I extend the examples to contrast motivation for understanding new concepts with motivation for explaining concepts that are already understood. I conclude with implications for teacher education and research.

Teachers' Conceptual Understanding of Instructional Representations

Instructional representations feature prominently in characterizations of teachers' professional knowledge. Introducing the seminal notion of pedagogical content knowledge (PCK), Shulman, (1986, p. 9) described two categories of teacher work: appraising students' conceptions and reasoning and selecting and using instructional representations. Because students often use representations to communicate their reasoning, both categories rely on teachers' knowledge of instructional representations.

Distinguishing PCK from disciplinary knowledge reveals a central problem of teacher knowledge: Disciplinary knowledge—the ultimate goal of instruction—is compressed and highly structured and thus insufficient to do the work of teaching; instead, teachers need knowledge that is "unpacked" (Ball & Bass, 2000). The challenge for teacher education is that PSTs often lack both disciplinary knowledge as well as PCK. PSTs who develop disciplinary knowledge in teacher preparation content courses may find that what they know is ‘packed’ and thus less useful for the work of teaching. In methods classes, PSTs may struggle to apply what they understand about mathematics to the problems of teaching and learning.

I draw on a Piagetian distinction to contrast representation to learn and representation of what has been learned. Learners who coordinate simple concepts to achieve a goal can—often unconsciously—encapsulate their coordinated activity as a new concept. This process is driven by *reflecting abstraction* in which knowledge is reorganized to a higher level of cognition (e.g., from physical actions to mental operations; Piaget, 2001). The new resulting concept can then be used to achieve similar goals as previously were achieved through the coordination, but without need of the previously coordinated concepts. Piaget (2001) distinguished reflecting abstraction from *reflected abstraction* which involves *consciously* reflecting on a complex concept to become aware of ways it might arise from reorganizing simpler concepts (Tillema & Hackenberg, 2011). Using the distinction as a lens to examine instructional representations has promise for untangling the central problem of teachers’ disciplinary knowledge.

The work of Kylie, a 4th grade student discussed by Simon, Placa, and Avitzur (2016), illustrates how representations support reflecting abstraction. Figure 1a shows how she solved a fraction task. Asked to find 1/6 given a region named 1/3, she first iterated the region to recreate the whole (Step 2), then partitioned 1/3 (Step 3) to find a part such that 6 parts would make the whole. On the third such task, Kylie found the required part without iterating to the whole or partitioning it; she went directly from Step 1 to Step 4 in Figure 1 by using whole number division to appropriately partition the initial part. This change evidenced reflecting abstraction because she learned a new concept that went beyond the simple concepts she had previously coordinated. Now “she could produce $1/mn$ from $1/n$ by partitioning $1/n$ into m parts” (p. 78), but she no longer needed to think explicitly about the common whole, even less to draw it out.

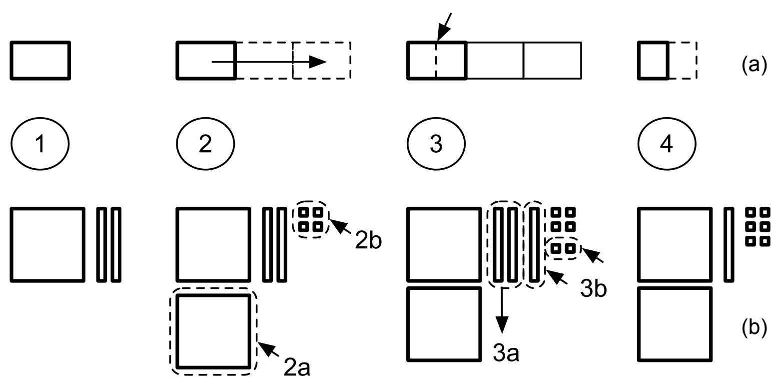

Figure 1. Sequences of instructional representation (a) from Kylie, a student, using drawings to find 1/6 given 1/3 by iterating to find the whole and then partitioning, and (b) from Alan, a PST, using base ten blocks to illustrate how he solved 120 + 96 by adding 100 and subtracting 4.

Reflecting abstraction undergirds mathematics learning but simultaneously it makes teaching difficult. Once a new, more powerful concept is constructed, the simple concepts from which it came—and from which someone else might be able learn—no longer seem relevant because the new concept changes how a problem is perceived. To illustrate this point, consider the instructional explanation offered by Alan, a PST who was asked to use base ten blocks to explain the solution of 120 + 96 to a student (Figure 1b). His said he would add 100 and subtract 4, and he illustrated this by moving in a 100-flat (Step 2a) and four 1-cubes (Step 2b). Then he said the answer was 216 without further explanation. He represented 216 by moving out two 10-longs (Step 3a) and moving in one 10-long and two 1-cubes (Step 3b). From an observer's perspective, the quantitative meaning of the 4 cubes changed dramatically; in Step 2 they represented 4 to be subtracted from 220 but in Step 3 they were part of the remaining 216. Alan gave no indication he was attending to quantitative meaning as he transformed the materials.

One possible explanation for these data is that Alan had already constructed a powerful concept for multi-digit addition; his knowledge was "packed." Because the answer was obvious to him, Step 3 was only a convenient way to show the answer, not an attempt to justify it. It was like a scene change between acts in a play—on stage but not meant to be seen. Alan's representational activity with base ten blocks was different in kind than Kylie's work on the first problem. Unlike Kylie's reasoning before the reflecting abstraction, Alan's reasoning did not depend on the instructional representation because of prior reflecting abstraction.

PSTs' explanations sometimes involve quantitatively incoherent use of instructional representations because after reflecting abstraction solutions do not depend on representation use. This observation clarifies the goals for teacher education: we must help PSTs value representations for showing how simple concepts underlie complex ones. Indeed, prior work has described how instructional representations can be an important catalyst for teacher learning (e.g., Kinach, 2002). Here is the novel insight: Teacher learning from representations can be characterized by *reflected abstraction* (Piaget, 2001). Just as instructional representations can help students learn concepts by providing a context for reflecting abstraction, I conjecture that representations can help teachers by providing a resource for reflected abstraction.

Teachers Motivation to Use Instructional Representations

In this section I start with the premise that motivation is critical to learning, and contribute a novel argument. In sum, the motivation to teach mathematics has a fundamentally different nature than motivation to learn mathematics because the need that drives each activity is different. This difference in motivation parallels the previous difference in representation.

I begin with the concept of *intellectual need*, and I paraphrase of a recent definition provided by Harel (2013). Suppose an individual encounters a mathematical problem that cannot be solved with their current state of knowledge. After some time working on the problem, suppose further that she is able to solve it. We would attribute new knowledge to her, and the intellectual need for that new knowledge is simply the problem that was at first unsolvable. She has an intellectual need for the new knowledge precisely because it enabled her to solve the problem.

To understand how the problem of teacher motivation is related to mathematical knowledge, it is helpful to return to the example of Kylie (Simon et. al, 2016). At first, Kylie said she had no idea how to solve the fraction tasks and had to draw out the whole; yet by the third task, her new concept made drawing unnecessary. Suppose that when Kylie grows up she becomes an elementary teacher. Responsible for teaching the same problem, what would *motivate* her to represent the whole by iterating 1/3 (Figure 1a, Step 2) since, because of her conceptual knowledge, partitioning 1/3 directly (Step 4) is now obvious? Even after reflected abstraction to gain conscious awareness of the fundamental role of the whole in her partitioning concept, the extra drawing is more work, takes more time, and no longer fulfills an intellectual need. Instead of being necessary for Kylie, drawing the whole is inefficient and laborious.

It follows from this thought experiment that representing simple component concepts once one knows the higher-level concept does not satisfy personal intellectual need. Instead of motivating PSTs by focusing on their own personal intellectual need, teacher educators might be more successful at motivating PSTs to use instructional representations if they can clarify and make explicit the pedagogical need that is part of their putative role as teachers.

I define *pedagogical need* as analogous to intellectual need. In view of specific knowledge in the curriculum, a teacher's pedagogical problem is engendering students' intellectual need for this knowledge and the knowledge itself. A teacher has a pedagogical need for the PCK that enables her to solve this pedagogical problem.

In the case of Kylie, the higher-order knowledge that made drawing the whole unnecessary came about in relation to an intellectual need engendered by the sequence of problems she encountered and her concurrent representational activity. The pedagogical problem in this case was how to help Kylie develop this knowledge, and the corresponding PCK includes understanding the sequence of tasks and the importance of drawn representations that lead Kylie to realize her intellectual need and the corresponding knowledge. Therefore, in order to gain motivation for using instructional representations, PSTs must understand how instructional representations can function as a bridge between students' initially simple ideas and the more complex ideas that are the goal of instruction. That is, they must come to understand how instructional representations solve pedagogical problems they may encounter.

Conclusion

I have argued that learning to use instructional representations of mathematical concepts may require a different kind of cognitive process—namely, reflected abstraction—than the cognitive process used to learn concepts themselves. In addition, I have proposed a link between teachers' knowledge and their motivation for using instructional representations by arguing that conceptual knowledge decreases teachers' intellectual need for representing foundational ideas. I propose pedagogical need as an alternative source of motivation for instructional representation.

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TAKING MEASURES TO COORDINATE MOVEMENTS: UNITIZING EMERGES AS A MEANS OF BUILDING EVENT STRUCTURES FOR ENACTING PROPORTION

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Rhythm is a means of production—a scheme for coordinating the enactment of real or imagined physical movements over time, space, material resources, and concerting participants. In activities requiring the coordination of two or more continuous *motor actions, rhythmic re-assembly of the actions creates a goal event structure mediating the enactment. Yet building that structure requires first unitizing continuity. Unitizing could thus be conceptualized as a cultural–historical strategy for supporting mundane routines by parsing, distributing, and codifying activity as a sequence of iterated actions of equivalent magnitude. Ipso facto, unitizing shifts us from naive to disciplinary activity: articulated rhythm is an ontogenetic achievement driving cognitive growth. We present empirical data of a student spontaneously measuring continuous actions as her means of organizing the enactment of a bimanual task designed for proportions.*

Keywords: Cognition, Measurement

Introduction

Mathematics learning activities designed in accordance with embodiment theories of cognition create opportunities for students to engage in the solution of sensorimotor problems prior to interpreting and representing their solutions formally in normative symbolic register (Lee, 2015). This heuristic design principle is grounded in a constructivist (Boom, 2009) and enactivist (Reid, 2014) consensus that concepts emerge through noticing repeated patterns in perceptual dynamics guiding motor action. Empirical work has corroborated this historical conjecture through combining eye-tracking and clinical data analysis (Abrahamson et al., 2016).

The objective of this paper is to contribute to scholarship on students' passage from sensorimotor action to mathematical reasoning. Our study's empirical context is an action-based embodied design, the Mathematical Imagery Trainer (Abrahamson, 2014), wherein participant students enact a challenging bimanual motor action related to the development of proportional reasoning. By micro-analyzing students' behaviors, we demonstrate spontaneous utilization of measurement units. We argue that emergent rhythmic enactment facilitated discretization that led to evoking measurement units and in turn building an event structure mediating the enactment.

Theoretical Background

Cultural–historical positions view the practice of measuring, along with its artifacts, routines, and discourse, as evolved to serve an essential means of mathematizing human action and thought (Malafouris, 2013). In particular, measure units enhance one's ability to estimate, compare, and calculate continuous quantities (Stavy & Babai, 2016). Cognitive-developmental psychology defines measuring as follows: "To measure is to take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of a whole: measurement is therefore a synthesis of sub-division and change of position" (Piaget, Inhelder, & Szerninska, 1960, p. 3). Measuring competently thus requires: (a) conserving the size of the unit; (b) iterating the unit; and (c) transitively, inferring the relative length of two objects by comparing them to a unit. When we imbue these measurement routines with the temporal dimension, we can discern the enactment of rhythmic actions. Indeed Radford (2015) found structured temporal qualities in analyzing students' performance in algebraic pattern-generating activity: meter, rhythmic grouping, prolongation, and theme. Sinclair, Chorney, and Rodney (2016) used rhythm as their focal analytic construct in investigating the mathematical activity of children interacting with a tablet application designed for learning number. They implicate rhythmic actions as the embodied origin of cognitive structure, preceding planning and reflection. In like vein, Bautista and Roth (2012) documented the role of rhythmical hand movements in students' haptic engagement with geometrical regularities in material solids (cf. Bamberger & diSessa, 2003).

In summary, embodiment perspectives on mathematical cognition conceptualize dynamical sensorimotor problem solving as constitutive of conceptual growth. Positioned within the embodiment paradigm, we present a case of a student who spontaneously evoked measurement operations as her means of regulating the enactment of a challenging bimanual motor task designed to support the development of proportional reasoning. Our objective is to enrich scholarship on the rhythmic qualities of mathematics learning by way of interpreting the circumstances and process leading to her rhythmic mathematization of continuous quantities.

Methods

The empirical context for this study was the Mathematics Imagery Trainer for Proportion (MIT-P; see Figure 1). Unlike earlier studies in this empirical context (Abrahamson & Trninic, 2011), in the current study no mathematical tools were offered, such as a grid or numerals.

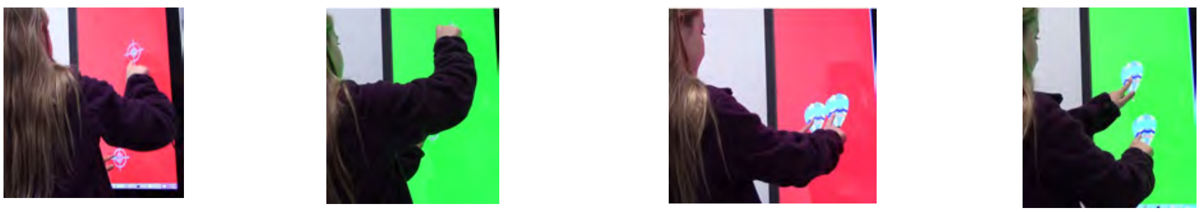

Figure 1. The Mathematical Imagery Trainer for Proportion (MIT-P). The student manipulates two cursors along vertical axes, one by each hand. The task is to make the screen green and then keep it green while moving your hands. The screen will be green only when the heights of the two cursors above the screen base relate by a particular ratio unknown to the user (e.g., here 1:2). Otherwise it will be red. Cursors may be either "stark" (e.g., generic targets; see on left) or "iconic" (e.g., hot air balloons; see on right).

K was an 11-year-old female student, one of 25 students participating voluntarily in a task-based semi-structured clinical interview (for details, see Rosen, Palatnik, & Abrahamson, 2016). The interview lasted in total 19 minutes: a general introduction (1 min.); and the problem-solving phase (18 min.), where she manipulated: (a) hot-air balloons (7 min.); (b) cars (4 min.); and (c) crosshair targets (7 min.). The interview took place in our lab and was audio–video recorded.

We located all the events where the student expressed new insight pertaining to her manipulation strategy. The interview was then parsed into episodes, running from each insight to the next. Episodes were further coded as: (a) either researcher- or self-initiated; and (b) discrete ("finding green" static co-locations) or continuous ("keeping green" while sliding the cursors).

Applying grounded micro-genetic analysis, we focused on the students' range of physical actions and multimodal utterance pertaining to the available media (Ferrara, 2014) as well as on the task-effectiveness of their actions. First, we attended to student actions that preceded their articulation of a new rule for "making green," searching in particular for patterns in the timing and sequencing of student hand movements through space (Sinclair et al., 2016). A notation system emerged for the most frequently used movements. For example, vertical bimanual movement with the right hand going up and the left going down was denoted as "↓,↑ ," and placing both fingers statically on the screen as "●,●." Second, we analyzed K's responses to our recurring question, "How would you explain your strategy for finding green to another person?"

Results: Spontaneous Evocation of Units of Measurement

In our analysis we will attempt to implicate the specific event of K evoking measure units as a formative moment in her progress from unreflective continuous movement to unit-based discrete movement and, through this, in her learning of proportion.

Before evoking measuring units, K had developed two different rhythmical patterns as her task solutions: continuous movement, where the right hand moves twice faster than left (↑ ↑x2): and discrete, where she placed her fingers together (●,●) then vertically apart (↑,↓), traversing along recurring screen locations: bottom, middle, top. In articulating each of her strategies, K implicitly evoked particular qualities of the situation, such as the distance between her hands, as things she noticed and aimed to control even as she was moving her hands. For instance: "Down here my hands were really close, and then up here they were a little apart, and then up here they were really apart." Hoping for greater specificity of her movement rules, the interviewer asked K, "Ok, do you have any sense of… kind of…how this [the distance between her hands] is changing? How much it is changing, how much faster it is moving?" Beginning at 05:29, where she was working with hot-air balloon cursors, K responded by performing a particular action pattern repeatedly, at a constant pace, at three screen locations: bottom, middle, and top, stating that the balloons were: (bottom) "touching each other"; (middle) "There's about a balloon between them….the length of the balloon"; and (top) "Two balloons [apart], maybe." At 6:45 K repeated: "Ammm… [Quick succession of demonstrations: bottom ●,●, ↓,↑; middle ●,●, ↓,↑; top ●,●, ↓,↑] kind of at the bottom, there… it goes *zero* balloons between them, in the middle there is *one* balloon between them, and at the top, *two* balloons between them. So *it grows by one* at a time." K thus spontaneously utilized an available virtual object as a measure unit.

When the interviewer asked her to show "how to keep the screen green," K first gestured and then moved her hands continuously, with one hand moving twice as fast [↑ , ↑ x2]. It is of note that she tried to use her insights from the previous enactment as landmarks, that is, to connect the bottom, middle, and top discrete solutions into a single continuous enactment, as follows:

> (7:15) K: I would say, like, start at the bottom, and put them close together. And then, move one hand up faster… Wait, actually, [inaudible] …and as I said, in the middle, they are separated like one balloon [inaudible], and at the top two balloons.

Thus, a qualitative scheme for finding and keeping green, "one hand moves faster than the other as it goes up," assimilated a quantitative scheme, "it grows by one at a time," to better serve K's goals. Our claim is that the rhythmic qualities of K's actions—iteration, grouping, stability—as well as the interviewer's prompt to quantify ("how much") catalyzed this process. We observed feedback loops, where movements were coordinated into action patterns, and those patterns in turn were iteratively repeated, both spatially (bottom, middle, and top of the screen) and temporally. The linear extents of the hands' respective displacements came to attention *as a result of experiencing/enacting the emergent rhythm*. Namely, the rapid, cyclic repetition of implicitly measured actions gave rise to rhythmic enactment. The unit of measure emerged as a spontaneous combination of a stable pattern of movements and relatively stable perceptual elements, driven by a task demand to reflect on her own actions. Later in the interview, K quickly reenacted the new quantitative scheme in the case of car icons as well as the stark icons.

Conclusion

Rhythm is a means for coordinating physical operations over time, space, and material (or virtual) resources into new sensorimotor schemes. In the absence of any explicit frame of reference, rhythmic enactment bootstraps discretization, thus leading to further evocation of measurement units, which in turn improve performance and are thus adopted and codified. K's actions evolved from independent, explorative, seemingly uncoordinated movements into a stable temporal–spatial choreography comprising a succession of coordinated, measured clusters of movements preserving a relational invariant (see also Sinclair, Chorney, & Rodney, 2016).

K succeeded in coordinating her actions to produce green effectively well before she was able to articulate quantitative properties or her actions. When she first constructed a quantitative scheme for these actions, K was conscious not of a static structure. Rather, she responded to epiphenomenal features in the rhythmic cadence of enacting these coordinated actions. As such, rhythmic enactment mediated a transitioning from naïve to scientific reasoning. The temporal qualities of K's rhythmic enactment across the continuous display assimilated spatial qualities of available objects (hot-air balloons) to deploy motor-action execution over imaginary discrete units of measure. K thus extracted a measure unit from the situation as her means of extending insights from discrete to continuous actions. Unitizing is thus an evolved strategy for enhancing the coordination of continuous action by distributing it over regulated cycles of iterated enactment over projected spatial extensions. Further research is needed to understand the interplay of rhythm, action, and discourse in the elicitation of unitizing operations.

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REFRAMING MATHEMATICS DISABILITY: EMERGING CRITICAL PERSPECTIVES

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Keywords: Equity and Diversity, Research Methods, Problem Solving

This poster, inspired by last year's working group on Rethinking Mathematics Intervention (Sheldon et al., 2016), offers a review of the literature into recent critical perspectives on mathematics disability. Mathematics disability is not a topic normally covered in mathematics education journals; it is mostly relegated to special education journals. (Sheldon, 2013; Lambert & Tan, 2016). We seek to create to bridge the divide between special education and math education by creating intersections between these areas of study.

The purpose of this literature review poster is to explore emerging critical perspectives on mathematics disability, with a particular focus on perspectives grounded in critical theories. In order to conduct the review, we consulted two recent exhaustive literature reviews (Lambert & Tan, 2016a; Lewis & Fisher, 2016), drew upon a database of 107 articles, conducted keyword searches, and contacted scholars interested in mathematics disability.

The review revealed three main themes. First, research related to mathematics disability has reflected methodological conservatism. Second, there is a significant literature calling for mathematics reform within Special Education (e.g. Woodward & Montague, 2002). Third, we found an emerging literature involving critical perspectives. In this emerging literature, mathematics education researchers view disability as a difference rather than as a deficit and propose looking at intersectional notions of disability and the ways in which they affect perceived ability (Lambert, 2016a).

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THEORY IN MATHEMATICS EDUCATION: INTRA-ACTION AND (RE)CONFIGURATION

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Keywords: Equity and Diversity, Research Methods, Standards

Using (post) qualitative methodology located within a critical postmodern theoretical frame, I will examine the intersection(s) and borders of the field of mathematics education and the posts (post-structural, new materialist, post-humanist, post-qualitative). The "field" in this research will be considered a fluid space that shifts and responds as "new" research is brought into it. The research questions that guide this study are:

1. What are the taken for granted, expectations and norms of these spaces?
2. How are post theories being used/taken up within mathematics education?
3. How are the borders and boundaries of the field of mathematics education shifting in response to these theories and methodologies?

Mapping the Machine

The poster will consider how research is counted or excluded from the body of mathematics education research recognizing that "counting practices are bound up with the production of natural and social orders" (Martin & Lynch, 2009, p. 245). I will map the intersections, "lines of flight" (Deleuze & Guatarri, 1980/1987), and ruptures in the research body and consider how these matter. Drawing on new materialist theories, (Barad, 2007), I will consider the measuring and counting apparatus (math education journals, citations, and conference proceedings) which are always already entangled with the object of measurement (Barad, 2007). I will consider, following Dolphijn and Turin (2013), how "measurements are the entanglement of matter and meaning" (p. 16) and how they are, therefore, "calling data into being" (St Pierre, 2013, p. 223).

The intra-action (Barad, 2007) of theory with mathematics education research will be tentatively mapped giving attention to the complications created in the intra-action that allow (re)configurations of school mathematics (de Freitas & Sinclair, 2014). I will consider places where the intra-action allows for gathering speed and attention and places where intra-action disrupts normative practices in mathematics education research.

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NUMBER THEORY AND INTRODUCTORY MATHEMATICS IN HIGHER EDUCATION

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Keywords: Curriculum, Technology, Post-Secondary Education

Due to sequential organization of university level mathematics curricula, non-math majors often do not get to experience abstract, higher level mathematics. We present preliminary results from working with students enrolled in an Elementary Statistics course to explore patterns in results related to the accessible, but still unproven, Collatz Conjecture from number theory. Not only does this work provide an account of where a conjecture from advanced mathematics is engaged by students in early, university-level mathematic courses but by analyzing the results from the recursive implementation of Collatz Conjecture (using the agent—based NetLogo programming language) using statistical methods from what they are studying, the work may illustrate how explorations in one area of mathematics can, at a more advanced level, be seen to interact in significant ways with other areas of mathematics.

Paul Erdös said "mathematics is not ready" to analyze the Collatz Conjecture. For our study students were able to generate, using a NetLogo program that we developed for this purpose, results from implementing this conjecture over a large number of user-specified values (e.g., all even numbers between 1 and 10,000). The students' results are displayed and analyzed using techniques from introductory statistics (e.g., linear regression). We present our results from applying a grounded theory methodology to analyze a series of semi-structured interviews conducted with students as they engaged with cycles of NetLogo-based exploration followed by the application of statistical methods to the results (Charmaz, 2008). The longer-term goal of this line of research is to find ways to understand how it might be possible and what it might mean to have students in introductory courses be able to explore topics from advanced mathematics.

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Chapter 14

Working Groups

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ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION: RESETTING AT THE CROSSROADS

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As the title suggests, this Working Group has a dual focus on issues of mathematics teaching and learning and issues of equity and diversity. Following on the topics discussed at the Working Group between 2009-2016, this year the focus is resetting and reestablishing the purpose of this group and supporting the development of new directions for equity-oriented research working groups. The sessions will focus on regrouping attendees interested in equity, generating and brainstorming new subtopics and potential projects, and working to establish standalone working groups dedicated to furthering research on equity. The purpose of this resetting is to encourage a move away from "big-tent" equity thinking and toward more productive working collectives.

Keywords: Equity and Diversity

Brief History

This Working Group originates from the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME scholars graduated from one of three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA) that comprised the DiME Center. The Center was dedicated to creating a community of scholars poised to address critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).The DiME Group (as well as subsets of that group) has engaged in important scholarly activities, including the publication of a chapter in the *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007), a one-day AERA Professional Development session examining equity and diversity issues in mathematics education (2008), a book on research of professional development that attends to both equity and mathematics issues with chapters by many DiME members and other scholars (Foote, 2010), and a book on teaching mathematics for social justice (Wager & Stinson, 2012) that also included contributions from several DiME members. In addition, several DiME members have published manuscripts in a myriad of leading mathematics education journals on equity in mathematics education. This working group provides a space for continued collaboration among DiME members and other colleagues interested in addressing the critical problems facing mathematics education.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Over time, the Working Group has encouraged building on and featuring senior scholars' work, including Marta Civil (Civil, 2007; Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Eric Gutstein (Gutstein, 2003, 2006; Gutstein & Peterson, 2013), Jacqueline Leonard (Leonard, 2007; Leonard & Martin, 2013), Danny Martin (Martin, 2000, 2009, 2013), Judit Moschkovitch (Moschkovich, 2002), Rochelle Gutiérrez (2002, 2003, 2008, 2012, 2013) and Na'ilah Nasir (Nasir, 2002, 2011, 2013; Nasir, Hand & Taylor, 2008; Nasir & Shah, 2011). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

Previous iterations of this Working Group at PMENA 2009 – 2013, and 2015-2016 have provided opportunities for participants to continue working together as well as to expand the group to include other interested scholars with similar research interests. Experience has shown that collaboration is a critical component to this work. These efforts to expand participation and collaboration were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group each of the past five years. Moving forward, we hope to "reset" the group toward providing opportunities for a new generation of scholars whose work intersects with issues of equity/inequity, diversity/inclusion, privilege/oppression, and justice in mathematics education research, practice, and development.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, prospective teacher education (primarily in mathematics methods classes), factors impacting student learning (including the learning of particular sub-groups of students such as African American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics. There continues to be too great a divide between research on mathematics teaching and learning and concerns for equity.

The Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes (and for whose benefit), and how these culturally bound distinctions afford and constrain opportunities for traditionally marginalized students to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions the Working Group will continue to consider are:

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?

- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?
- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?

Specific to the intent of this year's Working Group, we will organize around questions like the ones above in order to create specific, targeted working groups that are charged to address and act around such questions.

Plan for Working Group

Based on feedback from the previous year and the emergence of new working groups related broadly to "equity," this working group should shift toward a renewed focus on facilitating "collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education." However, we propose to reconfigure the working group toward being a catalyst for new spaces instead of a "destination" for the inclusion of equity discourse within the PME-NA organization. To put it differently, our vision for the working group should be to bring together attendees toward developing their own agendas and specific working groups related to equity-oriented themes--or toward themes that push the field beyond traditional equity discourses yet adhere to the needs and challenges of inequity within mathematics education.

Our plans for PMENA 2017 will proceed as follows. Each session will build on previous sessions, beginning with a facilitated conversation around resetting of the working group. The format for the sessions will include:

- DAY 1: Resetting, Norm-setting, and Brainstorming: On first day, we we lead attendees through introductory activities, collective norm-setting, and a series of small- and whole-group brainstorming activities that will generate new ideas and directions for the working group more broadly.
- DAY 2: Agenda-setting: On the second day, the major focus will be the development and support of new smaller sub-specializing groups based on the reported interests of attendees. We will work with and encourage these subgroups to establish possible common topics of interests, potential products, and planning for the next year to support the growth of their group and topic.
- DAY 3: Working working groups: On the third day, the newly established subgroups will "take flight" and initiate their yearly plan to support their chosen topics.

Previous Work of the Group

The Working Group met for productive sections since 2009. In 2009, participants identified areas of interest within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. Over the past five years subgroups met to consider potential collaborative efforts and provide support. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. In addition, graduate students had the opportunity to share research plans and get feedback. The following were topics covered in the subgroups:

- Teacher Education that Frames Mathematics Education as a Social and Political Activity
- Culturally Relevant and Responsive Mathematics Education
- Creating Observation Protocols around Instructional Practices
- Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms
- A Critical Examination of Student Experiences

As part of the work of these subgroups, scholars have been able to develop networks of colleagues with whom they have been able to collaborate on research, manuscripts and conference presentations.

As a result of the growing understanding of the interests of participants (with regard both to the time spent in the working group and to intersections with their research), we began to include focus topics for whole group discussion and consideration and continued to provide space for people to share their own questions, concerns, and struggles. With respect to the latter, participants have continually expressed their need for a space to talk about these issues with others facing similar dilemmas, often because they do not have colleagues at their institutions doing such work or, worse yet, because they are oppressed or marginalized for the work they are doing. These concerns, in part, informed the focus topics for whole group discussion and consideration. For example, in 2009 research protocols (e.g., protocols for classroom observation, video analysis and interviewing) were shared to foster discussions of possible cross-site collaboration. In 2012, the Working Group explicitly took up marginalization in the field of mathematics education with a discussion about the negotiation of equity language often necessary for getting published; this was done in the context of the 'Where's the mathematics in mathematics education' debate (see Heid, 2010; Martin, Gholson, & Leonard, 2010). Dr. Amy Parks was invited to join Working Group organizers to share reflections on their experiences. In 2013 the Working Group hosted its first panel in which scholars (Dr. Beatriz D'Ambrosio, Dr. Corey Drake, Dr. Danny Martin) shared their perspectives on the state of and new directions for mathematics education research with an equity focus. The success of prior panel discussions have encouraged us to use that format as a launching point for deepening conversations on lingering tensions in the field. We see this working group as questioning critical borders that persist within mathematics education.

Pre-Conference and Follow-up Activities

In order to best plan for working group facilitation and prepare attendees for working group participation, we plan to send out pre-conference communication, including a Qualtrics survey, to former and potential participants in order to gauge the topics and kinds of work being done or sought, as well as the resources and forms of support desired by participants.

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ADVANCING PEDAGOGIES OF ENACTMENT IN MATHEMATICS PROFESSIONAL EDUCATION: IMPLICATIONS FOR RESEARCH AND PRACTICE

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This working investigates the design and use of pedagogies of enactment in mathematics teacher education and professional development. These pedagogies, which feature coached rehearsals and enactments, provide a promising way to support preservice and practicing teachers in developing skill with core instructional practices, aligned with ambitious and equitable goals for instruction. Building on existing research and our discussions at PME-NA 38, we will engage in collective investigation around three central issues: articulating theories of and perspectives on teacher learning to frame research and practice, foregrounding issues of equity in this work, and conceiving partnerships to support and scale these efforts. These discussions will lead to immediate takeaways as well as plans for future collective efforts around research and practice.

Keywords: Instructional Activities and Practices, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development

Brief History of the Group

This working group is a continuation of a group that initiated at PME-NA 38 in Tucson, Arizona in 2016. The group is focused on advancing the use and study of "pedagogies of enactment" (Grossman, Hammerness, & McDonald, 2009), specifically the use of coached rehearsals of instructional activities (Kazemi, Ghousseini, Cunard, & Turrou, 2016; Lampert et al., 2013) toward the development of core or high-leverage practices (Ball & Forzani, 2009; McDonald, Kazemi, & Kavanagh, 2013). We, as a group, focus on this work in the context of secondary mathematics teacher preparation and professional development (PD).

Prior to the first meeting at PME-NA 38, a group of secondary mathematics teacher educators (TEs) and researchers (represented, in part, by the leaders of this working group) collaborated informally for five years to consider the work of secondary teacher preparation in relation to recommendations from Grossman and her colleagues for teacher education and the work of Lampert, Kazemi, Franke and their colleagues in the area of elementary mathematics education. As a result of this collaboration, small groups of teacher educators have developed tools for supporting the development of ambitious mathematics teaching that leverages students' work with the Standards for Mathematical Practice (NGA & CCSSO, 2010). These tools included a collection of "instructional activities" (IAs; Lampert & Graziani, 2009) focused on secondary mathematics topics that have been shared across institutions, resulting in focused data collection and inquiry. At PME-NA 38, working group participants were able to discuss and glean insights from current efforts in development and research.

In this proposal, we first outline some background information related to pedagogies of enactment. We then detail the outcomes of our group's initial work at PME-NA 38 and the efforts that have followed. From this, we propose a new set of focal issues that continue to press efforts to design, use, and research pedagogies of enactment moving forward: (1) articulating theories of and perspectives on teacher learning; (2) focusing on issues of equity; and (3) conceiving partnerships.

Finally, we outline how the sessions of the working group will be organized to promote participant engagement around these three focal issues.

Background Information

In this section we frame the efforts of this working group by specifying what we mean by "ambitious and equitable mathematics instruction" and providing an overview of practice-based and practice-focused approaches to teacher education and PD, specifically the use of coached rehearsals as a pedagogy of enactment for teacher development.

Ambitious and Equitable Mathematics Instruction

Teachers must ensure that each student has access to rigorous academic work to develop mathematical proficiency and meet the demands of an increasing mathematically, statistically, and technologically complex society (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2014). These expectations are summed up, in part, by the way in which students must engage with and develop a set of mathematical practices (NGA & CCSSO, 2010). These practices represent the skills individuals in mathematics-related fields utilize in their work and the way in which all individuals make sense of, reason about, and make decisions regarding mathematical and quantitative situations. These opportunities must deliberately be made available to all students, drawing upon students' diverse cultural and linguistic resources in the mathematics classroom and positioning mathematics as a human practice and a tool for social change (Gutiérrez, 2011).

To support these "ambitious and equitable" goals (Jackson & Cobb, 2010), mathematics teachers must enact "skilled practice" (Grossman & McDonald, 2008) to carry out the work in classrooms. Specifically, teachers need skill with a set of core practices (e.g., Grossman et al., 2009; Forzani, 2014) that represent integral aspects of ambitious and equitable mathematics teaching. In our collective work we argue that participation in pedagogies of enactment need not be preceded by changes in teachers' beliefs. Teachers learn to take up ambitious and equitable practices through engagement in pedagogies of enactment.

In our collective work, we focus on core practices that include: leading whole class discussions (Boerst, Sleep, Ball, & Bass, 2011; Chapin, O'Connor, & Anderson, 2009), eliciting and responding to students' reasoning through tasks and questioning (Lampert et al., 2013; Stein, Engle, Smith, & Hughes, 2008), building on student thinking (Leatham, Peterson, Stockero, & Van Zoest, 2015; Van Zoest, Leatham, Peterson, & Stockero, 2016), representing students' reasoning verbally and visually (NCTM, 2014), and steering instruction toward a clear and worthwhile mathematical point (Baldinger, Selling, & Virmani, 2016; Sleep, 2012). Teachers' capacities with these instructional practices have the potential to engage students in key practices of the discipline of mathematics. Efforts have been made to capture a set of core practices for teaching, such as the "high leverage practices" from TeachingWorks (2016) or the eight essential Mathematics Teaching Practices from NCTM's (2014) *Principles to Actions.* This working group focuses on a set of tools and approaches of teacher education and PD experiences that support teachers' development of these skills.

Pedagogies of Practice and Coached Rehearsals

There has been an increased focus on practice-based approaches to supporting teachers' professional learning. A practice-based approach uses "practice as a site of inquiry in order to center professional learning in practice" (Ball & Cohen, 1999, p. 19), and creates opportunities for teachers to examine the everyday aspects of teaching. Grossman, Compton, and colleagues (2009) have called for the need to organize practice-based approaches to professional education around what they call representations, decompositions, and approximations of practice, with the latter referring, "to opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (p. 2058), though we also consider how to approximate practice to support the development of practicing teachers in our work.

IAs serve as one form of an approximation of practice, containing core practices, pedagogical tools, and principles of high-quality teaching (Kazemi et al., 2009; Lampert & Graziani, 2009). IAs are designed to structure the relationship between the teacher, students, and content in order to put a teacher in position to engage in and develop skill with interactive practices around facilitating rich discussions about mathematics. IAs and the practices they contain serve as a focus of a set of activities that provide teachers the opportunity to both investigate and enact the work of teaching (Lampert et al., 2013; McDonald et al., 2013), often referred to as a "cycle of investigation and enactment." This consists of observing, decomposing, and planning the IA; rehearsing the IA in the teacher education setting with in-the-moment coaching from a TE; enacting the IA in a K-12 classroom setting; and using artifacts of practice such as video and student work to analyze instruction and make connections between teaching practices, student learning, and a broader vision of ambitious and equitable mathematics teaching.

In this working group, we focus on the "enactment" elements of this cycle of activities—both the coached rehearsals of IAs in a teacher education or PD setting and the enactments done in classrooms with students. In a rehearsal, the teacher (or teacher candidate) leads an IA with a particular problem or prompt and aligned goal with their peers serving as the students. The peers may be encouraged to "act like themselves" or may be assigned some standardized role to play, perhaps to put forward a common error. In-the-moment coaching, often by the TE or PD facilitator, is an important and distinguishing part of a rehearsal. The coach may pause the rehearsal to highlight a particular teacher-student or student-student interaction or teaching move that is worthy of discussion (Grossman et al., 2009; Kazemi et al., 2016). The rehearsing teacher is also permitted to pause the rehearsal to ask questions of the coach. The coach may also play the role of student, offering a strategic contribution as part of the rehearsal experience. The immediate feedback offered through coaching allows for a novel development process for all teachers, not just the individual rehearsing. Despite its potential benefits, coaching is complex work and requires the teacher educator to decide how to interrupt (e.g., by asking a question, suggesting a teaching move, highlight a successful move), how often to interrupt, and when to interrupt (Baldinger et al., 2016; Kazemi et al., 2016; Lampert et al., 2013).

Overview of Prior Work at PME-NA

The working group meetings at PME-NA 38 served as an important catalyst to the discussions and action of this group around the design, use, and research of pedagogies of enactment (specifically, coached rehearsals) in mathematics teacher education. The group originally focused on three focal issues: (1) issues of language around instruction and mathematics used in the work, (2) theoretical and methodological choices, and (3) considering multiple settings and boundaries. Over the three meeting sessions, more than 25 conference attendees participated in the discussions—raising questions, offering ideas and experiences, presenting emerging research ideas and efforts, and contributing to the conceptualization of next steps. The three meeting sessions were each organized around a different focal issue listed above.

The framing topic for the first day was, "Enactments of what? Examining choices about instructional practice and content," which provided an opportunity for participants to share and discuss the decisions TEs make regarding what gets focused on and worked on in rehearsals and enactments (both mathematically and instructionally) and what is informing those decisions. We organized a panel with individuals from three active lines of work in this area to prompt the discussion. One takeaway from this discussion was some clarification about the language used in this work, such as what is meant by "coached rehearsal" and how people are using "IA" and "practices" in different ways that shape the nature of their work with teachers (e.g., Arbaugh, Freeburn, Graysay, & Konuk, 2016; Baldinger et al., 2016; Campbell & Elliott, 2015; Campbell, Selling, & Baldinger, 2017). While language continues to be a challenge for researchers, TEs, and other stakeholders, we made some progress on identifying necessary agreement and other areas where reasoned differences were appropriate given one's context or focus. For example, we identified coached rehearsal as the pedagogy of practice we were focused on (though not the only one someone might be interested in). We also identified an IA as a necessary component of that work (e.g., the routine that gets rehearsed), though IAs have different scopes and forms.

The second day focused on the question, "What constitutes evidence of 'skilled practice' and its development?" This session featured a structured poster session in which eight individuals or groups presented ideas about their research or practice around the design, use, and impact of coached rehearsals and enactments. This session served as an important step to move beyond a broad, conceptual discussion of this area of work toward a more specific discussion about tools for practice and research. For example, presenters shared specific IAs they use in their setting, ways they structure their coaching, data collection tools they use (e.g., video, written performance tasks, interviews), and analytic frameworks.

The final session was organized around a discussion focused on "moving forward with an agenda." Participants reflected on ideas from the previous sessions and considered next steps—for themselves or in collaboration with others. Some takeaways from this session included:

- a clarification of "coached rehearsals" as one type of approximation of practice and enactment pedagogy, all embedded in a broader landscape of pedagogies of practice;
- acknowledgement of different approaches taken by individuals in the group and the need to clarify language;
- ambiguity around how teacher learning or development is being theorized to inform both practice and research in this area;
- interest from newcomers in incorporating coached rehearsals into elementary and secondary mathematics methods courses, and the value of speaking and working across elementary and secondary contexts;
- considering a distinction between individuals interested in doing the work in their teacher education practice and individuals interested in research and development efforts;
- a call for sharing resources for supporting activities around coached rehearsal, and;
- structural constraints (e.g., programmatic, school-university partnerships, logistical) that makes incorporating pedagogies of enactment difficult.

These takeaways shaped some of the early follow-up efforts and continue to frame the need for more engagement and collaboration. In part, this discussion highlighted the issue of "considering multiple settings and boundaries," which included the perceived divides between the university and school classrooms or between work with practicing teachers versus teacher candidates. We continue to explore those boundaries and look to reframe them in productive ways.

Focal Issues

In the time since PME-NA 38, working group participants continued the work in various ways. Some new connections supported the incorporation of coached rehearsals into more mathematics methods courses across institutions. Research efforts have continued, evident in a set of sessions at the 2017 Annual Meeting of the Association of Mathematics Teacher Educators (Campbell, Baldinger, & Selling, 2017; Jones & Campbell, 2017; Webb, Wilson, Duggan, & Bryant, 2017). This ongoing work has represented a range of scale—methods courses at a single institution; efforts to bridge the work across methods courses and field placements; designing PD for practicing teachers centered on pedagogies of enactment; efforts to collaborate among TEs, teacher candidates, and practicing teachers; and collaboration efforts across institutions.

From these efforts and takeaways from the previous working group, additional questions have emerged. We propose to address these emergent questions in three focal areas:

1. Articulating theories of and perspectives on teacher learning, development, and practice within the context of rehearsals and enactments;
2. More deliberately focusing on equity, both in thinking about equitable mathematics teaching as well as equitable and responsive mathematics teacher education and PD; and,
3. Conceiving partnerships (e.g., research-practice partnerships) that support and can be supported by efforts around pedagogies of enactment.

We look at all of these foci through two lenses: (1) advancing the work of TEs, and (2) enhancing research on improving teaching. Given the increasing interest in the field in the use of pedagogies of enactment to develop core practices, we see it as a crucial time to make strategic contributions, answer foundational questions, and plan for future collaborations and products.

Theories of and Perspectives on Teacher Learning, Development, and Practice

Much of the existing research around pedagogies of enactment has tended to focus on the rehearsals themselves. Challenges arise when thinking about how to conceptualize and track teachers' development within enactment cycles, across cycles, and across settings. In particular, researchers must make explicit the theories of teacher learning that guide the use of pedagogies of enactment. Past research has advanced that learning skilled professional practices entails developing deliberate practice (Ericsson, 2002), adaptive performance (Hatano & Inagaki, 1986), or disciplined perception (Grossman, Compton, et al., 2009). These ways of noticing and acting involve developing teaching identities for practice (Lampert et al., 2013). For examining the activity system in which learning professional practice takes place, researchers have made use of theoretical lenses such as activity theory (Campbell, 2014; Campbell & Elliott, 2015) and theories of knowing in practice (Cook & Brown, 1999; Lampert et al., 2013).

A compelling question to advance research on pedagogies of enactment is what are the linkages between underlying perspectives on learning and the tools and pedagogies employed within cycles, across cycles, and across settings? Thinking about these issues helps us consider how constellations of teacher learning tools and pedagogies support teachers' capacity to judge what instructional practices are called for and how to deploy them. Attention must also be paid to other outcomes of teacher development that are important to the various stakeholders invested in this work. Without explicit discussion of the theoretical constructs we employ we run the risk of leaving our goals unspecified, of designing instruments misaligned to our goals, and of making claims that are unwarranted given our research designs. A related challenge is identifying methodological tools that build on different theoretical framings that could allow us to systematically document evidence of learning, change, or growth in a range of data sources. As we and others look to make claims about the impact of pedagogies of enactment on teachers' practice, there needs to be more explicit discussion about the theoretical and methodological choices that contribute to research and development.

Focusing on Issues of Equity

A number of important questions have emerged around the relationship between pedagogies of enactment and efforts to help teachers develop more equitable instructional practices (Jackson & Cobb, 2010). Bartell and colleagues' (2017) framework linking equitable teaching and the Standards for Mathematical Practice begins to address this issue. In their research commentary, the authors advance a set of teaching practices that they have found to address inequalities within classrooms and invite researchers to investigate how this set of teaching practices leverages students' learning of mathematical practices. We consider how this framework might influence the use of pedagogies of enactment. We also draw Gutiérrez's (2011) work, framing our thinking about pedagogies of enactment using four dimensions of equity organized into a dominant axis of access and achievement and a critical axis of identity and power. The dominant axis identifies access as necessary, but not sufficient, to ensure achievement in the discipline of mathematics (meaning that which has been seen to ensure economic well-being and is of high value within policy and school mathematics). The critical axis of identity and power highlights the role of agency to meaningfully participate both within mathematics learning and in a democratic society. This axis brings into view that mathematics learning can be a lens for making sense of the world and seeing what and who is marginalized and valorized. Gutiérrez's work calls for us to consider equity in terms of social transformation across spaces and timescales.

In addition to focusing on the role pedagogies of enactment play in supporting the development of equitable instructional practices, we also ask questions about how these pedagogies provide equitable opportunities for learning in the context of mathematics teacher education. How can pedagogical tools such as coached rehearsals of IAs support all those who are working to improve their teaching practice? How can theories of equity and social justice that focus on the mathematics classroom be extended to apply to the teacher education setting?

Developing and Supporting Partnerships Around Practice and Research

To fully consider the implications of pedagogies of enactment, careful consideration must be given to the role of partnerships across contexts in the practice and research of these pedagogies. In particular, we must examine the systems in which these pedagogies are used. Recent efforts to improve educational systems focused on central problems of practice have gained traction in collaborative researcher and practitioner communities. These efforts form a family of models: Research Practice Partnerships (Coburn & Penuel, 2016; Coburn, Penuel, & Geil, 2013; Rosenquist, Henrick, & Smith, 2015), Design Based Implementation Research (DBIR), and Improvement Science, including Networked Improvement Communities. These approaches share common values and strategies that connect to problems of practice, such as bi--directional educational improvement activity (Kazemi & Hubbard, 2008), iterative collaborative design, development of learning and implementation theory, and system-level sustained improvement (Penuel, Fishman, Cheng, Sabelli, 2011). DBIR, for example, has four dimensions: (1) work on problems of practice – those that improve practice, (2) engage in iterative cycles of improvement, (3) authentic partnerships among practitioners and researchers, and (4) attention to system improvement (cohesiveness). This work requires cross institutional learning and attention to individuals, interactions, and organizations to build and coordinate knowledge, practice, and theory. These models help address key questions about the role of partnerships in this work.

Relevance to PME-NA 39

The focus of this working group also provides a contribution in the context of the theme of this year's conference: "Synergy at the Crossroads: Future Directions for Theory, Research, and Practice." We see the focus of this group as fitting with a number of the conceptions of "crossroads." For example, we see this working group as an intersection point, considering the intersections between theory, practice, and research around pedagogies of enactment. Based on the attendance at our past working group, we recognize that mathematics TEs are motivated to attend for a variety of reasons. Some TEs pursue questions to advance theory about teacher development, others pursue learning more about tools to improve teacher education practice, and finally, other TEs are interested in researching teaching and teacher development through the design and use of pedagogies of enactment. Some individuals are motivated to pursue multiple areas of interest, often negotiating how to manage multiple demands or make decisions about what to foreground or background. Because the working group's efforts sit at the intersection of design of innovation and research on innovation to improve practice, the group is positioned to advance and integrate conversations of theory, research and practice across settings and with multiple stakeholder groups—crossroads that are often viewed as a barrier. These interests and needs also serve in representing a crossroads as a potential change in route—reflecting on our past experiences as practitioners, researchers, and collaborators to think about next steps. As such, we will frame each session of the working group to attend to theory, research, and practice.

Plan for Engagement of the Working Group

The three working group sessions are structured around the focal issues outlined above.

Session 1. Examining Theoretical and Methodological Choices

The first session will open with introductions and then briefly frame the purposes and the structure of the working group. The remainder of this session will include a structured poster session that will provide participants with an opportunity to present the theoretical and methodological tools they use to frame their work around pedagogies of enactment, whether in practice or for research. Participants will then debrief and summarize these ideas in a whole group discussion, identifying similarities and differences and discussing implications.

Session 2. Connecting to Visions of Equity in Mathematics Teaching and Teacher Education

The second session will build on the previous day's work to consider how conceptions of teacher learning build on and contribute to visions of equity in mathematics teaching and teacher education. The session will focus on two main questions connected to our dual foci of research and practice. The first question informing practice is: How might recent visions of equity inform pedagogies of enactment? This question will help us think about and identify the tools and practices we employ as TEs to take up equity in authentic ways within pedagogies of enactment. Drawing on Gutiérrez's (2011) four equity dimensions, we will examine Bartell and colleagues' (2017) research framework for linking equitable teaching with the Standards for Mathematical Practice as well as our shared sense of "ambitious mathematics teaching." The second question framing the session, which foregrounds our research lens, is: What are the implications of these visions of equity on research questions, methodologies, and settings related to pedagogies of enactment? To prepare for this discussion, we will provide participants a short synopsis of Bartell et al. (2017) and Gutiérrez (2011) as well as access to the full text. We will take up these questions through initial small group discussions, where participants will be able to engage in careful investigation of these questions as they apply to each unique context. We will then have a broader discussion as a whole group to develop a shared understanding of the ways visions of equity play out in our work with pedagogies of enactment. This discussion with conclude with a summary of next steps, particularly as they relate to theoretical considerations and research.

Session 3. Supporting the Development of Research-Practice Partnerships

The final working group session will provide an opportunity to discuss next steps, with a focus on the initiation and support of research-practice partnerships in local contexts centered on the use of pedagogies of enactment across contexts. From our previous work, we recognize that an important next step in this work is to consider how efforts can be scaled, how work is done across contexts with multiple stakeholders, and how systemic change can be fostered. We will consider frameworks such as DBIR or research practice partnerships to guide our discussion, leveraging experiences and expertise from within the group shared through existing work and

This final working group session will also serve as an opportunity to consider and discuss follow-up activities that could build on the work begun during the working group. Participants will have the opportunity to connect with others about common interests to initiate plans for collaboration about practice or research. We will also discuss a proposed set of follow-up activities (see below) and invite recommendations on other next steps. We will identify participants who can take the lead in facilitating follow-up work.

Anticipated Follow-Up Activities

There are a number of potential follow-up activities to build on the work that will begin during the proposed sessions. We are hopeful that this working group would continue to serve as a catalyst for collaborative activities around designing and studying pedagogies of rehearsals and enactments. For some, this can entail beginning to incorporate rehearsals into a methods course or a PD effort. Others may ask more substantive practice-oriented questions of their use of rehearsals and enactments. Both of these efforts would benefit from opportunities for sharing resources and materials, such as a website, blog, or shared folder. Others will continue to engage in more substantial data collection and analysis to research teacher development, which may serve as pilot projects, eventually leading to proposals to fund collaborative research projects. This is particularly salient in the context of considering the development and support of research-practice partnerships. We are also currently conceptualizing a proposal for funding a focused conference to bring together scholars for focused work around pedagogies of enactment, and the working group will be an opportunity to further pursue or advertise those plans.

Another target activity is the collective dissemination of the ideas that are shared and created through the working group itself. For example, submitting a paper to a journal like *Mathematics Teacher Educator* that outlines similar or contrasting ideas around rehearsals and enactments could provide an appropriate forum and audience for these ideas. Alternatively, collectively writing a research commentary would allow us to focus on theoretical or methodological challenges in researching teacher development in practice-based activities. This could be submitted to a venue such as the *Journal for Research in Mathematics Education* that publishes commentaries. We will also continue to disseminate ideas at national conferences such as those of the Association of Mathematics Teacher Educators, the National Council of Teachers of Mathematics, and the National Council of Supervisors of Mathematics.

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CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

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Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. The inconsistencies in definitions and usages of the terms argumentation, justification, and proof highlight the need for scholarly conversations addressing these (and other related) constructs. Collaboration is needed to move toward, not one-size-fits-all definitions, but rather a framework that highlights connections among them and exploits ways in which they may be used in tandem to address overarching research questions. Working group leaders aim to facilitate discussions and collaborations among researchers and to advance our collective understanding and conveyed use of argumentation, justification and proof, particularly the relationships among these important mathematical constructs. The 2017 working group sessions will provide continued opportunities for participants to discuss existing definitions and descriptions, with increased focus on how these definitions and descriptions are used by researchers and practitioners within particular contexts and applications. Participants will examine data through a variety of lenses to investigate the use of particular conceptualizations and discuss implications of such use.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking

Brief History of the Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group (AJP-WG) met for the first time in 2015 at Michigan State University in East Lansing, Michigan during the 37th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education (PME-NA), and again the following year for the 38th Annual Meeting of PME-NA in Tucson, Arizona. The working group's primary focus is on the field's conceptualization of the interrelated objects and processes of argumentation, justification, and proof. Previous working groups at PME and PME-NA had focused on either proof or argumentation, but the present working group is the first to attend specifically to the connections among these three constructs.

During the working group's initial meeting (2015), attendees made progress on considering the interrelationships among argumentation, justification and proof, and we deepened our understandings of our own perspectives and the range of perspectives held by others in the group. The goal during this meeting was not consensus or deciding a best approach. Rather, we sought to better understand the complexity and diversity of individuals' perspectives with respect to their research agendas and professional practice. We were encouraged that our efforts were well received, with 46 scholars, including at least 10 graduate students, participating in the working group the first year.

The second gathering of the AJP-WG was held in 2016 at the annual meeting for PME-NA in Tuscon, AZ. Thirty-six scholars attended the sessions, which continued the group's focus on the interrelationship between and among the concepts and terms related to argumentation, justification, and proof. Specifically, participants considered their definitions for these concepts and how such definitions may change given different contexts and foci. Additional information about the first two meetings of the AJP-WG is included later in the proposal.

Summary of Focal Issues

There is a large and growing body of research in mathematics education focused on argumentation, justification, and proof. The research on proof, for example, includes studies on the role of proof in the discipline; proof in school mathematics and at the undergraduate level; what counts as a proof; proof schemes and categories; teachers' conceptions of proof; students' abilities to write valid proofs; and what teaching proof looks like in classrooms at various levels (e.g., Boero, 2007; Harel & Sowder, 2007; Reid & Knipping, 2010; Stylianou, Blanton, & Knuth, 2009). At the same time, researchers and policy documents have issued calls to engage K-12 students with disciplinary practices such as constructing viable arguments, justifying conclusions, critiquing the reasoning of others, and constructing proofs for mathematical assertions (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

Yet, as the field moves forward to maximize students' learning opportunities for engaging in these disciplinary practices, mathematics educators need to refine their notions of these terms in scholarly activities and in policy documents (Cai & Cirillo, 2014). How, when, and why decisions related to word choices are made (e.g., 'argument' versus 'proof') in curriculum materials, policy documents, and research is an open question. In fact, some researchers have hinted that these choices are not always purposeful. For example, Lynn Steen, a member of the 1989 NCTM Standards Committee, claimed that uncertainty about the role of proof in school mathematics caused NCTM in its 1989 *Standards* document to resort to, what he called, "euphemisms" such as "'justify,' 'validate,' 'test conjectures,' [and] 'follow logical arguments'" (Steen, 1999, p. 274). Rarely, he stated, did the document use the term 'proof.' Although Steen's comments were published more than 15 years ago, we argue that his proposition, that the role of proof (as well as argumentation and justification) in school mathematics is uncertain, continues to be true today.

One additional challenge of reading extant research or developing a research agenda related to these disciplinary practices is that the classifications offered differ according to the perspective of the researcher, the focus of the research, and the particular data being analyzed (Reid & Knipping, 2010). Only recently have we begun to see mathematics educators offering explicit definitions of these constructs in their work; this is ironic given the importance of definitions in the field of mathematics itself and particularly in the activity of proving.

Although proof has received more attention in extant research, argumentation as a concept seems to be garnering new prominence with increasing attention in the mathematics education literature (Conner, Singletary, Smith, Francisco, & Wagner, 2014; Staples & Newton, 2016; Stylianides, Bieda & Morselli, 2016) and as a process that is playing an important role in policy and curricular documents across many disciplinary fields. The notion of constructing and analyzing arguments appears in the most recent standards of four core K-12 disciplines – English, mathematics, social studies and science. For example, the *Common Core State Standards for Mathematics (CCSSM)* includes as one of its standards for mathematical practice, "Construct viable arguments and critique the reasoning of others" (NGA & CCSSO, 2010a) Similarly, *CCSS for English Language Arts* devotes a portion of an appendix to "The Special Place of Argument in the Standards," emphasizing argumentation as critical for success in college and careers (NGA & CCSSO, 2010b, Appendix A, pp. 24-25). The National Council of the Social Studies (NCSS) highlights argumentation as an aspect of historical thinking, with a focus on causation and argumentation (NCSS, 2013). A summary table of where and how references to arguments or argumentation appear in the policy documents is included in Table 1.

On a related note, although not an explicit Standard for Mathematical Practice, we find evidence in past and current mathematics standards documents that *justification* is considered an important mathematical practice. For example, in the 1989 *Standards*, it was noted by the authors that throughout the document, "verbs such as *explore, justify,…describe, develop,* and *predict* are used to convey this active physical and mental involvement of children in learning the content of the curriculum" (NCTM, 1989, p. 17). In particular, students are asked to "justify their answers and solution processes" (p. 29) as part of the Mathematics as Reasoning Standard. In *Principles and Standards for School Mathematics (PSSM)* (NCTM, 2000), geometry is positioned as a natural site for the development of students' "reasoning and justification skills" (p. 41). Justifying is also explicitly linked to the Reasoning and Proof, Communication, and Problem Solving Process Standards in *PSSM*. Finally, the authors of *CCSSM* (NGA & CCSSO, 2010a) consider "the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from" (p. 4) to be a hallmark of mathematical understanding. Looking across time, analyses of standards adopted prior to that of *CCSSM* suggest a relatively infrequent inclusion of the terms justify/justification in content standards language (Larnell & Smith, 2011), with a significant increase in such usage in *CCSSM* (Kosko & Gao, in press). Furthermore, and along the same lines as argumentation, justification as a topic of study has seen increased recent attention in the research literature (Lesseig, 2016; Lin & Tsai, 2016). Together, these examples demonstrate that justification has been considered to be important in school mathematics, with increased focus in recent years.

Table 1: Summary of References to Argument(ation) in Policy Documents

Standards	**Role in Standards**	**Specific Reference for Practice or Recommendation**
Common Core State Standards for Mathematics	One of eight Standards for Mathematical Practice that should be developed in students	3. Construct viable arguments and critique the reasoning of others. (p. 6)
Common Core State Standards for English Language Arts & Literacy	Argumentation identified as the first of the Standards' "Three Text Types"	The Special Place of Argument in the Standards in Appendix A: Research Supporting Key Elements of the Standards (p. 24)
Next Generation Science Standards	One of eight Science and Engineering Practices in the NGSS identified as essential for all students to learn	7. Engaging in argument from evidence (Appendix F, p. 13)
National Curriculum Standards for Social Studies	One of three factors related to "powerful" social studies teaching and learning	Teachers show interest in and respect for students' thinking and demand well-reasoned arguments rather than opinions voiced without adequate thought or commitment. (p. 13)

Highlights from the Year 1 Working Group Discussions

A focal activity during the initial working group sessions in 2015 was the development of Diagrams/Concept Maps in which each participant generated a representation of the relationships among argumentation, justification, and proof from his or her perspective. Discussion surrounding the relationships between constructs was supported by an initial presentation by Keith Weber on Day 1, followed by a panel presentation, moderated by Samuel Otten, with Kristen Bieda, AnnaMarie Conner, and Pablo Mejía -Ramos serving as our expert panelists (i.e., Bieda - justification; Conner - argumentation; and Mejía-Ramos – proof). The panelists shared how they conceptualized the interrelationships among argumentation, justification and proof; they explicated how they came to use the central construct they use in their research and why they felt that choice was productive for their work; and they offered their thoughts on the current state of the field and what we might need to tackle next in relation to these constructs. The final day of the 2015 working group sessions offered the opportunity to revisit participants' Diagrams/Concept Maps, though now potentially informed by additional perspectives and questions gained from the prior two sessions.

Three products were generated from the 2015 meeting. First, a pair of podcasts generated by Samuel Otten are available worldwide: The first podcast is of Weber's talk (http://mathed.podomatic.com/entry/2015-11-16T07_01_19-08_00), and the second podcast is the moderated panel discussion (http://mathed.podomatic.com/entry/2015-11-19T07_19_37-08_00). The second product was a white paper that was developed by the working group organizers, the panelists, and several other participants from the working group who volunteered to participate in the online publication (Cirillo et al., 2016). The white paper summarized the working group activities and discussions and also includes the set of 44 Diagrams/Concept Maps that were generated as well as annotations and analyses. At the time of this writing, the white paper has garnered over 230 reads since its online publication six months ago. The final product was a poster presentation for PME-NA 2016 that was based on the analyses of the Diagrams/Concept Maps (Strachota et al., 2016).

Weber's presentation highlighted different traditions and points of disagreement, for example, citing Reid's (2001) observation that research simultaneously suggests that secondary students struggle to construct proofs, while at the same time suggesting that primary children are capable of engaging in proof. Weber prompted the group to consider how different traditions may inform each other in order to advance the field collectively. In particular, he outlined three broad traditions in proving: *proving as problem solving, proving as convincing,* and *proving as socially embedded activity.* Each corresponds to a different focus for research and/or instruction. Weber offered two thought-provoking suggestions, both of which may help us understand the lack of convergence in results and definitions. One suggestion was that proof may not be a singular, easily defined concept, but rather a *cluster concept,* as used by Lakoff (1987). In this sense, there is no list or decision procedure to identify a proof, but rather there is a set of features associated with the concept, and many - but not all - apply in any one instance.

The second suggestion was that the features or properties associated with proof may be closely interrelated for mathematicians, but not for students. In particular, for mathematicians, a convincing argument and socially sanctioned argument are often one-in-the-same. For students, however, those are not tightly connected and may describe very different types of arguments. He suggested, "Perhaps much of the disagreement amongst mathematics educators is that they are using proof as a shorthand to denote things that are different to students but similar for mathematicians" (Cirillo et al., 2016, p. 7).

The panel discussion on our second day further raised awareness of how crucial it is to not only define one's terms, but also to specify the context of one's work. For example, Bieda and Conner both work closely with students and teachers in secondary settings, and in that context, they have found proof and proving to be terms that distance or invoke conceptions of end product and formality. Consequently in their work, they have chosen different focal constructs. Both Bieda and Conner articulated an explicit link of their work to students' proof-producing capacities at the tertiary levels, but they do not centralize that term in their research in school mathematics.

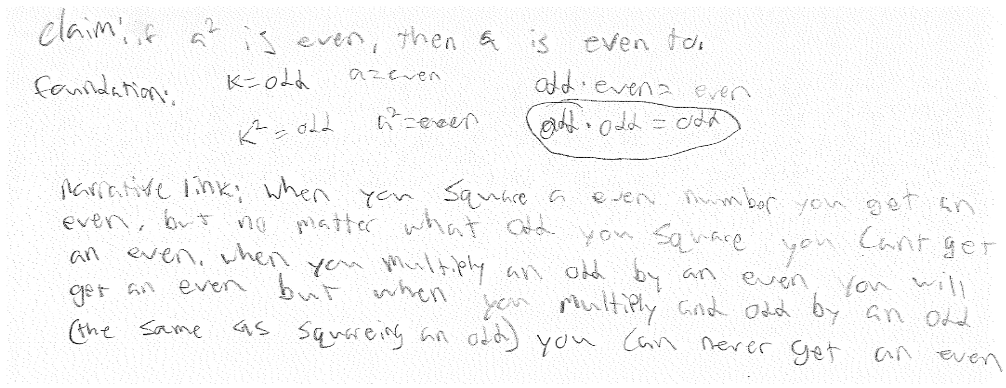
Our discussions and subsequent analyses of the Concept Diagrams/Maps offered additional information about the variety of ways these three important constructs are understood in relation to one another. During the gallery walk and subsequent analyses across the Concept Diagrams/Maps, we could discern little to no agreement about the relationship between justification and argumentation. It seemed that participants held these two constructs (justification and argumentation) either fully distinct from the set of things we call proof, or that proof was a very specific version of each of these. Some considered justification a subset of arguments; others positioned arguments as a subset of justifications; and still others had them as overlapping, but not concurrent, sets.

Proof (and proving) seemed to be of a different nature than argumentation and justification for our participants, and proof participants either positioned it at the far end of a continuum of the constructs, a plane above, or as a separate entity. Alternately, however, others considered proof to be a specific subset of arguments and justifications. Additional details can be found in the White Paper (Cirillo et al., 2016). A question raised in the discussion on Day 3 was whether proof was so valorized that we position it as the "desired end product" for all arguments, even when that might not be a productive or educative goal. This lack of not only convergence but general clarity provides an important opportunity for further exploration and raises questions about the consequences of these different concepts. It implores us to continue to work to develop a framework to connect these constructs and clarify not only our commitments and definitions but the interrelationships among these important ideas.

Highlights from the Year 2 Working Group Discussions

During the first session of the second meeting of the AJP-WG, Samuel Otten moderated panel presentations on differing applications of definitions provided by Eric Knuth, David Yopp, and Orit Zaslavsky. On the second day, participants examined data artifacts from different grade levels and were asked to consider whether and in what ways they would define features in the data with respect to argumentation, justification, and proof. All participants were provided two common artifacts (a transcript of a high school math class discussion & five samples of grades K-3 writing samples) as well as one additional artifact of choice (middle school, high school, or tertiary level artifact). Discussion surrounding the artifacts provided an opportunity for participants to consider how their definitions were affected by application to the differing data. It provided an arena in which to see how each definition served as a lens for viewing the artifacts and how these different lenses might lead one to position a work sample in different ways, depending on the type of mathematical activity represent in the work sample based on the definition. This discussion continued through to the final session. Further, the discussion surrounding the artifacts facilitated initial organization of three networking groups–one focused on reading and sharing journal articles and two focused on producing written products. One product available from this second AJP-WG meeting is a podcast of the panel discussion produced by Samuel Otten (https://www.podomatic.com/podcasts/mathed/episodes/2016-11-14T07_10_36-08_00). A second product is a white paper (available at https://www.researchgate.net/publication/317267228) summarizing the working group activities and discussions of the artifacts from year two (Staples et al., 2017). The final product is the organization of networking groups for continued collaboration related to, but independent from, the working group. Thus, activity across both years of the working group has facilitated the continued development of a community of mathematics education researchers who we anticipate will continue these discussions over several years.

During the panel discussion, our panelists shared how they used and defined justification, argumentation, and proof and how their definitions and applications of these constructs influenced, and were influenced by, their research questions or contexts. A key point arose from David Yopp's juxtaposition of his definition of proof with Stylianides' (2007) definition of proof in relation to a particular 8th-grade work sample. Yopp argued that by Stylianides' definition, the student's work was not a proof (violating two of three conditions). By Yopp's definition, however, the student's work was a proof. Yopp stated that a proof eliminates the possibility of counterexamples. Thus, from the student's perspective—and perhaps class's perspective—he had eliminated the possibility of counterexamples, thus providing proof of the claim (see Figure 1).


claim: if a^2 is even, then a is even too

foundation: k=odd a=even odd·even= even

k^2=odd a^2=even odd·odd = odd

narrative link: when you square a even number you get an even, but no matter what odd you square you cant get an even. when you multiply an odd by an even you will get an even but when you multiply and odd by an odd (the same as squareing an odd) you can never get an even.

Figure 1. Student work samples shared by David Yopp.

We then turned our attention to reviewing artifacts of K-16 classrooms, to provide a concrete opportunity to see how definitions might interact with context. Participants were offered two artifacts in common, from elementary grades work samples and a high school transcript, and then could choose a third artefact to review, from middle school, high school or tertiary level. Although it became apparent was that we offered too much in too short a time frame, our discussion was productive, and we recap a few of the key ideas offered here.

One question raised was how much context we needed to know in order to engage the question, is this a proof? Or is this an argument? This question was posed based on recognizing that both Yopp and Stylianides offered definitions of proof that dependent to a degree on a child's/class's conceptual sphere. In Stylianides' definition, this context element comes through with the criteria that all must be "known by" or "within the conceptual reach" of the community. In Yopp's, this context element comes through with the idea that one must eliminate all possible counterexamples, so if a student is not aware of, say, complex numbers, s/he does not have to offer a proof that accounts for all complex numbers. The student can only attend to and eliminate the possibility of counterexamples (actively) from his or her realm of possibility. Continuing with the general idea of context, questions were raised about how grade-level dependent a proof was, and whether one needed to know more about what was taken-as-shared in a class to evaluate whether an argument did or did not comprise a proof. One of the final points raised was that a proof is not a stand-alone entity; we need to know what the norms and assumptions are in order to understand if something is operating as a proof in that context.

In our final session, we sought to continue the discussion from the previous session and also organize toward "networking groups" for those interested in sustaining or advancing conversations with respect to argumentation, justification and proof during the year. Participants each wrote brief questions they were interested in pursuing and indicated a level of commitment they might have for the pursuit (e.g., discussion groups, reading groups, research group). Our time then was devoted to participants having conversations to connect with one another around shared areas of interest and similar levels of desired future commitment.

The networking time yielded three distinct follow-up groups. Each group also identified some goals for the upcoming year and shared these with the larger group. Please note that all participants in the working group were welcome to join these groups.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

AJP Journal Club. This group aims to read and discuss two articles per quarter related to issues of interest to its members. One goal is to investigate how authors attempt to make explicit their definitions of argumentation, justification, and proof relevant to their work.

Argumentation Research Commentary. This group aspires to write a research commentary that (a) describes research on argumentation in our field, and (b) explores reasonable guidelines for productive research on mathematical argumentation.

Op Ed piece. This group intends to pursue an op-ed piece making the case that the mathematics education community needs to take concerted efforts to eliminate two-column proof from geometry courses in the United States.

Focus for Year 3

Our focus for Year 3 is to generate interest and commentary for a book proposal that will engage members of the community in a more structured analysis of data using different perspectives and definitions of argumentation, proof, and justification. In doing so, we intend to continue to facilitate communication and collaboration among members of the mathematics education community with interests in the areas of argumentation, justification, and proof.

As we learned in Years 1 and 2, members of our community have widely differing perspectives on the definitions of and relationships among argumentation, justification, and proof (see Cirillo et al., 2015; Staples et al., 2016). Likewise, the consequences of these definitions and the contexts in which they are applied are varied, leading to a diverse set of conclusions and implications about the teaching and learning of argumentation, justification, and proof. This was illustrated in Year 2 of our working group, particularly by the conversations about David Yopp's artifact (see Figure 1).

In Year 2, we began to investigate how context, interpreted primarily as grade band, when combined with particular definitions, impacted our interpretation of data. We found that, to some extent, the written or verbal nature of the data also impacts our interpretations. Thus in Year 3 we will continue to dig into the interactions between context and definition in our examination of relevant data. We plan to build upon what we learned in Years 1 and 2 about different perspectives on argumentation (arguments), proving (proofs), and justifying (justifications). We intend to leverage ideas and relationships from our first two years into a book proposal in which chapters will highlight analyses of particular sets of data (artifacts) from different perspectives/definitions. This book will address how definitions, commitments, and contexts interact to produce different insights into the teaching and learning of mathematics. In particular, the book will highlight the importance of defining, distinguishing, and integrating argumentation, justification, and proof in mathematics education research and its applications. We will continue to facilitate communication and collaboration among members of the mathematics education community who are involved in research and scholarship using constructs that include argumentation, proof, and justification. As we explore the roles of definitions and the contexts in which we work in determining our definitions and how we use them, we will continue to facilitate connections among people with similar interests in hope of facilitating lasting collaborations to further knowledge related to argumentation, proof, and justification in the field.

Plan for the Working Group

Session 1: Looking at the Past and Toward the Future

In Session 1, we will begin with introductions, review our progress from Years 1 and 2, and hear reports from the three networking groups established in Year 2. We will then present our current idea for the book proposal and solicit feedback, input, and advice from participants.

In Year 2, we established the aforementioned networking groups with different purposes. During this session, we will invite the leaders and members of these groups to report on the progress and products accomplished during the past year. As appropriate, we will invite session attendees to join established groups or to create new ones.

During this session, we will present the rationale and structure of our proposed book, asking attendees to critique the ideas presented, make suggestions, and consider contributing to the book.

Session 2: Examining Data from One Context

Prior to the conference, working group leaders will analyze data and write an initial draft of two chapters for the book, analyzing data from a single context from at least two perspectives. In Session 2, we will focus on the data from this context, including transcript and written work. We will ask participants to examine the data (in small groups), encouraging them to make their assumptions about and definitions of argumentation, justification, and proof explicit. We will share snippets of these conversations with the whole group. We will then hear a short presentation from one group of researchers who previously analyzed the data. This group will describe the definitions of argumentation, justification, and proof used to analyze the data and present conclusions and implications from the analysis. Working group participants will be invited to critique the interpretations and offer alternative ways to examine or interpret the data.

Session 3: A Second Perspective on the Data

In Session 3, we will again focus on the data presented in Session 2. We will hear a short presentation from the second group of researchers with different commitments to argumentation, justification, and proof detailing their analyses and interpretations of the artifacts. Working group participants will be invited to critique the interpretations and offer alternative ways to examine or interpret the artifact. Participants will then compare and contrast the different conclusions together with their definitions and commitments, and we will facilitate a discussion about the differing affordances of the different ways to analyze the data, as well as any questions prompted by this examination.

During Session 3, we will revisit the plan for the book, soliciting feedback from participants on the structure and outline presented in the first session. We will finish Session 3 with an invitation to continue to participate in the work of the working group by proposing chapters with different analytic frames, writing commentaries on the analyses presented or on future chapters, or by participating in one of the networking groups discussed in Session 1.

Anticipated Follow-Up Activities

We anticipate that the products and follow-up activities from Year 3 will build on the activities and products from our two previous years. A major follow-up activity for the following year will be the development of a book proposal accompanied by drafts of several chapters for the book. A table of contents, introductory chapter and at least one chapter with accompanying commentary will be developed and revised for the submission of the book prospectus.

We will continue to encourage networking groups, which will likely shift and expand from Year 2. New working groups interested in collaborating on chapters for the book will be invited and encouraged. Additional networking groups with other related interests will also be encouraged.

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CRITICAL PERSPECTIVES ON DISABILITY AND MATHEMATICS EDUCATION

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Research on mathematics and disabilities traditionally has been conducted within a special education paradigm, which often implicitly or explicitly adopts a deficit model of the learner. The deficit model locates the "problem" within the individual student rather than in the social, discursive, political, or structural context. Instruction for these students tend to focus primarily on rote algorithms and calculation skills rather than the solving of rigorous, high cognitive demand problems. As mathematics education researchers and practitioners we are poised at the crossroads, ready to reclaim work traditionally relegated to special education. Our working group is composed of researchers and educators who draw upon critical theories, such as Disability Studies in Education, Critical Race Theory, and DisCrit, in order to offer an alternative vision of mathematics education based around a different conceptualization of disability and learning differences. We consider both the framing of these students as well as how to reframe classroom practice utilizing models like Complex Instruction (CI) and Universal Design for Learning (UDL) to create classrooms in which all students are able to access the curriculum in meaningful and rigorous ways.

Keywords: Equity and Diversity, Instructional Activities and Practices, Classroom Discourse

Overview of the Working Group

The purpose of this working group is to assemble researchers and practitioners who bring a critical lens to issues of mathematics education and disability. Historically, the consideration of disability in mathematics education has been examined from a special education lens, which pathologizes student difference and aims to "remediate" these perceived defects. There is a well-known epistemological difference between special education and mathematics education. We have come to a crossroads, where we must decide how to address this theoretical divide between special education and mathematics education. As mathematics educators we believe a critical perspective on disability, drawing upon disability studies, critical race theory, and discrit, can address this divide and offer an alternative vision of mathematics education and disability. Our working group is designed to create sustainable opportunities for researchers and practitioners interested in bringing a critical lens to understanding disability and difference in mathematics education. In this proposal we briefly present the history of this working group, the theoretical perspectives that this group draws upon, and our plans for collaboration.

History of the Working Group

Our PME-NA working group met for the first time last year. Fifteen researchers (faculty and graduate students) and 2 educators met during PME-NA 2016 in Tucson, AZ. In this first series of working group meetings, participating group members shared theoretical perspectives of disability they employed within their work in mathematics education, current projects they were engaged in related to disability and mathematics education, and critical issues members often found themselves confronting in taking up this complex work. Conversations that took place during this time were exciting. Many common interests were uncovered and shared among group members. At the conclusion of our time together at the conference, many of the group members were hopeful that the working group in subsequent years could serve as a foundation for continued work and conversation. This group has the potential to serve as a main platform and organizing structure as we embark upon this paradigm-shifting work.

The impact and work of this group continued through the year. We established an email list-serv which members have used regularly to share and solicit feedback on work and to organize our collective efforts. We are in the process of submitting an AERA conference grant to provide a multi-day venue for members of this working group to meet and work on an edited book project. We are also planning to submit a book proposal.

In this coming year we plan to continue and expand collaborations between members of this working group. We hope to provide time to learn with and from each other and to establish the foundation for collaborative work throughout the year. Although we are actively pursuing opportunities to bring this group together for a more extended period of time (through AERA/National Science Foundation (NSF) conference grants), maintaining our PME-NA working group is essential. It not only provides an opportunity to connect with others, engage in ongoing projects, but also to invite new researchers and educators to join our community. As we have experienced in other working groups, each year brings new members, and provides a synergistic place for an ever-growing number of researchers to come together to begin collaborations around these challenging issues. We hope this working group can provide a space for scholars who are taking a critical lens to explore disability related to mathematics education. Our efforts to recruit new members have been ongoing since our first meeting. We have promoted and advertised this group at national mathematics education conferences (AMTE & RCML), made announcements in Review of Disability Studies, promoted it on the Disability Studies in the Humanities listserv, and posted on various facebook groups (e.g., Teaching Disability Studies and Mathematics Education Researchers and STaR). In addition, many of the original members have reached out to their informal networks to invite others who have expressed an interest in joining this group.

Paradigm Shifts for Disabilities in Mathematics

Gathering together a group of researchers, graduate students, undergraduate students, and classroom teachers interested in developing an alternate paradigm around disability, we have begun to explore the different theoretical frameworks that we can use to analyze and change this situation within mathematics education. Heeding Lather's (1986) call for researchers to utilize multiple theoretical schemes in their work, our working group believes that this work requires the participation of multiple paradigms of critique. Our working group draws upon critical theories such as disability studies, critical race theory, and other poststructural theories (including queer studies and trans studies) in this analysis. A commonality among these perspectives is that each problematizes normativity. We welcome the participation of those with other perspectives as well. In the following sections, we describe a sampling of these perspectives in order to frame the work of the group.

Critical Theories

Although the notion of critique dates back to antiquity, the phrase critical theory originally referred to a body of work coming out of the Frankfurt School and Institute for Social Research in the late 1920s through the early 1940s. Critical theory seeks to reshape reality, not merely explaining things. As Karl Marx (1845) wrote almost a century earlier, "The philosophers have only interpreted the world, in various ways; the point is to change it." Critical theory has come to mean not just the work of the original critical theorists, but any theory that seeks to transform rather than merely explain society. The Stanford Encyclopedia of Philosophy gives three main criteria for a critical theory, that "it must explain what is wrong with current social reality, identify the actors to change it, and provide both clear norms for criticism and achievable practical goals for social transformation." All of the theoretical perspectives that our working group draws upon are in some ways "critical theories."

Disability Studies and Disability Studies in Education

Many of the members of our working group utilize disability studies (DS) and disability studies in education (DSE) as a mode of inquiry in order to questioning the taken-for-granted assumptions and practices in mathematics education and the intervention paradigm. DS and DSE provides a framework for exploring questions such as: who is labeling, who is being labeled, whose voices we value, and how do we advance more equitable practices for all students. Disability studies calls into question the medical/individual model of disability in which disability is seen as a deficit within an individual that requires "curing."

In contrast to the medical model, many disability studies scholars and activists have adopted a social model of dis/ability, which locates dis/ability in an inaccessible environment. Those who adopt the social model of dis/ability make a distinction between impairment, as any physical or mental limitation, and disability, as the "social exclusions based on, and social meanings attributed to, that impairment" (Kafer, 2013, p. 7). Kafer (2013), however, argues that such a sharp distinction between impairment and dis/ability is unhelpful because it "fails to recognize that *both* impairment and disability are social" (p. 7). In the book *Feminist Queer Crip*, Kafer suggests the term "political/relational model" to refer to perspectives recognizing that both impairment and dis/ability are socially constructed. Within this social framing of disability is the acknowledgement that individuals with disabilities should be included within the research process itself. Traditional approaches to researching disabilities are oppressive to individuals with disabilities, as the researchers determine the questions asked, the methods of data collection, and the meaning made of the data, with no input from the individuals with disabilities. The social model of disability acknowledges that these individuals have unique insights into their lived experiences and empowers them to engage directly in the research process.

In educational settings, this construction of dis/ability manifests in the double education system that splits general education and special education. Scholars have traced the ways in which special education "serves as a vehicle for preserving general education in the midst of ever increasing diversity" (Reid & Valle, 2004, p. 468, paraphrasing Dudley-Marling, 2001; also see Skrtic, 1991, 2005). Rather than using research-validated frameworks like Universal Design for Learning (UDL) and Complex Instruction (CI) to deliver rigorous, high-cognitive demand instruction to all mathematics students, the system of special education shunts certain students (especially students of color) into an inferior, segregated mathematics education, thus providing a band-aid to a broken general education system and preventing larger, more systematic changes. One line of research pursued by working group members involves developing understanding and theorizing the research divide between special education and mathematics.

Institutional schooling practices such as writing Individual Education Plans (IEPs) construct certain students as having disabilities; however, from a disability studies perspective, "the label of students with IEPs [can be viewed] not as an inherent and static determinant of individual ability, but as a school-based designation which reflects and recreates differential ability within the classroom" (Foote & Lambert, 2011, p. 250; also see Dudley-Marling, 2004; McDermott, Goldman & Varenne, 2006; Skrtic, 2005). Certain students are chosen for this assessment and intervention, and this selection process is not objective and often singles out those students who are not from a dominant cultural background.

Returning to the assumptions inherent in the concept of intervention, a disability studies perspective problematizes the taken-for-granted assumption that what is "wrong" with the situation requiring intervention is a pathology or deficit within students. Instead, the problem is located in the inaccessibility of the environment; in other words, what needs to be changed is not the student, but rather the environment to allow access for students who differ from one another. As Reid and Valle (2004) assert, "the responsibility for 'fitting in' has more to do with changing public attitudes and the development of welcoming classroom communities and with compensatory and differentiated instructional approaches than with individual learners (Shapiro, 1999). In other words, our focus is on redesigning the context, not on 'curing' or 'remediating' *individuals'* impairments" (p. 468). Different working group members have been addressing this in their current work in different ways. One scholar, draws upon a Vygotskian framing of disability and identifies the ways in which standard mediational tools (e.g., mathematical representations or symbols) are inaccessible to some learners. She reframes the word "remediation" as "re-mediation" to make an explicit move away from the deficit framing and toward a framing of disability in terms of access to mediational tools (Lewis, 2017). A second scholar, conceptualizes interventions as increasing *participation* rather than specific skills (Lambert & Sugita, 2016). That is, what interventions might contribute to more equitable participation and deeper engagement across students in mathematics classrooms? This has been explored through empirical research focused on equitable participation in a Cognitively Guided Instruction algebra routine (Foote & Lambert, 2011). Moreover, a political/relational model suggests that inaccessibility is embedded in the context of power relations. Finding ways to "intervene" to make the environment accessible, then, also requires analyzing the power relations involved in maintaining inaccessibility. A third scholar uses her work in learning trajectories to critique notions of "fixing" students and viewing difference as something that is "wrong" as opposed to a natural strength that can be leveraged in instruction (Robertson & Ne'eman, 2008). Specifically, she critiques a static interpretation of trajectory as an instructional directive to move children across levels or stages of a progression at the expense of paying attention to the reasoning children employ and working to support children to explore, revise, and advance that reasoning (Hunt, Westenskow, Silva, &Welch-Ptak, 2016). Such a use of trajectories could be viewed as an example of the social effects of difference that disable rather than natural biological variation (Siebers, 2008).

Critical Race Theory and DisCrit

Critical race theory (CRT) is another theoretical framework that informs the work of the group. According to CRT, racism is 'normal' rather than an anomaly in U.S. society (Delgado, 1995). Critical race theorists assert that the U.S. was founded on property rights, and specifically the fact that enslaved African Americans were considered property, rather than civil rights (Ladson-Billings & Tate, 1995). CRT reveals the way race and racism continue to structure U.S. society. In relation to educational interventions, critical race theorists have addressed the issue of over-representation of students of color in special education.

Often, however, these analyses leave ableist assumptions in place; similarly, DS perspectives often fail to adequately consider race. DisCrit is a perspective that acknowledges that racism and ableism are both "normalizing processes that are interconnected and collusive. In other words, racism and ableism often work in ways that are unspoken, yet, racism validates and reinforces ableism, and ableism validates and reinforces racism" (Connor, Ferri, & Annamma, 2016, ch. 1). Studies of administrators' and teachers' perceptions related to overrepresentation of students of color in special education have revealed that their perceptions tend to be rooted in "deficit thinking and infused with racial and cultural factors" (Connor, Ferri, Annamma, 2016, ch. 1; also see Abram et al., 2001 and Skiba et al, 2006). DisCrit perspectives, therefore, identify the individual problematic attitudes of teachers and administrators as one "accessible entry point for intervention" (Connor, Ferri, Annamma, 2016, ch. 1).

Summary of the Problem

This working group will investigate issues related to disability in mathematics education. Using multiple theoretical frameworks, the working group participants will analyze current practices in mathematics interventions, including the power relations involved, and develop and elaborate on alternatives. The working group participants will also plan ways to evaluate these alternatives in various educational settings and contexts.

Plan for Active Engagement of Working Group Participants

Session 1

In the first session, the organizers will introduce the rationale for the working group and its one-year history. We anticipate spending some time again hearing about how each of the members is theoretically addressing disability from a critical perspective in our ongoing work. The group will collaboratively refine the goals of the working group.

Session 2

In the second session, (based on input from session 1) participants will either:

- brainstorm potential collaborative endeavors,
- brainstorm how to position this critical work to make inroads into historically inhospitable venues for this work (e.g., Council for Exceptional Children)
- break up into subgroups to make progress on collective projects (e.g., NSF Conference grant, book proposal, collaborative work)
- discuss how to develop projects around member's PMENA presentations or
- discuss a shared reading (distributed to the list-serv before the conference and handed out during session 1) to push our thinking and shared understanding of critical perspectives.

Session 3

Session 3 will be devoted to planning our ongoing collaboration and distributing responsibilities for the group's shared endeavors (e.g., conference / book proposals, developing a shared online repository of research and teaching resources).

Plan for Sustainability: Anticipated Follow-up Activities

The working group sessions during the conference are designed to enable the participants to develop concrete plans for collaborative work beyond the end of the conference timeframe. We plan on continuing to communicate between working groups through our email list-serv. Specifically, the third session is allotted for developing specific plans for future collaborative work.

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DESIGNING AND RESEARCHING ONLINE PROFESSIONAL DEVELOPMENT

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In this working group, we consider design and research methodologies related to teacher learning in online professional development contexts. We describe the extant literature on online professional development, including a review of digital technologies and the applicability of this practice to teacher learning in education. We then describe an innovative project designed to support the development of middle school mathematics teachers in rural contexts, with a focus on three distinct forms of online learning: digitally communicated demonstration lessons, an online course, and online video coaching. Given recent technological advances and demands to support teachers in various contexts, we contend that researching and understanding these online models, as well as other online models is important for the broader field of mathematics education. As a result, the proposed discussion group will combine whole-group and subgroup time to converse about: (a) the challenges of online professional learning experiences, (b) research tools, methods, and analyses, (c) the connections among different projects and studies, and (d) future collaborations and research.

Keywords: Teacher Education-Inservice/Professional Development, Research Methods, Learning Theories

All teachers need access to high quality professional development in order to meet the needs of students and to teach rigorous mathematics as outlined in college and career-ready standards (Marrongelle, Sztajn, & Smith, 2013). Given the limited resources in some areas, including rural and urban school districts, online professional development has the potential to provide access to a wider range of teachers than what is possible face to face. Furthermore, given the propensity of millennials to seek online learning experiences, we feel that more attention needs to be given to the design, dissemination, and research of online professional development. Given the emerging importance and availability of online professional development, we propose a working group that will focus on the design, dissemination, and research on online professional development. The working group participants will analyze current practices in online professional development, including the technology affordances and limitations. Major themes that will be addressed are:

- affordances of online platforms,
- affordances and constraints of synchronous vs. asynchronous experiences,
- challenges related to scaling up high-quality online professional development, and
- methodologies used to research professional learning in online contexts.

As schools turn to digital learning contexts, it is inevitable that professional development will follow a similar trend. It is imperative to have research-based models that demonstrate how the features of high quality face-to-face professional development can be matched or augmented in online contexts. As an example of necessity, teachers in rural areas face constraints in terms of accessing the expertise and resources required for high-quality professional learning experiences, often because of lack of proximity to such resources as institutions of higher education and critical masses of teachers required to collectively reflect on problems of practice (Howley & Howley, 2005). Rural contexts are thus ideal sites for online professional development, which can be offered at a distance and can involve geographically dispersed participants (Francis & Jacobsen, 2013). At the same time, teachers in urban and suburban areas may have more regular access to professional development, but online formats afford conveniences and other applications that may not be available in face-to-face settings. Digital learning contexts provide opportunities for connections and visual supports that may otherwise not be accessible through more traditional forms of professional development. As a result, we consider it necessary to research and study online learning in these contexts and consider the importance of engaging in dialogue with mathematics educators and researchers about advancing technologies, specifically online professional learning, as related to mathematics professional development. This working group is intended to advance the practices of designing and researching online professional learning experiences by investigating the challenges of balancing high quality learning experiences and accessibility for teachers.

Below we provide an overview of the literature related to professional learning in online contexts. Then we describe one model of online professional development that is being implemented with funding from the National Science Foundation (NSF). We will devote part of the first session explaining the components of the project as a means of introducing possible models and methodologies to study online professional development, leaving opportunities over the next working sessions to incorporate discussion of other models and methodologies. We then discuss Focal Issues in the Psychology of Mathematics Education, and conclude with aims for the 2017 working group.

Overview of Literature Related to Online Professional Learning

Digital Technologies

Online professional learning experiences combine longstanding and emerging digital technologies to provide high-quality, interactive, content-focused professional development. Longstanding digital technologies (e.g., electronic learning management systems) have been used to implement online courses to design and implement professional development for the past couple of decades. Emerging digital technologies involve an internet-based platform to implement online video coaching, or other online communications, in ways that augment the interactivity of face-to-face coaching. Online video coaching emerges from the content-focused face-to-face coaching that the project personnel have engaged in over the last ten years.

Research shows that while online communication lacks some of the modalities (e.g., gestures, facial expressions) and spontaneity of face-to-face communication (Tiene, 2000), there are also affordances unique to its asynchronous and text-based nature. In online discussions, communication tends to be more exact and organized (Garrison, Anderson, & Archer, 2001; McCreary, 1990), involve more formal and complex sentences (Sotillo, 2000; Warschauer, 1995) and incorporate critical thinking, reflection, and complex ideas (Davidson-Shivers, Muilenburg, & Tanner, 2001; Marra, Moore, & Klimczak, 2004). Research on synchronous online communication – which can include text chat windows and shared space in learning management systems – shows that it is experienced as more social than asynchronous spaces (Chou, 2002). Synchronous sessions induce *personal participation*, which Hrastinski (2008) compared to *cognitive participation* in that personal communication in synchronous spaces "involves more intense interaction … while cognitive participation is a more reflective type of participation supported by asynchronous communication" (p. 499). Furthermore, synchronous communication fosters multiple communication channels based on emerging networks within the larger group, including the use of chat boxes and personal email during synchronous sessions (Haythornthwaite, 2000, 2001). Researchers have reported positive outcomes from professional development involving synchronous exchanges via typing (e.g. Chen,

Chen, & Tsai, 2009). However, synchronous verbal online discussions and group activities have not been a focus of research.

Online Professional Development in Education.

Despite the growing popularity of online professional development, there is a need for empirical research regarding its quality and effectiveness (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009). Prior research has not demonstrated advantages for online professional development in terms of teacher outcomes (cf. Fishman et al., 2013), in part due to the lack of online professional development contexts that involve teachers in sustained, intensive reflection on their practices. Furthermore, teacher learning in online spaces can be challenging, especially related to complex forms of learning. Sing and Khine (2006) found that a number of factors make it difficult for teachers to engage in complex or difficult forms of learning in an online context, such as teachers' roles as implementers rather than producers, cultural norms where disagreement is seen as confrontational, and the cognitive demands relative to the available teacher time. Teacher learning in online contexts is discussed in more detail below.

In order to illustrate professional learning in an online context, we present a model that the authors are currently using in a project situated in rural contexts. We present the model in order to begin the discussion of this model and other potential models, as well as the learning platforms and other features, such as the synchronous or asynchronous nature of learning in online environments.

A Model of Online Professional Development

The innovative online professional learning experiences in the author's project focus on the development of teacher capacity to enact ambitious, responsive instruction aligned with the rigorous content and practice elements of the Common Core State Standards for Mathematics (CCSSM). We use the term *professional learning experiences* to denote that the professional development we employ differs from traditional workshop or other models that are too short or fragmented to be effective (Garet, Porter, Desimone, Birman, & Yoon, 2001).

In the project, we identified three primary research goals. To study and understand: (a) the ways online-based professional development can help teachers improve their instructional practices and their ability to notice and respond to student thinking; (b) the characteristics of the feedback cycles in the online coaching, the role of video feedback, and the asynchronous components of feedback cycle; and (c) the features of the professional development model that would inform efforts to scale up the model, including the resource commitments, the requisite capacity of the course instructors and coaches, and the logistical requirements of the courses and coaching. We are currently in year one of four years of the project. The following describes the three online components of our project. In the working group, we envision these and other components used by other researchers serving as the catalysts for dialogue around online professional learning.

Demonstration "Fishbowl" Lessons

In order to address the challenges of engaging teachers in learning complex practices in an online context, we include a component aimed at initiating and reinforcing relationships between participants and project personnel and at helping participants to understand the types of learning experiences and design and feedback cycles that will be the core of the project. Research on lesson study (e.g., Amador & Weiland, 2016; Stigler & Hiebert, 1999) has led to an emphasis on demonstration lessons where teams of teachers collectively plan, enact, and reflect on lessons in ways that make public the features of the lessons and teachers' instructional practices (Saphier & West, 2009). Consequently, one component of our project is a collaborative classroom activity (a demonstration "fishbowl" lesson) that builds from the *studio classroom* model developed by the Teachers Development Group (2010), with features consistent with content-focused coaching (West

& Staub, 2003). For each lesson, a group led by project personnel plans a lesson around a cognitively demanding task. On the day of the lesson, project participants review the lesson plan and explore the task, the mathematics embedded in the task and anticipated student approaches to solving the task, and the related CCSSM practice and content standards, making necessary revisions to facilitate the lesson. The project personnel then enact the lesson while the rest of the group observes and collects evidence of student thinking and learning. The group then collectively reflects on the experience, with a focus on describing evidence for student understanding using the data gathered by the teachers and observers. This process is repeated regularly with participants.

In the beginning of the project all demonstration lesson activities are face-to-face. However, by the third demonstration lesson this model is moved to an online format of synchronous and asynchronous activities. The planning and debriefing portions are held via a video conferencing platform, Zoom, allowing for synchronous engagement in both whole group and small group discussions of the lessons. Asynchronous activities include watching and reflecting on the video recording of the demonstration lesson between the planning and debriefing sessions. Discussion in the working group will center on the affordances and constraints of the online model and possible modifications to ensure the intended professional development goals are met.

Online course - Orchestrating Mathematical Discussions

The second component of our project is the two online course modules, *Orchestrating Mathematical Discussions Parts One and Two*, aimed at orienting the participants toward high-leverage discourse practices that facilitate mathematically productive classroom discussions (Smith & Stein, 2011). In this course, the participants solve and discuss a series of high cognitive demand tasks, activities that will be accompanied by synchronous and asynchronous discussions around the *5 Practices for Orchestrating Productive Mathematics Discussions* (i.e. anticipating, monitoring, selecting, sequencing, connecting; Smith & Stein, 2011). The courses are designed to develop awareness of specific teacher and student discourse moves that facilitate productive mathematical discussions, to understand the role of high cognitive demand tasks in eliciting a variety of approaches worthy of group discussions, and to further develop participants' mathematical knowledge, particularly the rich connections around big mathematical ideas that are helpful to teach with understanding (Ball, 1991; Ma, 1999). The discourse moves are designed to expand student participation in the development of mathematical ideas in the classroom. These actions position students as actively constructing mathematical knowledge, and consequently develop productive mathematical dispositions and students' mathematical identities (Boaler & Staples, 2008; Chapin, O'Connor, & Anderson, 2003; Choppin, 2007a; 2007b; 2014; Herbel-Eisenmann, Steele, & Cirillo, 2013; O'Connor & Michaels, 1993).

In order to take advantage of the affordances of both asynchronous and synchronous characteristics of online communication, the course is embedded in a learning management system (LMS) that: allows for synchronous whole class and small group interaction; the sharing of artifacts, including those collectively developed in the LMS; and asynchronous discussion threads. In the online course modules in the LMS, the facilitator verbally presents a challenging task to the participants, which is viewed in the shared work space. The course instructor then assigns participants to virtual breakout rooms, in which the participants work synchronously in a common workspace, creating virtual white boards to share with the other groups. They can talk to each other, work simultaneously in the virtual space, and use the chat window to communicate. The course instructor can listen to and participate in these group discussions to determine when the groups are ready to present their solutions. The course instructor then closes the virtual breakout rooms, which automatically returns all participants to the main room to conduct a summary discussion of the different strategies, in effect modeling the practices in the *5 Practices* book. Asynchronously, the group can continue to go back and reflect and comment on the task and related solutions, as well as on the readings from the *5 Practices* book using discussions threads in the LMS. Participants are also encouraged to share resources, lesson plans, and student work as appropriate. The working group will discuss this format for online professional learning as well as other formats and tools that have proven beneficial for users.

Online Video Coaching

The third – and most innovative – component of our project's professional development program is the online video coaching that builds from models of content-focused coaching (West & Staub, 2003). More recently, thanks to the advent of improved internet-based software aimed at increasing collaboration around video data, the project personnel have begun conducting online video coaching with teachers. The coaching cycles are focused on identifying and unpacking the mathematics with the teacher, while anticipating likely student strategies, conceptions, and misconceptions. The coach helps the teacher identify evidence for demonstrating how students are thinking (from the video as well as from student artifacts) and make connections between different student approaches in order to help the teacher structure the summary discussion of the lesson.

The online coaching experiences involve synchronous and asynchronous components, with the goal of engaging participants in reflective or deliberative practice. The online coaching has features similar to face-to-face coaching, such as video conferencing conversations via Zoom, in which the coach and participant collaborate to plan lessons and reflect on the qualities of lessons. However, the online coaching includes an innovative component that involves asynchronous collaboration and feedback that structures the post-lesson collaborative reflection, features that augment or surpass the kind of feedback that can be given face-to-face. Teachers video-record themselves using Swivl, which allows them to place a camera (iPhone or other device) on a robot that tracks them around the room, allowing for teacher-focused video without the necessity of someone operating the camera. The video is automatically uploaded into a password-protected site and processed, and is immediately accessible to view and notate. The notation feature in Swivl allows the coach and the candidate to separately view and annotate the video. For example, a teacher can stop the video by hitting the pause button and type in a comment or question that is synced with the video, so that when the coach watches the video, she can read the comment during the point in the video referenced by the comment. The coach can do the same. The video can be viewed repeatedly, which allows for more thorough reflection and analysis. The notation provides for more in-depth and substantive feedback, pointing to specific instances of practice. The discussion group will focus on this model for professional coaching as well as other models or avenues for supporting individual teachers in online professional learning.

Researching Online Professional Learning Experiences

There is a dearth of research on online professional development, especially online professional development that is sustained and intensive. Similarly, while there have been 15 years of intensive efforts to implement coaching in schools, much of the research have revolved around the role and impact of coaches (Coburn & Russell, 2008; Penuel, Riel, Krause, & Frank, 2009), and less around the impact on reflective or deliberative practice. Although coaching has now been around for over ten years, there is limited research on the effectiveness of coaching in terms of improving teacher quality (Matsumura, Garnier & Spybrook, 2012). The greatest dearth of research involves online video coaching in education, as opposed to face-to-face video coaching, which has no peer-reviewed research yet associated with it.

Structure of the Working Group Sessions

Within this working group we propose to explore the following questions related to researching online professional learning experiences:

1. What are various platforms and models for online professional development?
2. What theoretical framework and methodologies are salient for researching online digital technologies and online professional learning experiences?
3. What data analysis methods are suited to the data captured in online environments?
4. In what ways can online professional learning experiences help teachers improve their instructional practices and their ability to notice and respond to student thinking?
5. In what ways can the characteristics of the feedback cycles in online coaching, the role of video feedback, and the asynchronous components of feedback cycle inform the design on online professional learning experiences to maximize teacher learning?
6. What features of the professional development model would inform efforts to scale up the model, including the resource commitments, the requisite capacity of the course instructors and coaches, and the logistical requirements of the courses and coaching?

Plan for Working Group:

In Session 1, the organizers will present brief reports on the Author's project and research design. Subgroups will be formed to explore current design and implementation efforts with online professional learning experiences from their own research and current efforts in the field.

During Sessions 2 and 3 we will provide the subgroups time to continue collaborating. Participation in the subgroup work times will involve: a) identifying the challenges of online professional learning experiences that are the most challenging and why, b) refine research tools, methods, and analyses, c) explore connections among different projects and studies, and d) discuss future collaborations and research. We will close Session 3 with time to review group progress and discuss next steps for our work as shown in Table 1. Meeting notes, work, and documents will continue to be shared and distributed via our Google Folder (set up for this Working Group). The use of Google documents allows members to create an institutional memory of activities during the working group that we will continue to use and add to following the 2017 working group. This shared folder will also provide a shared space for future collaborations and writing projects related to online professional learning experiences within the 2017 working group members.

Table 1. Overview of Proposed Working Subgroup Sessions

	Activities	**Guiding Questions**
Session 1	1. Introductions and Agenda 2. Brief Presentations of Authors' Project and Research Questions 3. Subgroup formation and initial work time - designing Online PD experiences	1. What research is being done related to online professional development? 2. Which aspects of online professional learning experiences are the most challenging and why? 3. Goals of attendees for attending this working group

Session 2	1. Overview of subgroup's work from previous day 2. Subgroup work time - engagement in online professional learning experiences 3. Brief sharing of work in subgroups	1. Identifying factors of engagement in online professional learning experiences 2. What platforms are used for online learning? What are the affordances of various platforms? 3. How can online learning support teacher learning? 4. What are the affordances and constraints of synchronous and asynchronous experiences?
Session 3	1. Overview of subgroup's work from previous day 2. Subgroup work time - researching online professional learning experiences 3. Brief sharing of work in subgroups 4. Final reflections – future collaborations and research	1. What theories and theoretical frameworks have informed the design of your research project(s)? 2. How might your work inform theory in researching online professional learning experiences? 3. What issues and challenges have you faced in designing studies in this area? 4. What challenges may exist for scaling up high-quality online professional development?

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DEVELOPING A RESEARCH AGENDA OF MATHEMATICS TEACHER LEADERS AND THEIR PREPARATION AND PROFESSIONAL DEVELOPMENT EXPERIENCES

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This working group will discuss mathematics teacher leaders and the assessment and evaluation of programs for their professional development. Mathematics teacher leaders (MTLs) are school-based, teacher leaders who are responsible for supporting other teachers with mathematics teaching and learning. Mathematics teacher leaders learn to lead professional development, coach other teachers, and work on school-level initiatives for mathematics. The challenge is that understanding of the role is still evolving as are the areas in which they need the most professional development. Many programs focus on developing their mathematical content knowledge for teaching, but they also need to learn about curriculum, assessment, pedagogy, and leadership for school change. This working group will start by discussion of existing tools and resources for documenting the work and development of MTLs. The discussion will continue with setting an agenda for research about their professional development.

Keywords: Assessment and Evaluation, Design Experiments, Teacher Education In-service/Professional Development

Defining Mathematics Teacher Leaders

The organizers of this working group find common purpose in considering the roles, responsibilities, knowledge and development of school-based mathematics teacher leaders (MTLs), specifically on their role as leaders of other teachers. An ongoing challenge and opportunity in research and development is the diverse interpretations and implementations of mathematics teacher leadership. Such leaders may have formal titles like “mathematics specialist” or “mathematics coach.” While in this role, MTLs may support teachers of mathematics, continue to teach whole mathematics classes, work with small groups of students or lead within the school in an informal capacity. This working group is focused on the MTLs leadership role with the purpose of examining the learning experiences and assessments that need to be created to understand their work. In some settings, MTLs are referred to as elementary mathematics specialists or coaches so we draw on literature and background for those roles in crafting this working group.

Knowledge & Skills

While MTLs are accomplished mathematics teachers, an additional layer of knowledge and skills is required to be a leader (e.g., Bitto, 2015). Notably, the Association of Mathematics Teacher Educators (AMTE) Standards for Elementary Mathematics Specialists (AMTE, 2010 & 2013) distinguish the role of teacher of mathematics from the role of teacher leader. This separation of roles recognizes that the knowledge and skills for being a teacher of mathematics are distinct from the knowledge and skills to be a mathematics leader. Specific leadership skills required of MTLs to support adult learning include: leading professional development, working one-on-one with teachers, collaborating with teacher professional learning communities, and creating school-level initiatives.

Brief History of the Working Group

The organizers originally convened at a meeting run by the Brookhill Institute in 2015 to examine research and create working groups of researchers to study MTLs, specifically mathematics specialists and coaches. Since then, the organizers have published a book about elementary mathematics specialists (McGatha & Rigelman, 2017), presented at conferences (McGatha & Rigelman, AMTE, 2017; Larsen, Bailey & Baker, National Council of Supervisors of Mathematics [NCSM], 2017), written book chapters (Baker, Bailey, Larsen, & Galanti, 2017) and published articles related to mathematics teacher leadership (McGatha, Davis, & Stokes, 2015). While this is the first organized working group of these leaders at a PME-NA conference, the proposed working group is a continuation of previous efforts and seeks to engage additional researchers in examining research and development projects related to MTLs. A significant goal in this effort is to build on the momentum that began at the 2015 Brookhill meeting and advance the mathematics teacher leadership research community. Significant research is still required that centers on MTLs including: their professional development, knowledge, and practices particularly in terms of their leadership development and their work with teachers both individually and in groups (McGatha et al., 2015).

Rationale for Working Group

In 2015, McGatha, Davis, and Stokes published an NCTM Research Brief that examined the current compilation of MTL literature. Comprised of 24 articles, an increase from the 9 articles available in 2009 (McGatha et al., 2015), the need for additional research on MTLs was illuminated. Specifically, the NCTM Research Brief included a review of literature centered on MTL experiences, roles and effectiveness to impact both mathematics teachers and students. Studies were placed on a continuum from less-directive to most-directive. In less-directive activities, MTLs encouraged teachers to reflect through activities that promoted the thoughtful practice of teaching mathematics. The opposite end of the continuum, most-directive, involves the MTL reflecting and doing most of the work. The teacher or collaborative team are passive participants, not active in the planning, taking the materials provided and accept being told what actions need to occur. In the middle of the continuum were actions, such as co-teaching and co-planning, that involved the MTL and classroom teacher collaborating to improve instruction and student learning. McGatha, et al., (2015) concludes that more research needs to be conducted on the work of the MTL while coaching individual and groups of teachers, their roles as leaders in their schools, how systems might establish and support the MTL, and what are the actions that lead to transforming instruction and productive student outcomes.

The 2015 NCTM Research Brief guided A Critical Analysis of Emerging High-Leverage Practices for Mathematics Specialists (Baker, Bailey, Larsen & Galanti, 2017) which was presented at the 2017 AMTE annual conference. In this book chapter, the authors present an analysis of the 24 articles through two distinct lenses that aim to identify effective practices for MTL: 1) the NCTM CAEP Standards (2012); and 2) the potentially productive coaching practices which included engaging in mathematics, examining student work, analyzing classroom video, rehearsing, lesson study or studio day, co-teaching, and modeling (Gibbons & Cobb, 2012). The analyses revealed that co-teaching and modeling were the practices most frequently mentioned in MTL research, with eleven and twelve mentions respectively. With respect to the role and actions of the MTL, engaging in mathematics and lesson study was discussed in five articles each (Alloway & Jilk, 2010; Campbell & Malkus, 2011; Chval, et al., 2010; Gibbons, 2015; Polly, 2012), two of the research articles employed analyzing classroom video (Alloway & Jilk, 2010; Campbell & Malkus, 2011) and rehearsing was not mentioned at all. Although the frequency of MTL actions speaks to the need for additional research, the research on MTLs did not provide enough specific information about the practices used so that they might be replicated to provide evidence of impact or transform instruction.

Baker et al. identified a need for MTL researchers to develop and use a common language to achieve the clarity and cohesiveness required to advance the work of the MTL.

The work of the MTL and transforming instruction takes time with the leaders requiring support in their work (Borko, Koellner & Jacobs 2014; Campbell, 2012; Chval, et al., 2010). Campbell (2011) posits that stakeholders on a global scale (e.g. policymakers, mathematics educators, mathematicians, professional developers) need to come together with research to further the work of MTLs in a "non-competitive sense of mutual obligation and responsibility." A support structure must exist that includes the opportunity to share the implementation and research challenges.

Background and Theoretical Perspective

Elementary Mathematics Specialists: A Movement to Advance MTLs

A single MTL can have an incredible impact on the development and effectiveness of others. Leaders in mathematics education at all levels of the school or district organization are crucial for ensuring attainment of high-quality school mathematics programs. In response to such leadership needs Elementary Mathematics Specialist (EMS) programs and job roles have been created across the United States. Many have made the case that practicing elementary school teachers are not adequately prepared to meet the demands for increasing student achievement in mathematics (NCTM, 2000; National Mathematics Advisory Panel, 2008; National Research Council, 1989). The main reason given to explain this insufficiency is that most elementary teachers are generalists—that is, they study and teach all core subjects, rarely developing in depth knowledge and expertise in teaching elementary mathematics.

More recently, the National Mathematics Advisory Panel (2008) noted that ―the use of teachers who have specialized in elementary mathematics teaching could be a practical alternative to increasing all elementary teachers' content knowledge (a problem of huge scale) by focusing the need for expertise on fewer teachers. Building on the Panel's recommendations, Wu (2009) advocated for mathematics specialists by suggesting the "problem of scale" could be addressed by utilizing a smaller cadre of well-prepared teachers to focus on mathematics at the elementary grades. Over the past two decades, others made similar recommendations (Battista, 1999; Conference Board of the Mathematical Sciences, 2001, p. 11; Learning First Alliance, 1998; National Council of Teachers of Mathematics, 2000, pp. 375–376; Reys & Fennell, 2003).

Beginning in 2009, under the leadership of then President Barbara Reys, AMTE began its advocacy for Elementary Mathematics Specialist certification and programs. They developed the EMS Standards (2010 & 2013) to support establishment of state certifications and development of programs. From there, they collaborated with ASSM, NCSM, and NCTM to develop a position statement, host two state certification conferences, and a conference for those with research connected to EMS. In this time, the number of states with EMS certification has increased from 9 to 20 which has resulted in an increase in the number of programs for EMS.

Together with AMTE, the Association of State Supervisors of Mathematics (ASSM), NCSM, and NCTM (2010) endorsed and recommended the use of elementary mathematics specialists to "enhance the teaching, learning, and assessing of mathematics in order to improve student achievement" (p. 1). Furthermore, they recommended that "every elementary school have access to an EMS [elementary mathematics specialist]. Districts, states/provinces, and higher education should work in collaboration to create: (1) advanced certification for EMS professionals; and (2) rigorous programs to prepare EMS professionals" (AMTE, ASSM, NCSM, & NCTM, 2010, p. 1).

The hope of state-level EMS certification would be to provide formal recognition, opportunities, and incentives for teachers to increase their knowledge and skill to teach or to lead others in teaching mathematics in elementary classrooms. With a formal certificate program, school and district administrators would be better positioned to create EMS positions and identify qualified personnel, thus, improving support for their teachers and students. Expectedly, with the increase in the call for mathematics specialist and increase in formal certification programs, the need for additional evaluation and research about mathematics specialist is critically needed. Articulating the knowledge and skills needed by EMS professionals is a necessary step in initiating state-level certification and program development. Although there is substantive and ongoing research on the effectiveness of EMS "as coaches," research related to elementary mathematics teachers is almost nonexistent (McGatha 2009; McGatha et. al., 2015).

Numerous publications and presentations have resulted from the work to develop Elementary Mathematics Specialist programs that now spanned more than two decades. Notably, EMSs have demonstrated the potential to positively influence teachers' instructional practices and beliefs in results from both qualitative and quantitative research (e.g., Baldinger, 2014; Campbell, 1996; Race, Ho, & Bower, 2002); however, some studies have noted limited changes in teachers' beliefs and practices even after the coaching process (e.g., Ai and Rivera, 2003; Olson & Barrt, 2004). EMSs have also shown to be effective on positively impacting students' mathematics learning and achievement (e.g., Balfanz, Mac Iver, & Byrnes, 2006; Campbell & Malkus, 2011). Large-scale projects focused on the relationship between EMSs and students' mathematics achievement provide evidence that these instructional leaders may provide schools the avenue needed for improved student performance (Balfanz et. al., 2006; Brosnan & Erchick, 2010; Campbell, 1996; Campbell & Malkus, 2011; Foster & Noyce, 2004). However, study limitations and the narrow number of empirical studies requires additional research to corroborate these promising findings.

Assessment and Evaluation of Elementary Mathematics Specialists

Many EMS programs are in their infancy and while the curriculum may exist the assessment and evaluation plan may not be fully developed. This is due in part to the various roles of an EMS as an elementary mathematics teacher, mathematics teacher leader, mathematics intervention specialist, and mathematics coach (McGatha & Rigelman, 2017) and the lack of specificity of the high leverage practices associated with the varied roles (Baker et al., 2017).

In a late 2014 survey of AMTE members (n=70), only 21% of the programs represented by survey respondents had program evaluation systems in place. Those without evaluation systems reported that their program was new and that program evaluation was under development. Respondents describe their EMS programs as aligned with state competencies for EMS professionals, NCTM/NCATE/CAEP EMS Standards, and/or AMTE EMS Standards. Of the programs with evaluation systems, 92% of the respondents commented on their program evaluation as tied to periodic reviews at the local, state, or national level.

Of those programs with evaluation plans in place, only two (17%) characterized their evaluation as innovative. Typical of the EMS program evaluation models described was an assessment process parallel to other advanced degree programs in the department/school (e.g., key assessments in particular courses, course evaluations, action research projects). Some programs use a portfolio as a measure of outcomes, while others use mathematics content knowledge assessments. A few commented on gathering data pre- and post-program on teacher beliefs and their ability to analyze classroom practice. Those programs with the most robust and for program evaluation described their work as based on their grant evaluation.

EMS program developers can learn from the various recommendations to teacher education programs about how to best assess teacher knowledge and skills. Researchers discuss the need to include assessments of practical knowledge, determine through multiple measures (AMTE, 2017; Beijard & Verloop, 1996; Darling-Hammond, 2006; Dwyer, 1993, NCTM, 1995). Some describe the potential of a teaching portfolio (Beijard & Verloop, 1996; Darling-Hammond, 2006; Zeichner,

2001) as a performance-based measure which could provide evidence of teacher leaders learning in and from their practice (Ball & Cohen, 1999). Program developers need to consider both the possibilities and limitations of the various tools they select for candidate and program assessment.

Session Organization and Plan for Engagement

This working group aims to explore the current avenues of and intentionally develop approaches for studying the improvement of mathematics teacher leaders, their preparation programs and professional development experiences. We first focus on the examination of existing assessment and evaluation tools, as well as the recommendations from the professional literature regarding high-quality candidate and program assessment. After generating questions of practice, participants will break into two groups to develop a plan of research that both builds capacity among and advances the research centered on MTLs.

Session 1: Examination of Assessment and Evaluation Tools

The first session will begin with introductions that share the areas of interest and expertise of each participant. Session leaders will provide a brief overview of the current status of the evaluation and assessment of mathematics teacher leader candidates and programs. Afterwards, session leaders will facilitate a large group discussion that connects the current status of candidate and program assessment with participants' research. This discussion will result in a list of collaboratively-generated research questions. Sample anticipated questions from the small group collaboration are:

- What might it look like to align EMS candidate and program assessment with high quality assessment principles?
- What are examples of key assignments, rubrics, and other instruments that have effectively measured the knowledge, skills, and practices of EMSs? To which aspects of the EMSs' role and/or knowledge needs do they attend? What are our development needs?
- Often, when programs are developed with support of grant funding, they develop robust candidate and program evaluation to comply with funders' expectations. What might it look like to maintain a robust assessment and evaluation system after the grant is over?
- What are some ways that programs can effectively maintain access to their graduates so they can determine efficacy of the EMS and/or programs that prepared the EMS?
- In what ways is the work of a MTL supported or hindered by various stakeholders? How can advocacy of the position of a MTL increase?
- Which aspects of the work of an MTL are most impactful for teachers and students?

Group members will then break into small groups to explore existing assessment and evaluation tools for not only mathematics teacher leaders and programs, but also other school personnel and university programs in which there is intersectionality and overlap in roles and responsibility (e.g. reading specialists, school administrators, special educators). During this time participants will determine how the provided tools might be adapted for their current research contexts and aligned with the group generated research questions. The session will conclude with small groups sharing their conversations and session leaders recording the tools into a matrix that connects the group-generated research questions with purposefully identified tools for one of two categories: Tools to Evaluate and Assess Mathematics Teacher Leaders and Tools to Evaluate and Assess Preparation Programs and Professional Development.

Session 2: Developing a Common Research Agenda

Initial whole group conversations during the second session will center on participant reflections from the matrix created on the first day. Participants will then determine their research interest and split into two to three small groups to: 1) allow participants to flexibly meet their needs based on context; and 2) align their research interests with a population they have access to. Session leaders will divide among the groups based on their own research experiences and interests, as well as the size of the groups. The group discussions will explore the following questions aimed at developing a common context, language and purpose:

- What access or relationships do you have to either candidates or programs (e.g. individuals, schools, districts, universities)
- Which of the research questions generated from the first session are of interest?
- What research tools are available or adaptable that will support the exploration of the identified research questions?
- In what ways can we align our research interests with the available populations and tools?

The session will conclude with each group sharing their conversations and questions as they develop a common research agenda.

Session 3: Designing Research Studies

In the third session, participants will build on the collaborative work they began in the second session and further design a research study. Session leaders will facilitate group discussions that support the participants in developing an outline of a research study. Conversations will center on creating an outline of a research study that includes the following aspects: research question, participants/location, specific evaluation or assessment tool that will be used/modified during the study, identified research activities, a timeline for research activities with set goals and a method of communication. After each group shares the outline of their research study, the session will conclude with the session leaders providing information on the anticipated follow-up activities and possible research products and outcomes.

Anticipated Follow-Up Activities

It is anticipated that the participants from this working group will continue to meet virtually over the course of the year using Google Tools, synchronous online platforms, social media and phone conferencing to develop several research products. Although the participants will influence the direction of the research, session leaders are prepared to develop the following products with participants: a database or list of the mathematics teacher leader and program assessment and evaluation tools, collaborative share results of studies at mathematics education conferences (e.g. AMTE, NCSM, NCTM-Research) or in journal submissions, the exploration of grant funding to advance the collaboration, submitting a symposium for PME-NA 2018, and developing a proposal for the continuation of the working group at PME-NA 2018.

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) WORKING GROUP

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Embodied cognition is growing in theoretical importance and as a driving set of design principles for curriculum activities and technology innovations for mathematics education. The central aim of the EMIC (Embodied Mathematical Imagination and Cognition) Working Group is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education. A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities, practices, and emerging technologies that count as mathematical. EMIC builds upon our 2015 and 2016 working groups with a specific focus on examining the embodied nature of mathematical collaboration. In particular, we view collaboration as a type of crossroad that brings together people and artifacts, from which EMIC communication and activities can emerge.

Keywords: Classroom Discourse, Cognition, Informal Education, Learning Theory

Motivations for This Working Group

Recent empirical, theoretical and methodological developments in embodied cognition and gesture studies provide a solid and generative foundation for the establishment of **a regularly held Embodied Mathematical Imagination and Cognition (EMIC) Working Group** for PME-NA. The central aim of EMIC is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education, including, but not limited to, studies of mathematical reasoning, instruction, the design and use of technological innovations, learning in and outside of formal educational settings, and across the lifespan.

The interplay of multiple perspectives and intellectual trajectories is vital for the study of embodied mathematical cognition to flourish. Partial confluences and differences have to be maintained throughout the conversations; this is because instead of being oriented towards a single and unified theory of mathematical cognition, EMIC strives to establish a philosophical/ educational "salon" in which entrenched dualisms, such as mind/body, language/materiality, or signifier/signified are subject to an ongoing and stirring criticism. A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities and emerging technologies that count as mathematical, and envision alternative forms of engagement with mathematical ideas and practices (e.g., De Freitas & Sinclair, 2014). This broadening is particularly important at a time when schools and communities in North America face persistent achievement gaps between groups of students from many ethnic backgrounds, geographic regions, and socioeconomic circumstances (Ladson-Billings, 1995; Moses & Cobb, 2001; Rosebery, Warren, Ballenger & Ogonowski, 2005). There also is a need to articulate evidence-based findings and principles of embodied cognition to the research and development communities that are looking to generate and disseminate innovative programs for promoting mathematics learning through movement (e.g., Ottmar & Landy, 2016; Smith, King, & Hoyte, 2014). Generating, evaluating, and curating empirically validated and reliable methods for promoting mathematical development and effective instruction through embodied activities that are engaging and curricularly relevant is an urgent societal goal.

Past Meetings and Achievements of the EMIC Working Group

The first meeting of the EMIC working group took place in East Lansing, MI during PME-NA 2015, and has been continuing to expand ever since into a website (https://sites.google.com/site/emicpmena), a second EMIC working group in Tucson, AZ, at PME-NA 2016, and a sister pre-conference workshop event at the upcoming Computer Supported Collaborative Learning conference (Williams-Pierce et al., accepted). It has a somewhat longer origin, dating back to the 2007 AERA symposium, "Mathematics Learning and Embodied Cognition." By now, several research programs have formed to investigate the embodied nature of mathematics (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards, Ferrara, & Moore-Russo, 2014; Lakoff & Núñez, 2000; Ottmar & Landy, 2016; Radford 2009; Nathan, Walkington, Boncoddo, Pier, Williams, & Alibali, 2014; Soto-Johnson & Troup, 2014; Soto-Johnson, Hancock, & Oehrtman, 2016), demonstrating a "critical mass" of projects, findings, senior and junior investigators, and conceptual frameworks to support an on-going community of likeminded scholars within the mathematics education research community.

Since our first meeting at PME-NA 2015, some of **our collaborative accomplishments** include:

1. Creating a contact list with names and emails of attendees, and other interested scholars who could not attend PME-NA 2015 or 2016
2. Developing a group website using the Google Sites platform to support ongoing interactions throughout the year, and regularly adding additional resources/activities
3. Joint submission of an NSF DRK-12 by members who first met during the 2015 EMIC sessions
4. Some senior members joining a junior member's NSF ITEST and Cyberlearning grant proposals
5. Submission to the IES CASL program to study the role of action in pre-college proof performance in geometry (Funded 2016-2020 for Nathan & Walkington)
6. Submitting a proposal for the continuation of the EMIC WG to PME-NA 2016
7. Examining the potential for an NSF Research Coordination Network (RCN)
8. Application for a grant from Association for Psychological Science (APS) to develop a better website and offer stipends for contributors
9. Proposing a pre-conference workshop to CSCL 2017 on the embodied tools to promote STEM education, which was accepted as a full-day event (Williams-Pierce et al., accepted)

Current WorkingGroup Organizers

As the WorkingGroup has matured and expanded, we have a broadening set of organizers for the coming year that represent a range of institutions and theoretical perspectives (and is beyond the limit of six authors in the submission system). This, we believe, enriches the WorkingGroup experience and the long-term viability of the scholarly community. The current organizers for 2017 are (alphabetical by first name):

- Candace Walkington, Southern Methodist University
- Carmen J. Petrick Smith, University of Vermont
- Caro Williams-Pierce, University at Albany, SUNY
- David DeLiema, University of California, Berkeley
- David Landy, Indiana University
- Dor Abrahamson, University of California, Berkeley
- Erin Ottmar, Worcester Polytechnic Institute
- Hortensia Soto–Johnson, University of Northern Colorado
- Martha W. Alibali, University of Wisconsin-Madison
- Mitchell J. Nathan, University of Wisconsin-Madison
- Rebecca Boncoddo, Central Connecticut State University

Focal Issues in the Psychology of Mathematics Education

Emerging, yet influential, views of thinking and learning as embodied experiences have grown from several major intellectual developments in philosophy, psychology, anthropology, education, and the learning sciences that frame human communication as multi-modal interaction, and human thinking as multi-modal simulation of sensory-motor activity (Clark, 2008; Hostetter & Alibali, 2008; Lave, 1988; Nathan, 2014; Varela et al., 1992; Wilson, 2002). These views acknowledge the centrality of both unconscious and conscious motor and perceptual processes for influencing conscious awareness, and of embodied experience as following /producing pathways through social and cultural space. As Stevens (2012, p. 346) argues in his introduction to the *JLS* special issue on embodiment of mathematical reasoning,

> it will be hard to consign the body to the sidelines of mathematical cognition ever again if our goal is to make sense of how people make sense and take action with mathematical ideas, tools, and forms.

Four major ideas exemplify the plurality of ways that embodied cognition perspectives are relevant for the study of mathematical understanding: (1) Grounding of abstraction in perceptuo-motor activity as one alternative to representing concepts as purely amodal, abstract, arbitrary, and self-referential symbol systems. This conception shifts the locus of "thinking" from a central processor to a distributed web of perceptuo-motor activity situated within a physical and social setting. (2) Cognition emerges from perceptually guided action (Varela, Thompson, & Rosch, 1991). This tenet implies that things, including mathematical symbols and representations, are understood by the actions and practices we can perform with them, and by mentally simulating and imagining the actions and practices that underlie or constitute them. (3) Mathematics learning is always affective: There are no purely procedural or "neutral" forms of reasoning detached from the circulation of bodily-based feelings and interpretations surrounding our encounters with them. (4) Mathematical ideas are conveyed using rich, multimodal forms of communication, including gestures and tangible objects in the world.

Alongside these theoretical developments have been technical advances in multi-modal and spatial analysis, which allow scholars to collect new sources of evidence and subject them to powerful analytic procedures, from which they may propose new theories of embodied mathematical cognition and learning. Just as the "linguistic turn" in the social sciences was largely made possible by the innovation that enabled scholars to collect audio recordings of human speech and conversation

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

in situ, growth of interest in multi-modal aspects of communication have been enabled by high quality video recording of human activity (e.g., Alibali et al., 2014; Levine & Scollon, 2004), motion capture technology (Hall, Ma, & Nemirovsky, 2014; Sinclair, 2014), developments in brain imaging (e.g., Barsalou, 2008; Gallese & Lakoff, 2005), multimodal learning analytics (Worsley & Blikstein, 2014), and data logs generated from embodied math learning technologies that interacts with touch and mouse-based interfaces (Manzo, Ottmar, & Landy, 2016).

Theme: The Crossroads of Collaboration

Inspired by the PME-NA 2017 theme, we will specifically focus on the ways in which people can influence one another. Examples that we will use during the Working Group include: the interactions of two middle school-aged friends playing a mathematics game together and using both physical and digital gesture to augment their spoken communication (Williams-Pierce, 2016); a teacher guiding the movements of a learner exploring ratios (Abrahamson & Sánchez-García, 2016); pre-service teachers using their distributed gestures to explore a mathematical conjecture and establish its truth and justification using embodied and extended cognition (Walkington, Woods & Nathan, under review); pre-service teachers interacting with designed dynamic algebraic notations as a means of engaging embodied aspects of mathematical derivation (Jacobson, Landy, & Ottmar, in prep); having students and teachers play and create embodied technology games to teach mathematics and computational thinking (Arroyo & Ottmar, 2016), and pairs of elementary students working together on a series of body-based tasks centered on angle concepts (Smith et al., 2016).

Through these examples, we will explore questions such as: when students meet within a common arena, how might an activity design motivate them to develop mathematical notions through making pragmatic ideas mutually intelligible, and how do they accomplish this feat? What are the roles of more knowledgeable members of the community in facilitating this process? During the conference, participants in our EMIC workshop will engage in dedicated activities and guided reflections as a basis for exploring the interpersonal and interactive crossroads of goals, action, and discourse as these play out in the emergence of mathematics learning. This investigative effort will be crafted so as to align with recent developments in embodiment literature, whereby scholars are struggling to model individual sensorimotor learning within established cultural practices, norms, and values.

Plan for Active Engagement of Participants

Our formula from PME-NA 2015 and 2016 proved to be effective: By inviting participants into math activities at the beginning of each session, we were rapidly drawn into those very aspects of mathematics that we find most rewarding. We plan to facilitate collaborative EMIC activities, followed by group discussions (and we now have many activities and members who can trade off in these roles!) that will help us all to “pull back” to the theoretical and methodological issues that are central to advancing math education research. Within this structure of beginning with mathematical activities and facilitated discussions, on **Day 1** we plan to begin with activities that forefront collaboration around EMIC activities, with four different groups engaging in different activities. These activities will serve as the foundation for a broad group discussion about the varied roles of collaboration in EMIC. See Figure 1 below for examples of collaborative activities from PME-NA 2016.

Figure 1. Collaborative activities. Martha Alibali and two participants form a triangle together with their arms (left); a group just finished jointly assembling a large icosahedron in an activity facilitated by Dor Abrahamson and Leah Rosenbaum (right).

In previous years, we have found that the full first session will generally be taken up by introductions and and a round of activities followed by discussion. If there is additional time, we will begin brainstorming new collaborative EMIC activities - if there is not, then we will ask attendees to jot down any new activity ideas they have to share at the following session.

On **Day 2**, we will begin the session with technology-based collaborative activities, with four stations that pairs of participants rotate through. Examples of two of those stations are in Figure 2. Continuing with the routine established in Day 1, a full group discussion will follow, with a particular focus on designing EMIC digital contexts to support collaboration.

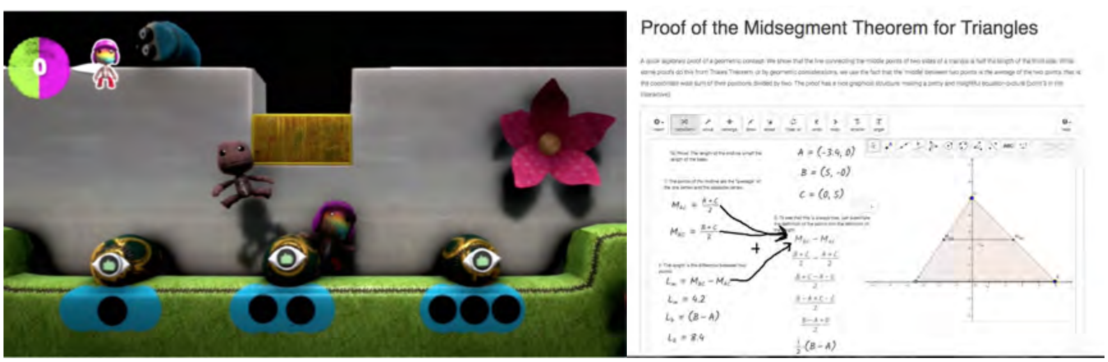

Figure 2. Digital collaborative activities. *Rolly's Adventure* by Caro Williams-Pierce (left); and *Graspable Math* by David Landy and Erin Ottmar (right).

After the discussion, we will discuss different EMIC activity ideas that participants began jotting down the day before, with the goal of developing additional collaborative activities that can be used in various research and learning contexts. The final activities will be shared on the EMIC website.

Day 3 is agenda-setting day, where we all discuss how we will keep the momentum going, such as developing an NSF Research Coordination Network (RCN), as a potential complement to the PME-NA Working Group. The RCN is not intended to promote any one particular research program, but rather to build the networked community of international scholars from which many fruitful lines of inquiry can emerge. Commensurate with the aims of the RCN, we will explore ways to

> share information and ideas, coordinate ongoing or planned research activities, foster synthesis and new collaborations, develop community standards, and in other ways advance science and education through communication and sharing of ideas.

Another example is to develop a proposal for a special issue of the *Journal of Research in Mathematics Education* that focuses on sharing the different theoretical perspectives, research activities, and operationalization of EMIC by the working group members.

In order to find common ground for the RCN submission and the JRME special issue, we may perform a live concept mapping activity that is displayed for all participants to explore the range of EMIC topics and identify common conceptual structure. We will discuss different general foci, such as teacher professional development with EMIC, designing EMIC games or museum exhibits, etc. Then, harkening back to the four major ideas that we developed earlier, sample seed topics for organizing this activity will be explored, such as:

1. Grounding Abstractions
    a. Conceptual blending (Tunner & Fauconnier, 1995) & metaphor (Lakoff & Núñez, 2000)
    b. Perceptuo-motor grounding of abstractions (Barsalou, 2008; Glenberg, 1997; Ottmar & Landy, 2016; Landy, Allen, & Zednik, 2014)
    c. Progressive formalization (Nathan, 2012; Romberg, 2001) & concreteness fading (Fyfe, McNeil, Son, & Goldstone, 2014)
    d. Use of manipulatives (Martin & Schwartz, 2005)
2. Cognition emerges from perceptually guided action: Designing interactive learning environments for EMIC
    a. Development of spatial reasoning (Uttal et al., 2009)
    b. Math cognition through action (Abrahamson, 2014; Nathan et al., 2014)
    c. Perceptual boundedness (Bieda & Nathan, 2009)
    d. Perceptuomotor integration (Ottmar, Landy, Goldstone, & Weitnauer, 2015; Nemirovsky, Kelton, & Rhodehamel, 2013)
    e. Attentional anchors and the emergence of mathematical objects (Abrahamson & Bakker, 2016; Abrahamson & Sánchez–García, 2016; Abrahamson et al., 2016; Duijzer et al., 2017)
    f. Mathematical imagination (Nemirovsky, Kelton, & Rhodehamel, 2012)
    g. Students' integer arithmetic learning depends on their actions (Nurnberger-Haag, 2015).
3. Affective Mathematics
    a. Modal engagements (Hall & Nemirovsky, 2012; Nathan et al., 2013)
    b. Sensuous cognition (Radford, 2009)
4. Gesture and Multimodality
    a. Gesture & multimodal instruction (Alibali & Nathan 2012; Cook et al., 2008; Edwards, 2009)
    b. Bodily activity of professional mathematicians (Nemirovsky & Smith, 2013; Soto-Johnson, Hancock, & Oehrtman, 2016)

c. Simulation of sensory-motor activity (Hostetter & Alibali, 2008; Nemirovsky & Ferrara, 2009)

We will also discuss the implications of this work and the different areas of the concept map for teaching, and discuss ideas for bridging the gap between research and practice.

Finally, we will introduce the EMIC website (publicly available at https://sites.google.com/site/emicpmena/home). On this website, we have a list of members with their emails and bios, information about our PME-NA presence, and short personal introduction videos. We've also created a space for members to share information about their research activities – particularly for videos of the complex gesture and action-based interactions that are difficult to express in text format. In addition, we have a common publications repository to share files or links (including to ResearchGate or Academia.edu publication profiles, so members don't have to upload their files in multiple places). At our 2015 working group, some junior members expressed particular interest in this literature support for their pending theses, while more senior members were eager to share and organize the emerging body of work on embodied math education. We've also linked the Google Sites platform directly to a Google Group, so members can participate in online forums (or the linked listserv), and discuss cutting edge topics, share in-progress working papers for review, or advertise for conferences, special issues, or other EMIC-relevant opportunities.

Follow-up Activities

We envision an emergent process for the specific follow-up activities based on participant input and our multi-day discussions. At a minimum, we will continue to develop a list of interested participants and grant them all access to our common discussion forum and literature compilation. Those that are interested in the NSF RCN plan will work to form the international set of collaborations and articulate the intellectual topics that will knit the network together; and those that are interested in the JRME special issue proposal will outline a specific timeline for progressing. One additional set of activities we hope to explore is to introduce educational practitioners at all levels of administration and across the lifespan to the power and utility of the EMIC perspective.

In the past three years, we have seen a great deal of progress. This is perhaps best exemplified by coming together of the EMIC website, the ongoing collaborations between members, and the proposals here and to CSCL, which each draws across multiple institutions. We thus will strive to explore ways to reach farther outside of our young group to continually make our work relevant, while also seeking to bolster and refine the theoretical underpinnings of an embodied view of mathematical thinking and teaching.

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EXAMINING SECONDARY MATHEMATICS TEACHERS' MATHEMATICAL MODELING KNOWLEDGE FOR TEACHING

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This 3rd year working group addresses the research question: What knowledge do secondary teachers need to possess to advance and foster mathematical modeling abilities in students? In the previous sessions, definitions and exemplar task distinguishing modeling mathematics and mathematical modeling were made. A task analysis framework, with three distinct features – openness, authenticity, and complexity, was created to investigate differences between mathematical modeling tasks and other problems. The group identified characteristics for both mathematical and modeling knowledge. Researchers considered the content and real world knowledge required when using different types of mathematical models, e.g. mechanistic and deterministic. The type of pedagogical knowledge needed for using different types of modeling tasks in classrooms still needs considerable research. While pedagogical knowledge discussions occurred during previous sessions, the nature of this knowledge was fully developed. This will be the primary focus of this year's work so that a more complete Mathematical Modeling Knowledge for Teaching (MMKT) can be defined to inform teacher training programs. Progress on grant writing, national survey distribution, and current journal writing from previous years' work will be reported. New research questions will be developed during these sessions.

Keywords: Modeling, Mathematical Knowledge for Teaching, Teacher Knowledge, Teacher Education-Inservice/Professional Development

The Common Core State Standards for Mathematics (CCSSM), launched in 2009, are the current academic standards in 42 states. The CCSSM includes modeling mathematics as both a content standard and a mathematical practice (National Governors Association & Council of Chief State School Officers, 2010). The CCSSM, as well as the standards from other states such as Texas (TEA, 2015) and Virginia (VDOE, 2017), specifically include *mathematical modeling* in their high school mathematics content and *modeling with mathematics* in their elementary and middle grades mathematics materials. Mathematical modeling (MM) and modeling with mathematics (MWM) differ in problem authenticity, degree of task openness, and complexity of mathematics and decision making needed. Modeling with mathematics (MWM) activities can either be mathematical in context or consider real life situations whereas mathematical modeling problems are drawn from authentic real life phenomena. MWM require the problem solver to make fewer decisions than mathematical modeling situations indicating that the MWM tasks are less open, in that, their problem statements reduce the number of assumptions, variables, and relationships needed. Because of their less complex nature, MWM tasks allow the student fewer mathematical approaches available, which limits the number of solutions students can generate.

Mathematical modeling provides a decision-making process to address the messy, ill-posed problems found in real life. For example, when asked to identify the stopping distance of a vehicle (NGA & CCSSO, 2010), the student must identify a problem statement in the less structured world environment, make simplifying assumptions, identify needed variables, and recognize constraints in order to produce and manipulate a mathematical model that may describe or make predictions about the real world situation (Jensen, 2007). The model's success depends on the decisions made by the student, and these decisions are influenced by the student's experiences, which means that each student may travel a different solution path and produce a different mathematical model for the same situation (Confrey & Maloney, 2007).

Because few teacher education programs include specific training to develop secondary teachers' mathematical modeling abilities, both as mathematicians and as teachers of mathematical modeling, implementing the academic standards surrounding mathematical modeling is challenging (Goos, 2014). From discussions in previous working group sessions, mathematics educators and researchers have confirmed this challenge and expressed concern about the limited availability of training materials including access to previously used mathematical modeling problems with potential solutions and pedagogical guidance for planning, implementing and assessing learning when using these ill-posed problems. This proposed working group will continue to collect mathematical modeling resources and share mathematical modeling experiences to build understanding about the mathematical and modeling content knowledge teachers' need. Research attention will be focused on the pedagogical demands that surround these types of problems. It is the intent of this research to more fully understand teacher knowledge surrounding mathematical modeling in order to increase the body of research on this topic and inform teacher training programs, which may ultimately promote a richer environment for student learning so that students may better understand the world that surrounds them.

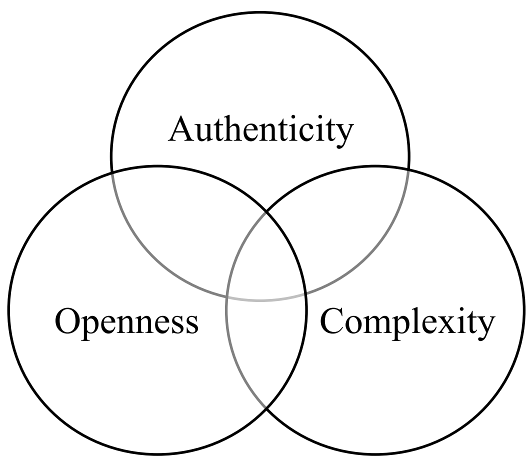
Progress From Previous Working Group Sessions

The fundamental research question for this working group asks, "*What knowledge do secondary teachers need to possess to advance and foster mathematical modeling abilities in students?* This knowledge includes the facts and skills a person acquires either through education or life experiences (Oxford University Press, 2016), and this knowledge expands through increased exposure to different types of modeling activities (Doerr, 2007). Through collaborative discussions of published research regarding mathematical modeling activities in K-12 education, basic definitions for modeling with mathematics and mathematical modeling were founded, then the working group considered the types of knowledge needed by teachers when working with mathematical modeling problems and the group considered that this knowledge may change according to the problem context, which draws upon both conceptual and experiential knowledge of the world as well as the breadth of mathematical topics potentially needed for the problem-solving process, and the type of modeling situation the teachers experienced, which requires a specific set of mathematical skills to answer questions that are descriptive, explanatory, or predictive. For example, the knowledge and skills needed to answer "how far will a vehicle travel when coming to a stop" may vary if the question was more predictive, such as "what is the likelihood the vehicle will stop in 20 feet". To provide insight into the complexity of mathematical modeling knowledge, the researchers facilitated discussions with session participants about the cognitive demands surrounding both the context of mathematical modeling tasks, which resulted in the development of a task analysis framework, and the purpose of the mathematical modeling question, which introduced session participants to different types of mathematical modeling situations, e.g. descriptive or analytic, qualitative or quantitative, and deterministic or mechanistic.

During the process of distinguishing modeling applications and modeling with mathematics from mathematical modeling problems, the mathematical modeling task analysis framework (Figure 1) was created and operational definitions for authenticity, complexity, and openness were stated to aid in revealing the cognitive demands surrounding the problem's context. The framework also provided a method to promote individual and group participation while analyzing mathematical modeling tasks and discussions found in research literature.

Both *authenticity* and *complexity* have two components. *Authenticity* measures the closeness of fit in the mathematical modeling experience between the classroom and the research field with regards to the data sources, computer techniques and software, and modeling approach (Vos, 2011). *Authenticity* also addresses the genuineness of the real life context in being situated in a subject area outside of mathematics (Niss, 1992) and originating from reality outside the classroom (Lesh & Lamon, 1992) in that it describes a real event, asks a meaningful question, and produces a realistic solution (Palm, 2007). *Modeling complexity* depends on the amount of assistance the students need, the number of techniques available, and the number of modeling techniques needed to reach a solution (Stillman, 2000). *Mathematical complexity* relies on the degree of mathematical sophistication used in creating the mathematical model (Jensen, 2007). *Openness* describes problems with multiple solutions, interpretations, answers, and new questions (Abrams, 2001, p.18).

Figure 1. Mathematical modeling task analysis framework (Groshong & Park, 2015).



Given that mathematical models can describe, explain, or predict events, it is possible that the way students and teachers think about these different models changes with each type of mathematical model, so studying modeler's experiences in different modeling situations may reveal gaps in our understanding of mathematical modeling knowledge. Each type of mathematical model can be classified by its characteristics, properties, and features (Groshong, 2016). When reading about mathematical modeling in literature, the same mathematical model can be described using several terms, which can be very confusing for students, educators, and researchers (Figure 2). *Empirical* models are sometimes referred to as data models because of their reliance on data whereas *mechanistic* models are also called theoretical models as they originate with theoretical statements. *Deterministic* models are fixed and generate nonrandom outputs, but their counterparts, *stochastic* models, include randomness in their output (Edwards & Hamson, 2007).

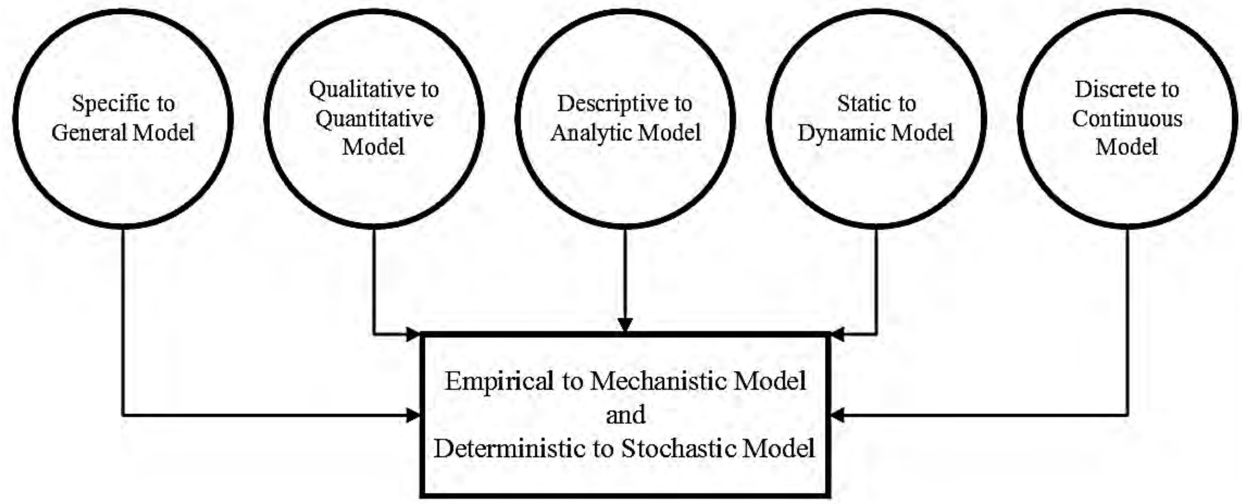


Figure 2. Different types of mathematical models (Groshong, 2016).

When the problem statement requires a single solution to meet exact criteria, a *specific model*, is produced; but when the model needs to address multiple yet similar situations, *general models* are the result. *Quantitative models* are mathematical expressions for predicting or explaining phenomena, and *qualitative models* use logical arguments explaining patterns and trends. In the initial modeling stages, *descriptive models* provide narratives or representations to state assumptions, define variables, and describe real-life issues (Tam, 2011), but *analytic models* capture the full modeling process and generate exact solutions using sophisticated mathematical methods. When time is held constant, *static models* describe this equilibrium condition, but when the situation changes with the passage of time, *dynamic models* examine the changing conditions. *Continuous models* allow variables to assume any value between two endpoints acceptable, but *discrete models* consider variables to have distinct, finite, and countable values.

Mathematical Modeling Knowledge for Teaching Framework

Adapting the 3-d representation of mathematical modeling competency by Jensen (2007), this framework defines three types of knowledge needed by teachers when planning, implementing, and assessing student learning with mathematical modeling activities (Figure 3). Each category of knowledge can be increased through formal and personal learning as well as experience. *Mathematical Knowledge* includes mathematical content and skills and the knowledge needed to apply mathematics to relevant situations. *Modeling Knowledge* consists of the understanding and ability to navigate the mathematical modeling process using a variety of mathematical methods and in different modeling situations. *Pedagogical Knowledge* comprises the broad set of information and practices teachers need in order to prepare for and facilitate modeling lessons guiding and assessing student learning throughout the activity (Groshong & Park, 2015).

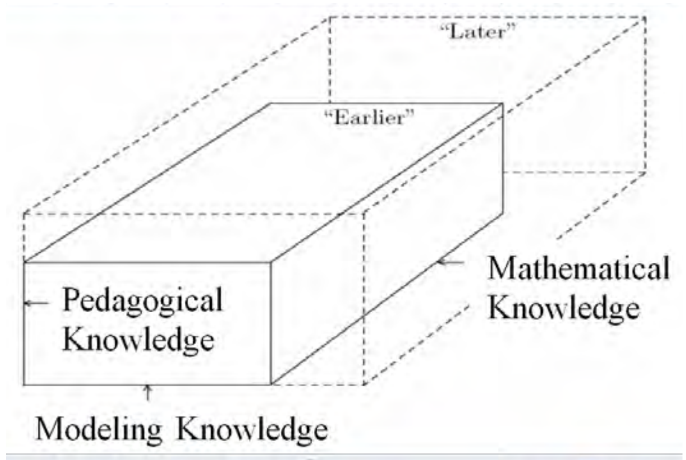


Figure 1. Mathematical modeling knowledge for teaching framework – MMKT (Groshong, Gomez, & Manouchehri, 2015; Groshong & Park, 2015).

In the previous working group session, it was noted that difficulties in one area of knowledge influenced performance in other areas of knowledge causing the participant to struggle with progressing through the modeling process. Throughout the time spent on this task analysis, participants frequently shifted their discussions from being a mathematician, a modeler, and a teacher. The historic activity was initially proposed by Wagner (1976) and asks the modeler to determine the maximum area a sofa can take as the sofa moves around a right angled hallway. The authentic context strikes a chord of familiarity with modelers in that most people have experienced the challenge of moving furniture. The openness of this problem challenges mathematics, computer science, and engineering students. A high school variation of the problem is found in Gould, Murray, and Sanfratello (2012) and provides students more structure by providing fixed dimensions for the sofa and hallway and asking students to determine if it is possible to move a fixed length, width, and height sofa around a fixed dimension hallway corner. Students need to generate and test a specific model. Although providing students with prescribed dimensions reduces the number of decisions students need to make and reduces the generality of the model, it does not reduce the complexity of the problem, as there are still many solution approaches that can be made, as was demonstrated in the discussion by the working group participants. One participant asked when the teacher should allow students to construct physical or computer models of the situation versus solving it mathematically. This led to the pedagogical discussion surrounding the importance of learning objectives to guide both mathematical and modeling skill and knowledge development. As researchers, the need to spend considerable time studying the breadth and depth of pedagogical knowledge required by teachers became apparent.

Mathematical Knowledge for Teaching

Areas of the mathematical modeling knowledge for teaching may be linked to the more generalized mathematical knowledge for teaching (MKT), which distinguishes teachers' mathematical content knowledge from their pedagogical knowledge (Ball, Thames, & Phelps, 2008). The working group researchers are making considerable progress in connecting an extensive literature review that defines sub-areas of MKT to the pedagogical discussions that occur during the task analysis activities focusing on areas of distinction between the mathematical knowledge for teaching other mathematics topics from the specific mathematical modeling knowledge for teaching when working with modeling activities.

In Ball, Thames, and Phelps' (2008) framework, the mathematical content knowledge, which has been demonstrated by participants during the task analysis as important features in the mathematical knowledge component of MMKT, are *common content knowledge* (CCK) that describes the average mathematics needs of educated adults as they carry on their lives; *specialized content knowledge* (SCK) that separates the mathematical knowledge of citizens from that of teachers who needed to present information to students, explain concepts, and predict areas of difficulties; and, *horizon content knowledge* (HCK) links mathematics to other academic subjects and across grade levels.

In working with the "moving the couch" problem, participants demonstrated CCK in qualitative statements of length and width explaining their experiences with moving furniture of fixed dimensions in fixed spaces. SCK was observed as participants described their problem solving steps as they progressed through the mathematical modeling process suggesting that SCK may be a component of both mathematical and modeling knowledge in the MMKT. The teacher's understanding of how mathematics fits in the world, including the depth of encyclopedic knowledge the breadth of factual knowledge from other disciplines that inform the real life situation (Stillman, 2000), is involved in HCK. In the couch problem, engineering comments about constructing scaled, physical models of the situation were evidence of HCK in using bodies of knowledge from other disciplines to aid mathematical problem solving.

Currently, working group participants are including mathematical modeling activities in teacher training programs with the intent of identifying mathematical and modeling content knowledge along with the experiential knowledge of the world and encyclopedic knowledge of other disciplines with regards to the CCK, SCK, and HCK deliberated during the workshops. Research is continuing in the area of needed content knowledge when employing different types of mathematical models, and reports on this progress may encourage other researchers to identify features of CCK, SCK, and HCK as they specifically apply to mathematical modeling events.

This year's working group focus will consider in detail Ball, Thames, and Phelps' (2008) *pedagogical content knowledge* (PCK), which may aid in defining the MMKT framework. With regards to mathematical modeling in the classroom *knowledge of content and students* (KCS) may be demonstrated not only as teachers predict, recognize, and assist students in overcoming mathematical barriers but also as teachers assist students in prevailing over modeling obstacles, which can impede students' mathematical modeling performance. *Knowledge of content and teaching* (KCT) can be evident in selecting appropriate mathematical modeling task when teachers take into account the advantages and disadvantages of using or the appropriate sequencing of open and complex problems. *Knowledge of curriculum* (KC) may include teachers' knowledge of where mathematical modeling fits in their course curriculum as well as in the courses that precede or follow and in other disciplines. In considering teachers' PCK, the amount and influence of experiential and encyclopedic knowledge of the real-world and of other academic disciplines needed by teachers and students in order to be successful mathematical modelers is still an area that needs extensive research.

Working Group Plan

The working group plans to continue expanding on our understanding of mathematical content and modeling knowledge through year-long research. The focus of the working group sessions will be directed towards discussing the mapping of the PCK knowledge subcategories outlined by Ball, Thames, and Phelps (2008) to mathematical, modeling, and pedagogical content areas of the MMKT in hopes of recognizing common areas or identifying gaps to increase our understanding of the scope of knowledge needed by teachers in order to promote student learning through rich mathematical modeling experiences. Discussions will also consider whether this PCK changes with the type of modeling task used in classroom instruction. Thus, the primary discussion topic will address the following research question: *What pedagogical knowledge is needed for working with different types of mathematical modeling tasks?*

Working Group Plan for Active Engagement of Participants

Participants are eagerly invited to join this working group to actively participate in defining this important research area. To continue examining this research question, we propose that members of the working group will divide into smaller groups, in order to provide an intellectual support system, to continue critiquing the definitions, taxonomies, and MMKT framework; to discuss the suggested focus questions; and to generate new research questions. The working group is committed to the goals of PME-NA while emphasizing collective reflection and collaborative inquiry, addressing challenges, and expanding the field of research. The working group will meet three times during the conference and will be invited to continue meeting virtually during the course of one year.

Session 1: Foundation

To welcome new participants and lay a solid foundation for future work, the first session will review the definitions of applications, modeling with mathematics, and mathematical modeling determined by the previous year's work; and review the task analysis framework and the MMKT. A report of current research projects, the status of publications, and the dissemination of the secondary mathematics teachers' survey will be given. Goals of this years' research along with an outline of session activities will be presented. Since it has been successful in furthering member participation and solidifying common understanding with regards to definitions, taxonomies, and frameworks, members will again be provided a mathematical modeling task and using the task analysis framework, they will identify areas of mathematical and modeling knowledge. The final part of the session will include sharing contact information, providing participants access to the group's on-line information, and encouraging the formation of research groups based on the interests of the participants.

Session 2: MMKT Framework – Pedagogical Knowledge - Introduced

The second session will continue the work from the previous session to further explain the MMKT as defined with mathematical knowledge, modeling knowledge, and pedagogical knowledge components. Employing a sample task and the task analysis framework as a vehicle to elicit understanding of the different knowledge types, participants will discuss the role of the teacher in selecting tasks, such as the exemplar, planning learning experiences, implementing activities, and assessing learning. The next step will be a discussion mapping Ball, Thames, and Phelp's (2008) MKT and a potential MMKT framework more clearly focusing on the pedagogical knowledge subcategories. Participants will revisit last year's lively discussion about the influence of the real world contextual knowledge and experiences in modeling and, then, consider the placement of the knowledge of the real world, with regards to its influence on pedagogy, in the MMKT framework.

Session 3: MMKT Framework – Pedagogical Knowledge - Detailed

The last session will summarize the previous days' findings, articulate research goals, and outline a working plan for designing research projects to be conducted during the upcoming year. Participants will be encouraged to join the ongoing research discussions virtually.

Post-Conference

To promote the working group's efforts, the results of all sessions and meetings will be documented and disseminated to all members. Following the conference, participants will be invited to continue discussing research interests in this area through a virtual meetings as a platform for suggesting and reviewing research, posing and critiquing mathematical modeling tasks, posting and discussing teacher and student solutions, developing an archive of useful and productive mathematical modeling tasks, and writing scholarly summaries of findings.

Conclusion

While this working group is still young, significant progress has been made in formalizing definitions, grounding work on theoretical principles, summarizing literature for research gaps, establishing a clear research focus, and developing active research projects. By articulating clear definitions and considering the influence a taxonomy of mathematical modeling types may have on teacher and student learning, a theoretical basis for areas of mathematical, modeling, and pedagogical knowledge has been outlined. Research projects, grants, and a large-scale survey have been developed and are in the process of being implemented. This research and these working group sessions are needed to expand understanding and inform the field of research in mathematical modeling teacher knowledge.

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EXPLORING AND EXAMINING QUANTITATIVE MEASURES

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The purpose of this working group is to continue to bring together scholars with an interest in examining the use of and access to large-scale quantitative tools used to measure student- and teacher-related outcomes in mathematics education. The working group session will focus on (1) updating the workgroup on the progress made since the first working group at PME-NA in Tucson, Arizona, specifically focusing on the outcomes of the Validity Evidence for Measurement in Mathematics Education conference that took place in April, 2017, in San Antonio, (2) continued development of a document of available tools and their associated validity evidence, and (3) identification of potential follow-up activities to continue this work. The efforts of the group will be summarized and extended through both social media tools and online collaboration tools to further promote this work.

Keywords: Assessment and Evaluation; Research Methods

Introduction

There is value in the knowledge that large-scale quantitative research can bring to the field in terms of generalizability to educational practice when appropriately conducted (American Statistical Association, 2007; Hill & Shih, 2009). The American Statistical Association's report (2007) on *Use of Statistics in Mathematics Education Research* states:

> If research in mathematics education is to provide an effective influence on practice, it must become more cumulative in nature. New research needs to build on existing research to produce a more coherent body of work… Studies cannot be linked together well unless researchers are consistent in their use of interventions; observation and measurement tools; and techniques of data collection, data analysis, and reporting. (pp. 4-5).

As education has shifted more towards data driven policy and research initiatives in the last 25 years (Carney, Brendefur, Thiede, Hughes, & Sutton, 2016; Hill & Shih, 2009), the data for policy-related aspects are often expected to be quantitative in nature (e.g., end-of-course assessments and numerical value of reform-oriented teaching). Funding agencies encouraging research (i.e., National Science Foundation and Institute of Education Sciences) often request proposals to employ quantitative measures with sufficient validity evidence (see http://ies.ed.gov/ and http://www.nsf.gov/).

Measure (instrument) quality strongly influences the quality of data collected and relatedly, findings of a research study (Gall, Gall, & Borg, 2007). Measures with a clearly defined purpose and supporting validity evidence are foundational to conducting high quality large-scale quantitative work (Newcomer, 2009). There are few syntheses of quantitative tools for mathematics educators to employ and even fewer discussions of the validity evidence necessary to support the use of measures in a particular context. Syntheses of measures for use in mathematics education can be found in the literature but these are typically not intended as a comprehensive analysis. For example, Carney et al. (2015a) conducted a brief review of self-report instructional practice survey scales applicable to mathematics education. Boston, Bostic, Lesseig, & Sherman (2015b) conducted a review of three widely known classroom observation protocols to assist mathematics educators in determining the appropriate tool for their particular research question and context. Both reviews provided a background on existing measures and their associated validity evidence in relation to a new measure under development. It is important that this type of work continues and is encouraged by the field. Thus, this working group aims to increase conversation around quantitative tools for use on a large-scale with this working group. We share three goals for this proposed working group: (a) To bring together scholars with an interest in examining the research on quantitative tools and measures for gathering meaningful data; (b) To spark conversations and collaboration across individuals and groups with an interest in large-scale tools and those conducting research on student- and teacher-related outcomes; (c) To generate products to disseminate widely across the field of mathematics education scholars.

Related Literature

Historical Context, Terms, and Rationale for Working Group

The National Mathematics Advisory Panel (2008) found that only a "small proportion of those [reviewed] studies have met methodological standards. Most ….failed to meet standards of quality because they do not permit strong inferences about causation or causal mechanisms" (pp. 2-7). Sound methodology is guided by appropriate measure or instrument choice. Good research takes on quantitative, qualitative, and at times both methodologies to become mixed-methodologies (Hill & Shih, 2009; Cresswell, 2012). Our focus for this proposal is quantitative-inclusive methodologies, specifically focusing on measures and tools associated with them, to support mathematics educators use of and need for quantitative tools that may be used in large-scale studies.

Near the core of any methodology is the measure or instrument used to collect data (Newcomer, 2009). The American Psychological Association, National Council on Measurement Education, and American Educational Research Association ([APA, NCME, AERA] 2014; 1999) provide clear guidelines regarding measurement validity and reliability. At a minimum, sufficient evidence for five variables must be shared related to validity: (1) content evidence, (2) evidence for relationship to other variables, (3) evidence from internal structure, (4) evidence from response processes, and (5) evidence from consequences of testing (AERA, APA, & NCME, 1999, 2014; Gall et al., 2007). Unfortunately, "evidence of instrument validity and reliability is woefully lacking" (Ziebarth, Fonger, & Kratky, 2014, p. 115) in the literature. Validation studies of quantitative measures are noticeably absent from mathematics education journals, which present the challenge of determining whether an instrument is appropriate for a given study much less whether it will generate valid and reliable data for analysis (Hill & Shih, 2009). Hill and Shih (2009) reported that eight of 47 studies published in the *Journal for Research in Mathematics Education* provided any evidence related to validity and the majority provided only psychometric evidence. Our goal for this literature review is to present a need for a working group at PME-NA 39 that will bring individuals together from North America to conduct more syntheses and further explore needed areas of tools that can be used to study both student- and teacher- related measures in large-scale research by mathematics educators.

Examining Student-Focused Measures

Quantitative measures of student's mathematics content knowledge, problem solving, beliefs, and other factors have been employed across various contexts. We share an initial set of literature to frame the thinking for working group participants. Moreover, we welcome those that have interests not necessarily listed in this section.

Mathematics content knowledge. Students' mathematics content knowledge has been assessed in large-scale studies using end-of-course (high-stakes) measures during the last decade, Trends in Mathematics and Science Study (TIMSS), and National Assessment for Educational Progress (NAEP). Researchers who developed the PISA and NAEP report the validation process; however, the end-of-course measures are often shrouded by commercial entities (e.g., American Institutes of Research and Pearson). The latter group makes examining the quality of the measures for content knowledge problematic. Broadly speaking, it is challenging for researchers aiming to make decisions regarding use of items (or previously used measures) without syntheses describing measure qualities as well as similarities and differences across measures. Thus, a measure may claim to measure students' (at one grade- or developmental-level) content knowledge but how is content knowledge defined for each measure?

Beliefs. Students' beliefs of mathematics, mathematics teaching, and usefulness of mathematics for the real world have been examined in various ways. Students taking the NAEP assessment also responded to questions designed to measure their perceptions of mathematics (Dossey, Mullis, Lindquist, & Chambers, 1988). In the survey created by Dossey and colleagues, students responded to several Likert scale items regarding their attitudes and beliefs about mathematics. Similarly, Lazim, Osman, and Salihin (2004) created a mathematics belief questionnaire that had four belief dimensions: "[about] the nature of mathematics, about the role of teachers, about teaching and learning mathematics, and about their competency in mathematics" (p. 5). Again, the instrument consisted of Likert scale items self-reported by the students. The authors claim they achieved high reliability after the development of the survey but it was not reported. Hence, greater examination of these instruments is needed to benefit mathematics education research.

Examining Teacher-Focused Measures

A couple articles have provided syntheses of the literature related to quantitative teacher-focused measures. We explore three sets here: observation protocols (of instruction), teachers' content knowledge, and teachers' beliefs. Again, we use this as a starting point and welcome interests within teacher-focused measures that are not necessarily represented within this frame.

Observation protocols. In 2015, Boston and colleagues compared the Reformed Teaching Observation Protocol, Mathematical Quality of Instruction, and Instructional Quality Assessment. A key finding of the study was that these three unique large-scale teacher-related observation protocols provided three unique lenses into teachers' instruction (Boston et al., 2015b). The authors encouraged the field of mathematics education to execute further work to closely examine other observation tools and share syntheses of relevant literature.

Teachers' content knowledge. The components of the Mathematical Knowledge for Teaching (MKT) construct (Ball, Thames, & Phelps, 2008) can serve as a useful tool for exploring and examining quantitative measures of teachers' knowledge. Quantitative measures designed for teacher certification purposes (e.g., the Praxis series) tend to focus on the component of common content knowledge, ignoring other important components of the MKT framework often deemed important to mathematics educators. Other assessments are designed specifically with the intent of measuring teachers' knowledge of particular content areas (e.g., Knowledge of Algebra for Teaching measure, McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012) or grade bands (e.g., Diagnostic Teacher Assessment in Mathematics and Science, Saderholm, Ronau, Brown, & Collins, 2010). The most commonly used quantitative measures for teachers' content knowledge in mathematics come from the Learning Mathematics for Teaching (LMT) project (2005). The LMT assessments aims to measure teachers' content and pedagogical knowledge for teaching and are parsed into different content areas (e.g., K-6 geometry, 6-8 Number and Operations, and 4-8 proportional reasoning; LMT, 2005). A review of the NSF database for measures of teachers' math content knowledge for teaching (a) generating quantitative data, (b) with reliability and validity evidence, and (c) could be used in large-scale studies resulted in 16 measures, 11 of which were part of the set from the LMT series. While tools such as the NSF database or the National Council for Teachers of Mathematics Handbook Chapter "*Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts*" (Hill, Sleep, Lewis, & Ball, 2007) provide a brief summary of some potential measures a mathematics education researcher could use to examine teachers' knowledge, it does not provide a comprehensive synthesis that might aid in determining which measure to use for a given research question, much less describe the validity evidence associated with the measure. Again, there is no available synthesis of available tools to measure teachers' knowledge of mathematics.

Beliefs. Philipp (2007) defines beliefs as "held understandings, premises, or propositions about the world that are thought to be true. …Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual" (p. 259). Beliefs and attitudes are different; they are related and at times have been discussed synonymously in the literature (Philipp, 2007). One of the oldest and still used measures is the Fennema-Sherman Mathematics Attitude scale (see Fennema & Sherman, 1976). This measure uses a Likert-scale to assess respondents' attitudes towards several domains. The study describes four Likert-scale self-report measures and accurately suggests the limited scope of self-report measures with regards to validity evidence. The Integrating Mathematics and Pedagogy (IMAP, 2004; see also Ambrose, Clement, Philipp, & Chauvot, 2004) is a web-based survey with open-ended items. This measure overcame the challenges of Likert scales, the lack of context for an overall score, and that respondents may give an opinion when one is not naturally held (Ambrose et al., 2004). A search of academic journals for measures of mathematics teachers' beliefs provided numerous hits but few are found in mathematics education journals, much less a synthesis of those available with validity and reliability evidence to be used in studies with large data samples. Put simply, no syntheses of measures in this are shared.

Session Organization and Plan for Engagement

The purpose of continuing this working group is to reconvene individuals from the previous meetings held at PME-NA 38, as well as include new participants across North America, interested in the appropriate use of quantitative tools in mathematics education that can be used in studies with large samples to examine student- and teacher-related outcomes. The primary goal of this group is to bring together scholars with an interest in examining the research on quantitative tools and measures for gathering meaningful data, and to spark conversations and collaboration across individuals and groups with an interest in synthesizing the literature on large-scale tools used to measure student- and teacher-related outcomes.

The sequencing of the activities for the purposes of this working group will begin with a review of the products and outcomes from the previous working group meetings and the Validity Evidence for Measurement in Mathematics Education (V-M^{2}Ed) conference, a conference funded by the National Science Foundation that brought together researchers from different theoretical and methodological perspectives to contextualize current conceptions of validity within the field of mathematics education. The organizers of the working group also led the V-M^{2}Ed conference. This segues into further growing the products developed at these meetings. We primarily focus on two of the main themes for PME-NA 39:

1. Crossroads as access.
2. Crossroads as a place of community.

Prior Work

The idea for this working group proposal started at PME-NA 37 (2015). We explored interest across the field from potential attendees before writing this proposal. We sought feedback from colleagues using the Association Mathematics Teacher Educators' (AMTE) bulletin board feature as well as the Service, Teaching, and Research (STaR) list-serv. An interest survey was shared broadly with both groups (i.e., AMTE and STaR members) to gather an idea of the level of interest in this idea. Twenty-six people expressed interest, including from individuals who could not attend AMTE's 2016 annual meeting. We held a follow-up meeting at AMTE to meet with fourteen individuals who expressed interest and were attending AMTE's annual meeting. A majority of those at the AMTE follow-up meeting shared that they planned to attend the working group if accepted for PME-NA 38 (2016). The proposal for PME-NA 38 was accepted and in total, 27 different individuals attended the meetings and 12 were present for all three meetings. We received numerous inquiries for future meetings and continuing our work in face-to-face as well as online mediums. Although there are numerous mathematics education conferences, all of which include quantitative and/or measurement researchers, there is no specific conference that brings them together. This working group serves as a "crossroads as a place of community" (https://www.conf.purdue.edu/landing_pages/pme-na/submission.aspx) because it not only provides space for this group of researchers to meet, but PME-NA's working group is the only conference format which allows for this type of work to happen.

To that end, we plan on organizing the sessions in the following manner to address our two primary goals for the PME-NA 39 working group session.

Session 1

The first session of PME-NA 39 will focus on what the working group has accomplished in the past year, beginning with the PME-NA 38 working group sessions. Specifically, we will re-visit our generated definitions of the terms "quantitative tools" and "large-scale," as well as the framework that we used for organizing our discussions around quantitative tools that can be used with large samples to examine student- and teacher-related outcomes. During PME-NA38, the working group leadership and attendees created an initial instrument database that includes quantitative measures that have validity evidence. We will also summarize the work and outcomes of the V-M^{2}Ed conference for those in the group that were unable to attend. That conference was held April 1-2, 2017 just prior to the National Council of Teachers of Mathematics' Research Conference. The goal of this review session is to update all of the participants about the status of the work so that the entire group can move forward together on the tasks of Sessions 2 and 3: building the criteria for a future repository of quantitative measures.

Session 2

The focus of the second session is to decide two key aspects of the repository: (1) What are the necessary and sufficient criteria for including an instrument in the database? (2) What information should be presented to the user of this database? Both of these aspects build from the work of the previous year. The second session will begin with a discussion of the criteria necessary for including an instrument in a database of quantitative measures. This future database addresses the conference theme of "crossroads as access" by providing researchers access to quantitative tools as well as the guidance to use these tools appropriately. Moreover, access is distinctly grounded in use of tools that have met some or all of the standards for evaluation (AERA, APA, NCME, 2014). This unique grounding in validity evidence and arguments assures access and rigor to users of the database. We are offering the field opportunities to approach research questions in different ways. Group facilitators will offer two examples for the larger group to discuss as a means to explore criteria for including an instrument and how results might appear to a user.

Session 3

The third session will primarily be a working session, focusing on placing instruments within the database. Logistically, attendees will divide into small-group teams based on interest, with each group working on their own tools and then presenting to the whole group towards the end of the session. At PME-NA 38, we started to create an instrument database during the third session as a result of the working group. At the time, we had not considered necessary and sufficient criteria for including an instrument nor the associated validity evidence. The purpose was merely to include instruments. Thus, our working group makes progress on our broad goal as well as sub-goals specific to this proposal. By the end of this third session, we intend to have a draft database of some instruments and their associated validity evidence. We do not anticipate this will be comprehensive at this time; the work will continue after PME-NA 39. We plan to conclude session 3 with a discussion of anticipated follow-up activities to determine the level of interest and commitment from the group in continuing with this work.

Anticipated Follow-up Activities

As a result of our working group discussion and document development, we anticipate several potential follow-up activities. Participants will greatly influence the specific follow-up activities; however, we outline a potential progression of activities to guide discussion of potential 'next-steps'.

One outcome of the working group sessions is a draft database inclusive of the available tools and their associated validity evidence. An anticipated outcome will be to determine how this should be further refined and later distributed. For instance, if attendees are interested and willing to continue this work then we will generate plans to move it forward and become more available to the broad scholarly community.

We see several possible venues for further conversations and work related to access to quantitative tools in mathematics education that can be used with studies of large-scale samples to examine student- and teacher-related outcomes. First, we anticipate using both social media tools (e.g., creating a Facebook group) and online collaboration tools (e.g., Google hangouts and documents) to promote these syntheses. Second, we anticipate using mathematics education conferences venues to further the conversations and synthesis work around the project. More specifically, we plan on proposing to continue the PME-NA 40 (2018) working group. In addition, we anticipate submitting for a symposium at the 2018 annual meetings of the Association of Mathematics Teacher Educators and National Council of Teachers of Mathematics Research Conference. Finally, there is potential to apply for grant funding through an NSF proposal to provide the means to actually create an instrument database, which connects with the aim of this working group as well as the V-M^{2}Ed conference.

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IMPROVING PRE-SERVICE SECONDARY MATHEMATICS CLINICAL EXPERIENCES THROUGH CO-PLANNING AND CO-TEACHING

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The purpose of this working group is to examine the use of co-planning and co-teaching in clinical experiences of pre-service, secondary mathematics teachers, especially during their internship. We have developed an Apprenticeship Model for Learning where mentor teachers and pre-service teachers co-plan and co-teach focusing on learning, and our network improvement community is currently using improvement science to study co-planning and co-teaching during the internship experience. This working group is appropriate for those interested in (or already engaged in) co-planning or co-teaching as part of secondary mathematics clinical experiences. Our sessions will 1) introduce specific co-teaching and co-planning strategies, 2) consider the challenges and complexities of implementing co-planning and co-teaching, 3) provide an overview of our productivity so far, and 4) allow time for participants to share their ideas and work. We would also like to set up a plan to continue research efforts related to co-planning and co-teaching by arranging consistent follow-up meeting times to support research efforts, grant writing, and future submissions for working groups, conference proposals, and articles for publication.

Keywords: High School Education, Instructional Activities and Practices, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development

Introduction

One of the most important areas of pre-service teacher (PST) development is the clinical experiences component. However, there is often a disconnect between theoretical course work for PSTs and what the PST encounters in the field of practice (Darling-Hammond, 2006), leaving interns and novice teachers to make connections on their own. This disconnect is exacerbated by adherence (implicit or explicit) to very dissimilar theories of learning. Other areas of disconnect include school-university differences in philosophy regarding the teaching and learning of mathematics, insufficient opportunities for professional learning that might assist novice teachers implement new standards and curriculum materials, and challenges with the recruitment and retention of mathematics teachers for high-needs schools (Sears et al., 2017). Further, many mentor teachers often are uncertain about how to mentor (Anderson, 2007), many interns believe they are ill-prepared to teach mathematics (Ingersoll, 2012), and university faculty may be less than effective in the role of change agent (Veal and Rikard, 1998).

Based on the experiences of the working group members, one of the most difficult pieces for PSTs to master (or even become marginally competent in) is the workable lesson plan – not the plan that one submits to the mathematics methods instructor, but the one enacted in the mathematics classroom. Following close behind, in terms of difficulty, are the moves and decisions the novice teacher makes in the classroom during instruction. Given these challenges, a number of universities are using co-teaching and co-planning as an integral part of their teacher preparation (Sears et al., 2017).

In addition to the possible efficacy of co-teaching to assist interns with the minute-by-minute decision-making process and the potential of co-planning to expose some of the mystery of the lesson planning process, there are other benefits for using these particular innovations. Notification and subsequent dialog about co-teaching and co-planning address the problem of insufficient communication between school and university personnel (Zeichner, 2010). The innovation provides a solid reason for mentors and interns to gather for common professional development (Grady et al., 2016), which sets the stage for goal setting, provides common professional learning opportunities, and casts university personnel in leadership roles for providing new ideas to be used during the internship experience. Simply, the professional development needed to implement co-teaching and co-planning provides a reason for mentors, interns, and university faculty members to come together, develop common language and goals, and deepen their understanding of the differences and similarities within their varied roles. When implemented properly, co-teaching and co-planning provides a platform for transparency into the implicit decisions for planning and teaching made by mentor teachers, making them explicit for the PST. Thus, there is potential for what is often unseen to be seen.

Related Literature

Background and Rationale for Working Group

The current members of the proposed working group are part of the Mathematics Teacher Educator Partnership (MTEP), itself a working group of university faculty members who are addressing the challenges in secondary mathematics teacher preparation from recruitment to induction. The MTEP operates under the Association of Public and Land Grant Universities and is part of the work of the Science and Mathematics Teaching Imperative. Organized in 2012, MTEP "provides a coordinated research, development, and implementation effort for secondary mathematics teacher preparation programs to promote research and best practices in the field" (http://www.aplu.org/projects-and-initiatives/stem-education/mathematics-teacher-education-partnership/). Members of the partnership are organized into Research Action Clusters (RAC), including the Clinical Experiences RAC within which the members of this group work. Given the Clinical Experiences group is large and there are many different types of experiences, one portion of the group operates as the Co-Teaching and Co-Planning sub-RAC. This sub-group has worked over the past three years to identify issues, develop tentative solutions, and then research the efficacy of its work. The research is done using the Carnegie model for continuous improvement, particularly through network improvement communities (more about that later in the paper).

The proposed working group at PME-NA 39 will allow for the current group to expand, bringing new ideas and providing additional opportunities to test these ideas. Certainly, there are mathematics education researchers doing work with co-teaching and co-planning whose universities are not part of MTEP, and this working group will provide a platform for cross-collaboration. Work done with the group is at the heart of what PME-NA promotes, helping develop a deeper and better understanding of the psychological aspects involved in mathematics teacher preparation.

Co-Teaching

Co-teaching began in special education as a new way for general-education and special-education teachers to work together in the classroom (Friend et al., 2010). The ideas have since come to be more widely used in other settings, particularly in teacher education programs as PSTs and their mentors seek to find new ways to engage in full-time internships (Bacharach et al., 2010).

Table 1: Co-Teaching Strategies (Adapted from Bacharach, Heck, Dahlberg, 2010; Murawski and Spencer, 2011)

Strategy	Definition	Benefits	Concerns
One Teach, One Observe	One teacher leads instruction, while the other teacher gathers specific information.	Extra set of eyes; provides data about instruction or student learning; easy to implement.	Easy to become a habit; must agree in advance what is to be observed.
One Teach, One Assist	One teacher works with the whole class, while the other assists individual students or groups of students.	Provides assistance to individual students; easy to implement; may provide a "voice" to share student concerns.	Too easy to become a habit and for one teacher to always feel like an assistant; changing roles is essential.
Station Teaching	Students divided into three or more groups; students rotate through multiple stations; teachers facilitate individual stations or circulate among stations.	Smaller groups are better for instruction, assessment, and class management; allows for differentiation, movement, and hands-on activity.	Teachers may need to use space differently; class management and transition needs to be structured; independent stations need to be carefully planned.
Parallel Teaching	Each takes half the class. Groups may be doing the same or different content in the same or different ways. Groups do not switch during lesson.	Smaller groups better for instruction, assessment, and class management; teachers have their own groups; interns teach same lesson/mirror teacher.	Teachers need to be willing to use their space differently; both teachers need to plan for their group; class management needs to be structured.
Alternative Teaching	One teacher works with large group of students, other teacher works with smaller group (re-teaching, pre-teaching, or enrichment).	Good for smaller and more specific group work; good for addressing IEP/504 goals; teachers can plan separately.	DO NOT always pull the same kids; need place for group to meet; watch noise levels; plan how to integrate group back into class.
Team Teaching	Both teachers presenting. This may take the form of debates, modeling information, compare/ contrast, or role-playing.	Demonstrates parity and collaboration between teachers; good for modeling; fun for role-playing.	Takes willingness to "share the stage"; both need to feel comfortable in front of the class.

More recently, members of the Clinical Experiences Research Action Cluster of the Mathematics Teacher Education Partnership have worked to adapt the ideas for use in secondary mathematics classrooms and begun to study the effectiveness of both co-teaching and co-planning in these settings (Sears et al., 2017; Grady et al., 2016).

Co-teaching is more than just a collection of strategies for how two teachers work together in a single classroom setting; especially in an internship setting, it is a paradigm shift in how what it means to be an intern and a mentor teacher. Many traditional internship programs tend to have sink-or-swim structure, in which interns may go directly from observing their mentor to taking on entire responsibility for teaching of a class (Bediali, 2010). In the co-teaching model, the emphasis is on shared responsibility for instruction from the beginning to the end of the internship. The level of responsibility may shift over time and even vary from day to day as different co-teaching strategies are employed but both teachers maintain an identity as a teacher in the classroom.

The literature on co-teaching provides some specific strategies for engaging in co-teaching. Table 1 provides a synopsis of these strategies, along with some potential benefits and concerns of each.

Co-Planning

Similar to co-teaching, we view co-planning as an environment where two or more teachers actively engage in planning together. When co-planning together teachers need to agree on instructional goals for a lesson, as well as the timeline for how the instruction will occur. Co-planning involves analyzing student learning, as well as development of instructional tasks to reach learning goals. The teachers must work together to select instructional tools to utilize and create their own learning tools when necessary. Another important aspect teachers must agree upon is an assessment plan to ascertain student understanding in relation to the learning goals.

Research identifies co-planning as an integral component for successful co-teaching (e.g., Howard & Potts, 2009; Magiera, Smith, Zigmond, & Gebauer, 2005). As described above, the literature defines specific ways that two teachers may share instructional responsibilities, organizes their physical space, and articulates the role each co-teacher might assume during co-teaching. Conspicuously absent from the literature base are similar strategies for co-teachers to co-plan effectively to support co-teaching. This is a major concern when considering pre-service teachers entering their internship experience. The internship experience marks a crossroads in a pre-service teacher's program where their skills in planning for instruction are put to the test. Due to their lack of experience, interns are likely to have more difficulty than experienced teachers, such as their mentor teacher, being flexible and attentive to student needs as they plan for instruction (Borko, Livingston, & Shavelson, 1990; Leinhardt & Greeno, 1986; Livingston & Borko, 1989). Another issue that arises as the intern must shift from planning several lessons during methods courses, where their access to students varies, and planning for daily instruction to promote student learning. Co-planning has the potential to bridge the gap between these expectations.

Realizing the need to provide a bridge for the gap described, the mathematics educators at one of the presenting institutions developed six specific co-planning strategies. These strategies, summarized in Table 2, operationalize the process of co-planning between an intern and mentor teacher to provide structure and support for intern development.

Table 2: Co-Planning Strategies

	Definition	Benefits	Concerns
One Plans, One Assists	Each co-teacher brings a portion of the lesson, although one clearly has the main responsibility. The team works jointly on final planning.	Better instructional materials; intern sees how a good lesson can be improved; final planning done jointly.	Initial planning done separately may not mesh well; critical that intern not remain in assistant role.
Partner Planning	Co-teachers take responsibility for about half of the components of the lesson plan. Then they complete the plan collaboratively.	It is efficient; each teacher provides initial planning for only part of a lesson.	Pieces of lesson may not mesh well; requires initial visioning together.
One Reflects, One Plans	Mentor thinks aloud about the main parts of the lesson and the intern writes the plan.	Lesson content is a reasonable fit; Intern is not planning blindly; provides transparency early in planning process.	May be a gap between what the mentor spoke out loud and what the intern heard; excessive use of this strategy may not support intern development.
One Plans, One Reacts	One co-teacher plans and the other makes suggestions for improvement.	Provides opportunity for good feedback and discussion of lesson plan elements, primarily for the intern; gives interns space for creativity in initial plans.	Provides response after the fact instead of in real time; initial approach may be off base; one may feel like an assistant.
Parallel Planning	Each member of the co-teaching team develops a lesson plan, and the two bring them together for discussion and integration.	Allows for compare and contrast of examples and points of emphasis; gives both teachers opportunity for creativity in planning.	Duplicate work done; teachers may become heavily invested in their own plan, making collaboration difficult.
Team Planning	Both teachers actively plan at the same time and in the same space with no clear distinction of who takes leadership.	Resulting lesson plan may be better than a plan done independently by either; may be more efficient because feedback and collaboration happen in real time.	One co-teacher, likely the intern, may be less prepared to contribute than the other; requires a very high level of trust and communication.

Theoretical Approach to Examine Co-Planning & Co-Teaching

Our work with co-teaching and co-planning during pre-service teachers' internship experiences is grounded in Lave's (1991) construct of situated learning. As interns go out into the field, their learning moves from a predominately academic experience to an apprenticeship in a community of practice. In such a setting the working relationship between intern and mentor teacher becomes a major determining factor in the intern's ability to participate productively and collaboratively in the practice of classroom teaching. In our work we consider ways to expand traditional visions of this working relationship between intern and mentor, envisioning mentor and intern as collaborators in classroom planning and instruction.

In a co-planning and co-teaching internship, the instructional responsibilities for the mentor and the intern change over time (Figure 1). As Figure 1 indicates, the mentor initially assumes more instructional responsibility, with the intern gradually increasing their level of responsibility.

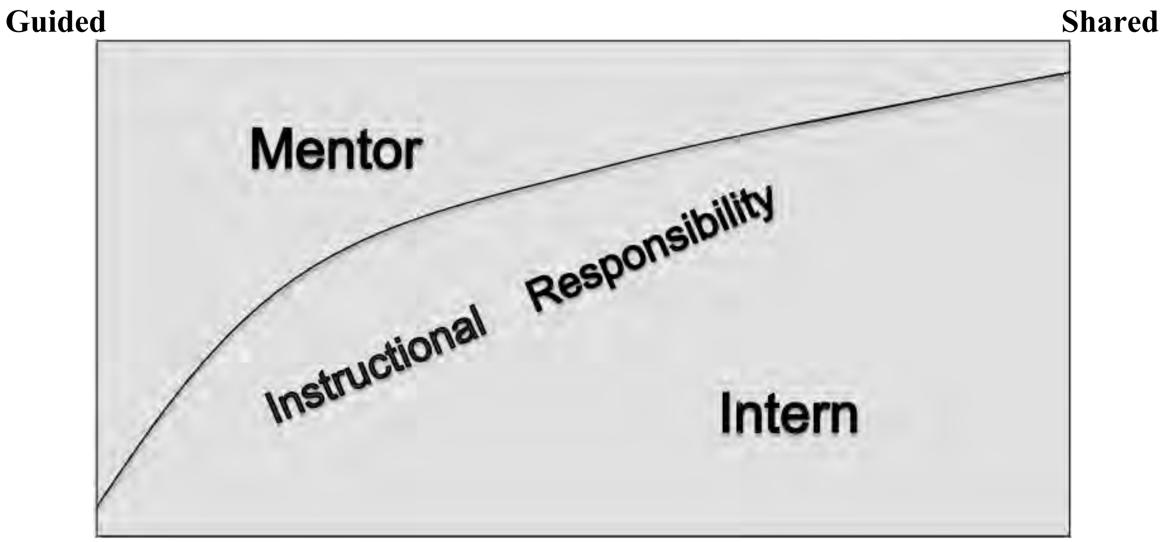

Figure 1: Instructional Responsibilities for Co-Teaching Internship (Brosnan, Jaede, Brownstein, & Stroot, 2014).

Notice that the intern eventually takes on the majority of instructional responsibilities; however, within a co-planning/co-teaching paradigm the mentor remains an active, participating teacher throughout the internship experience.

A common issue with the implementation of innovative ideas in education is that the research so often lags behind the pace of implementation. By the time we have examined the effectiveness of an innovation, it has often either been abandoned because there were perceived issues with implementation or fully adopted because serious concerns had not yet been identified. Improvement science research methods, with their emphasis on short improvement cycles, offer an alternative that may better meet some of our needs as educators. Improvement science is designed to study practices, make modifications, and provide a cycle of continual improvement (Bryk, Gomez, Grunow and LeMahieu, 2015). Improvement science is based on three questions, "1. What specifically are we trying to accomplish? 2. What change might we introduce and why? 3. How will we know that a change is actually an improvement?" (Bryk et al., 2015, p. 114). The process of inquiry is based in the Plan-Do-Study-Act cycle, in which plans are made for possible improvements in a system, the plan is enacted and studied, and then decisions are made about adjustments to make for the next PDSA cycle. This iterative process provides a mechanism for modifying interventions and research methods as the study progresses. At this time, the co-planning/co-teaching research group from the Mathematics Teacher Education Partnership has enacted several PDSA cycles, each of which has resulted in modifications to the implementation of co-planning/co-teaching and each increasing the scale on which the research is being conducted.

Connections to PME-NA

Our working group objectives fit well with the overall goals of PME-NA by promoting and stimulating interdisciplinary research and the exchange of scientific information in the area of clinical experiences for PST, as we involve mathematics teacher educators, mathematics teachers, and pre-service teachers, all working to learn more about the psychological foundation necessary for a prospective mathematics teacher to develop the particular knowledge and skills to be a successful mathematics teacher. With respect to the 2017 PME-NA theme – Synergy at the Crossroads: Future Directions for Theory, Research, and Practice – the combined work of the group has the potential of pushing our efforts further that the sum of the individual parts. We clearly are attempting to move theory into practice, research this effort, refine our theory and practice, and then research its impact. The working group format has the potential to greatly increase the scale of our work as other partners come on board. With respect to crossroads, our work is at an intersection, where we are ready to move from the state highway to the interstate, allowing more participants to test drive the work done so far and help push the thinking forward.

Session Organization and Participant Engagement

Presenting institutions have a range of experience from a few years to more than a decade related to co-teaching and co-planning, and we invite anyone interested in co-teaching or co-planning, regardless of prior work in these areas. The sequence of sessions will provide working group participants with the opportunity to engage in discussions related to co-teaching and co-planning during internship experiences. Paradigm shifts from more traditional internship experiences to co-teaching, as well as co-planning with a focus on student learning will frame the first two sessions. Specific co-teaching and co-planning strategies will be shared, and presenting institutions will discuss current co-teaching and co-planning training materials and research efforts. Working group participants will be invited to share what their institutions are doing (or may be interested in doing) related to co-teaching and co-planning as part of each session.

Session 1: Transitioning from Traditional Internship Experience to Co-Teaching

The first session will begin by contrasting the role of the intern and mentor teacher in traditional versus co-teaching internship settings. Presenting institutions will share their experiences navigating the crossroads between these two models, explain where they are currently in the process of implementing co-teaching, and discuss issues related to the process of shifting paradigms, including buy-in from all stakeholders involved in the process. The session will continue with a review of specific co-teaching strategies, followed by sharing current training materials utilized by presenting institutions to train university supervisors, mentor teachers, and interns. The session will conclude with discussion of the data collection and instruments we are currently using to capture co-teaching during the internship experience. During each phase, particular emphasis will be placed on having working group participants share where they are in the process of co-teaching, as well as how we can coordinate our efforts to support each other.

Session 2: Focus on Student Learning & Co-Planning

The second session will focus on co-planning as a critical support for co-teaching. The session will begin with our description of planning for student learning. Planning for learning pushes interns and mentor teachers to think deeply about instructional decisions to focus on their students. We utilize three guiding questions to frame co-planning for student learning; 1) What do students need to learn? (content), 2) How will you know if they have learned?

(formative/summative assessment), and 3) What tasks/activities will students engage in to ensure learning happens? (pedagogy). The session will continue with a description of six co-planning strategies developed at one of the presenting institutions, as well as co-planning training materials utilized with university supervisors, mentor teachers, and interns. The session will culminate with a discussion of how presenting institutions are collecting data to capture co-planning during the internship experience, and where each institution is currently in navigating the shift toward co-planning. During each portion of the session, working group participants will be asked to share their ideas about co-planning, where they are in relation to co-planning during the internship experience. A by-product of this session will be to document strategies to support each other in the implementation of co-planning during co-teaching or traditional internship experiences.

Session 3: Mine, Yours, and Ours

The third session will be a working session incorporating the information and ideas gleaned from the first two sessions. We will consider what presenting and participating institutions are doing in relation to co-teaching and co-planning to review the research questions currently under study, and we will also discuss what tools are being used for training and research. This process will inform our discussion of possible gaps in our work and will provide participants in the working group an opportunity to identify further research questions related to co-teaching and co-planning. The goal of this session will be to bring these pieces together to create a collective set of training and research materials to support work at presenting institutions and working group participants that allow for the individualities of each context. One product will be a draft PDSA cycle for data collection and research across interested institutions. We will also discuss potential next steps including, but not limited to, collecting/sharing training and data collection materials, future working group sessions at other conferences, and grant opportunities to support our work.

Anticipated Follow-up Activities

One of the first follow-up activities will be to create a collective space to share and refine training and research materials. Potential tools for this include Dropbox and Google Drive, as well as other suggestions from working group participants. A second activity will be periodic virtual meetings to check in about data collection and research. A third activity will be to submit future working group proposals to AMTE and PME-NA to continue our work with co-teaching and co-planning. One longer-term goal will be to utilize data collected during the 2016-2017 and 2017-2018 PDSA cycles to submit research symposium proposals at conferences. These data results may also be utilized to submit as research manuscripts, as well as grant proposals to continue work across institutions.

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MAPPING THE LEARNING PROGRESSION IN EARLY MATHEMATICAL MODELING

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This working group will engage PMENA members to consider ways in which early introduction to mathematical modeling can promote 21st century skills such as critical thinking, creativity, collaboration and communication, as well as connect to interdisciplinary topics in STEM. In particular, we will gather interested individuals to deepen our understanding of the learning progression of mathematical modeling that can connect elementary to secondary mathematics education. Second, we will discuss efforts to design and implement professional development that introduces K-8 teachers to mathematical modeling. Finally, based on the interests of the participants, we will devote work time to finding synergistic collaborative topics to pursue for future research and practice.

Keywords: Mathematical Modeling, Elementary Education, Teaching Practices, Professional Development, Learning Progressions

Overview of the Working Group

This is a new working group that will build on PMENA's long tradition of working groups on *Models and Modeling.* Our goal is to broaden the access of mathematical modeling to elementary grades and advance the field's collective understanding of the interrelated processes of mathematical modeling in the elementary grades and beyond. Although there has been a long history of mathematical modeling at PME and PMENA, the focus has primarily been on middle, high school and university levels. We believe it is critically important to understand the learning progression of mathematical modeling from elementary to secondary grades to ensure coherence and rigor in the mathematics curriculum. Implementing mathematical modeling in the elementary grades is not just going "light" with the high school math modeling curriculum. Instead we advocate integrating aspects of mathematical modeling in the early grades effectively to enhance student learning and to help build their competency in real-world problem solving using their current mathematical knowledge. The latter content knowledge is expected to evolve as students continue to learn new mathematics as they progress towards high school and beyond.

The working group leaders come from a three university collaboration working with school districts (with diverse populations) to understand the nature of mathematical modeling in the elementary grades. In our design-based implementation research, each university site worked with the collaborating district's teacher leaders to co-plan the professional development. Teachers became co-designers of the mathematical modeling curriculum for the elementary classrooms. In our project, we engaged elementary teachers in considering mathematical modeling using real world tasks that contained several of the following attributes: (a) Open-endedness; (b) Problem-posing; (c) Creativity and choices; d) Iteration and revisions.

Why focus on early grades? In addition to the direct benefits of modeling, the elementary school environment affords many advantages that complement work in mathematical modeling. Elementary students often rely on using concrete referents such as objects, drawings, diagrams, and actions or pictures to help conceptualize and to construct carefully formulated arguments to solve a problem. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades (CCSSO 2010). Young students have high potential to become fluent – native speakers, thinkers and dreamers of mathematics. Thinking creatively may come more easily to children first learning and exploring mathematical concepts. Kindergarten students can use manipulatives to independently solve traditional multiplication or division problems they have never seen before, which is evidence that students come with knowledge--we don't have to wait to incorporate modeling activities until we have "shown them how" to do everything. Because early grade teachers are generalists, they can address several subjects simultaneously through modeling activities. Mathematical Modeling is of interest and relevance to the mathematics education community especially because it connects to the need for professional development focused on mathematical modeling in the elementary grades.

Our researchers used Design-Based Implementation Research methodology, DBIR (Fishman, Penuel, Allen, Cheng, & Sabelli, 2013) to examine the design of our professional development and to study and enhance our design through feedback from our iterative implementation cycles. DBIR was a method of choice for our study because it has (1) a focus on problems of practice from multiple stakeholders' perspectives; (2) a commitment to iterative, collaborative design; (3) a concern with developing theory and knowledge related to both classroom learning and capacity for sustaining change in systems (Fishman, Penuel, Allen, Cheng, & Sabelli, 2013, p. 136).

Through our work, we are gaining a better sense of teaching practices and classroom routines that support modeling. We are contributing to the understanding of what is possible in early elementary grades and how these processes support the development of critical 21st century skills. As we continue in our research to consider what constitutes the practice of Mathematical Modeling (MM) and how it could be implemented in classrooms at different grain size, we invite the larger PMENA community to build on this knowledge. Over the past decades, working group leaders have individually and in subgroups, been theorizing about as well as collecting, analyzing, and reporting on data relating to mathematics modeling. This Working Group builds on and extends the work of previous Model and Modeling tradition by discussing current work from leading scholars from diverse perspectives.

Relevance to Psychology of Mathematics Education

The purpose of this working group is to invite individuals across the research community interested in synthesizing the literature and collaborating on research focused on mathematical modeling along the developmental continuum. Our goal of mapping a learning progression of mathematical modeling from K-12 education, particularly starting from elementary to middle grades is critically important to provide coherence in the mathematics curriculum.

The primary focus for this working group will be around the following three goals:

1. Bring together scholars with an interest in examining research with meaningful data consisting of student MM artifacts and teachers' content and modeling competencies.
2. Map the learning progression for mathematical modeling and task design for K-6 mathematics education and beyond.
3. Begin dialogue and collaboration among individuals and groups conducting research on student- and teacher-related outcomes related to implementing mathematical modeling, ways mathematical modeling promotes 21st century skills, and interdisciplinary skills in STEM.

Related Research

The complexity of the modern world places more demand and importance in developing students' abilities to deal with demands of our society (e.g. Gravemeijer, Stephan, Julie, Lin, & Ohtani, 2017). These abilities include interdisciplinary problem solving, techno-mathematical literacy, flexibility in applying numerical and algebraic reasoning, thinking critically, and constructing, describing, explaining, manipulating and predicting complex systems (English, 2013). Mathematical modeling (MM) is seen as a powerful tool for advancing students understanding of mathematics and for developing an appreciation of mathematics as a tool for analyzing critical issues in the real-world, that is, the world outside of the mathematics classroom (Greer & Mukhopadhyay, 2012). Traditionally, MM has been implemented primarily in secondary schools, but recent research examines this approach with elementary students to promote their problem solving and problem-posing abilities (e.g. English, 2010). MM provides the opportunity for students to solve genuine problems and to construct significant mathematical ideas and processes instead of simply executing previously taught procedures and is important in helping students understand the real world (English, 2010).

It must be pointed out that the phrases *mathematical modeling* and *modeling mathematics* are used in different ways. Cirillo, Pelesko, Felton-Koestler, and Rubel (2016) succinctly describe *modeling mathematics* as the use of representations to communicate mathematical concepts or ideas. The central characteristic of *modeling mathematics* is that the process "begins in the mathematical world, rather than in the real world (Cirillo et al., 2016, p. 4). For example, Lesh, Post, and Behr (1987) describe five representations that support students in understanding mathematical concepts or ideas: pictures, manipulatives, written symbols, oral language and real-world situations. For Lesh et al. (1987), the real-world situations provided a context for the problems; the representations began in the mathematical world. A form of mathematical modeling instruction introduced by Lesh & Doerr, 2003, Model-eliciting activities (MEAs), incorporate client-driven, real-life contexts and open-ended problem solving. *Mathematical modeling* is a process that starts in the real world and makes sense of non-mathematical situations in a mathematical format (English, Fox and Watters, 2005). The *mathematical modeling* process involves both the creation and the continuous modification of models of empirical situations to both understand them better and enhance decision-making in real-time. As students create and modify mathematical models to understand and solve real-world problems, they engage in a cyclical process of generating and validating their model and results. The figure below illustrates a cycle used with elementary students to help organize reasoning in mathematical modeling by using terms comprehensible to young math modelers.

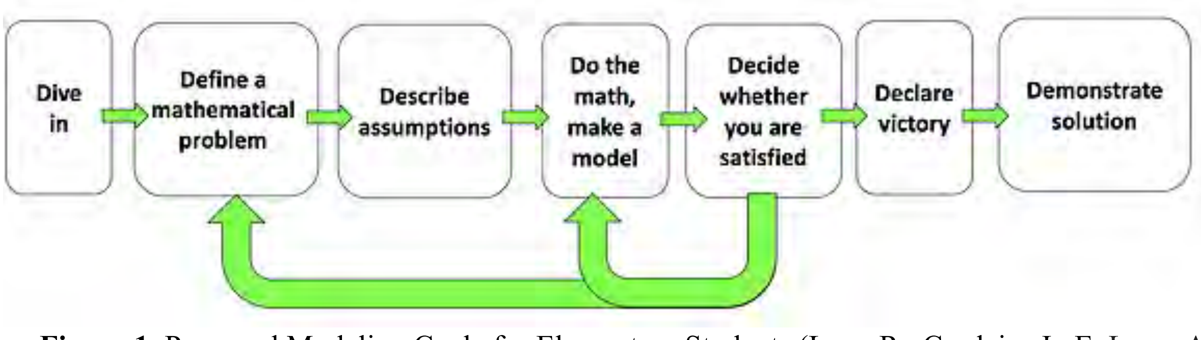

Figure 1. Proposed Modeling Cycle for Elementary Students (Levy R., Cordeiro J., E. Lane, A. Sierra, Sinclair D., Yang L. & Matson K., 2016).

So what does mathematical modeling look like in the elementary grades? One of the ways, the researchers in this working group approached MM in the elementary grades was to immerse students in a real world situation within their local context that was relatable to and personally meaningful. To keep the initial problem open-ended, students were encouraged to develop the habit of mind of being problem posers by identifying the many questions around the real phenomenon, then defining a mathematical problem that can be solved by way of mathematics. After the identification process of the problem, the modeler makes assumptions, eliminates unnecessary information, and identifies important quantities in order to form the model. This real-world model becomes a mathematical model when the processes are replaced by mathematical symbols, relations and operations. It should be noted that there can be several mathematical models for a given real-world situation. Next, the model is solved mathematically and results are translated back to the real-world and interpreted in the original context. The problem solver then validates the model by checking whether the solution is appropriate or reasonable for the purpose. This process of making assumptions, identifying variables, formulating the model, interpreting the result, and validating the model is iterative in nature and is modified or changed and repeated until a satisfactory solution has been obtained and communicated (Blum, 2002). It is important to note that teachers play a crucial role in MM. The teacher must be able to: (a) provide opportunities for students to acquire mathematical competencies and make connections between the real world and mathematics; (b) maintain the high cognitive demand of the MM process; and (c) provide classroom management that is learner-centered (Blum & Ferri, 2009).

Previous work with elementary school children demonstrated it is feasible for them to develop a disposition towards realistic mathematical modeling (Lieven & De Corte, 1997). One of the issues in implementing MM at the elementary level is that MM can be difficult for both teachers and students to implement (Blum & Ferri, 2009). MM can be difficult for teachers to implement as they must be able to merge mathematical content and real-world applications while teaching in a more open-ended and less predictable way (Blum & Ferri, 2009). It can be a challenge for students because each step of the modeling process presents a possible cognitive barrier (Blum & Ferri, 2009). As stated in the Common Core Standards for Mathematical Modeling, *"Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. These real-world problems tend to be messy and require multiple math concepts, a creative approach to math, and involves a cyclical process of revising and analyzing the model"* (Carter et. al., 2009).

In a previous PMENA report, Suh, Matson, Williams and Seshaiyer (2016) reported the challenges and affordances of mathematical modeling in the early grades.

Teacher Challenges.The challenges teachers faced when implementing mathematics modeling in the elementary grades included: a) Novelty and ambitious nature of the *modeling process*-When implementing MM in the classrooms for the first time, teachers found it was difficult to move students through the full process as it was a novel approach and students had never been introduced to creating and validating their mathematical models; b) *Managing discourse*- Another difficulty encountered by teachers in the MM process was in defining their role as facilitators. The teachers commented that "...it is really difficult as a teacher to help students find a direction to go with their solution but not direct or guide them toward a teacher goal."; and c) *Constraints around mandated standards*-Participants acknowledged that MM takes time to implement in the classroom and that additional class time to implement these tasks would be helpful. An additional challenge noted by teachers was that mathematical modeling didn't go the way they expected it to and they wrestled with the need to meet state standards.

Affordances of Mathematical Modeling. The main affordances our teacher-designers mentioned were that mathematical modeling provided opportunity for content to be covered without direct instruction, had interdisciplinary connections, and provided mathematical relevance, and student engagement: a) *Content covered without direct instruction*- When teachers implemented MM in their classrooms for the first time they were amazed at the amount of content that could be covered without direct instruction. Students could see how the mathematics could serve their needs as they used the mathematics they learned while other times, the mathematics related to future learning objectives which allowed them to revisit their model as their learning progressed; b) *Interdisciplinary opportunities*-Another positive take-away from implementing MM in these teachers' classrooms for the first time was how MM created a space where content covered was interdisciplinary connecting to social studies, STEM and language arts; c) *Relevance*-By providing authentic tasks for students to grapple with through the MM process, mathematics became relevant to the students; d) *Student engagement*-A number of our teachers indicated how engaged their students were in their MM tasks. Mathematical modeling inspired these teachers' endeavors and provided pictures of practices that served as the proof of concept they needed to sustain their professional commitment to mathematical modeling.

Support Teachers Need. The three main areas of support teacher-designers requested were access to MM resources, pictures of practice, time and collaboration with like-minded teachers: a) *Resources and pictures of practice*-Teachers indicated a desire to use MM in their classrooms but indicated a need for a bank of open-ended MM lessons and new ideas for continuing to create these lessons; b) *Time*-Teachers expressed the need for more time to work through and become comfortable with implementing the modeling process in their classrooms. Teachers noted it was only in working through the MM cycle several times that they felt comfortable with the process and felt their students were able to understand the whole MM process; and c) *Teacher collaboration*-Teachers indicated a desire to continue to work with a cohort to build MM lessons; to observe other teachers implement MM in their classrooms; and to work alongside a colleague who valued MM and with whom they could share ideas.

Other related research the team will share include, Carlson, Wickstrom, Burroughs & Fulton's (2016) work, *A Case for Mathematical Modeling in the Elementary School Classroom*, where they provide a teaching framework for MM using the "organize - monitor - regroup" cycle to support the teachers' work in engaging young students in modeling. Wickstrom, Carr and Lackey (2017) will showcase an engaging article using mathematical modeling to explore Yellowstone National Park. Suh, Matson, & Seshaiyer (2017) will also share ways in which mathematical modeling enhanced students creativity, collaboration, critical thinking and communication skills and exposed students to interdisciplinary themes of service learning and STEM integration.

Plan for Active Engagement of Participants

The working group will meet three times during the conference and virtually during the course of one year. In each session, PMENA members will engage in mathematical modeling while sharing their perspectives in teaching and learning mathematics, considering synergistic areas fruitful for future research and practice, and finding collaborators within our group.

Session 1: Exploring the Nature of Mathematical Modeling in the Early Grades

The first session will focus on better understanding the nature of mathematical modeling in the elementary grades while considering the student perspective and recognizing the importance of teachers knowing their students and the contexts that are meaningful to their students. We will examine how mathematical modeling used by K-6 teachers demonstrates the interdisciplinary nature of mathematical modeling, the diversity of mathematical approaches taken by student modelers, and the multiple pathways the teacher can use to elicit students' mathematical thinking. Exemplar tasks that emphasized local contexts and tapped into students' funds of knowledge and student artifacts will be shared to illustrate the child's perspective and the developmental progression. These topics will facilitate group discussions exploring the learning progression for mathematical modeling thinking and habits of mind that can develop for emergent mathematical modelers from an early grade.

Session 2: Identifying the Knowledge of Content and Pedagogy Needed for Mathematical Modeling in the Elementary Grades

In our second session, we will focus on clearly defining modeling teaching practices and competencies needed for mathematical modeling and outlining research goals and objectives to monitor the enactment of these practices. We will detail classroom routines, such as the "organize - monitor - regroup" cycle (Carlson, et al. 2017), and the Pedagogical Practices for Mathematical Modeling (Suh, Matson, & Seshaiyer, in press) as we share designed activities and lesson vignettes to solicit more ideas around high leverage MM teaching practices. We will explore what mathematical knowledge is needed to "successfully" facilitate mathematical modeling tasks in elementary grades.

Session 3: Finding the Synergy Between Mathematical Modeling and the 21st Century Skills Frameworks and PBLs in STEM

The third session will outline several 21st century skill frameworks and teaching approaches and how mathematics educators, researchers and practitioners can find a synergistic way to bring important process skills without overwhelming teachers and students. We will discuss the ways elementary teachers can make connections between the problem-based ways they have engaged students in mathematical modeling and STEM. The teachers are able to take advantage of interdisciplinary opportunities across the subjects they teach and find complementary connections between subjects and common classroom practices that support MM.

Anticipated Follow-up Activities and Goals of Working Group

In the spirit of exploring the theme of the *Synergy at the Crossroads: Future Directions for Theory, Research, and Practice,* each session will engage participants to share their research interests related to mathematical modeling and form groups that might pursue research collaboratively based on the interests of the participants. Some of the questions include:

- What defines successful mathematical modeling lessons at different grade levels?
- What can we learn from teachers who implement MM regularly in their classrooms?
- How can we support teachers enacting MM through lesson plans and other resources?
- How can we map out the learning progression of MM across grade levels?
- How and what can we learn about models elicited from student artifacts from MM tasks?
- What do "successful" modeling practices look like in our elementary mathematics classrooms? How are they similar or different from practices in secondary classrooms?
- What does it mean to "see the math" in the components of mathematical modeling?

- How do teachers select and/or develop modeling problems? How can PLCs or Teacher Study Groups help teachers anticipating how students will answer the MM questions?

Our goal is for the working group leaders to propose an edited handbook or a special issues journal venue for mathematical modeling where participants interested in submitting manuscripts can work together to provide a comprehensive research trajectory documenting the progression of mathematical modeling from emergent levels to more sophisticated levels of modeling.

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MODELS AND MODELING WORKING GROUP

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Modeling continues to be a central feature of mathematical teaching and learning from both researcher and teacher perspectives. It is one of eight core mathematical practices called out in the Common Core State Standards for Mathematics; 2016 APME yearbook was dedicated to the theme of modeling; and international volumes reporting on new developments in Europe are expected out in 2017. The Models and Modeling Working Group at PME-NA has provided a venue for discussing and collaborating on research projects fundamental to this area since the first PME-NA conference in 1978. We propose to convene this Working Group at PME-NA 39 to build on energetic discussions that took place at PME-NA 37 and 38 and to establish a collaborative framework for shared research projects that integrate interested newcomers with established researchers within the Models & Modeling Perspective.

Keywords: Modeling, Problem Solving, Design Experiments

The Models and Modeling Working Group has been a significant presence in the community of PME-NA since the Conference was inaugurated in 1978. It has used the working-group format not only to support substantive research efforts, but also to grow collaboration and mentoring relationships between researchers in this field. An important historical purpose of the Working Group has been to pursue innovations in design-based research – in particular to discuss and extend the ways in which a focus on models and modeling can be used both to support learning in mathematics, science and engineering, and to study such learning processes in action. Moreover, we found that many of our participants in the Group meeting in PME-NA 38 were engaged in work that addressed teacher-identified problems of practice, and we expect that opportunities for such innovations will continue to be foregrounded for participants in this year's meeting.

Early in its history, the Group focused heavily on the design and analysis of particular activities that enabled groups of learners to engage in a deep form of modeling and that produced an auditable trail of thinking, exposing their thought processes to teacher and researcher observers. In this phase of the field's development, a primary effort involved elaborating design principles for these activities as learning environments and documenting the idea development they promoted. Gradually over time, however, researchers associated with the Group have expanded their perspective to consider implementations and curricular sequences that have longer time-duration, and that integrate models and modeling into the experience of learning mathematics in more extensive ways. Several broad patterns in this more extensive and disseminated approach to modeling in the curriculum have emerged, and there is no sense that we have yet exhausted the space of possibilities. These broader perspectives open both exciting opportunities and significant challenges. On the one hand, new questions can be researched, opening the way for new forms of contact and interaction with classroom practice; on the other, the approach raises new challenges at the level of methodology, data analysis, and forms of evidence that are convincing backings for claims about learner activity.

We propose convening the Group at PME-NA 39 to continue work begun at PME-NA 37 and expanded at PME-NA 38. Based on our experience in these previous working group sessions, we propose a work-session structure that can serve two dual purposes: (a) integrating newcomers to Models and Modeling as a research area and (b) making substantive progress in concrete work – planning collaborative research and framing a shared writing project. For this Working Group, these two goals are both essential: we do not propose to gather as a closed expert group (Broadening exposure to this kind of learning design is important to the individuals of the Group as well as to the group as a collective whole). And we also do not aim only at providing an initial introduction to the forms of learning enabled by Model-Eliciting Activities (There are urgent problems and opportunities of research and practice which we see the Working Group as providing essential means for addressing). In the following sections, we provide a very brief overview of the field of research represented by the Models and Modeling Perspective; we outline patterns in research efforts that have extended modeling activities over longer timescales; we identify key themes that the Group will address in its discussions; and we describe our plan of work in detail, illustrating how these goals are addressed as well as how we plan to productively integrate newcomers to the Group over the three working sessions of the Conference.

The Models and Modeling Perspective (M&MP)

For nearly forty years, M&MP researchers and educators have engaged in design research directed at understanding the development of mathematical ideas among groups of learners. A key principle behind this work has been that learners' ideas develop through, and in relation to conceptual entities called *models*, which we define as follows:

> conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently (Lesh & Doerr, 2003, p. 10)

As conceptual systems expressed through representational media, models can provide illumination into how students, teachers, and researchers adapt, formulate, and apply relevant mathematical concepts (Lesh, Doerr, Carmona, & Hjalmarson, 2003). An early finding was that, under appropriate conditions, groups of learners can be supported in rapidly producing and expressing such models. Once expressed, these productions can become objects of reflection by learners and collaborative groups, and they can form the basis for rich exploratory communication. In particular, when individuals and groups encounter problem situations with specifications that demand a model-rich response, their models are observed to grow through relatively rapid cycles of development toward solutions that satisfy these specifications.

It should be noted that the above account is not an account of *development* (of learners), nor even yet an account of *learning*. Instead, it is an account of *idea development*, as observed in the discourse and other representations produced by groups of learners as they iteratively work to mathematize and formulate a solution that meets the needs of a concrete *client* in a realistic setting. This note is important, in order to distinguish the analytical work of the early M&MP from accounts of microgenesis of learning and development, for instance those expressed in the debates about whether learners have "theories" about the world (as held by the so-called "theory-theory," (McCloskey, 1983), or whether their knowledge is better represented as a loose and context-dependent assemblage of knowledge "pieces" (as held by the Knowledge-in-Pieces perspective, (diSessa, 1993; diSessa, Sherin, & Levin, 2016) as well as by some connectionist approaches to modeling knowledge (e.g., Minksy, 1986). The M&MP was initially agnostic to these debates, as the questions at hand had to do with the interactional achievement of small groups when exposed to particular kinds of realistic problem. In fact, it was an important apparent paradox of this research that some features of *idea development* during these activities have appeared to mimic or shadow aspects of cognitive development as reported by Piaget, van Heile, or other accounts (Lesh & Harel, 2003). We defer discussion of the relation of the M&MP to such questions of learning and development until the section on longer instructional sequences on modeling, below.

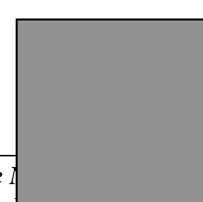
Thus, originally, the M&MP tradition was focused squarely on very local conceptual development (Lesh & Harel, 2003): that is, on investigating the nano-evolution of ideas in teachers and students. Thus, the resources and tools produced were first and foremost designed to *study* idea development (as opposed on one hand to serving teaching or curricular goals and on the other hand to explaining the phenomena of group productions by reference to a theory of individual cognition or learning). The results of this work include a body of *Model-Eliciting Activities* (MEAs), in which students are presented with authentic, real-world situations where they repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to generate a structurally significant product—that is, a model, comprising conceptual structures for solving the given problem. These activities differ markedly from some "problem-solving" settings, which emphasize applications. In contrast, MEAs give students the opportunity to create, apply and adapt scientific and mathematical models in interpreting, explaining, and predicting the behavior of real-world systems (Zawojewski, 2013). Extensive research with MEAs has produced accounts of learning in these environments (Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly, & Post 2000), design principles to guide MEA development (Doerr & English, 2006; Hjalmarson & Lesh, 2007; Lesh, et. al., 2000; Lesh, Hoover, & Kelly, 1992) and accounts and reflections on the design process of MEAs (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008).

Example MEA: The Shadows Problem

Students are usually introduced to the Shadows Problem by reading a "math rich newspaper article" that describes an exhibit at a nationally-known children's museum. The newspaper article focuses attention on the optical phenomena of the exhibit and places it in a mathematical and historical context. The students' challenge in the Shadows Problem is to assist exhibit designers in a local museum who want to create a similar experience. Thus, these 'clients' are interested in ways in which the optical perception one particular shape (a square) can be produced through shadows cast by different shapes and light sources. Below is the statement of the problem as given to students:

> A local museum is interested in building an interactive show on the topic of optical illusions. Inspired by a shadows exhibit that they saw at the Indianapolis Children's Museum (ICM), they are interested in creating something similar for one of their show's stations. The exhibit at the ICM used a small flashlight that had a point source of light, and it showed surprising shadows that different shapes could cast. Their museum designers' goal for this station is to make square shadows using a point source of light and shapes like the ones shown below (similar shapes are included in your packet). For the purposes of the exhibit, the "stranger" the shape that can cast a square shadow, the better.

> Your Task: Write a letter to the museum's show designers, explaining, for as many as possible of the shapes: (a) exactly how to hold the light and shapes so that they make square shadows on a wall (like the one shown here), and (b) which, if any, of the shapes



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> can *never* make a square shadow – no matter how you tilt the shape and the light with respect to the wall. If it is impossible to make a completely square shadow with a given shape, explain why. If you have additional ideas about the show, include them as well.

Student groups iteratively develop solutions to this problem in the time allotted—usually 50-60 minutes for this MEA. Afterwards, the teacher may choose to organize a structured "poster session" event. In one version of this activity structure, one member of each 3-person group hosts a poster presentation showing the results of their group. The other two students use a Quality Assurance Guide to assess the quality of the results produced by other groups in the class. These instruments are submitted to the teacher and contribute to assessment in various ways, providing evidence for the achievements of both individuals and groups.

MEAs like the Shadows problem present learners with situations in which familiar procedures and constructs are *applicable* but also *insufficient*. That is, on the one hand they are accessible to learners from a wide range of levels of ability, experiences, or knowledge (upper elementary school through graduate school). On the other hand, learners encountering these problems find that they have no ready-made solution they can apply to address the client's needs. As a result, learners engage in sense-making and solution-construction processes that put them off balance in comparison to typical school-mathematics tasks. Indeed, this uncertainty is part of the design of MEAs, illuminating fundamental conceptual issues associated with the core mathematical structures involved.

MEA Design Principles

Historically, as individual MEAs emerged, an intense period of design research ensued to establish these activities as a compelling genre of learning tasks that would (a) stimulate mathematical thinking representative of that which occurs in contexts outside of artificial school settings (Lesh, Caylor, & Gupta, 2007; Lesh & Caylor, 2007); (b) enable the growth of productive solutions through rapid modeling cycles; and (c) leave behind researchable traces of learners' ways of thinking during the process. This line of work produced the notion of Thought-Revealing Artifacts and Model-Eliciting Activities (MEAs) (English et al., 2008; Kelly & Lesh, 2000; Kelly, Lesh & Baec, 2008). The success of MEAs as research tools was both enabled by and illustrated through the articulation of a set of six design principles for such activities (Hjalmarson & Lesh, 2007; Lesh & Harel, 2003; Lesh et al., 2000); these principles indicate the key structural and dynamical elements in MEAs as contexts for problem solving. Table 1, below, also indicates "touchstone" tests for whether each of these six principles has been realized in a given implementation setting.

Table 1: Six Design Principles for MEAs

Principle	Touchstone Test for its Presence
Reality Principle	Students are able to make sense of the task and perceive it as meaningful, based on their own real-life experiences.
Model Construction Principle	To solve the problem, students must articulate an explicit and definite conceptual system (model).
Self-Evaluation Principle	Students are able to judge the adequacy of their in-process solution on their own, without recourse to the teacher or other "authority figure".
Model Generalizability Principle	Students' solutions are applicable to a whole range of problems, similar to the particular situation faced by the "client" in the MEA.
Model-Documentation Principle	Students generate external representations of their thinking during the problem-solving process.

Simplest Prototype Principle	The problem serves as a memorable representative of a kind of mathematical structure, which can be invoked by groups and by individuals in future problem solving.

Nested Levels of Modeling: Multi-Tiered Design Research

In parallel with learner-focused research using MEAs, researchers also have observed that *teachers'* efforts to understand their students' thinking involve yet another process of modeling: In this case, teachers engage in building models of student understanding. Although these teacher-level models are of a different category from student-level models, students' work while engaged in MEAs does provide a particularly rich context for teachers' modeling processes. Following this line of inquiry, the M&MP community has also produced tools and frameworks that can be useful to teachers in making full use of MEAs in classroom settings, while also providing *researchers* with insights into *teachers'* thinking.

Finally, at a third level of inquiry, researchers' own understandings of the actions and interactions in *curricular activity systems* (Roschelle, Knudsen, & Hegedus, 2010) involving students, teachers, and other participants in the educational process can also be studied through the lens of model development. Multi-tier design experiments in the M&MP tradition have done precisely this, involving researcher teams in self-reflection and iterative development as well (Lesh, 2002). Therefore, multi-tier design research involves three levels of investigators— students, teachers, and researchers—all of whom are engaged in developing models that can be used to describe, explain, and evaluate their own situations, including real-life contexts, students' modeling activities, and teachers' and students' modeling behaviors, respectively.

From Single Activities to Curricular Materials Supporting Modeling at Larger Timescales

Over the past 10 years, M&MP researchers have continued this direction of work in their own teaching and in partnerships with K-12 classroom teachers. Within the domain of statistical thinking in particular, this effort has produced resources and tools sufficient to support *entire courses* in several versions and including accompanying materials related to learning and assessment aimed at both student and teacher levels. Because the courses supported by these materials were designed explicitly to be used as research settings, for investigating the interacting development of students' and teachers' ways of thinking, the materials were modularized so that important components could be easily modified or rearranged for a variety of purposes in different implementations. In particular, by selecting from and adapting *the same basic bank of materials*, parallel versions of the course have been developed for: (a) middle- or high-school students, (b) elementary and secondary preservice teachers, and (c) in-service teachers. When these courses have been taught by M&MP researchers familiar with the underlying theory, they have produced impressive gains (e.g., Lesh, Carmona, & Moore, 2009).

Moving beyond a single MEA and sustaining modeling over longer timescales will bring M&MP research more directly into contact with theories of learning and development (including the questions mentioned in the first section, above). This will spur theoretical and methodological reflections on topics such as the relations of *models* to constructs seen to mediate perception and understanding in emerging experts (e.g., *coordination classes* [diSessa & Sherin, 1998] or *disciplined perception* [Stevens & Hall, 1998]).

Approaches to Longer-Timescale Modeling Research

At a more pragmatic level, however, ongoing investigations are exploring ways of integrating M&MP work into larger time-scale curriculum structures. And as might be expected, different approaches to sequencing and integrating modeling activities within curricular structures to expand the scale of modeling reveal different aspects of students' and teachers' thinking. In broad terms, we distinguish these approaches on the basis of the relationship that they create between individual MEAs and the structure of the larger instructional unit or course (Eames, Brady, Jung, & Glancy, in review).

In a common form of initial curricular integration, MEAs are deployed in a *series*, allowing students and the teacher to experience the type of learning characteristic of MEAs several times. Here, each new iteration offers opportunities for teachers to revise their approaches to facilitating MEAs, based on their emerging expectations of how students will engage with problems, their experiences of implementing them, and their knowledge of students' mathematical thinking.

This "series" experience of modeling through MEAs is how many teachers first experience activities from within the M&MP. However, our research has led us to ask how ideas and practices that emerge in these settings can build toward supporting longer-term disciplinary goals, norms, and concepts. These are critical issues in helping modeling to move beyond "island" activities and impact the core experience of mathematics learning for students. Here, we identify two approaches to modeling at this course-level scale that have emerged.

In the first, students' work in MEAs is seen to produce rich but idiosyncratic mathematical ideas and products that need to be *unpacked* and placed into relationship with each other and with more canonical concepts, practices, and procedures from the discipline. Thus, a family of activities is elaborated around the MEA, focusing on refining and extending student modeling work. These "Model Development Sequences" or *MDSs* (e.g., Brady, Eames, & Lesh, 2015; Brady, Lesh & Sevis, 2015; Doerr & English, 2003; Lee et al, 2016; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003) provide opportunities for classroom groups to reflect on and refine the thinking they have done in MEAs. Research over the last 15 years on MDSs has helped to elaborate a suite of *tools* and *activity types* along with cases of their use to support model exploration, extension, adaptation, and analysis.

An alterative approach to course-level modeling has emerged through work in engineering education. Hamilton, Lesh, Lester, & Brilleslyper (2008) have argued that MEAs themselves connect mathematics and engineering, and many researchers have found success using MEAs to develop engineering concepts in K-12 and undergraduate settings (e.g., Diefes-Dux, Moore, Zawojewski, Imbrie & Follman, 2004, English & Mousoulides, 2011, Moore, Miller, Lesh, Stohlmann, & Kim, 2013, Yildirim, Shuman, & Besterfield-Sacre, 2010). In a project coming out of this tradition, larger-timescale modeling units were developed with an engineering focus, engaging learners in extended inquiry on a theme. This engagement was patterned on the shape of inquiry cycles, similar in nature to the modeling cycles that characterize student work within a single MEA, though at a much larger time scale. Thus, this approach aims to scale up qualities of the MEA experience itself to help structure the dynamics of a project that extends over weeks of sustained work. In doing so, researchers hope to see how model development continues over longer time scales along as well as how effectively the MEA design principles extend to larger course or time units.

Research and Discussion Themes to Guide the Working Group

Our working group in Tucson brought together 25 participants from the US, Canada, and Mexico. Participants were interested in wide range of grade levels, engaging in a diverse array of modeling projects. In their work, they applied a variety of interpretive lenses and frameworks to the activity of modeling. Throughout the conference, and particularly on the final day, we had significant participation from teachers and researchers from Mexico, interested in bringing both materials and research publications from the M&MP to Spanish-speaking teachers and researchers in Mexico and across Latin America. Leadership from the Tec of Monterrey and supporting enthusiasm from both Canada and the US gave evidence that this could be a strong area for future work.

One of the principal efforts of the 2016 working group was to identify possible topics and areas for future collaborative research and writing, as well as to imagine structures that would support and encourage such collaborations. At the level of research themes, a large number of possible synergies were identified, including the following questions or areas of study. This list was not generated to be exhaustive but rather to reflect identified common areas of interest during the immediate discussions of the working sessions themselves.

Student-Level Modeling

How might we gain analytic purchase on topics such as students' motivation and engagement; creativity and innovation; patterns in discourse and interpersonal dynamics in small group work; and student's epistemologies and beliefs about the nature of mathematics?

Teacher Professional Development and Implementation Issues

What knowledge and competencies do teachers need, to facilitate modeling activities? To pursue larger-timescale modeling sequences along any of the three approaches described above? How do modeling practices change in the presence of software environments that provide dynamic representations? How do the MEA design principles help teachers to conceptualize STEM integration in math? What is the impact of decisions to sequence model-based activities in different ways, and to sequence modeling activities with more traditional activities in different ways? What are the differences in student modeling behavior across different grade bands? Is group work necessary for modeling activities?

Social Justice, Equity, and Cultural Relevance

What community and cultural connections are possible within modeling task? How can modeling tasks be designed or adapted to increase cultural relevance? What connections can MEAs have with social justice concerns? Is it possible to establish client-type relations with community members to structure modeling tasks?

Classroom Structures, Interactions, and Behavior

How do individual and group learning interact? How do enactments of "the same" MEA in different classrooms illuminate classroom 'cultures' and local communities? What mathematical practices and competencies do students take away from repeated exposures to modeling tasks? How do elementary-school, middle-school, and high-school classroom contexts differ with respect to MEAs (including procedures and supports for setup, facilitation, and poster-session sharing)? How do age differences affect the design and implementation of activities that surround MEAs?

The Development of Mathematical Concepts and Practices

How do we describe the relation between modeling sequences and standards for content and practices? Is it possible to use MEA tasks repeatedly (multiple encounters with the same task) to gauge changes in the mathematics tools and content that learners bring? How does developing a solution to an MEA relate to content mastery?

Assessment Issues

Can modeling tasks be a context where it is possible to discern evidence not only of students' appropriation of the modeling practice but also of connections between this practice and other practices? Similarly for connections between content areas? What evidence do teachers need to evaluate student learning/growth in MEAs?

These themes and questions, illustrate the breadth of the common ground that 2016 participants found in their research interests. This has prompted us to organize the 2017 meeting around elaborating specific shared research and writing projects.

Concrete High-Leverage Actions Identified by the Group as Desirable Projects

In addition to identifying themes of common interest, participants at the Tucson meetings identified activities and projects to pursue in future working group sessions, including:

1. Opportunities to familiarize themselves with more of the curricular materials and research from the M&MP (Concretely: establishing guided pre-reading for before the working group, along with opportunities to discuss during the conference, though perhaps extending outside of the three time-limited sessions)
2. Time and supports to establish a collaborator network for one or more shared writing projects. Two ideas discussed were:
3. Implementing the same MEA across two or more contexts and collaboratively analyzing the data generated.
4. Formulating a successor volume to *Beyond Constructivism*
5. Related to project 2a, the group identified a need for identifying solutions to institutional barriers to cross-institutional collaboration and exploring/proposing solutions (specifically, writing IRB proposals that may allow data sharing, including video)
6. Creating an online repository of materials and tools for both research and teaching in the M&MP.
7. Developing translations and localizations of materials and research papers into Spanish, to support the growth of research and practice in Mexico and elsewhere in Latin America.

In the time since the PME-NA 2016 meeting, the Group has gotten a preliminary start on possible project 2a, above, supporting planning and implementation of an MEA by a participant at the Tucson sessions, collecting data to allow cross comparisons with two other implementations with contrasting learner populations. Analysis of these data will provide a ‘proof of concept’ for this proposed line of collaborative research, as well as offering an indication of what is to be gained in such a comparative study. Further work to test the viability of other suggested strands of work will continue in advance of the Indianapolis meeting, so that the Group can “hit the ground running.”

Working Group Session Outline: Advancing our Agendas While Building Community Capacity

The working group will meet in three sessions over the course of the conference. As preparatory work continues, the precise contents of each of these sessions will be more clearly defined, but the broad outlines are given in this section.

The concrete research projects identified in the section above will be a primary focus of the 2017 meetings. During the summer months, the facilitator group will meet to plan activities to address the desires expressed by participants in Tucson. Then, approximately 2 months before the Conference, we will send out a draft agenda for comment to our email list of all Tucson attendees, plus collaborators and researchers in the field that have participated in the past (approximately 60 names). This will enable final refinement of the agenda as well as circulation of the “pre-reading” requested in point #1, above. Collaboration prior to, during, and after the Conference will be supported by a Google Group (Models and Modeling), a Working Group wiki, and a website for hosting materials and collaborative products (https://sites.google.com/view/modelsandmodelingpmena/home).

In addition to these project goals, our agenda will also ensure that we serve the “building capacity” objective of the working group, inviting newcomers to the M&MP and providing them with an engaging introduction to the experience of MEAs.

Session One

The capacity-building objective for this session is to introduce newcomers to the M&MP

tradition and its approach to research. At the same time, more experienced participants want to get an early start on defining collaborative work toward the several high-leverage projects that the group has identifies. Thus, in Session 1 we will be divided into two groups:

The Newcomer Group will have the opportunity to experience an MEA as a student. We will use the Pelican Nesting Ground MEA that we used in 2016, to enable newcomers to discuss their experience with any participant from the Tucson meeting, ensuring a common ground of experience. This Newcomer Group will be facilitated by a researcher who has experience with this MEA and has seen a variety of learner responses to the task. By the end of Session 1, each modeling team within the Newcomer Group will have developed their solution and prepared a presentation for Session 2.

In the meantime, the More Experienced Group will gather to discuss collaborative efforts toward the projects of Shared Implementation of a particular activity or joint production of an edited volume. Email discussions will likely lead to a group-level decision about which of these two possible projects to pursue during the Conference. By the end of Session 1, each project team of the More Experienced Group will have prepared a plan for collaborative work to be presented in Session 2.

Session Two

The first half of this day will be dominated by presentations. The Newcomer and More Experienced groups will be re-joined as a larger community. Newcomers will present their solutions to the MEA as well as reflect briefly on the experience and their modeling process. The More Experienced Group teams will present the proposed collaborations for comment and review by the larger group. By the end of this day, newcomers will select a collaboration group to join with, with either the objective of participating in the project or getting a deeper understanding of the practice of M&MP research by watching the discussion unfold. By the end of Session 2, small working groups will have formed and begun to dig into their project work.

Session Three

Project working groups will continue to press forward on their agendas during this session, which we leave flexibly structured. By the end of the session, each group will publish to the wiki their "roadmap" for collaborative work. Time permitting, we will have working groups report out verbally to allow other groups visibility into progress and plans.

Throughout our agenda for the Conference sessions, we will be particularly attentive for opportunities to foster mentoring relations between established names in the M&MP tradition and newcomers. One of the signature strengths of the M&MP has been the generosity of leading names in the field, who act as advisors to the Working Group as well as periodic participants. We anticipate attendance by one or more of these senior leaders in Indianapolis, and we will take the opportunity to organize activities for younger scholars to interact with and learn from them.

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SPECIAL EDUCATION AND MATHEMATICS WORKING GROUP

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This working group was formed to create collaborative opportunities for researchers/practitioners from math and special education, with an intention to move forward the teaching and learning of mathematics involving students with learning disabilities/difficulties in mathematics (LDM). This working group was driven by following premises: (1) students with LDM are capable of and need to develop conceptual understanding of mathematics, and (2) special education instruction needs to transition toward this focus. Participants will (a) continue to develop and refine collaborative research agenda for the group, (b) brainstorm specific research questions that will address that agenda, and (c) continue the dissemination effort.

Keywords: Instructional Activities and Practices, Learning Trajectories, Mathematical Knowledge for Teaching, Equity and Diversity

About five to ten percent of school-age children have been identified as having mathematics disabilities, whereas students whose math performance was ranked at or below the 35 percentile are often considered at risk for learning disabilities or for having learning difficulties in mathematics (LDM). Recent National Assessment of Educational Progress (NAEP, 2015) data indicates that mathematics performance gap between students with disabilities and their same-age peers has not been closing—in fact, it seems to get wider. The purpose of our working group (WG) is to explore issues of research around the intersection of mathematics education and special education. Substantial work exists that focuses on development of mathematical cognition and reasoning of students in general education. However, much less is known about the mathematical development of students with disabilities or how to support the learning of these students. The absence of research addressing this subset of students may be due in part to theoretical orientation of the field of special education, which emphasizes explicit teaching of targeted skill-sets, rather than exploratory models such as inquiry-based learning. In addition, it seems that many special educators consider "research-validated" curriculum (or intervention) to largely include studies that follow large-scale, Randomized Control Trial (RCT) approaches (Woodward & Tzur, 2017). Often, the role of pedagogical content knowledge of interventionists or how the intervention helps students construct the concept or develop conceptual understanding of mathematical ideas are lost in "data crunching" of RCT studies.

This WG was formed five years ago to create sustainable opportunities for collaborative work between researchers and practitioners from both the fields of math education and special education, with an intention to promote the teaching and learning of mathematics education involving students with LDM. Understanding how students with disabilities /difficulties develop mathematics concepts and skills, an important dimension of the psychology of mathematics education, has several implications for both research and practice.

First, practitioners in both general and special education can benefit from diverse perspectives and gain a richer understanding of how students with LDM learn mathematical concepts. Second, active study of the development of mathematics concepts and skills for students with disabilities provides both researchers and practitioners with mechanisms for moving toward a methodological focus on pedagogy rooted in assessment of what students with disabilities are capable of learning.

For the purposes of continuing the conversation around mathematics in special education, this group is concerned with students who have significant issues with mathematics, including:

- students with learning disabilities specific to mathematics (MD);
- students with cognitive differences in how they understand and process numbers;
- students who are placed in special education and have difficulties with mathematics.

We refer to these students as having learning disabilities or difficulties in mathematics (LDM) in the remainder of this paper. while acutely recognizing those difficulties may reflect, in part, inadequate teaching (Tzur, 2013).

History of the Working Group

Our PME-NA/PME working group has met five times; each year our group had good participation of both returning and new members. In 2012, 15 researchers (faculty and graduate students) and 2 practitioners met during PME-NA in Kalamazoo, MI. This first meeting was specifically focused on better understanding mathematical learning disabilities (MD). The working group began with a discussion of the issues around identification and definition of MD. In particular, the group discussed the unique characteristics of students with MD (e.g., slow speed of processing despite average reasoning; fundamental issues with number sense; over learning of procedural knowledge at the expense of mathematical reasoning) and implications for instruction and assessment. We took up a theoretical stance that positioned disability as an issue of diversity and considered the origin of the disability as the inaccessibility of instruction rather than a defect within the individual. Most importantly, within this stance on diversity we agreed to a critical premise that underlies our work, namely, rejection of deficit models of those capable learners and focusing on how research and practice can be integrated to support their mathematical abilities (Woodward & Tzur, 2017). Members shared videotapes of various students with MD solving problems in assessment and teaching situations and discussed the need for teachers to target and teach toward the specific mathematical strengths and weaknesses demonstrated by the student. We further discussed at what point(s) the learning paths of students with MD may differ from what is documented among students in general education, how existing developmental trajectories may or may not fit the population of students with MD, and the need to expand or further document current trajectories to include students with MD. Moreover, discussions focused on issues surrounding motivation related to the design and use of instruction, mathematical tools, and mathematical tasks. A rich discussion was held concerning the nature and sequencing of mathematical tasks, the use of concrete and pictorial representations and the extent to which they are (or not) supportive of the abstraction of mathematical concepts for this population. All WG participants agreed about the need for increased research to inform the creation of adequate, empirically grounded, practitioner tools and resources.

During the first year of our WG, our focus was specifically on MD- those students with a biological and cognitively-based difference in how their brain processes numerical information. Based on our discussions during the first year of our working group we decided to expand from a narrow focus on MD to a more inclusive focus on students in special education who struggle with mathematics (i.e., learning disabilities/difficulties in mathematics, LDM). This stance not only avoids the definitional issues at the forefront of the field of special education (i.e., the lack of assessments to accurately identify students with MD and the resulting conflation of low achievement and MD) but also more accurately reflects the diversity of interests of the members of this group—and how their own work seemed to provide a way forward.

In the second year of our WG (2013), 14 participants focused on collaboration that was a result of the progress made during the first year. Specifically, two faculty members worked together on a teaching experiment about fraction knowledge, a compliment to a 2012 funded National Science Foundation (NSF) CAREER project (Hunt, 2012). Their collaboration resulted in each bringing unique expertise; the mathematics education scholar brought insight into the mathematical thinking of the student, while the special education scholar brought insight into learning differences and opportunities. The goal of the teaching experiment was to document how the foundational scheme of unit fractions (1/n) evolves in the mathematical activity of two cases of students with learning disabilities. The students' evolving conceptions were supported by constructivist-oriented pedagogy. Video data segments (i.e., each girl's conceptualization of the multiplicative nature of and inverse relation ($1/m > 1/n$ if $m < n$) among unit fractions; the girls' solutions to novel problems) from this project served as starting points for discussions in the subsequent PME-NA working group meeting. Specifically, WG members used the video segments and descriptions of the collaboration as a springboard for discussing possible research questions and methods of data analysis to employ in future, collaborative work. The proposed WG is designed to foster further collaboration of this nature.

In 2014, the WG met at the joint PME and PME-NA conference in Vancouver. We continued to expand (25 members), by the opportunity to include members from the international group of PME. During that meeting, two working group members (one from math education and one from special education) shared a multiplicative reasoning assessment tool resulting from their NSF-funded research project (Xin, Tzur, & Si, 2008). Upon examining this instrument, the WG discussed alternative ways for assessing students with LDM and implications for intervention development. Our collaboration in Vancouver yielded two main accomplishments. First, as a group we identified three research subgroups: (a) cognitive characteristics of students with LDM, (b) interventions for students with LDM, and (c) teacher preparation or professional development, that represented the interests of the members. Each research subgroup identified pertinent research questions and an agenda for further collaboration.

Second, as a group, we proposed an idea for developing a proposal for a special issue to be published in an influential special education journal to address the research around the intersection of math and special education. Later in the year, members of the working group developed the proposal for the special issue and worked extensively with the editors of a special education journal *Learning Disabilities Quarterly* (LDQ). The quality of our proposal led to the acceptance of this special issue proposal by the co-Editors of LDQ. Two members from the working group served as co-Guest Editors of this special issue and identified potential contributing works from the working group members. In addition, we invited a well-known scholar from the field of special education for co-authoring a commentary paper as part of this special issue. To date, all five papers of this special issue series, as well as an introduction and final commentary papers (Xin & Tzur, 2016; Woodward & Tzur, 2017) have been recently published in the 2016 and 2017 issues of LDQ.

In 2015 PME-NA, we continued and expanded collaborations between members of this WG, by focusing discussions around two central themes: (a) math concept development and corresponding methodologies for studying its emergence in students with special needs, and (b) framing research questions and designing a research plan around this topic. We invited interested researchers and educators to the WG sessions. We had several new members joining, including international scholars. During that meeting, equity was raised as a new focal point. Following 2015 PME-NA, two members of this working group, along with a new member, formed a new collaboration; and they have been working on writing a practitioner piece pertinent to differentiating instruction for diverse students—that article has been accepted to MTMS.

In 2016, the Math and Special Education Working Session met twice during PME 40 held in Australia. Nine people from five continents attended these sessions, while another three people actively participate in our projects throughout the year. On the first day there were several new members to the group. Thus, we introduced each other and explained the history and goal of the working session. We then discussed a project that we started at PME 38 and have recently concluded, namely, the special series of *Learning Disabilities Quarterly* (LDQ). Our discussions yielded a number of questions that we would like to explore (e.g., challenges and strategies of collaborative work between professionals/scholars from the field of math education and the field of special education) and consider questions from an international perspective.

On the second day of that PME working session, we discussed our next big project for the group. We decided to produce a book that teacher educators can use for teaching undergraduate and graduate students about the intersection between mathematics education and special education. We are committed to working on this book project throughout the year and look forward to discussing our progress on this project at PME-NA-39 in Indianapolis, Indiana.

Issues Relating to Psychology of Mathematics Education

Historically, special education researchers and teachers focused almost exclusively on students' mastery of procedural skills, such as basic number combinations and ability to execute mathematical algorithms (Jackson & Neel, 2006; Fuchs et al., 2005; Geary, 2010; Swanson, 2007; Kameenui & Carnine, 1998). A recent literature review comparing instructional domains for students with disabilities found that the majority of special education research addressed basic computation and problem solving, with the primary focus placed on mnemonics, cognitive strategy instruction (e.g., general heuristic four-step strategy: read, plan, solve, and check), or curriculum-based measurement (Van Garderen, Scheuermann, Jackson, & Hampton, 2009). Instructional practices either focused on task analysis (breaking up skills into decontextualized steps that need to be memorized and followed), flash cards, or general heuristics that do not help with domain knowledge learning and concept development (Cole & Washburn-Moses, 2010). In particular, the procedure-driven instruction and primary focus on rote memorization skills seem to result in students' incomplete and inaccurate understanding of fundamental mathematical concepts, as well as a lack of retention and/or transfer (Baroody, 2011).

Importance of Both Conceptual and Procedural Knowledge

Crucial for rich mathematical understandings that enable retention and transfer of fundamental concepts is the iterative development of conceptual understanding along with procedural proficiency (Rittle-Johnson, Siegler, & Alibali, 2001; Rittle-Johnson & Koedinger, 2005). Rittle-Johnson and Alibali (1999) noted that conceptual knowledge supports procedural generalization. In particular, conceptual knowledge could aid children in mindfully avoiding the use of procedures that fail to work in novel situations. Additionally, an ability to understand and manipulate different mathematical representations to conceptually navigate a mathematical context contributes to conceptual understanding and procedural skill (Ball, 1993; Kaput, 1987; Rittle-Johnson et al., 2001). It seems that any investigation into mathematical cognition, whether related to disability or not, must fundamentally engage with issues of conceptual understanding (Hunt & Empson, 2014).

A focus on procedural skills limits students with disabilities' access to the general education curriculum, which is a requirement of the Individuals with Disabilities Educational Improvement Act (Maccini & Gagnon, 2002). In mathematics, access to the general education curriculum means addressing problem-solving, mathematical modeling, higher order reasoning, and algebra readiness as required by the new Common Core Standards (CCSSI, 2012). To accomplish these Standards, mathematics educators need to actively engage students in making conjectures, justifying and questioning each other's ideas, and operating in ways that lead to deep levels of mathematical understanding (Kazemi & Stipek, 2001; Lampert, 1990; Martino & Maher, 1999; Yackel, 2002).

Pedagogy Based on Conceptual Diagnosis

A pedagogical approach to be explored and advanced during this WG's meeting is one that focuses on *promoting conceptual learning in students with LDM*. This approach is rooted in a constructivist stance (Piaget, 1985; von Glasersfeld, 1995), particularly the notion of assimilation, which stresses the need to build instruction on what students already know and are able to think/do. That is, teaching needs to be sensitive, relevant, and adaptive to students' available ways of operating mathematically (Steffe, 1990). To this end, teachers must learn how to: (a) diagnose students' available conceptions, and (b) design and use learning situations that both reactivate these conceptions and lead to intended transformations in these conceptions.

Building on Simon (2006)'s core idea of hypothetical learning trajectories, Tzur (2008) has articulated such an adaptive pedagogy, which revolves around the Teaching Triad notion: (a) students' current conceptions, (b) goals for students' learning (intended math), and (c) tasks/activities to promote progression from the former to the latter. Key here is that in designing every lesson one proceeds from conceptual diagnosis of the mathematics students are capable of thinking/doing. That is, assessment methods need to focus on dynamic (formative) inquiry into student understandings, as opposed to on testing correct and incorrect answers per se. Such day-to-day diagnosis, which a teacher conducts via engaging students in solving tasks and probing for their reasoning processes, gives way to selecting goals for students' intended learning. Building on such diagnosis, a mathematics lesson begins with problems that students can successfully solve on their own, which Vygotsky (1978) referred as the Zone of Actual Development (see also Tzur & Lambert, 2011). Recent studies of mathematics teaching in China (e.g., Jin, 2012) revealed a strategic, targeted method, *Bridging*, which is geared specifically toward both: (a) bringing forth mathematical conceptions the teacher supposes all students know, and (b) directing their thinking to the new, intended ideas.

Exemplar Research Activities with Students with LDM

Multiplicative Reasoning Project. From 2008 to 2015, two members of this working group (one from math education and one from special education) have been working collaboratively on a federal funded grant project (Xin, Tzur, & Si, 2008). This project integrated research-based practices from mathematics education and special education and was aimed to promote multiplicative reasoning and problem solving of elementary students with LDM. As an outcome of this collaborative project, the research team has developed an intelligent tutor, PGBM-COMPS. The intelligent tutor draws on three research-based frameworks: a constructivist view of learning from mathematics education (Steffe & D'Ambrosio, 1995), data (or statistical) learning from computer sciences (Sebastiani, 2002), and Conceptual Model-based Problem Solving (COMPS) (Xin, 2012) that generalizes word-problem underlying structures from special education.

Rooted in a constructivist perspective on learning (Piaget, 1985; von Glasersfeld, 1995), the PGBM part of the intelligent tutor focused on how the aforementioned student-adaptive teaching approach, which tailors goals and activities for students' learning to their diagnosed available conceptions, can foster advances in multiplicative reasoning. This approach eschews a deficit view of students with learning disabilities. Rather, it focuses on and begins from what students do know and uses task-based activities to foster transformation into advanced ways of knowing. On the other hand, intelligent computer systems can play an important role in students' learning by effectively modeling their thinking and dynamically recommending tasks tailored to their conceptual profiles. Going hand-in-hand, the COMPS part of the program (Xin, 2012) generalizes students' understanding of multiplicative reasoning to the level of mathematical models. At this stage, students no longer rely on concrete or semi-concrete models for problem solving; rather, the mathematical models directly drive the solution plan.

The collaborative research team has conducted several piloting studies to field test the PGBM-COMPS intelligent tutor with elementary students with LDM. The preliminary studies have shown promising results—participating students with LDM who interacted with this intelligent tutor not only enhanced their problem solving skills on a researcher-designed criterion test but also a norm-reference standardized test (Xin et al., 2017). In addition, the results of these studies have shown success in promoting students' conceptual advances (e.g., concept of number, multiplicative reasoning). In fact, a paper resulting from a randomized control trial (RCT) study of this project has been published as part of the special series (Xin & Tzur, 2016) produced by this working group. As a follow up, the research team has continued on this line of work and recently embarked on a new NSF supported project (Xin, Kastberg, & Chen, 2015), which focuses on additive reasoning.

Fraction Project. Another working group member is documenting learning trajectories of elementary school children with LDM as they come to understand fractions as quantities (Hunt & Empson, 2014). In its first year, the goal of this work was to produce models of children's key developmental understandings (KDU; see Simon, 2006), or critical transitions in how children may conceive of a mathematical idea along a carefully sequenced combination of tasks and varying instructional guidance necessary to grow conceptual knowledge not yet well formed (Daro, Mosher, and Corcoran, 2011). During the proposed WG, this researcher will illustrate varying levels of a students' informal notions of fractions, how the mathematical ideas can be elicited, the grappling of ideas a student might experience, and how more solidified notions of mathematics form through a student's activity (i.e., external manipulation or representation; internal mental activity; actions; strategies for problem solving). This research aims to map trajectories that can assist educators looking to individualize instruction for students with LDM and improve students' conceptual understanding of fractions.

As part of that grant's Year 1 activities, the research team has documented an initial trajectory from semi-structured interviews with fifty 2nd, 3rd, 4th, and 5th graders with LDM. Interviews followed a protocol that established a basis for questioning while maximizing researcher flexibility to fully examine student thinking. Information pertaining to the children's specific needs was also collected to allow for an examination of any trends occurring across similar cognitive profiles. The constructed trajectory is also being tested with a smaller subset of four students and an expanded group of tasks from which the trajectory is based. Data collected from mini-interventions will undergo analysis (Siegler, 2006) to confirm the robustness of the preliminary trajectory. That grant's Year 2 activities will use a teaching experiment methodology, much like those used in the collaborative pilot that resulted from this WG, to document how children with LDM construct conceptions of fractions.

As forgoing, the proposed WG participants will use artifacts from projects described above as possible starting points to further explore possible applications of student-adaptive pedagogy (conceptual diagnosis based) as well as conceptual development trajectories in the design of effective/efficient assessment and intervention programs for students with LDM. We believe such approaches are complimentary and have the potential to constitute core methodologies for teaching and studying the conceptual understandings of students with LDM. In a similar way, this working group provides a venue to give and receive feedback on ongoing cutting-edge empirical work, which is reshaping how students with LDM are researched.

Plan for Working Group

The aim of this proposed WG is to continue the productive collaboration and conversation about the intersection of mathematics education and special education. One of the major goals of this year's working group sessions is to draft a proposal for publishing a book that university professors and teacher educators can use for teaching undergraduate and graduate students about the intersection between mathematics education and special education.

This working group intends to accomplish the following:

- identify the content of the book and identify the primary audience for the book
- develop a table of content for this book proposal;
- Identify book titles with which our book will compete and discuss how our book will differ from these titles. What are the selling points of our book?
- Discuss the logistics of collaborations to carrying out the identified tasks, and
- Discuss further collaborations leading to additional publication and funding opportunities.

These goals are further outlined across sessions as follows:

Session 1: Introduction and Progress-to-Date
GOAL: Identify participant's affinity for established sub-groups and identify potential new sub-group possibilities.

- Prior members will briefly introduce the working group's history and describe the collaborations that have emerged in prior years, supposing everybody reads this proposal.
- Participants will each introduce themselves and their current research and interest in students with LDM.
- New sub-groups may be formed among participants on the basis of common (research) interests to be shared.
- Begin discussion of the tentative ideas for the book.

Session 2: Book Proposal Development
GOAL: Brainstorm and draft book proposal.

- Identify the contents and possible sections of the book as well as its primary audience.
- Develop a table of content for this book proposal and list key words to assist readers
- Identify other book titles with which ours may compete; discuss how our book will differ from these titles. What are the "selling points" of our proposed book?

Session 3: Continue the Ongoing Collaboration
GOAL: Establish next steps for both the sub-groups and the whole working group.

- Within the group or sub-groups we will:
    - Articulate the overarching research agenda for the group/sub-group
    - Articulate potential research questions that the group/subgroup would like to address through collaborative work.
    - Explore a variety of methodological and analytic approaches that can be leveraged to address the research questions.
- Discuss the logistics of collaborations to carrying out the identified tasks, and

- Determine what our next whole group meeting will entail (e.g., PMENA working group for the following year)

Anticipated Follow-Up Activities

Throughout the year, the members of this WG are working on research problems of common interest. They will contribute to a common website in which they will update other members of the WG about the progress of the various research collaborations. We will continue our effort in disseminating the collaborative work resulting from this working group to broaden its impact in the field of mathematics education for students with LDM.

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